The Complexity of Equivalence and Containment for Free Single Variable Program Schemes

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Abstract

Non-containment for free single variable program schemes is shown to be NP-complete. A polynomial time algorithm for deciding equivalence of two free schemes, provided one of them has the predicates appearing in the same order in all executions, is given. However, the ordering of a free scheme is shown to lead to an exponential increase in size.

This research was supported in part by the office of Naval Research under contract N00014-76-0018 and by Aarhus University, Aarhus, Denmark.

1. Introduction

Much work in the theory of program schemes has gone into the investigation of decidability properties for different classes of schemes [G,M]. In the cases where a problem is decidable, a natural question is to determine the complexity of the decision procedure. Some of those questions were answered in [CHS] where it was shown that noncontainment and nonequivalence for single variable program schemes and for monadic linear recursion schemes are NP-complete.

In this paper we investigate the complexity of these two problems for the class of free single variable program schemes. The requirement of freedom (i.e. absence of pieces of code which cannot possibly be executed), is a very natural one if we want to consider schemes which are models of real programs. Although most real programs have more than one variable, we show that even in the single variable case the equivalence problem is difficult.

We show that the noncontainment problem for free schemes remains NP-complete. We do not know the complexity of the equivalence problem for free schemes (except that inequivalence is in NP), but we can reduce it to the problem of determining equivalence of acyclic schemes involving only predicates and terminal assignment statements. We present a partial solution to the equivalence problem by showing that if one of the schemes has all predicates appearing in the same order, then there is a polynomial time algorithm. However, we show that there are schemes in which ordering the predicates causes an exponential increase in size, indicating that preprocessing by ordering one of the schemes cannot lead to a polynomial time algorithm.

The paper is organized in 5 sections. In section 2 we introduce the notion of a B-scheme, which is an acyclic single variable program scheme containing only predicates and terminal assignment statements. Section 3 contains the proof that noncontainment for free B-schemes is NP-complete as well as the polynomial time algorithm for the case where one scheme is ordered. In section 4 we present an unordered B-scheme with no small equivalent ordered scheme, and in section 5 we show that equivalence for the full class of free single variable schemes is decidable in polynomial time if and only if the equivalence problem for free B-schemes is decidable in polynomial time.

Although this is a paper about program schemes, some of the results, notably the exponential blow-up in section 4, are of interest in their own right. Since these results are formulated in terms of standard concepts from graph theory, no particular knowledge from program scheme theory is required.

2. Preliminaries

A B-scheme is a labeled rooted dag whose vertices have outdegree 2 or 0. Vertices with outdegree 2 are called tests and are labeled with Boolean variables; vertices with outdegree 0 are called leaves and are labeled by function symbols. One edge from a test is labeled T, the other F. $|S|$ denotes the number of nodes in scheme S. A B-scheme is free if there is no path from the root to a leaf which contains two or more tests with the same label.

Let S be a B-scheme. A B-assignment A (assignment for short) is a mapping from the Boolean variables of S to {true, false}. t(A) is the path constructed by starting at the root and selecting the edge labeled T (F) whenever encountering a test labeled b where A(b) = true (false). The value mapping Val maps pairs of schemes and assignments to function symbols and is defined as follows:

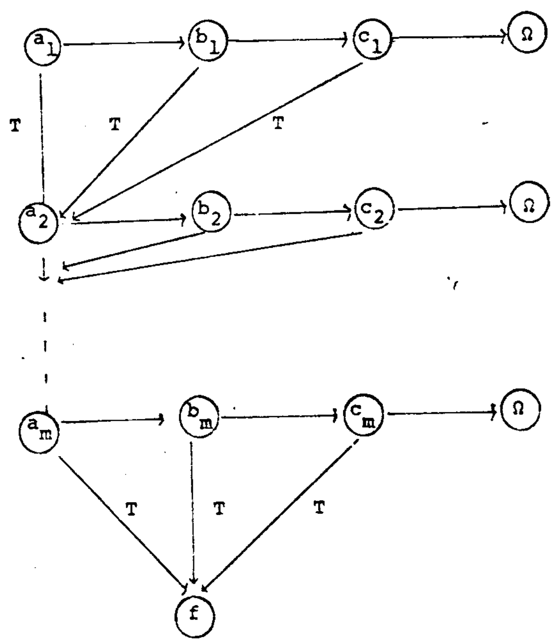
Val(S,A) = f iff the leaf reached by the path t(A) has label f.

The B-schemes S_1 and S_2 are equivalent, ($S_1 \equiv S_2$), if and only if for each assignment A, whose domain contains all Boolean variables in S_1 and S_2, $Val(S_1,A) = Val(S_2,A)$. One function symbol Ω is designated as a special symbol and represents the undefined function. S_1 is contained in S_2, ($S_1 \subseteq S_2$), if and only if for each assignment A whose domain contains all Boolean variables in S_1 and S_2, either $Val(S_1,A) = \Omega$ or $Val(S_1,A) = Val(S_2,A)$.

restriction that $u_1^i = u_2^i = \ldots = u_{p_i}^i = \bar{v}_1^i = \ldots = \bar{v}_{q_i}^i$.

Let $F' = (a_1+b_1+c_1)(a_2+b_2+c_2)\ldots(a_m+b_m+c_m)$. Then the schemes S_1 and S_2 are

S_1:



We note that if the leaves in a B-scheme are replaced by a HALT-statement, then we obtain the switching schemes of [CHS].

3. Containment and equivalence for free B-schemes

Here we show that the containment problem for free B-schemes is NP-complete, and that in certain cases we can find polynomial time algorithms for equivalence.

Theorem 3.1: The set

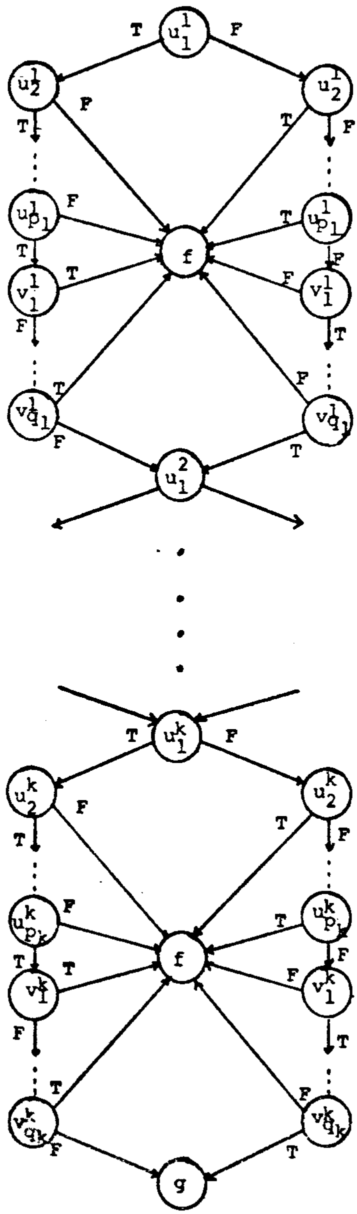
$$BNCONT = \{(S_1,S_2) \mid S_1 \text{ and } S_2 \text{ are free B-schemes and } S_1 \not\leq S_2\}$$

is NP-complete.

Proof: The usual guess and check method shows that BNCONT is in NP.

To show that BNCONT is NP-hard we reduce 3-CNF satisfiability to it. Let F be a 3-CNF formula with variables $x_1,x_2,\ldots x_k$, and let x_i appear uncomplemented in F p_i times and complemented q_i times. Let $u^i_1,u^i_2,\ldots,u^i_{p_i}$ be new variables and replace every uncomplemented occurrence of x_i in F by a distinct u^i. Similarly let $v^i_1,v^i_2,\ldots v^i_{q_i}$ be new variables and replace every complemented occurrence of x_i by a distinct v^i. Let F' be the formula obtained by replacing every x_i. We will construct two schemes S_1 and S_2 such that $S_1 \not\leq S_2$ iff the original formula F is satisfiable. Intuitively, when $S_1 \not\leq S_2$, S_1 will force the satisfiability of the formula F' and S_2 will enforce the

S_2:



Now if the original formula F was satisfiable we can find an assignment A such that $Val(S_2,A) = g$ and $Val(S_1,A) = f$, so $S_1 \not\leq S_2$. Conversely, if $S_1 \not\leq S_2$, then there is an assignment A such that $Val(S_1,A) = f$ and $Val(S_2,A) = g$. But $Val(S_2,A) = g$ only if, for each i, $u_1^i=u_2^i=\ldots=u_{p_i}^i=\bar{v}_1^i=\ldots=\bar{v}_{q_i}^i$. Hence assigning to each x_i the value $A(u_1^i)$ satisfies F. Since S_1 and S_2 can be written down in time polynomial in the length of F, BNCONT is NP-hard. ■

We now turn to the equivalence problem for free B-schemes. First we show that if the two schemes are ordered, then there is a polynomial time algorithm for deciding equivalence.

Definition 3.2: A B-scheme with Boolean variables $b_1 \ldots b_k$ is ordered if whenever a test labeled b_i is a predecessor of a test labeled b_j then $i<j$. ■

In the proof of the next theorem we use the observation that if a scheme is ordered, then the size of the finite automaton accepting the interpreted value language [G] is polynomial in the size of the scheme.

Theorem 3.3: There is a polynomial time equivalence algorithm for ordered schemes.

Proof: Let S_1 and S_2 be schemes in which the Boolean variables $b_1 \ldots b_k$ appear. We will construct deterministic finite automata M_1 and M_2 from S_1 and S_2 such that $S_1 \equiv S_2$ iff deterministic finite automata can be done in polynomial time [AHU], there is a polynomial time algorithm for ordered schemes. ■

We close this section by proving that Theorem 3.3 remains true in the case where just one scheme is ordered. The method can be characterized as "graph pushing".

<u>Definition 3.4</u>: Let S be a free B-scheme and b a Boolean variable. Then S[b=true] is the scheme obtained from S by setting b to be true. More precisely:

1. For each vertex v labeled b in S, do the following. Delete v and any edges connected to it. Let u be the vertex such that (v,u) was labeled T. If v was the root, make u the root. Otherwise for each vertex w such that (w,v) was in S, insert edge (w,u) and give it the label of (w,v).
2. Delete any inaccesible vertices.

S[b=false] is defined analogously. ■

<u>Lemma 3.5</u>: Let S_1 and S_2 be free B-schemes. Then $S_1 \equiv S_2$ if and only if

$$S_1[b=true] \equiv S_2[b=true] \text{ and } S_1[b=false] \equiv S_2[b=false]$$

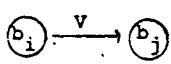
<u>Proof</u>: Immediate. ■

We now present a polynomial time algorithm which solves the equivalence problem for two free B-schemes, provided one is ordered.

$L(M_1) = L(M_2)$. M_i will accept the string $V_1V_2 \ldots V_k f$ (where V_j is either T or F and f is a function symbol) iff $Val(S_i, A) = f$ where A is the assignment

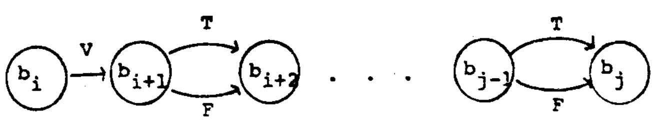
$$A(b_i) = \begin{cases} \text{true if } V_i = T \\ \text{false if } V_i = F. \end{cases}$$

M_i is constructed as follows. We extend S_i so that every Boolean variable is tested on every path from root to leaf. We may need to add extra tests if (1) the root is not labeled b_1, (2) there is an edge from a test labeled b_i to a test labeled b_j, and $j>i+1$, or (3) there is an edge from a test labeled b_i to a leaf, and $i<k$. For example in the second case the edge



$j>i+1$, V=T or F

is replaced with



We add a new accepting node and for each leaf labeled f an edge labeled f from the leaf to the accepting node. Then the resulting graph is the state graph of M_i; nodes are states, edge labels are state transitions, the test labeled b_1 is the start state, and the accepting node the only accepting state.

Since the Boolean variables are ordered it is clear that $L(M_1) = L(M_2)$ iff $S_1 \equiv S_2$. Since M_1 and M_2 can be computed in time polynomial in the size of S_1 and S_2, and equivalence of

Algorithm 3.6:

Input: Free B-scheme S_1 and ordered B-scheme S_2.

Output: "Yes" if the schemes are equivalent, "No" otherwise.

```
begin
    comment L is a list of pairs of graphs which must be
        equivalent in order that S1 and S2 be equivalent;
    initialize L to (S1,S2);
    repeat
       let n be a node of S1 all of whose predecessors have
          been marked and let v be the subgraph with root n;
       let (v,v1),...,(v,vm) be all the pairs of graphs on
          L in which v occurs;
       comment since v1,v2,...,vm are subgraphs of an ordered
          scheme, the method in Theorem 3.3 can be used to
          test their equivalence;
       if ¬ (v1≡v2≡...≡vm) then output ("No") and halt;
       if v is a leaf then
          comment since v is trivially ordered, the method
              in Theorem 3.3 can again be used to test
              equivalence of v and v1;
          if ¬ (v≡v1) then
              output ("No") and halt;
       else
       A: add to L the pairs (v',v1[b=true]) and (v",v1[b=false])
              where b is the label of v's root n and v'(v")
              is the subgraph of S1 reachable via n's
              outgoing T-edge (F-edge)
       fi;
       remove the pairs (v,v1),...,(v,vm) from L;
       mark n;
    until all nodes of S1 have been marked;
    output ("Yes") and halt;
```

Theorem 3.7: Algorithm 3.6 works correctly and runs in polynomial time.

Proof: It follows from Lemma 3.5 that the property

$$P: S_1 \equiv S_2 \iff \forall (v, v_i) \in L : v \equiv v_i$$

is an invariant for the loop. To show correctness then, it is sufficient to note that P is true intially and that when the algorithm stops, one of the following is true:

a) all nodes have been marked, the list L is empty and the answer is "Yes".

b) not all nodes have been marked, there is a pair (v, v_i) on L such that $v \not\equiv v_i$ and the answer is "No".

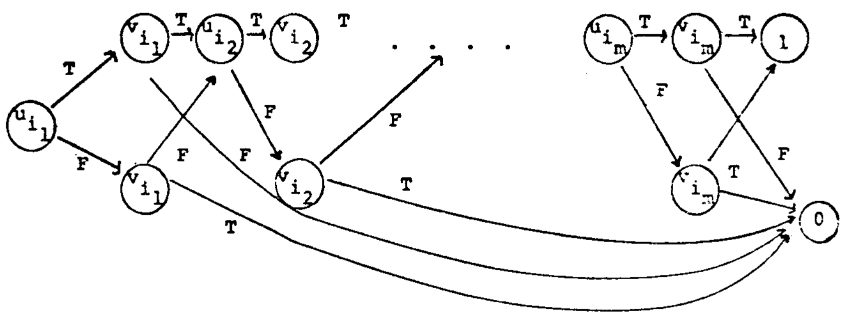
To see that the algorithm runs in polynomial time observe that the loop is executed at most $|S_1|$ times and each execution of the loop requires at most $|S_2|$ equivalences of ordered schemes which can be done in polynomial time by Theorem 3.3. ∎

Note that the freedom of S_1 guarantees that the graph v'(v") in the statement labeled A in the algorithm is equal to v[b=true](v[b=false]).

4. A scheme with no small equivalent ordered scheme

Here we construct a free B-scheme S_0 whose smallest ordered equivalent has size "exponential" in $|S_0|$. First we need some extra notation.

Let S be a B-scheme. A partial B-assignment (partial assignment for short) is a partial mapping from the Boolean varaibles of S to {true,false}. Two partial assignments A_1 and



Note that if A satisfies all equalities then the node labeled 1 is reachable via A.

The scheme S_0 is now constructed in two stages

a) The base of S_0 is a complete binary tree with n-1 interior nodes labelled with $u_1,\ldots,u_{n-1}$. The leaves are numbered from 0 to n-1.

b) The i'th leaf is replaced by the column C_i, obtained as follows. Remove from the set of equalities $\{u_1=v_{(1+i)\bmod n}, u_2=v_{(2+i)\bmod n},\ldots,u_{n-1}=v_{(n-1+i)\bmod n}\}$ all equalities involving variables that occur on the path from the root to leaf i, and construct C_i from the remaining equalities. Note that the sets of equalities are just cyclic permutations of equalities between $\{u_1,\ldots,u_{n-1}\}$ and $\{v_1,\ldots,v_n\}$.

The following facts about S_0 are evident

a) S_0 is free and has $n-1+3(n-1-\log n)\cdot n+2n<3n^2$ nodes.

b) No equality constraint appears more than once.

A_2 are consistent if they have the same value whenever they are both defined. The <u>union</u> of two consistent partial assignments A_1 and A_2, $A_1 \cup A_2$, is defined to be

$$(A_1 \cup A_2)(b) = \begin{cases} A_1(b) & \text{if } A_1(b) \text{ is defined} \\ A_2(b) & \text{if } A_2(b) \text{ is defined} \\ \text{undefined otherwise} \end{cases}$$

A partial assignment A_1 is an <u>extension</u> of A_2 if for each Boolean variable b, $A_2(b)$ defined implies $A_1(b) = A_2(b)$.

Let S be a scheme. A partial assignment A determines a path from the root to a node which is either a leaf or a test with a label on which A is not defined. Nodes on this path are said to be <u>specified</u> by A. Any node specified by some extension of A is said to be <u>reachable</u> via A. Note that the path determined by A can not be extended arbitrarily by an extension of A since certain tests not on the path may already be specified by A.

Assume that n is a power of 2. The scheme S_0 will contain 2n-1 Boolean variables $u_1,\ldots,u_{n-1},v_1,\ldots,v_n$. We say that a partial assignment A satisfies an equality $u_i=v_j$ if $A(u_i)$ and $A(v_j)$ are both defined and are equal. Given a set of equalities $\{u_{i_1}=v_{i_1},\ldots,u_{i_m}=v_{i_m}\}$ we construct the scheme, called a <u>column</u>, shown below

c) Every path from the root to a leaf labeled 1 is missing log n variables among the v's.

Now let S_1 be an ordered B-scheme which is equivalent to S_0, and let Y be the $\sqrt{n/2}$ Boolean variables which come first in the ordering. We shall show that there are "exponentially" many assignments to variables in Y which compute different functions of the remaining variables. Since each of these different functions must be represented by different nodes in S_1, S_1 must have "exponentially" many nodes.

Relabel the variables such that $Y = \{y_1, \ldots y_{\sqrt{\frac{n}{2}}}\}$ and let the remaining variables be $Z = \{z_1, \ldots z_{2n-1-\sqrt{n/2}}\}$. Call a column in S_0 <u>acceptable</u> if there is no equality $y_i = y_j$ between two elements of Y appearing in the column. There are at most $(\sqrt{\frac{n}{2}})^2 = \frac{n}{2}$ unacceptable columns. Call an assignment A to variables in Y <u>acceptable</u> if there is some acceptable column reachable via A.

Now we show the key result of this section, that if two acceptable assignments are "a little different" then they can be extended such that one of them specifies a node labeled 1 and the other a node labeled 0.

<u>Lemma 4.1</u>: Let A_1 and A_2 be acceptable assignments (to the variables in Y) which differ in more than log n variables. Then there is an assignment A to the variables in Z such that $Val(A_1 \cup A, S_0) \neq Val(A_2 \cup A, S_0)$.

Proof: Since A_1 and A_2 are acceptable assignments, we can always reach acceptable columns via A_1 and A_2. There are two cases to consider:

1) Assume that some acceptable column C is reachable via both A_1 and A_2. There are 2 log n variables which do not appear in C. Half of them are u's which appear on the path from the root to the column. The other half consists of v's. A_1 and A_2 cannot differ on the variables on the path from the root to C since C is reachable via both A_1 and A_2. Thus even if A_1 and A_2 differ on all the log n u's missing from column C, there is at least one variable, $y_i \epsilon Y$, which appears in an equality of C on which A_1 and A_2 differ. (The variable y_i may be either a u or a v, we don't care which.) The equality in which y_i appears must be of the form $y_i = z_j$, $z_j \epsilon Z$ since the column is acceptable, that is, the column has no equality between two y's. Since S_0 is free, z_j does not appear on the path from the root to C. Hence we can find an assignment A to the variables in Z such that $A_1 \cup A$ and $A_2 \cup A$ both specify C and $A_1 \cup A$ satisfies all equations in C. However, $A(z_j) = A_1(y_i) \neq A_2(y_i)$ so $\mathrm{Val}(A_1 \cup A, S_0) = 1$ and $\mathrm{Val}(A_2 \cup A, S_0) = 0$.

2) Assume that there is no acceptable column C which is reachable via both A_1 and A_2. We first find a partial assignment A to the variables in Z such that $A_1 \cup A$ specifies a column which can be satisfied by some extension, A', of $A_1 \cup A$. Then we show that we can choose the extension A' such that it satisfies the cloumn specified by $(A_1 \cup A)$ but the column specified by $(A_2 \cup A) \cup A'$ is not satisfiable.

Before we can show that there are many acceptable assignments which differ by more than log n of the variables we prove the following lemma which states that the total number of acceptable assignments is big.

Lemma 4.2: Let S be a B-scheme whose graph is a complete binary tree, with 2^k-1 interior nodes labeled with variables $u_1,\ldots,u_{2^k-1}$ and 2^k leaves labeled over $\{0,1\}$. Let M be any subset of the variables of size m and let the number of leaves labeled 1 be g. Call an assignment to the variables in M acceptable if a leaf labeled 1 is reachable from it, and denote by $A(m,g,k)$ the number of acceptable assignments. Then $A(m,g,k) \geq 2^m g/2^k$.

Proof: The proof is by induction on k, the height of the tree.

Basis: The result is immediate for k=0.

Induction step: Assume that $A(m,g,k-1) \geq 2^m g/2^{k-1}$ and consider complete binary trees with 2^k leaves. Let the number of leaves labeled 1 in the left subtree be g_ℓ and in the right subtree g_r. Let the number of variables from M in the left subtree be ℓ and in the right subtree r. There are two cases to consider.

1) The root is not labeled with a variable in M, hence $\ell+r = m$. Now

$$A(m,g,k) = 2^\ell A(r,g_r,k-1) + 2^r A(\ell,g_\ell,k-1) - A(r,g_r,k-1)\, A(\ell,g_\ell,k-1)$$

and using the inductive hypothesis

Let C_1 be an acceptable column reachable via A_1 and let A be the minimal partial assignment such that $A_1 \cup A$ specifies C_1 and all equations in C_1 involving variables in Y are satisfied (this is always possible since A_1 is acceptable, S_0 is free and no $y_i = y_j$ appears in C_1). A is now defined for at most $|Y| + \log n = \sqrt{n}/2 + \log n$ variables. Perform the following step while $A_2 \cup A$ does not specify some column: let z_k be the label of the last node specified by $A_2 \cup A$. Extend A by setting z_k to be false, and if $z_k = z_e$ appears in C_1, extend A to set z_e to false. (Setting z_k and z_e to true would work equally well.) This process terminates after adding at most 2 log n variables to A, after which $A_2 \cup A$ specifies some column C_2 (C_2 is not necessarily acceptable). Note that all equalities in C_1 involving variables in $A_1 \cup A$ are still satisfied. There are at least $(n - \log n - |A|)/2 = (n - \log n - \sqrt{n}/2 - 3 \log n)/2$ equalities in C_1 all of whose variables are unassigned by $A_1 \cup A$. There are only 2 log n variables not appearing in C_2, thus there is a $z_i = z_j$ in C_1, z_i and z_j not assigned in $A_1 \cup A$, and $z_i = x_e$, some x_e, is in C_2. x_e is not z_j by the construction of S_0. Now by extending A so that all equalities in C_1 are satisfied, and $A(z_i) = A(z_j) \neq (A_2 \cup A)(x_e)$, we can ensure that $A_1 \cup A$ satisfies C_1 whereas $A_2 \cup A$ does not satisfy C_2. This completes the proof of the lemma. ∎

$$
\begin{aligned}
A(m,g,k) &\geq 2^{\ell}(2^{r}g_r/2^{k-1}) + 2^{r}(2^{\ell}g_{\ell}/2^{k-1}) \\
&\quad - (2^{r}g_r/2^{k-1})\ (2^{\ell}g_{\ell}/2^{k-1}) \\
&= 2^{\ell+r}[(g_{\ell}+g_r)/2^{k-1} - g_{\ell}g_r/2^{2(k-1)}] \\
&= 2^{m}[g/2^{k} + g/2^{k} - g_{\ell}g_r/2^{2(k-1)}] \\
&\geq 2^{m}g/2^{k} \text{ as } g_{\ell},\ g_r \leq 2^{k-1}
\end{aligned}
$$

2) The root is labeled with a variable from M. Then $\ell+r+1 = m$ and

$$
\begin{aligned}
A(m,g,k) &= 2^{\ell}A(r,g_r,k-1) + 2^{r}A(\ell,g_{\ell},k-1) \\
&\geq 2^{\ell}(2^{r}g_r/2^{k-1}) + 2^{r}(2^{\ell}g_{\ell}/2^{k-1}) \\
&= 2^{\ell+r}(g_{\ell} + g_r)/2^{k-1} \\
&= 2^{m}g/2^{k}
\end{aligned}
$$

■

Now we can prove that any ordered scheme equivalent to S_0 must be big.

Theorem 4.3: Let S_1 be an ordered B-scheme which is equivalent to S_0. Then

$$|S_1| \geq 2^{m-(\log^2 n+1)/2} \quad \text{where} \quad m = \sqrt{n/2}$$

Proof: From the discussion preceding Lemma 4.1 we know that S_0 contains at least $n/2$ acceptable columns. Since Y contains m variables there are at least $A(m, n/2, \log n)$ acceptable assignments to variables in Y. From Lemma 4.1 we know that if two of these assignments differ by more than log n of the variables then they must lead to two different nodes in S_1. Now there are at most $\binom{m}{i}$ assignments to m variables which differ from a given assignment in i variable values. Hence there can be at most $\sum_{i=0}^{\log n} \binom{m}{i} < \sum_{i=0}^{\log n} m^i < m^{\log n+1}$ assignments which differ from a given assignment by at most log n variables. Therefore, there are at least $A(m,n/2,\log n)/m^{\log n+1}$ acceptable assignments which differ by more than log n variables and hence $|S_1| \geq A(m,n/2,\log n)/m^{\log n+1}$. By lemma 4.2 we now get

$$
\begin{aligned}
|S_1| &\geq (2^m \cdot (n/2)/2^{\log n})/m^{\log n+1} \\
&= 2^{m-1}/2^{(\log n+1)\log m} \\
&= 2^{m-1-(\log n+1)(\log n-1)/2} \quad \text{(recall that } m = \sqrt{n/2}) \\
&= 2^{m-\frac{1}{2}(\log^2 n+1)}
\end{aligned}
$$

and the theorem is proved. ∎

5. Extension to single variable program schemes

In this section we show that the equivalence problem for free single variable program schemes (free Ianov schemes) is polynomial time equivalent to the equivalence problem for free B-schemes.

A single variable program scheme (an I-scheme) is a rooted directed graph (not necessarily acyclic) whose nodes have outdegree 0,1 or 2. Nodes with outdegree 2 are tests and are labeled with Boolean variables. Nodes with outdegree 0 and 1 are called function nodes and are labeled with function symbols. Only vertices with outdegree 0 may be labeled with Ω. Edges

$p\ell(S_2,A)$. Thus for example two function nodes are 0-equivalent iff they have the same label.

The next lemma, the proof of which we leave to the reader, states that k-equivalence can be determined from (k-1)-equivalence and some equivalence tests on B-schemes.

<u>Lemma 5.1</u>: Let S be a free I-scheme with function nodes n_1 and n_2. Let v_i be the B-scheme whose root is the descendant of n_i, i=1 or 2 (v_i may be simply a function node). Label each leaf ℓ in v_i by its equivalence class $[\ell]_{k-1}$ in the (k-1)-equivalence relation. Then n_1 and n_2 are k-equivalent if and only if n_1 and n_2 are (k-1)-equivalent and $v_1 \equiv v_2$, where the last equivalence is of B-schemes. ■

<u>Theorem 5.2</u>: Let S be a free I-scheme with t nodes. Given an oracle for determining equivalence of free B-schemes, there is a polynomial time algorithm for determining if two function nodes in S are k-equivalent for all k.

<u>Proof</u>: It follows trivially from the preceeding lemma that two nodes are k-equivalent for all k if and only if they are t-equivalent. Since 0-equivalence is easy to determine (the nodes must have the same label), we can use Lemma 5.1 to compute k-equivalence for k = 1,2,...,t. At most t^2 B-scheme tests are made for each value of k, hence at most t^3 B-scheme tests are made altogether. ■

Having shown how to handle k-equivalence for all k we now leaving tests are labeled with T and F as in B-schemes. An I-scheme is <u>free</u> if every B-scheme which is a subgraph is free.

We shall only be interested in the behaviour of our schemes under Herbrand interpretations (free interpretations [G]) where the values of the Boolean variables can change after each function step. We extend the notion of B-assignments in the following way. Let F be a set of function symbols. An <u>I-assignment</u> A maps elements from $(F-\{\Omega\})^*$ into B-assignments. The interpretation of A(w) is the mapping defining the values of the Boolean variables in state w (the state after computing the functions in w). The path determined by A in S is the obvious generalization of the trace t(A) defined for B-schemes.

The proof that we can determine equivalence of free I-schemes in polynomial time given an oracle for equivalence of free B-schemes uses a procedure which is very similar to the minimization procedure for deterministic finite automata on p. 124-127 in [AU].

Let F be a set of function symbols, and denote by $(F-\{\Omega\})^{*k}$ the set of all strings over $F-\{\Omega\}$ of length k or less. A <u>k-assignment</u> is defined as a I-assignment except that its domain is $(F-\{\Omega\})^{*k}$ rather than $(F-\{\Omega\})^*$.

The path label $p\ell(S,A)$ for I-scheme S and k-assignment A, is the string of function symbols appearing along the path determined by A. (The string may be of length less than k if the path reaches a leaf.) Let function nodes n_1 and n_2 appear in S, and let S_1 and S_2 be the (sub)-schemes with n_1 and n_2 as roots. Then n_1 is <u>k-equivalent</u> to n_2 if for each k-assignment A, $p\ell(S_1,A) =$

define what it means for two I-schemes to be equivalent.

Let S be an I-scheme and A an I-assignment (i.e. A maps elements from $(F-\{\Omega\})^*$ to B-assignment). The <u>value mapping</u> Val is defined as follows.

$$\mathrm{Val}(S,A) = \begin{cases} \text{the function symbols on the path determined by A if the path is finite and does not end in } \Omega \\ \Omega \text{ otherwise} \end{cases}$$

Two I-schemes S_1 and S_2 are <u>equivalent</u> if $\mathrm{Val}(S_1,A) = \mathrm{Val}(S_2,A)$ for all I-assignments A. It is clear that this definition means equivalence under all Herbrand interpretations (free interpretations) and it is well known that this implies equivalence under all interpretations [G].

We would like to show that two schemes are equivalent iff their root nodes are k-equivalent for all k. Unfortunately this is not quite true; the problem is that the schemes may both compute Ω but do so in different ways.

A free I-scheme is <u>compact</u> if from every non-leaf node there is a path to a leaf not labeled Ω.

<u>Lemma 5.3</u>: There is a polynomial time algorithm to transform any free I-scheme into an equivalent compact free scheme.

<u>Proof</u>: Immediate. ■

<u>Lemma 5.4</u>: Two free compact I-schemes S_1 and S_2 are equivalent iff their roots n_1 and n_2 are k-equivalent for every k.

Proof: It is clear that if n_1 and n_2 are k-equivalent for all k, then S_1 is equivalent to S_2. Conversely, suppose S_1 is equivalent to S_2 and let k be the smallest value for which there is a k-assignment A such that $p\ell(S_1,A) \neq p\ell(S_2,A)$. Not both of $p\ell(S_1,A)$ and $p\ell(S_2,A)$ can end in Ω, so assume $p\ell(S_1,A)$ does not.

We can extend A to an ℓ-assignment A', $\ell \geq k$ with $A'(w) = A(w)$ for all w, $|w| \leq k$, such that A' defines a path to a leaf not labeled Ω in S_1. Now since the k^{th} symbol on the path defined by A' in S_2 is different from the k^{th} symbol on the path in S_1, and $Val(S_1,A') \neq \Omega$, we must have S_1 not equivalent to S_2, a contradiction. ■

Now the following theorem is an immediate corollary of the preceding lemmas.

Theorem 5.5: There is a polynomial time algorithm to decide equivalence of free I-schemes if and only if there is a polynomial time algorithm to decide equivalence of free B-schemes. ■

We close this section with the remark that non-inclusion for I-schemes is NP-complete. Inclusion for I-schemes is defined exactly as for B-schemes with "I-assignment" replacing "B-assignment". That the problem is NP-hard is clear from Theorem 3.1. That it is in NP is shown in [CHS].

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