# The Complexity of Partition Functions

Andrei Bulatov

*School of Computing Science, Simon Fraser University, Burnaby, Canada*
*abulatov@cs.sfu.ca*

Martin Grohe

*Institut für Informatik, Humboldt-Universität, Berlin, Germany*
*grohe@informatik.hu-berlin.de*

**Abstract**

We give a complexity theoretic classification of the counting versions of so-called $H$-colouring problems for graphs $H$ that may have multiple edges between the same pair of vertices. More generally, we study the problem of computing a weighted sum of homomorphisms to a weighted graph $H$.

The problem has two interesting alternative formulations: First, it is equivalent to computing the partition function of a spin system as studied in statistical physics. And second, it is equivalent to counting the solutions to a constraint satisfaction problem whose constraint language consists of two equivalence relations.

In a nutshell, our result says that the problem is in polynomial time if the adjacency matrix of $H$ has row rank 1, and #P-hard otherwise.

*Key words:* counting complexity, partition function, graph homomorphism, constraint satisfaction

## 1 Introduction

This paper has two different motivations: The first is concerned with constraint satisfaction problems, the second with "spin-systems" as studied in statistical physics. A known link between the two are so-called $H$-colouring problems. Our main result is a complete complexity theoretic classification of the problem of counting the number of solutions of an $H$-colouring problem for an undirected graph $H$ which may have multiple edges, and actually of a natural generalisation of this problem to weighted graphs $H$. Translated to the world of constraint satisfaction problems, this yields a classification of the problem of counting the solutions to constraint satisfaction problems for two equivalence relations. Translated to the world of statistical physics, it gives a classification of the problem of computing the partition function of a spin system.

Let us describe our result from each of the different perspectives: Let $H$ be a graph, possibly with multiple edges between the same pair of vertices, e.g. a *multi-graph*. An *$H$-colouring* of a graph $G$ is a homomorphism from $G$ to $H$. Both the decision problem, asking whether a given graph has an $H$-colouring, and the problem of counting the $H$-colourings of a given graph, have received considerable attention [5,6,9,11,12]. Here we are interested in the counting problem. Dyer and Greenhill [5] gave a complete complexity theoretic classification of the counting problem for undirected graphs $H$ without multiple edges; they showed that the problem is in polynomial time if each connected component of $H$ is complete bipartite without any loops or is complete with all loops present, and #P-hard otherwise. Here we are interested in counting $H$-colourings for multi-graphs $H$. Note that, as opposed to the decision problem, multiple edges do make a difference for the counting problem. Let $H$ be a multi-graph with vertex set $\{1, \ldots, k\}$. $H$ is best described in terms of its adjacency matrix $A = (A_{ij})$, where $A_{ij}$ is the number of edges between vertices $i$ and $j$. Given a graph $G = (V, E)$, we want to compute the number of homomorphisms from $G$ to $H$. Observe that this number is

$$Z_A(G) = \sum_{\sigma: V \to \{1,\ldots,k\}} \prod_{e=\{u,v\} \in E} A_{\sigma(u)\sigma(v)}. \tag{1}$$

Borrowing from the physics terminology, we call $Z_A$ the *partition function* of $A$ (or $H$). We denote the problem of computing $Z_A(G)$ for a given graph $G$ by $\mathrm{EVAL}(A)$. Of course if we define $Z_A$ as in (1), the problem is not only meaningful for matrices $A$ that are adjacency matrices of multi-graphs, but for arbitrary square matrices $A$. We may view such matrices as adjacency matrices of weighted graphs (omitting edges of weight 0). We call a symmetric matrix $A$ *connected* (*bipartite*) if the corresponding graph is connected (*bipartite*, respectively).

We prove the following classification result:

**Theorem 1** *Let $A$ be a symmetric matrix with non-negative real entries.*

*(1) If $A$ is connected and not bipartite, then* $\mathrm{EVAL}(A)$ *is in polynomial time if the row rank of $A$ is at most 1; otherwise* $\mathrm{EVAL}(A)$ *is #P-hard.*
*(2) If $A$ is connected and bipartite, then* $\mathrm{EVAL}(A)$ *is in polynomial time if the row rank of $A$ is at most 2; otherwise* $\mathrm{EVAL}(A)$ *is #P-hard.*
*(3) If $A$ is not connected, then* $\mathrm{EVAL}(A)$ *is in polynomial time if each of its connected components satisfies the corresponding condition stated in (1) or (2); otherwise* $\mathrm{EVAL}(A)$ *is #P-hard.*

Note that this generalises Dyer and Greenhill's [5] classification result for graphs without multiple edges, whose adjacency matrices are symmetric 0-1 matrices.

Our proof builds on interpolation techniques similar to those used by Dyer and Greenhill, recent results on counting the number of solutions to constraint satisfaction problems due to Dalmau and the first author [1], and a considerable amount of polynomial arithmetic. Even though we present the proof in the language of constraint satisfaction problems here, in finding the proof it has been very useful to jump back and forth between the $H$-colouring and constraint satisfaction perspective.

Let us now explain the result for constraint satisfaction problems. A *constraint language* $\Gamma$ on a finite *domain* $D$ is a set of relations on $D$. An instance of the problem CSP($\Gamma$) is a triple $(V, D, \mathcal{C})$ consisting of a set $V$ of *variables*, the domain $D$, and a set $\mathcal{C}$ of *constraints* $\langle s, \rho\rangle$, where $\rho$ is a relation in $\Gamma$ and $s$ is a tuple of variables whose length matches the arity of $\rho$. A *solution* is a mapping $\sigma : V \to D$ such that for each constraint $\langle (v_1, \ldots, v_r), \rho\rangle \in \mathcal{C}$ we have $(\sigma(v_1), \ldots, \sigma(v_r)) \in \rho$. There has been considerable interest in the complexity of constraint satisfaction problems [16,14,7,2,3], which has mainly been driven by Feder and Vardi's [7] *dichotomy question*, asking whether for all languages $\Gamma$ the problem CSP($\Gamma$) is either solvable in polynomial time or NP-complete. A similar dichotomy question can be asked for the problem #CSP($\Gamma$) of counting the solutions for a given instance [4,1].

We consider constraint languages $\Gamma$ consisting of two equivalence relations $\alpha, \beta$. Suppose that $\alpha$ has $k$ equivalence classes and $\beta$ has $\ell$ equivalence classes. Then $\Gamma$ can be described by a $(k \times \ell)$-matrix $B = (B_{ij})$, where $B_{ij}$ is the number of elements in the intersection of the $i$th class of $\alpha$ and the $j$th class of $\beta$. We show that, provided that the matrix is "indecomposable" (in a sense made precise in Section 2.2), the problem #CSP($\Gamma$) is in polynomial time if the row rank of $B$ is 1 and #P-hard otherwise. There is also a straightforward extension to "decomposable" matrices (see Corollary 15 for the precise statement). In [1], it has been shown that if #CSP($\Gamma$) is in polynomial time, then $\Gamma$ has a so-called *Mal'tsev polymorphism*. The result of this paper provides a further necessary condition for $\Gamma$ to give rise to a counting problem solvable in polynomial time.

We can generalise our result for CSP whose language consists of two equivalence relations to *weighted CSP*, where each domain element $d$ carries a non-negative real weight $\omega(d)$. The weight of a solution $\sigma : V \to D$ is defined to be the product $\prod_{v \in V} \omega(\sigma(v))$, and the goal is to compute the weighted sum over all solutions (see Theorem 14 for the precise statement of our result). As an important intermediate step, we even prove our classification result for weights that are polynomials with integer coefficients.

Let us finally describe the connection with statistical physics. Statistical physics explains properties of substances, such as gases, liquids or crystals, using probability distributions on certain states of the substance. In one of the standard models, a substance is considered as a conglomeration of particles (atoms or molecules) viewed as a graph $G = (V, E)$, also called a *lattice*, in which adjacent vertices represent particles interacting in a non-negligible way. Every particle may have one of $k$ *spins*; the interaction between neighbouring particles can be described by a *spin system*, which is just a symmetric $k \times k$-matrix $K = (K_{ij})$. The entry $K_{ij}$ of $K$ corresponds, in a certain way, to the energy that a pair of interacting particles, one of which has spin $i$, the other one has spin $j$, contributes into the overall energy of $G$. A *configuration* of the system on a graph $G = (V, E)$ is a mapping $\sigma : V \to \{1, \ldots, k\}$. The *energy* of $\sigma$ is the sum $H(\sigma) = \sum_{e=\{u,v\}\in E} K_{\sigma(u)\sigma(v)}$. Then the probability that $G$ has configuration $\sigma$ is $\frac{1}{Z}\exp(-H(\sigma)/cT)$, where $Z = \sum_\sigma \exp(-H(\sigma)/cT)$ is the *partition function* and $T$ is a parameter of the system (the *temperature*) and $c$ is a constant. As is easily seen, this probability distribution obeys the law "the lower energy a configuration has, the more likely it is". Observe that $Z = Z_A(G)$ for the matrix $A$ with

$$A_{ij} = \exp(-K_{ij}/cT).$$

Thus EVAL($A$) is just the problem of computing the partition function for the system described by $A$. Dyer and Greenhill in [5] dealt with spin systems in which certain configuration are prohibited and the others are uniformly distributed, while our results are applicable to arbitrary spin systems.

The article is organised as follows: We start with a few general preliminaries in Section 2. In Subsection 2.2, we introduce our terminology concerning decompositions of matrices (into *blocks* or *connected components*) and make a few simple observations about these decompositions. In Section 3, we prove the tractability part of our main theorem, which is fairly easy. As a matter of fact, we prove a slightly more general result that also includes matrices that are not symmetric. In Section 4, we introduce counting constraint satisfaction problems and their weighted version for constraint languages that consist of two equivalence relations. We then show how these problems can be described by matrices and how they relate to evaluating the partition function of these matrices. In Section 5, we state our main results in their full generality. The tractability parts of these results follow from the results of Section 3, so it remains to prove the hardness results. Section 6 is devoted to the hardness proof. The organisation of this proof is laid out at the beginning of the section.

## 2 Preliminaries

### 2.1 Graphs and Matrices

$\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ denote the real numbers, rational numbers and integers, respectively, and $\mathbb{Q}[X]$ and $\mathbb{Z}[X]$ denote the polynomial rings over $\mathbb{Z}$ and $\mathbb{Q}$ in an indeterminate $X$. Throughout this paper, we let $\mathbb{S}$ denote one of these five rings.

The *degree* of a polynomial $p(X)$ is denoted by $\mathsf{deg}(p)$.

For every set $S$, $S^{m\times n}$ denotes the set of all $m\times n$-matrices with entries from $S$. For a matrix $A$, $A_{ij}$ denotes the entry in row $i$ and column $j$. The row rank of a matrix $A \in \mathbb{S}^{m\times n}$ is denoted by $\mathsf{rank}(A)$. The *transpose* of $A$ is denoted by $A^\top$. A matrix $A \in \mathbb{S}^{m\times n}$ is *non-negative* (*positive*), if, for $1 \leq i \leq m, 1 \leq j \leq n$, the leading coefficient of $A_{ij}$ is non-negative (positive, respectively).

*Graphs* are always undirected, unless we explicitly call them *directed graphs*. Graphs and directed graphs may have loops and multiple edges. The *in-degree* and *out-degree* of a vertex in a (directed) graph are defined in the obvious way and denoted by $\mathsf{indeg}(v)$, $\mathsf{outdeg}(v)$, respectively.

Our model of real number computation is a standard model, as it is, for example, underlying the complexity theoretic work on linear programming (cf. [10]). We can either assume that the numbers involved in our computations are polynomial time computable or that they are given by an oracle (see [15] for a detailed description of the model). However, our results do not seem to be very model dependent. All we really need is that the basic arithmetic operations are polynomial time computable. Our situation is fairly simple because all real numbers we encounter are the entries of some matrix $A$, which is always considered fixed, and numbers computed from the entries of $A$ using a polynomial number of arithmetic operations. Instances of the problem EVAL($A$) are just graphs, and we do not have to worry about real numbers as inputs of our computations.

We assume that the reader is familiar with the basics of the complexity theory of counting problems, in particular with the class #P. All reductions in this article are *polynomial time Turing reductions*. We call two problems *polynomial time equivalent* if they are reducible to one another (by polynomial time Turing reductions). The problem of evaluating a partition function such as (1) (on page 2) is in #P if $A$ is a non-negative integer matrix; for such matrices our #P-hardness results are actually #P-completeness results. For other matrices, the partition function cannot be evaluated in #P simply because its values are not necessarily integral. For all matrices $A$ we consider, the partition function

$A$ can still be evaluated in $\mathrm{FP}^{\#\mathrm{P}}$, the class of all function problems in the closure of #P under polynomial time Turing reductions. It is common in the area (e.g. [13,5]) to refer to such results as #P-completeness results anyway, but to avoid confusion we refrain from doing so and just state them as hardness results.

### *2.2 Block Decompositions*

Let $B \in \mathbb{S}^{k\times\ell}$. A *submatrix* of $B$ is a matrix obtained from $B$ by deleting some rows and columns. For non-empty sets $I \subseteq \{1,\dots,k\}$, $J \subseteq \{1,\dots,\ell\}$, where $I = \{i_1,\dots,i_p\}$ with $i_1 < \dots < i_p$ and $J = \{j_1,\dots,j_q\}$ with $j_1 < \dots < j_q$, $B_{IJ}$ denotes the $(p \times q)$-submatrix with $(B_{IJ})_{rs} = B_{i_r j_s}$ for $1 \le r \le p, 1 \le s \le q$. A *proper submatrix* of $B$ is a submatrix $B' \neq B$.

**Definition 2** *Let $B \in \mathbb{S}^{k\times\ell}$.*

*(1) A* decomposition *of $B$ consists of two proper submatrices $B_{IJ}$, $B_{\overline{I}\,\overline{J}}$ such that*
   *(a) $\overline{I} = \{1,\dots,k\} \setminus I$,*
   *(b) $\overline{J} = \{1,\dots,\ell\} \setminus J$,*
   *(c) $B_{ij} = 0$ for all $(i,j) \in (I \times \overline{J}) \cup (\overline{I} \times J)$.*
   *$B$ is* indecomposable *if it has no decomposition.*
*(2) A* block *of $B$ is an indecomposable submatrix $B_{IJ}$ with at least one non-zero entry such that $B_{IJ}, B_{\overline{I}\,\overline{J}}$ is a decomposition of $B$.*

Indecomposability may be viewed as a form of "connectedness" for arbitrary matrices. For square matrices there is also a natural graph based notion of connectedness.

Let $A \in \mathbb{S}^{k\times k}$ be a square matrix. A *principal submatrix* of $A$ is a submatrix of the form $A_{II}$ for some $I \subseteq \Big\{1,\dots,k\Big\}$. Instead of $A_{II}$ we just write $A_I$. The *underlying graph* of $A$ is the (undirected) graph $G(A)$ with vertex set $\{1,\dots,k\}$ and edge set $\Big\{\{i,j\} \mid 1 \le i,j \le n \text{ such that } A_{ij} \neq 0 \text{ or } A_{ji} \neq 0\Big\}$. Note that we define $G(A)$ to be an undirected graph even if $A$ is not symmetric.

**Definition 3** *Let $A \in \mathbb{S}^{k\times k}$.*

*(1) The matrix $A$ is* connected *if the graph $G(A)$ is connected.*
*(2) A* connected component *of the matrix $A$ is a principal submatrix $A_C$, where $C$ is the vertex set of a connected component of $G(A)$.*

**Lemma 4** *A connected symmetric matrix is either indecomposable or bipartite. In the latter case, the matrix has precisely two blocks which are each others transposes.*

Note that by permuting rows and columns a connected bipartite symmetric matrix can be transformed into a matrix

$$\begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix},$$

where $B$ and hence $B^\top$ are indecomposable. The rows of the two blocks $B$ and $B^\top$ correspond to the two parts of the bipartition of the graph of the matrix.

There is another useful connection between indecomposability and connectedness. For a matrix $B \in \mathbb{S}^{k\times\ell}$, let

$$\mathsf{bip}(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathbb{S}^{(k+\ell)\times(k+\ell)}.$$

Note that $\mathsf{bip}(B)$ is the adjacency matrix of a weighted bipartite directed graph. The following lemma is straightforward.

**Lemma 5** *Let $B \in \mathbb{S}^{k\times\ell}$ and $A = \mathsf{bip}(B)$. Then for every block $B_{IJ}$ of $B$ there is a connected component $A_C$ of $A$ such that $A_C = \mathsf{bip}(B_{IJ})$, and conversely for every connected component $A_C$ of $A$ there is a block $B_{IJ}$ of $B$ such that $A_C = \mathsf{bip}(B_{IJ})$.*

*In particular, $B$ is indecomposable if, and only if, $A$ is connected.*

## 3 The Tractable Cases

In this section, we shall prove the tractability part of Theorem 1. Even though the theorem only speaks about symmetric matrices and (undirected) graphs, it will be useful to generalise partition functions to directed graphs and prove a slightly more general result.

Let $A \in \mathbb{S}^{k\times k}$ be a square matrix that is not necessarily symmetric and $G = (V, E)$ a directed graph. For every $\sigma : V \to \{1, \dots, k\}$ we let

$$\omega_A(\sigma) = \prod_{(u,v)\in E} A_{\sigma(u)\sigma(v)},$$

and we let

$$Z_A(G) = \sum_{\sigma:V\to\{1,\dots,k\}} \omega_A(\sigma).$$

Note that if $A$ is symmetric, $G = (V, E)$ a directed graph, and $G_U$ the underlying undirected graph, then $Z_A(G_U) = Z_A(G)$. (where $Z_A(G_U)$ is defined as in (1) on page 2). Thus by EVAL($A$) we may denote the problem of computing $Z_A(G)$ for a given directed graph, with the understanding that for symmetric $A$ we can always consider the input graph as undirected.

**Theorem 6** *Let $A \in \mathbb{S}^{k\times k}$ be a matrix.*

*(1) If each connected component of $A$ has row rank 1, then* EVAL($A$) *is in polynomial time.*
*(2) If $A$ is symmetric and each connected component of $A$ either has row rank at most 1 or is bipartite and has row rank at most 2, then* EVAL($A$) *is in polynomial time.*

**PROOF.** Let $A_1, \dots, A_\ell$ be the connected components of $A$. Then for every graph $G$ with connected components $G_1, \dots, G_m$ we have

$$Z_A(G) = \prod_{i=1}^{m} \sum_{j=1}^{\ell} Z_{A_j}(G_i).$$

Thus without loss of generality we may assume that $A$ is connected.

(1) If $\mathsf{rank}(A) \leq 1$ there are numbers $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{R}$ such that for $1 \leq i, j \leq k$ we have:

$$A_{ij} = a_i \cdot b_j$$

(the $b_j$ can be chosen to be the $A_{1j}$ and $a_i = A_{i1}/A_{11}$). Let $G = (V, E)$ be a directed graph and $\sigma : V \to \{1, \dots, k\}$. Then

$$\omega_A(\sigma) = \prod_{(v,w)\in E} A_{\sigma(v)\sigma(w)} = \prod_{(v,w)\in E} a_{\sigma(v)} b_{\sigma(w)} = \prod_{v\in V} a_{\sigma(v)}^{\mathsf{outdeg}(v)} b_{\sigma(v)}^{\mathsf{indeg}(v)}.$$

Thus

$$Z_A(G) = \sum_{\sigma:V\to\{1,\dots,k\}} \omega_A(\sigma) = \sum_{\sigma} \prod_{v\in V} a_{\sigma(v)}^{\mathsf{outdeg}(v)} b_{\sigma(v)}^{\mathsf{indeg}(v)} = \prod_{v\in V} \sum_{i=1}^{k} a_i^{\mathsf{outdeg}(v)} b_i^{\mathsf{indeg}(v)}.$$

The last term can easily be evaluated in polynomial time.

(2) Again we assume that $A$ is connected. The case not covered by (1) is that $A$ is symmetric and bipartite with $\mathsf{rank}(A) = 2$, so let us assume that $A$ has these properties. Then there are $k_1, k_2 \geq 1$ such that $k_1 + k_2 = k$ and a matrix $B \in \mathbb{S}^{k_1\times k_2}$ with $\mathsf{rank}(B) = 1$ and

$$A = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}.$$

Let $G = (V, E)$ be a graph. If $G$ is not bipartite then $Z_A(G) = 0$, therefore, we may assume that $G$ is connected and bipartite, say, with bipartition $V_1, V_2$. Let $G_{12}$ be the directed graph obtained from $G$ by directing all edges from $V_1$ to $V_2$, and let $G_{21}$ be the directed graph obtained from $G$ by directing all edges from $V_2$ to $V_1$. Recall that

$$\mathsf{bip}(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathbb{S}^{k \times k}.$$

We have

$$Z_A(G) = Z_{\mathsf{bip}(B)}(G_{12}) + Z_{\mathsf{bip}(B)}(G_{21}).$$

Since EVAL($\mathsf{bip}(B)$) is in polynomial time by Theorem 6(1), this shows that $Z_A(G)$ can be computed in polynomial time. $\square$

## 4 Constraint Satisfaction Problems

In this section, we study counting constraint satisfaction problems and their weighted version for constraint languages that consist of two equivalence relations. We show how these problems can be described by matrices and how they relate to evaluating the partition function of matrices. The results of this section will enable us to translate results back and force between partition functions of graphs and counting constraint satisfaction problems. We start by introducing a weighted version of counting constraint satisfaction problems and a "partition function" that is defined on the instances of such problems.

Recall that a *constraint language* $\Gamma$ on a *domain* $D$ is a set of relations on $D$. The pair $(D, \Gamma)$ is occasionally called the *template* of the constraint satisfaction problem CSP($\Gamma$). An *instance* of CSP($\Gamma$) is a triple $(V, D, \mathcal{C})$ consisting of a set $V$ of *variables*, the domain $D$, and a set $\mathcal{C}$ of *constraints* $\langle s, \rho \rangle$, where $\rho$ is a relation in $\Gamma$ and $s$ is a tuple of variables whose length matches the arity of $\rho$. A *solution* is a mapping $\sigma : V \to D$ such that for each constraint $\langle (v_1, \dots, v_r), \rho \rangle \in \mathcal{C}$ we have $(\sigma(v_1), \dots, \sigma(v_r)) \in \rho$. #CSP($\Gamma$) is the problem of counting the number of solutions for a given instance $\mathcal{P}$. We shall now define a weighted version of this problem. Let $D$ be a domain and $\omega : D \to \mathbb{S}$; we call $\omega$ a *weight function* on $D$. Slightly abusing notation, we also use $\omega$ to denote the weight of a solution $\sigma : V \to D$ for an instance $\mathcal{P} = (V, D, \mathcal{C})$ of some CSP with domain $D$: The weight of $\sigma$ is defined by

$$\omega(\sigma) = \prod_{v \in V} \omega(\sigma(v)).$$

For every constraint language $\Gamma$ with domain $D$ and every weight function $\omega : D \to \mathbb{S}$ we define a function $\mathcal{Z}_{\Gamma,\omega}$ from the instances of CSP($\Gamma$) to $\mathbb{S}$ by letting

$$\mathcal{Z}_{\Gamma,\omega}(\mathcal{P}) := \sum_{\sigma} \omega(\sigma)$$

where the sum ranges over all solutions $\sigma$ for $\mathcal{P}$. We denote the problem of computing $\mathcal{Z}_{\Gamma,\omega}$ by WCSP$(\Gamma, \omega)$. The triple $(D, \Gamma, \omega)$ is called the *weighted template* of the problem WCSP$(\Gamma, \omega)$.

Observe that the problem WCSP$(\Gamma, \omega)$ has exactly the same instances as the problems CSP$(\Gamma)$ and #CSP$(\Gamma)$. In particular, instances of WCSP$(\Gamma, \omega)$ do not depend on $\omega$. Thus we often introduce instances of WCSP$(\Gamma, \omega)$ as instances of CSP$(\Gamma)$ or #CSP$(\Gamma)$.

### *4.1 CSPs with two Equivalence Relations*

Our main results on (weighted) constraint satisfaction problems are concerned with constraint languages consisting of two equivalence relations, which we usually denote by $\alpha$ and $\beta$. In this subsection, we associate certain matrices with constraint languages consisting of two equivalence relations and describe the corresponding CSP in terms of these matrices. This will enable us in the next subsection to establish a connection between such CSP and the problem of computing the partition function of graphs.

So suppose that $\alpha$ and $\beta$ are equivalence relations on a domain $D$. Let $C_1, \dots, C_k$ be the equivalence classes of $\alpha$ and $D_1, \dots, D_\ell$ the equivalence classes of $\beta$. We define a matrix $B(\alpha, \beta) \in \mathbb{Z}^{k\times\ell}$ by

$$B(\alpha, \beta)_{ij} = |C_i \cap D_j|.$$

Conversely, for every non-negative integer matrix $B \in \mathbb{Z}^{k\times\ell}$ there are equivalence relations $\alpha_B, \beta_B$ on the domain

$$D_B = \Big\{1, \dots, \sum_{\substack{1\le i\le k\\ 1\le j\le \ell}} B_{ij}\Big\}$$

such that $B = B(\alpha_B, \beta_B)$. We fix such relations $\alpha_B, \beta_B$ and call $(D_B, \{\alpha_B, \beta_B\})$ the *canonical template* for $B$.

We never need an explicit definition of $\alpha_B$ and $\beta_B$, but for example, we can define the relations as follows: For $1 \le i \le k$ we let $m_i = \sum_{j=1}^{\ell} B_{ij}$, and we let $m_0 = 0$. We define $\alpha_B$ in such a way that its equivalence classes are

$$C_i = \{m_{i-1} + 1, \dots, m_i\}$$

for $1 \le i \le k$. For $1 \le i \le k$ we let $n_{i0} = m_{i-1}$ and, for $1 \le j \le \ell$,

$$n_{ij} = n_{i(j-1)} + |B_{ij}|.$$

We define $\beta_B$ in such a way its equivalence classes are

$$D_j = \bigcup_{i=1}^{m} \{n_{i(j-1)} + 1, \ldots, n_{ij}\}$$

for $1 \leq j \leq \ell$. Then for $1 \leq i \leq k, 1 \leq j \leq \ell$ we have

$$C_i \cap D_j = \{n_{i(j-1)} + 1, \ldots, n_{ij}\}$$

and thus $|C_i \cap D_j| = B_{ij}$. Therefore, $B = B(\alpha_B, \beta_B)$.

We give similar definitions for weighted problems. Again, let $\alpha$ and $\beta$ are equivalence relations on a domain $D$, and let $C_1, \ldots, C_k$ be the equivalence classes of $\alpha$ and $D_1, \ldots, D_\ell$ the equivalence classes of $\beta$. Let $\omega : D \to \mathbb{S}$ be a weight function. We define a matrix $B(\alpha, \beta, \omega) \in \mathbb{S}^{k \times \ell}$ by

$$B(\alpha, \beta, \omega)_{ij} = \sum_{d \in C_i \cap D_j} \omega(d).$$

The crucial fact is that essentially the problem $\mathrm{WCSP}(\{\alpha, \beta\}, \omega)$ only depends on the matrix $B(\{\alpha, \beta\}, \omega)$. This is made precise in the next lemma.

**Lemma 7** *Let $(D, \{\alpha, \beta\}, \omega)$ and $(D', \{\alpha', \beta'\}, \omega')$ be two weighted templates, where $\alpha, \beta$ and $\alpha', \beta'$ are equivalence relations on $D, D'$, respectively. Suppose that*

$$B(\{\alpha, \beta\}, \omega) = B(\{\alpha', \beta'\}, \omega').$$

*Then the problems $\mathrm{WCSP}(\{\alpha, \beta\}, \omega)$ and $\mathrm{WCSP}(\{\alpha', \beta'\}, \omega')$ are equivalent in the following sense: If $\mathcal{P} = (V, D, \mathcal{C})$ is an instance of $\mathrm{CSP}(\{\alpha, \beta\})$ and $\mathcal{P}' = (V, D', \mathcal{C}')$ is the instance of $\mathrm{CSP}(\{\alpha', \beta'\})$ obtained from $\mathcal{P}$ by replacing each constraint $\langle (u, v), \alpha \rangle$ by $\langle (u, v), \alpha' \rangle$ and each constraint $\langle (u, v), \beta \rangle$ by $\langle (u, v), \beta' \rangle$, then*

$$\mathcal{Z}_{\{\alpha, \beta\}, \omega}(\mathcal{P}) = \mathcal{Z}_{\{\alpha', \beta'\}, \omega}(\mathcal{P}').$$

*In particular, $\mathrm{WCSP}(\{\alpha, \beta\}, \omega)$ and $\mathrm{WCSP}(\{\alpha', \beta'\}, \omega')$ are polynomial time equivalent.*

The proof of this lemma is straightforward, but to familiarise the reader with the technical notions, we shall give it nevertheless. We need one more definition: For every matrix $B \in \mathbb{S}^{k \times \ell}$, the *canonical weighted template*

$$(D_B^w, \{\alpha_B^w, \beta_B^w\}, \omega_B^w)$$

is defined as follows:

- The domain is $D_B^w = \{1, \ldots, k\} \times \{1, \ldots, \ell\}$,
- the equivalence relation $\alpha_B^w$ is equality on the first component,
- the equivalence relation $\beta_B^w$ is equality on the second component,

- the weight function $\omega^w_B : D^w_B \to \mathbb{S}$ is defined by $\omega^w_B((i,j)) = B_{ij}$.

Then clearly $B = B(\alpha^w_B, \beta^w_B, \omega^w_B)$. Thus by Lemma 7, which we will prove soon, for every weighted template $(D, \{\alpha, \beta\}, \omega)$, where $\alpha$ and $\beta$ are equivalence relations on $D$ with $B(\alpha, \beta, \omega) = B$, the problems WCSP$(\{\alpha, \beta\}, \omega)$ and WCSP$(\{\alpha^w_B, \beta^w_B\}, \omega^w_B)$ are equivalent. In the following, we write $\mathcal{Z}_B$ instead of $\mathcal{Z}_{\{\alpha^w_B, \beta^w_B\}, \omega^w_B}$ and WCSP$(B)$ instead of WCSP$(\{\alpha^w_B, \beta^w_B\}, \omega^w_B)$.

Recall that for every instance $\mathcal{P} = (V, D^w_B, \mathcal{C})$ of CSP$(\{\alpha^w_B, \beta^w_B\})$ (and thus of WCSP$(B)$) we have

$$\begin{aligned}
\mathcal{Z}_B(\mathcal{P}) &= \sum_{\substack{\sigma: V \to D^w_B \\ \text{solution}}} \omega^w_B(\sigma) \\
&= \sum_{\substack{\sigma: V \to D^w_B \\ \text{solution}}} \prod_{v \in V} \omega^w_B(\sigma(v)) \\
&= \sum_{\substack{\sigma: V \to \{1,\dots,k\} \times \{1,\dots,\ell\} \\ \text{solution}}} \prod_{v \in V} B_{\sigma(v)}
\end{aligned}$$

**PROOF of Lemma 7.** Without loss of generality we may assume that

$$(D', \alpha', \beta', \omega') = (D^w_B, \alpha^w_B, \beta^w_B, \omega^w_B).$$

Let $\mathcal{P}$ be an instance of CSP$(\{\alpha, \beta\})$ and $\mathcal{P}'$ the instance of CSP$(\{\alpha', \beta'\})$ obtained from $\mathcal{P}$ as described in the statement of the lemma. We shall prove that

$$\mathcal{Z}_{\{\alpha,\beta\},\omega}(\mathcal{P}) = \mathcal{Z}_B(\mathcal{P}').$$

The crucial observation is that for every solution $\sigma' : V \to D' = \{1, \dots, k\} \times \{1, \dots, \ell\}$ of $\mathcal{P}'$, its weight $\omega'(\sigma')$ is precisely the sum of the weights $\omega(\sigma)$, where the sum ranges over all solutions $\sigma : V \to D$ that map each variable $v \in V$ with $\sigma'(v) = (i,j)$ to the intersection of the $i$th equivalence class of $\alpha$ and the $j$th equivalence class of $\beta$.

Let us make this precise: Let $C_1, \dots, C_k$ and $D_1, \dots, D_\ell$ be the equivalence classes of $\alpha$ and $\beta$, respectively. For every $\sigma : V \to D$, we let $F(\sigma) : V \to \{1, \dots, k\} \times \{1, \dots, \ell\}$ be the mapping defined by

$$F(\sigma)(v) = (i,j) \iff \sigma(v) \in C_i \cap D_j.$$

We observe that $\sigma$ is a solution for $\mathcal{P}$ if and only if $F(\sigma)$ is a solution for $\mathcal{P}'$. To see this, let $\langle (u,v), \alpha \rangle \in \mathcal{C}$ be a constraint of $\mathcal{P}$. Then $\langle (u,v), \alpha' \rangle$ is a constraint of $\mathcal{P}'$. If $\sigma$ is a solution of $\mathcal{P}$, then $\sigma(u)$ and $\sigma(v)$ are in the same equivalence class of $\alpha$, that is, there is some $i$ such that $\sigma(v), \sigma(v) \in C_i$. But $\sigma(v), \sigma(v) \in C_i$ implies that $F(\sigma)(u) = (i,j)$ and $F(\sigma)(v) = (i,j')$ for some $j, j' \in \{1, \dots, \ell\}$. Hence, recalling that $\alpha' = \alpha^w_B$ is the equality relation on the first component, $F(\sigma)(u)$ and $F(\sigma)(v)$ are in the same equivalence class of $\alpha'$. Essentially the same argument shows that, conversely, if $F(\sigma)(u)$ and $F(\sigma)(v)$ are in the same equivalence class of $\alpha'$, then $\sigma(u)$ and $\sigma(v)$ are in the same equivalence class of $\alpha$. Thus $\sigma$ satisfies the constraint $\langle (u,v), \alpha\rangle \in \mathcal{C}$ if and only if $F(\sigma)$ satisfies the corresponding constraint $\langle (u,v), \alpha'\rangle$. Constraints involving $\beta$ are dealt with similarly.

Now let $\sigma' : V \to \{1, \ldots, k\} \times \{1, \ldots, \ell\}$ be a solution of $\mathcal{P}'$. Then we have

$$\begin{aligned}
\omega'(\sigma') &= \prod_{v \in V} B_{\sigma'(v)} \\
&= \prod_{v\in V} \sum_{\substack{d \in C_i \cap D_j \\ \text{where } \sigma'(v)=(i,j)}} \omega(d) \qquad\qquad \text{(because } B = B(\alpha,\beta,\omega)) \\
&= \sum_{\substack{\sigma: V \to D \\ F(\sigma)=\sigma'}} \prod_{v \in V} \omega(\sigma(v)) \\
&= \sum_{\sigma \in F^{-1}(\sigma')} \omega(\sigma).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{Z}_B(\mathcal{P}') &= \sum_{\substack{\sigma' : V \to \{1,\ldots,k\}\times\{1,\ldots,\ell\} \\ \text{solution of } \mathcal{P}'}} \omega'(\sigma') \\
&= \sum_{\substack{\sigma' : V \to \{1,\ldots,k\}\times\{1,\ldots,\ell\} \\ \text{solution of } \mathcal{P}'}} \sum_{\sigma \in F^{-1}(\sigma')} \omega(\sigma) \\
&= \sum_{\substack{\sigma : V \to D \\ \text{solution of } \mathcal{P}}} \omega(\sigma) \\
&= \mathcal{Z}_{\{\alpha,\beta\},\omega}(\mathcal{P}).
\end{aligned}$$

This completes the proof of the lemma. □

Note that for a non-negative integer matrix $B \in \mathbb{Z}^{k\times\ell}$ we have defined both a canonical template $(D_B, \{\alpha_B, \beta_B\})$ and a canonical weighted template $(D_B^w, \{\alpha_B^w, \beta_B^w\}, \omega_B^w)$, and they are not the same (that is, $(D_B, \{\alpha_B, \beta_B\}) \neq (D_B^w, \{\alpha_B^w, \beta_B^w\})$). However, it is easy to see that they define equivalent constraint satisfaction problems:

**Corollary 8** *For every non-negative integer matrix $B \in \mathbb{Z}^{k\times\ell}$ the problems* #CSP($\{\alpha_B, \beta_B\}$) *and* WCSP($B$) *are equivalent (in the sense that each instance yields the same result).*

**PROOF.** Define a weight function $\omega : D_B \to \mathbb{R}$ on the canonical template for $B$ by letting $\omega(d) = 1$ for all $d \in D_B$. Then we have $B(\alpha_B, \beta_B, \omega) = B$, and the problems $\mathrm{CSP}(\{\alpha_B, \beta_B\})$, $\#\mathrm{CSP}(\{\alpha_B, \beta_B\})$, and $\mathrm{WCSP}(\{\alpha_B, \beta_B\}, \omega)$ have the same instances. Furthermore, for each instance $\mathcal{P}$ we have

$$\begin{aligned}
\mathcal{Z}_B(\mathcal{P}) &= \mathcal{Z}_{\{\alpha_B, \beta_B\}, \omega}(\mathcal{P}) && \text{(by Lemma 7)} \\
&= \sum_{\substack{\sigma: V \to D_B \\ \sigma \text{ solution of } \mathcal{P}}} \omega(\sigma) \\
&= \sum_{\substack{\sigma: V \to D_B \\ \sigma \text{ solution of } \mathcal{P}}} 1,
\end{aligned}$$

which is precisely the number of solutions of $\mathcal{P}$. $\square$

The following useful lemma is an immediate consequence of the definitions.

**Lemma 9** *Let $B, B' \in \mathbb{S}^{k \times \ell}$ be such that $B'$ is obtained from $B$ by permuting rows and/or columns. Then $\mathcal{Z}_B = \mathcal{Z}_{B'}$.*

### *4.2 Back and Forth between CSP and $H$-colouring*

The next lemma shows that weighted CSP for two equivalence relations are equivalent to evaluation problems for weighted bipartite graphs.

**Lemma 10** *Let $B \in \mathbb{S}^{k \times \ell}$. Then the problems* $\mathrm{WCSP}(B)$ *and* $\mathrm{EVAL}(\mathsf{bip}(B))$ *are polynomial time equivalent.*

**PROOF.** The proof is based on the observation that if $G = (V, E)$ is a bipartite graph with bipartition $V_1, V_2$, then we can define two natural equivalence relations $\alpha, \beta$ on the set of edges by letting $e, e'$ be $\alpha$-equivalent if they have a common endpoint in $V_1$ and $\beta$-equivalent if they have a common endpoint in $V_2$. Conversely, if $E$ is a set and $\alpha$ and $\beta$ are two equivalence relations on $E$, then we can define a bipartite graph $G$ with edge set $E$ by letting the vertices of $G$ be the equivalence classes of $\alpha$ and $\beta$ and letting an edge connect the $\alpha$-class and $\beta$-class that it belongs to.

Let

$$A = \mathsf{bip}(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathbb{S}^{(k+\ell) \times (k+\ell)}.$$

We first reduce $\mathrm{EVAL}(A)$ to $\mathrm{WCSP}(B)$. Let $G = (V, E)$ be a directed graph. Observe that $Z_A(G) = 0$ unless there is a bipartition $V_1, V_2$ of $V$ such that $E \subseteq V_1 \times V_2$ (that is, all edges are directed from $V_1$ to $V_2$). In the following, we assume that there is such a bipartition $V_1, V_2$.

We have to construct an instance $\mathcal{P}$ of WCSP$(B)$ such that $Z_A(G) = \mathcal{Z}_B(\mathcal{P})$. Let $(D, \{\alpha, \beta\}, \omega)$ be the canonical weighted template for $B$. We let $\mathcal{P}$ be the following instance of WCSP$(B)$:

- The variables of $\mathcal{P}$ are the edges of $G$.
- The domain is $D = \{1, \ldots, k\} \times \{1, \ldots, \ell\}$, the domain of the canonical weighted template.
- For all edges $e, e' \in E$ that have the same endpoint in $V_1$, there is a constraint $\langle (e, e'), \alpha \rangle$.
- For all edges $e, e' \in E$ that have the same endpoint in $V_2$, there is a constraint $\langle (e, e'), \beta \rangle$.

We now show how to associate with every $\sigma : V \to \{1, \ldots, k + \ell\}$ such that $\omega_A(\sigma) \neq 0$ a solution $\sigma^* : E \to D$ for $\mathcal{P}$.

Let $\sigma : V \to \{1, \ldots, k + \ell\}$ such that $\omega_A(\sigma) \neq 0$. Then $\sigma(V_1) \subseteq \{1, \ldots, k\}$ and $\sigma(V_2) \subseteq \{k + 1, \ldots, \ell\}$. We define $\sigma^* : E \to D$ by letting $\sigma^*((u, v)) = (\sigma(u), \sigma(v) - k)$ for every edge $(u, v) \in E$. It is not hard to see that $\sigma^*$ is a solution for the instance $\mathcal{P}$. For example, if $e = (u, v), e' = (u, v') \in E$ are edges that have the same endpoint in $V_1$ then $\sigma^*(e)$ and $\sigma^*(e')$ have the same first coordinate $\sigma(u)$ and therefore are in relation $\alpha$ of the canonical weighted template. Thus the constraint $\langle (e, e'), \alpha \rangle$ is satisfied. Conversely, every solution for $\mathcal{P}$ is of the form $\sigma^*$ for some $\sigma$ with $\omega_A(\sigma) \neq 0$. Furthermore, we have

$$\omega_A(\sigma) = \prod_{(v,w) \in E} B_{\sigma(v)\sigma(w)} = \omega_B(\sigma^*).$$

Thus

$$Z_A(G) = \sum_{\sigma} \omega_A(\sigma) = \sum_{\sigma^*} \omega_B(\sigma^*) = \mathcal{Z}_B(\mathcal{P}).$$

This yields a reduction from EVAL$(A)$ to WCSP$(B)$.

To reduce WCSP$(B)$ to EVAL$(A)$, let $\mathcal{P} = (V, D, \mathcal{C}))$ be an instance of WCSP$(B)$. Let

$$\alpha' := \{(u, v) \in V^2 \mid \langle (u, v), \alpha \rangle \in \mathcal{C}\} \text{ and } \beta' := \{(u, v) \in V^2 \mid \langle (u, v), \beta \rangle \in \mathcal{C}\}.$$

Without loss of generality we may assume that $\alpha'$ and $\beta'$ are equivalence relations. To see this, just note that since $\alpha$ and $\beta$ are equivalence relations, every solution $\sigma : V \to D$ also satisfies all constraints of the form $\langle (u, v), \alpha \rangle$, where $(u, v)$ is in the reflexive symmetric transitive closure of $\alpha'$, and $\langle (u, v), \beta \rangle$, where $(u, v)$ is in the reflexive symmetric transitive closure of $\beta'$. Let $C_1, \ldots, C_k$ be the equivalence classes of $\alpha'$ and $D_1, \ldots, D_\ell$ the equivalence classes of $\beta'$. Let $G$ be the directed graph defined as follows: The vertex set is $\{1, \ldots, k + \ell\}$, and for $1 \leq i \leq k$, $1 \leq j \leq \ell$ there are $|C_i \cap D_j|$ edges from $i$ to $(k + j)$. It is easy to see that $\mathcal{Z}_B(\mathcal{P}) = Z_A(G)$. This yields a reduction from WCSP$(B)$ to EVAL$(A)$. $\square$

The following lemma is needed to derive the hardness part of Theorem 1 from the hardness results on weighted CSP.

**Lemma 11** *Let $A \in \mathbb{S}^{k\times k}$. Then* WCSP$(A)$ *is polynomial time reducible to* EVAL$(A)$.

**PROOF.** Let $A' = \mathsf{bip}(A)$. By Lemma 10, it suffices to prove that EVAL$(A')$ is reducible to EVAL$(A)$.

Let $G = (V, E)$ be a directed graph. If $G$ is not bipartite with all edges directed from one part to the other, then $Z_{A'}(G) = 0$. Therefore, we assume that there is a partition $V_1, V_2$ of $V$ such that $E \subseteq V_1 \times V_2$. We claim that

$$Z_{A'}(G) = Z_A(G). \tag{2}$$

Note that for every $\sigma' : V \to \{1, \ldots, 2k\}$ with $\omega_{A'}(\sigma') \neq 0$ we have $\sigma'(V_1) \subseteq \{1, \ldots, k\}$ and $\sigma'(V_2) \subseteq \{k + 1, \ldots, 2k\}$.

For $\sigma : V \to \{1, \ldots, k\}$, let $f(\sigma) : V \to \{1, \ldots, 2k\}$ be defined by $f(\sigma)(v_1) = \sigma(v_1)$ and $f(\sigma)(v_2) = \sigma(v_2) + k$ for all $v_1 \in V_1$, $v_2 \in V_2$. Then $\omega_A(\sigma) = \omega_{A'}(\sigma')$. Moreover, $f$ is one-to-one, and for every $\sigma' : V \to \{1, \ldots, 2k\}$ with $\omega_{A'}(\sigma') \neq 0$ there exists $\sigma : V \to \{1, \ldots, k\}$ such that $\sigma' = f(\sigma)$. This proves (2). $\square$

We close this section with another lemma which will be used later.

**Lemma 12** *Let $B \in \mathbb{S}^{k\times \ell}$. Then* EVAL$(B \cdot B^\top)$ *is polynomial time reducible to* WCSP$(B)$.

**PROOF.** By Lemma 10, it suffices to show that EVAL$(B\cdot B^\top)$ is polynomial time reducible to EVAL$(\mathsf{bip}(B))$.

For a given graph $G = (V, E)$, let $G' = (V', E')$ be the digraph obtained from $G$ by replacing every edge by two edges pointing to a new vertex. More precisely, let $V' = V \cup V_E$, where $V_E = \{v_e \mid e \in E\}$, and $E' = \{(u, v_{(u,v)}), (v, v_{(u,v)}) \mid (u, v) \in E\}$.

Observe that for every mapping $\sigma' : V' \to \{1, \ldots, k + \ell\}$ with $\omega_{\mathsf{bip}(B)}(\sigma') \neq \emptyset$ we have $\sigma(V) \subseteq \{1, \ldots, k\}$ and $\sigma(V_E) \subseteq \{k + 1, \ldots, \ell\}$. Thus

$$Z_{\mathsf{bip}(B)}(G') = \sum_{\sigma' : V' \to \{1,\ldots,k+\ell\}} \omega_{\mathsf{bip}(B)}(\sigma')$$

$$
\begin{aligned}
&= \sum_{\sigma:V\to\{1,\ldots,k\}} \sum_{\sigma_E:V_E\to\{k+1,\ldots,\ell\}} \prod_{e=(u,v)\in E} B_{\sigma(u)\sigma_E(v_e)-k} B_{\sigma(v)\sigma_E(v_e)-k} \\
&= \sum_{\sigma:V\to\{1,\ldots,k\}} \prod_{e=(u,v)\in E} \sum_{i=1}^{\ell} B_{\sigma(u)i} B_{\sigma(v)i} \\
&= Z_{B\cdot B^\top}(G).
\end{aligned}
$$

Thus the mapping $G \mapsto G'$ yields a polynomial time reduction from EVAL$(B \cdot B^\top)$ to EVAL$(\mathsf{bip}(B))$. □

## 5 The results

We are now able to state the main results of the paper in their most general form.

**Theorem 13** *Let $A$ be a symmetric matrix with non-negative entries from $\mathbb{S}$.* EVAL$(A)$ *is in polynomial time if the row rank of each non-bipartite connected component of $A$ is at most* 1 *and the row rank of each bipartite component is at most 2. Otherwise* EVAL$(A)$ *is* #P*-hard.*

Note that for $\mathbb{S} = \mathbb{R}$, Theorem 13 is equivalent to Theorem 1.

**Theorem 14** *Let $B \in \mathbb{S}^{k\times\ell}$ be a non-negative matrix.* WCSP$(B)$ *is in polynomial time if and only if the row rank of each block of $B$ is at most 1. Otherwise* WCSP$(B)$ *is* #P*-hard.*

The difficult parts of these theorems are the hardness results. They follow from Theorem 16, to be stated and proved in the next section. We now show how to prove the theorems using Theorem 16 and the results of Sections 3 and 4.

**PROOF of Theorem 13 and Theorem 14.** The hardness part of Theorem 14 is precisely Theorem 16. The tractability part of Theorem 13 follows from Theorem 6. It remains to prove the hardness part of Theorem 13 and the tractability part of Theorem 14.

For the former, let $A$ be a symmetric matrix with non-negative entries from $\mathbb{S}$ that either has a non-bipartite connected component of row rank at least 2 or a bipartite connected component of row rank at least 3. By Lemma 4, in both cases $A$ has a block of row rank at least 2. Then by Theorem 16, WCSP$(A)$ is #P-hard. Hence by Lemma 11, EVAL$(A)$ is #P-hard.

To prove the tractability part of Theorem 14, let $B \in \mathbb{S}^{k\times\ell}$ be a non-negative matrix such that the row rank of every block of $B$ is at most 1. Then by Lemma 5, the row rank of every connected component of the matrix $\mathsf{bip}(B) \in \mathbb{S}^{(k+\ell)\times(k+\ell)}$ is at most 1. Hence by Theorem 6(1), EVAL($\mathsf{bip}(B)$) is in polynomial time. By Lemma 10, it follows that WCSP($B$) is in polynomial time. □

Making use of Corollary 8 we derive a classification result for the counting constraint satisfaction problem.

**Corollary 15** *Let $\alpha, \beta$ be equivalence relations on a set $D$. $\#\mathrm{CSP}(\{\alpha, \beta\})$ is in polynomial time if and only if the row rank of each block of $B(\alpha, \beta)$ is at most 1. Otherwise $\#\mathrm{CSP}(\{\alpha, \beta\})$ is #P-hard.*

## 6 The Main Hardness Theorem

**Theorem 16** *Let $B \in \mathbb{S}^{k\times\ell}$ be non-negative such that at least one block of $B$ has row rank at least 2. Then* WCSP($B$) *is #P-hard.*

### *6.1 Outline of the proof*

Before we prove Theorem 16, we give a brief outline of the proof. Let $B \in \mathbb{S}^{k\times\ell}$ be a non-negative matrix such that at least one block of $B$ has row rank at least 2.

*Step 1: From numbers to polynomials (Subsection 6.2).*

In this first step of the proof we show that we can assume that all non-zero entries of $B$ are powers of some indeterminate $X$. More precisely, we prove that there is a matrix $B^*$ whose non-zero entries are powers of $X$ such that $B^*$ also has a block of row rank at least 2 and WCSP($B^*$) is polynomial time reducible to WCSP($B$). The construction is based on a lemma, which essentially goes back to [5], stating that the problem WCSP($B$) is equivalent to the problem of counting all solutions of a given weight. For simplicity, let us assume here that all entries of $B$ are non-negative integers; additional tricks are required for real matrices. We can use the lemma to filter out powers of a particular prime $p$ from all entries of $B$. This way we obtain a matrix $B'$ whose non-zero entries are powers of a prime $p$. Using a technique which corresponds to "thickening" in the graph context (cf. [13,5]), we can replace the entries of this matrix by arbitrary powers, and by interpolation we can then replace $p$ by the indeterminate $X$. This gives us the desired matrix $B^*$.

From now on, we assume that all non-zero entries of $B$ are powers of $X$.

*Step 2: Further preparations (Subsections 6.3–6.6).*

The second step consists of a sequence of reductions that further simplify the structure of the matrix $B$. At the end of these reductions, $B$ satisfies a set of *General Conditions*, which imply that it has a cell structure as the matrix displayed in Figure 2 (on page 43), where the $*$-cells contain powers of $X$ greater than 1. (More precisely, we prove that there is a matrix $B'$ of the desired form such that the weighted CSP of $B'$ is reducible to that of $B$.)

All reductions carried out in step 2 are some form of "gadget constructions", and neither of them is particularly difficult. However, there are a lot of them. In the following, we outline the main (sub)steps in more detail:

**Step 2a: Expanding the constraint language (Subsection 6.3).** We show that we can expand the constraint language underlying our problems without increasing the complexity. Of course a larger constraint language makes it easier to prove hardness.

**Step 2b: Permutable Equivalence Relations (Subsection 6.4).** We review a result due to [1] stating that the counting CSP for a language consisting of two *non-permutable* equivalence relations is hard and adapt the result to our weighted context.

**Step 2c: Eliminating the 0-Entries (Subsection 6.5).** We show that we can assume our matrix to be positive.

**Step 2c: Re-arranging the 1-Entries (Subsection 6.6).** We show that we can arrange the 1-entries of the matrix in order to obtain a matrix of the desired form.

*Step 3: Proving Hardness (Subsections 6.7 and 6.8).*

In this step, we give the actual hardness proof for matrices $B$ of the form obtained in Step 2.

**Step 3a: Separate Ones (Subsection 6.7).** We first consider the case that $B$ has at least two cells containing 1-entries (cf. Figure 2 on page 43). It is not hard to see that in this case we may assume that all diagonal entries of $B$ are 1s. Essentially, we show that we can reduce the problem EVAL($A$) for a symmetric non-singular $2 \times 2$-matrix $A$ to WCSP($B$). For such matrices $A$ the problem EVAL($A$) is #P-hard by a reduction from the problem of counting MAXCUTs of a graph.

**Step 3b: All 1s together (Subsection 6.8).** This part of the proof is the hardest, and it is difficult to describe on a high level. We assume that all entries of $B$ are positive and that a principal submatrix in the upper left corner of $B$ contains all the 1s. We define a sequence $B^{[k]}$, for $k \geq 1$, of matrices that are obtained from $B$ by some construction on the instances that is remotely similar to "stretching" and "thickening" (cf. [13,5]), but more complicated. We show that WCSP($B^{[k]}$) is reducible to WCSP($B$) for all $k$.

The entries of the $B^{[k]}$ are polynomials with integer coefficients (no longer just powers of $X$ as the entries of $B$). Employing a little bit of complex analysis, we prove that for some $k$, $B^{[k]}_{11}$ has an irreducible factor $p(X)$ such that the multiplicity of $p(X)$ in $B^{[k]}_{11}$ is higher than in all other entries in the first row and column, and the multiplicity in the corresponding diagonal entries is also sufficiently high. Using similar tricks as in Step 1, we can filter out the powers of this irreducible polynomial $p(X)$. We obtain a matrix whose weighted CSP is #P-hard by Step 3a.

### *6.2 From numbers to polynomials*

Let $q$ be an arbitrary element of the ring $\mathbb{S}$. A *$q$-matrix* is a matrix $B$ such that all non-zero entries of $B$ are powers of $q$. We are mainly interested in $X$-matrices, where $X$ is an indeterminate. (We view $X$-matrices as matrices over the ring $\mathbb{Z}[X]$). Note that $X$-matrices are always non-negative.

In this section, we shall prove the following lemma:

**Lemma 17 (X-Lemma)** *Let $B \in \mathbb{S}^{m \times n}$ be a non-negative matrix that has a block of row rank at least 2.*

*Then there exists an $X$-matrix $C \in \mathbb{Z}[X]^{m \times n}$ such that $C$ has a block of row rank at least 2 and* WCSP($C$) *is reducible to* WCSP($B$).

The proof consists of a sequence of lemmas; it will be completed at the end of the section. We decided to first prove the lemma for $\mathbb{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Z}[X], \mathbb{Q}[X]\}$ and state all intermediate lemmas in this section only for integer and rational matrices. The proof for real matrices is very similar, but requires one additional idea. It will be given in one chunk at the end of this section.

While our main purpose in this section is a proof of the X-Lemma 17, along the way we obtain other useful results. In particular, the Prime Elimination Lemma 24 will be used later.

We shall frequently use the standard interpolation technique based on the following well known lemma.

**Lemma 18 (Lemma 3.2, [5])** *Let $w_1, \dots, w_r$ be known distinct nonzero constants. Suppose that we know values $f_1, \dots, f_r$ such that*

$$f_s = \sum_{i=1}^{r} c_i w_i^s$$

*for $1 \le s \le r$. the coefficients $c_1, \dots, c_r$ can be evaluated in polynomial time.*

We need more general versions of some results of [5]. First of all we show that a transformation similar to 'thickening' can be devised for weighted CSP. For every matrix $B \in \mathbb{S}^{m\times n}$ and every $\ell \ge 0$, we let $B^{(\ell)}$ denote the matrix whose entries are $(B_{ij})^\ell$.

**Lemma 19** *For every matrix $B \in \mathbb{S}^{m\times n}$ and every $\ell \ge 0$, the problem* $\mathrm{WCSP}(B^{(\ell)})$ *is polynomial time reducible to* $\mathrm{WCSP}(B)$.

**PROOF.** Note that the canonical weighted template for $B^{(\ell)}$ has the same domain $D$ and the same equivalence relations $\alpha, \beta$ as the canonical weighted template for $B$ and the weight function $\omega^{(\ell)}$ defined by $\omega^{(\ell)}(d) = \omega(d)^\ell$.

Take an instance $\mathcal{P} = (V, D, \mathcal{C})$ of $\mathrm{WCSP}(B^{(\ell)})$. Then

- replace every $v \in V$ with $v_1, \dots, v_l$ and denote the resulting set by $V'$;
- replace every constraint $\langle (u, v), \alpha\rangle$ with $\langle (u_1, v_1), \alpha\rangle$;
- replace every constraint $\langle (u, v), \beta\rangle$ with $\langle (u_1, v_1), \beta\rangle$;
- for $v \in V$ and $1 \le i, j \le l$, include the constraints $\langle (v_i, v_j); \alpha\rangle$, $\langle (v_i, v_j); \beta\rangle$;
- denote the resulting set of constraints by $\mathcal{C}'$ and the problem $(V', D, \mathcal{C}')$ by $\mathcal{P}'$.

Clearly, $\mathcal{P}'$ is an instance of $\mathrm{WCSP}(B)$ with $\mathcal{Z}_B(\mathcal{P}) = \mathcal{Z}_{B^{(\ell)}}(\mathcal{P}')$. $\square$

We occasionally denote matrices over a polynomial ring such as $\mathbb{Q}[X]$ by $B(X)$, just to emphasise that the entries of the matrix are polynomials in $X$. Then for every element $a$ of the underlying ring, by $B(a)$ we denote the matrix obtained by substituting $X$ by $a$ in each entry.

The proofs of the following two lemmas are straightforward:

**Lemma 20** *For every matrix $B(X) \in \mathbb{Q}[X]^{m\times n}$ there is positive integer $a$ such that* $\mathsf{rank}(B(a)) = \mathsf{rank}(B(X))$.

**Lemma 21 (Substitution Lemma)** *For every matrix $B(X) \in \mathbb{Q}[X]^{m\times n}$ and every $a \in \mathbb{Q}$, the problem* WCSP$(B(a))$ *is polynomial time reducible to* WCSP$(B(X))$.

For a matrix $B \in \mathbb{S}^{m\times n}$ and an instance $\mathcal{P} = (V, D, \mathcal{C})$ of WCSP$(B)$, we define a set $P_B(\mathcal{P})$ of *potential weights for* $\mathcal{P}$ by

$$P_B(\mathcal{P}) = \Big\{ \prod_{\substack{1\le i\le m\\ 1\le j\le n}} (B)_{ij}^{m_{ij}} \mid 0 \le m_{ij} \le |V| \text{ for } 1 \le i \le m, 1 \le j \le n \text{ such that } \textstyle\sum_{\substack{1\le i\le m\\ 1\le j\le n}} m_{ij} = |V| \Big\}.$$

Then

$$\{\omega_B(\sigma) \mid \ \sigma \text{ is a solution of } \mathcal{P}\} \subseteq P_B(\mathcal{P}).$$

Note that for fixed $B$ the size of $P_B(\mathcal{P})$ is polynomial in the size of $\mathcal{P}$ and that $P_B(\mathcal{P})$ can be computed in polynomial time. Also note that

$$\mathcal{Z}_B(\mathcal{P}) = \sum_{w\in P_B(\mathcal{P})} w \cdot N_B(\mathcal{P}, w), \tag{3}$$

where $N_B(\mathcal{P}, w)$ denotes the number of solutions $\sigma$ of $\mathcal{P}$ such that $\omega_B(\sigma) = w$. Let COUNT$(B)$ denote the following problem:

| |
|---|
| *Input:* WCSP$(B)$-instance $\mathcal{P}$, $w \in \mathbb{S}$. |
| *Objective:* Compute $N_B(\mathcal{P}, w)$. |

**Lemma 22** *Let $B \in \mathbb{S}^{m\times n}$. Then the problems* WCSP$(B)$ *and* COUNT$(B)$ *are polynomial time equivalent.*

**PROOF.** The following proof mainly follows the proof of Lemma 3.3 from [5]. As is noticed above the set $P_B(\mathcal{P})$ can be constructed in polynomial time. Thus (3) yields a polynomial time reduction from WCSP$(B)$ to COUNT$(B)$.

To prove the converse, let $\mathcal{P}, w$ be an instance of COUNT$(B)$. Suppose that $w_1, \ldots, w_t$ are the non-zero elements of $P_B(\mathcal{P})$. If $w \notin P_B(\mathcal{P})$ then $N_B(\mathcal{P}, w) =$

0. Assume now that $w = w_j \in P_B(\mathcal{P})$. For $1 \le \ell \le t$, consider the number $\mathcal{Z}_{B^{(\ell)}}(\mathcal{P})$. We have

$$\mathcal{Z}_{B^{(\ell)}}(\mathcal{P}) = \sum_{\substack{\sigma \text{ a solution} \\ \text{to } \mathcal{P}}} \prod_{v \in V} \omega_B^\ell(v) = \sum_{w \in P_B(\mathcal{P})} w^\ell N_B(\mathcal{P}, w).$$

If $\mathbb{S}$ is a numerical ring then we complete the proof applying Lemma 18. If $\mathbb{S}$ is a polynomial ring then denote the value $\mathcal{Z}_{B^{(\ell)}}(\mathcal{P})$ by $f_\ell(X)$ and notice the equations above can be represented in the matrix form

$$\begin{pmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_t(X) \end{pmatrix} = \begin{pmatrix} w_1(X) & w_2(X) & \cdots & w_t(X) \\ w_1^2(X) & w_2^2(X) & \cdots & w_t^2(X) \\ \vdots & \vdots & & \vdots \\ w_1^t(X) & w_2^t(X) & \cdots & w_t^t(X) \end{pmatrix} \cdot \begin{pmatrix} N_B(\mathcal{P}, w_1) \\ N_B(\mathcal{P}, w_2) \\ \vdots \\ N_B(\mathcal{P}, w_t) \end{pmatrix}. \tag{4}$$

On the one hand, the determinant of the square matrix is Vandermonde. Since all $w_1, \ldots, w_t$ are distinct and non-zero, it is also non-zero. On the other hand, this determinant is a non-zero polynomial; let us denote it by $d(X)$. Therefore, there is an integer

$$a \le t \cdot \max_{1 \le j \le t} \mathsf{deg}(w_j) + 1$$

such that $d(a) \neq 0$. Substituting $a$ instead of $X$ in (4), we obtain a numerical matrix equation of the form

$$\begin{pmatrix} f_1(a) \\ f_2(a) \\ \vdots \\ f_t(a) \end{pmatrix} = \begin{pmatrix} w_1(a) & w_2(a) & \cdots & w_t(a) \\ w_1^2(a) & w_2^2(a) & \cdots & w_t^2(a) \\ \vdots & \vdots & & \vdots \\ w_1^t(a) & w_2^t(a) & \cdots & w_t^t(a) \end{pmatrix} \cdot \begin{pmatrix} N_B(\mathcal{P}, w_1) \\ N_B(\mathcal{P}, w_2) \\ \vdots \\ N_B(\mathcal{P}, w_t) \end{pmatrix}.$$

with a regular matrix, which we can solve to find the desired value $N_B(\mathcal{P}, w_j)$. $\square$

We now give a sequence of lemmas that contain statements for both numerical and polynomial matrices. The statements are essentially the same for both, but require a slightly different phrasing and slightly different proofs. We always state the modifications required in the polynomial case in square brackets.

**Lemma 23** *Let $B \in \mathbb{Q}^{k \times \ell}$ [$B \in \mathbb{Q}[X]^{k \times \ell}$] be a non-negative matrix with at least one block of rank at least 2. Then there is a non-negative matrix $C \in \mathbb{Z}^{k \times \ell}$ [$C \in \mathbb{Z}[X]^{k \times \ell}$] satisfying the same condition and such that* WCSP$(C)$ *is polynomial time reducible to* WCSP$(B)$.

**PROOF.** Let $N$ be the least common denominator of entries of $B$ [of coefficients of entries of $B$]. We set $C$ to be the matrix with entries $C_{ij} = N \cdot B_{ij}$. Since for any WCSP$(B)$-instance $\mathcal{P}$

$$\mathcal{Z}_C(\mathcal{P}) = N^{|V|}\mathcal{Z}_B(\mathcal{P}),$$

the problems WCSP$(B)$ and WCSP$(C)$ are polynomial time equivalent. $\square$

**Lemma 24 (Prime Elimination Lemma)** *Let $B \in \mathbb{Z}^{m\times n}$ [$B \in \mathbb{Z}[X]^{m\times n}$] be a non-negative matrix, and $p$ a prime number [an irreducible polynomial]. Let $C$ be the matrix obtained from $B$ by replacing all entries divisible by $p$ with* 0*. Then there is a polynomial time reduction from* WCSP$(C)$ *to* WCSP$(B)$*.*

**PROOF.** We shall reduce WCSP$(C)$ to COUNT$(B)$; this is sufficient by Lemma 22. Given an instance $\mathcal{P} = (V, D, \mathcal{C})$ of WCSP$(C)$, we first compute the set

$$P_C(\mathcal{P}) = (P_B(\mathcal{P}) - \{w \mid w \text{ divisible by } p\}).$$

Then for each $w \in P_C(\mathcal{P}) - \{0\}$, we compute the number $N_B(\mathcal{P}, w)$ using an oracle to COUNT$(B)$. Then we compute

$$\mathcal{Z}_C(\mathcal{P}) = \sum_{w \in P_C(\mathcal{P})} N_B(\mathcal{P}, w) \cdot w.$$

$\square$

Let $p$ be a prime number [an irreducible polynomial]. For an integer $a \in \mathbb{Z}$ [a polynomial $a \in \mathbb{Z}[X]$] we let

$$a|_p = \begin{cases} p^{\max\{k \mid k \geq 0,\ p^k \text{ divides } a\}}, & \text{if } a \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For a matrix $B \in \mathbb{Z}^{m\times n}$ [a matrix $B \in \mathbb{Z}[X]^{m\times n}$] we let $B|_p$ be the matrix with entries $(B|_p)_{ij} = (B_{ij})|_p$.

**Lemma 25 (Prime Filter Lemma)** *Let $B \in \mathbb{Z}^{m\times n}$ [$B \in \mathbb{Z}[X]^{m\times n}$] be a non-negative matrix, and $p$ a prime number [an irreducible polynomial]. Then* WCSP$(B|_p)$ *is polynomial time reducible to* WCSP$(B)$*.*

**PROOF.** Let $\omega$ be the weight function of the canonical weighted template for $B$ and $\omega'$ the one corresponding to $B|_p$. Let $\mathcal{P}$ be an instance of WCSP$(B|_p)$. Note that for every solution $\sigma$ of $\mathcal{P}$ we have $\omega'(\sigma) = (\omega(\sigma))|_p$. Then by (3),

$$\mathcal{Z}_{B|_p}(\mathcal{P}) = \sum_{w \in P_B(\mathcal{P})} w|_p \cdot N_B(\mathcal{P}, w).$$

Thus WCSP($B|_p$) is polynomial time reducible to COUNT($B$) and hence, by Lemma 22, to WCSP($B$). $\square$

**Lemma 26 (Prime Rank Lemma)** *Let $B \in \mathbb{Z}^{m\times n}$ [$B(X) \in \mathbb{Z}[X]^{m\times n}$] be a non-negative matrix which has a block of rank at least 2. Then there is some prime number [irreducible polynomial] $p$ such that there is a block of $B|_p$ of rank at least 2.*

**PROOF.** Suppose that for all primes [irreducible polynomials] $p$ every block of the matrix $B|_p$ has rank at most 1. We shall prove that any two rows from the same block of $B$ are linearly dependent, which is impossible.

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be two rows from a block of $B$. Then we have

$$a_i = \prod_p a_i|_p \quad \text{and} \quad b_i = \prod_p b_i|_p$$

for all $i$, where the (finite) products are taken over all primes [irreducible polynomials] $p$ dividing an entry of $B$. Since for every prime [irreducible polynomial] $p$ the rank of $B|_p$ is 1, there are $\lambda_p, \mu_p \in \mathbb{Z}$ [$\mathbb{Z}[X]$, respectively] such that $\lambda_p a_i|_p = \mu_p b_i|_p$ for $1 \leq i \leq n$. Let

$$\Lambda = \prod_p \lambda_p \quad \text{and} \quad M = \prod_p \mu_p.$$

Then

$$\Lambda a_i = \Lambda \cdot \prod_p a_i|_p = \prod_p \lambda_p a_i|_p = \prod_p \mu_p b_i|_p = M \cdot \prod_p b_i|_p = M b_i$$

for $1 \leq i \leq n$. This shows that indeed $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent. $\square$

Recall that for every $q \in \mathbb{S}$, a $q$-matrix is a matrix whose non-zero entries are powers of $q$.

**Lemma 27 (Renaming Lemma)** *Let $p \in \mathbb{Z}[X] \setminus \{0, 1, -1\}$ and $B \in \mathbb{Z}[X]^{m\times n}$ a $p$-matrix. Let $q \in \mathbb{Z}[X]$, and let $C$ be the matrix obtained from $B$ by replacing powers of $p$ with corresponding powers of $q$, that is, by letting*

$$C_{ij} = \begin{cases} q^l, & \text{if } B_{ij} = p^l \text{ for some } l \geq 0, \\ 0, & \text{if } B_{ij} = 0. \end{cases}$$

*Then* WCSP($C$) *is polynomial time reducible to* WCSP($B$).

**PROOF.** Let us denote $Y$ be an indeterminate and $C'$ the matrix obtained from $C$ by replacing powers of $p$ with corresponding powers of $Y$. Let $\ell_{\max}$ be maximum such that $Y^{\ell_{\max}}$ is an entry of $C'$. For every instance $\mathcal{P}$ of WCSP$(C')$ with, say, $m$ variables, $Z_{C'}(\mathcal{P})$ is a polynomial in $Y$ of degree at most $m \cdot \ell_{\max}$. Using an oracle to WCSP$(B)$, we can evaluate this polynomial for $Y = p$. By Lemma 19, we can actually evaluate the polynomial for $Y = p^i$ for all $i \geq 0$. Since $p \notin \{-1, 0, 1\}$, this gives us sufficiently many distinct values to interpolate and compute the coefficients of the polynomial. Then we can also compute its value for $Y = q$, that is, $Z_C(\mathcal{P})$. $\square$

We are now ready to prove the X-Lemma.

**PROOF of the X-Lemma 17.** We first prove the lemma for $\mathbb{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Z}[X], \mathbb{Q}[X]\}$. Let $\mathbb{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Z}[X], \mathbb{Q}[X]\}$ and $B \in \mathbb{S}^{m \times n}$. If $\mathbb{S} \in \{\mathbb{Q}, \mathbb{Q}[X]\}$, we first apply Lemma 23. Thus without loss of generality we may assume that $\mathbb{S} \in \{\mathbb{Z}, \mathbb{Z}[X]\}$. By the Prime Filter Lemma 25 and the Prime Rank Lemma 26, we may assume that $B$ is a $p$-matrix for some prime [irreducible polynomial] $p$. Now we can apply the Renaming Lemma 27 with $q = X$. This completes the proof of the X-Lemma for $\mathbb{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Z}[X], \mathbb{Q}[X]\}$.

It remains to prove the lemma for real matrices (see [15] for details about the model of real computation we use). The proof is very similar to the proof for integer matrices, the only difference being that we have to replace the primes involved there by suitable real numbers forming what we will call l-basis for the matrix $B$.

Let $B \in \mathbb{R}^{k \times \ell}$ be a non-negative matrix. Let $a_1, \ldots, a_m$ be the positive entries of $B$. For $1 \leq i \leq m$, let $r_i = \ln a_i$. We view $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and are interested in the subspace $L_B$ generated by $r_1, \ldots, r_m$. Let $1 \leq i_1 < \ldots < i_n \leq m$ such that $r_{i_1}, \ldots, r_{i_n}$ form a basis of $L_B$. For $1 \leq j \leq n$, let $b_j = a_{i_j}$. We call $\{b_1, \ldots, b_n\}$ an *l-basis* for $B$. Note that every positive entry of $B$ has a unique representation $b_1^{q_1} \ldots b_n^{q_n}$, where $q_1, \ldots, q_n \in \mathbb{Q}$. Since for every positive $s \in \mathbb{Z}$ the problem WCSP$(B^{(s)})$ is reducible to WCSP$(B)$, we may actually assume without loss of generality that every positive entry of $B$ has a unique representation $b_1^{\ell_1} \ldots b_n^{\ell_n}$, where $\ell_1, \ldots, \ell_n \in \mathbb{Z}$. Also note that if $B$ has a block of row rank at least 2 then so does $B^{(s)}$ for every $s \geq 1$. Observe that for every instance $\mathcal{P}$ and $w \in P_B(\mathcal{P})$, $w \neq 0$, we have $\ln w \in L_B$. Thus $w$ has a unique representation $b_1^{m_1} \ldots b_n^{m_n}$ with $m_1, \ldots, m_n \in \mathbb{Z}$. Thus the elements of an l-basis play the same role as that played by primes or irreducible polynomials for integer computations.

For $1 \leq s \leq n$ and $a = b_1^{\ell_1} \ldots b_n^{\ell_n}$ we let $a|_{b_s} = b_s^{\ell_s}$, and we let $A|_{b_s}$ be the matrix with $(A|_{b_s})_{ij} = (A_{ij})|_{b_s}$ if $A_{ij} > 0$ and $(A|_{b_s})_{ij} = 0$ if $A_{ij} = 0$.

Analogously to the Prime Filter Lemma 25 we can prove that for $1 \leq s \leq n$

the problem $\mathrm{WCSP}(B|_{b_s})$ is polynomial time reducible to $\mathrm{WCSP}(B)$. Here we use the fact that for every instance $\mathcal{P}$ every element of the set $P_B(\mathcal{P})$ has a unique representation in terms of our l-basis.

Analogously to the Prime Rank Lemma 26 we can prove that if $B$ has a block of row rank at least 2 then for some $b \in \{b_1, \ldots, b_n\}$ the matrix $B|_b$ has a block of row rank at least 2.

To complete the proof, assume that $B$ has a block of row rank at least 2 and let $b \in \{b_1, \ldots, b_n\}$ such that $B|_b$ also satisfies this condition. Let $X$ be an indeterminate and $C$ the matrix obtained from $B|_b$ by replacing each entry $b^\ell$ by $X^\ell$. Let $\mathcal{P}$ be an instance with $m$ variables. We want to compute the polynomial $q(X) = \mathcal{Z}_C(\mathcal{P})$, which is a polynomial of degree at most $m \cdot \ell_{\max}$ in $X$, where $\ell_{\max}$ is the maximum such that $X^{\ell_{\max}}$ is an entry of $C$. Observing that for $0 \leq r \leq m$ we have

$$\mathcal{Z}_{(B|_b)^{(r)}}(\mathcal{P}) = q(b^r),$$

we can compute the coefficients of $q$ by Lemma 19 and interpolation. This completes the proof of the X-Lemma. $\square$

### *6.3 Expanding the Constraint Language*

Clearly, proving hardness of a CSP becomes easier if the constraint language gets richer. In this section, we will show that the constraint language of our weighted CSP with two equivalence relations can be expanded by certain relations without increasing the complexity. Specifically, for every element of the domain we will add a unary relation that consists only of this element. This will enable us to specify partial solutions in an instance by adding constraints that ensure that certain variables get mapped to specific domain elements. A different perspective on these unary relations is that we add "constants" for the domain elements to our language. Furthermore, if $B$ is a square matrix, we will add a binary relation that contains the diagonal elements of $B$, or more precisely, the elements $(i, i)$ of the canonical weighted template.

The results of this subsection are twofold. First, and most importantly, we show how to reduce the problems over the richer languages to those over the basic language (just consisting of two equivalence relations). For the added constants (unary relations for all domain elements), this will be done in the Constant Reduction Lemma 32. For the language with the relation that contains the diagonal elements, the situation is slightly more complicated, because we only have the extension of the language for square matrices, but would like to apply it to all matrices. In the Symmetrisation Lemma 33, we show that the weighted CSP of the square matrix $B \cdot B^\top$ over the expanded language is reducible to the the weighted CSP of $B$ over the basic language. This shows that it is sufficient to prove hardness for the weighted CSP of a symmetric matrix over the expanded language.

But now we are facing a new problem: We have taken $B$ to be an $X$-matrix, but this does not mean that the symmetric matrix $C = B \cdot B^\top$ is also an $X$-matrix. We could apply the X-Lemma again to the matrix $C$ and would obtain an $X$-matrix $C'$ such that the weighted CSP for $C'$ is reducible to that of $C$. But it is not clear that the reduction also works for the problems over our expanded language. Therefore, we will have to prove an extended version of the X-Lemma (the Extended X-Lemma 36) that also works for the richer language.

Let us now define the extensions of our constraint language. Let $B \in \mathbb{Z}[X]^{n\times m}$ be a matrix and $(D, \{\alpha, \beta\}, \omega)$ be the canonical weighted template for $B$. Recall that $D = \{1, \dots, m\} \times \{1, \dots, n\}$.

For $d \in D$, let $\kappa_d$ be the unary one-element relation $\{(d)\}$ and $K(D)$ the set $\{\kappa_d \mid d \in D\}$. We let $\mathrm{WCSP}^K(B)$ be the weighted CSP over the language consisting of $\alpha$ and $\beta$ and all the $\kappa_d$, that is,

$$\mathrm{WCSP}^K(B) = \mathrm{WCSP}(\{\alpha, \beta\} \cup K(D), \omega).$$

Furthermore, if $m = n$, let $\theta$ be the unary relation $\{(i, i) \mid 1 \le i \le m\}$ on $D$ consisting of all 'diagonal' elements, and let

$$\mathrm{WCSP}^{KD}(B) = \mathrm{WCSP}(\{\alpha, \beta\} \cup K(D) \cup \{\theta\}, \omega).$$

To better understand the problems $\mathrm{WCSP}^K(B)$ and $\mathrm{WCSP}^{KD}(B)$, let us describe the corresponding "partition functions" $\mathcal{Z}_{\Gamma,\omega}$ directly. To simplify the notation, we let

$$\mathcal{Z}_B^K = \mathcal{Z}_{\{\alpha,\beta\}\cup K(D),\omega}$$

be the partition function of $\mathrm{WCSP}^K(B)$ and

$$\mathcal{Z}_B^{KD} = \mathcal{Z}_{\{\alpha,\beta\}\cup K(D)\cup\{\theta\},\omega}$$

the partition function of $\mathrm{WCSP}^{KD}(B)$.

First, let $\mathcal{P} = (V, D, \mathcal{C})$ be an instance of the problem $\mathrm{WCSP}^K(B)$ (or equivalently, an instance of the problem $\mathrm{CSP}(\{\alpha, \beta\} \cup K(D))$). Note first that if $\mathcal{C}$ contains constraints $\langle v, \kappa_d \rangle$ and $\langle v, \kappa_{d'} \rangle$ for $d \ne d'$, then the instance $\mathcal{P}$ has no solutions and thus $\mathcal{Z}_B^K(\mathcal{P}) = 0$. From now on, we only consider instances that do not have constraints $\langle v, \kappa_d \rangle$ and $\langle v, \kappa_{d'} \rangle$ for $d \ne d'$. For each such instance

$\mathcal{P} = (V, D, \mathcal{C})$, let $\varphi_{\mathcal{P}}$ be the partial mapping from $V$ to $D$ defined by

$$\varphi_{\mathcal{P}}(v) = \begin{cases} d & \text{if } \langle v, \kappa_d \rangle \in \mathcal{C}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Note that the instance $\mathcal{P}$ is completely specified by the partial mapping $\varphi_{\mathcal{P}}$ and the *reduct* of $\mathcal{P}$ to the language $\{\alpha, \beta\}$, that is, by the instance $\mathcal{P}|_{\{\alpha,\beta\}} = (V, D, \mathcal{C}|_{\{\alpha,\beta\}})$ with

$$\mathcal{C}|_{\{\alpha,\beta\}} = \{\langle (u, v), \gamma \rangle \in \mathcal{C} \mid u, v \in V, \gamma \in \{\alpha, \beta\}\}.$$

Furthermore, we have

$$\mathcal{Z}^K_B(\mathcal{P}) = \sum_{\substack{\sigma: V \to D \\ \sigma \text{ solution of } \mathcal{P}}} \omega(\sigma) = \sum_{\substack{\sigma: V \to D \text{ with } \varphi_{\mathcal{P}} \subseteq \sigma \\ \sigma \text{ solution of } \mathcal{P}|_{\{\alpha,\beta\}}}} \omega(\sigma).$$

Here we write $\varphi_{\mathcal{P}} \subseteq \sigma$ to denote that $\varphi_{\mathcal{P}}$ is a restriction of $\sigma$.

Conversely, observe that if $\mathcal{P}' = (V, D, \mathcal{C}')$ is an instance of $\mathrm{WCSP}(B)$ and $\varphi$ is a partial mapping from $V$ to $D$, then there is exactly one instance $\mathcal{P}$ of $\mathrm{WCSP}^K(B)$ such that $\varphi_{\mathcal{P}} = \varphi$ and $\mathcal{P}|_{\{\alpha,\beta\}} = \mathcal{P}'$. Thus $\mathrm{WCSP}^K(B)$ is the problem of computing the weighted sum over all solutions extending a given partial solution for an instance $\mathcal{P}'$ of $\mathrm{WCSP}(B)$.

The problem $\mathrm{WCSP}^{KD}(B)$ has a less intuitive meaning, because the diagonal of the matrix $B$ is somewhat arbitrary in the sense that it depends on a specific order of the equivalence classes of $\alpha$ and $\beta$. But having a distinguishable diagonal will be extremely useful later, because it will help us to "pin down" specific entries of the matrix that ultimately cause the hardness of the weighted CSP. Let us briefly discuss how instances of $\mathrm{WCSP}^{KD}(B)$ relate to their reducts over $\{\alpha, \beta\}$. Let $\mathcal{P} = (V, D, \mathcal{C})$ of the problem $\mathrm{WCSP}^{KD}(B)$ (or equivalently, an instance of the problem $\mathrm{CSP}(\{\alpha, \beta\} \cup K(D) \cup \{\theta\})$). Let $\varphi_{\mathcal{P}}$ be defined as above and

$$U_{\mathcal{P}} = \{v \in V \mid \langle v, \theta \rangle \in \mathcal{C}\}.$$

Then $\mathcal{P}$ is completely specified by $\varphi_{\mathcal{P}}$, $U_{\mathcal{P}}$, and and the reduct $\mathcal{P}|_{\{\alpha,\beta\}}$. Conversely, for all instances $\mathcal{P}' = (V, D, \mathcal{C}')$ of $\mathrm{WCSP}(B)$, partial mappings $\varphi$ from $V$ to $D$, and subsets $U \subseteq V$, there is exactly one instance $\mathcal{P}$ of $\mathrm{WCSP}^K(B)$ such that $\varphi_{\mathcal{P}} = \varphi$, $U_{\mathcal{P}} = U$, and $\mathcal{P}|_{\{\alpha,\beta\}} = \mathcal{P}'$.

Our first goal will be to prove that $\mathrm{WCSP}^K(B)$ is reducible to $\mathrm{WCSP}(B)$. The proof relies on a result from [1], which we state as Lemma 31. In addition, we need several lemmas similar to those in the previous subsection. The proofs are usually completely analogous to those of the corresponding lemmas in the previous subsection. We usually need versions of the lemmas for $\mathrm{WCSP}^K(B)$

and $\mathrm{WCSP}^{KD}(B)$. The modifications required for the KD-version are stated in square brackets.

**Lemma 28** *For every matrix $B \in \mathbb{Z}[X]^{m\times n}$ [with $m = n$] and every $\ell \geq 0$, the problem $\mathrm{WCSP}^K(B^{(\ell)})$ [$\mathrm{WCSP}^{KD}(B^{(\ell)})$] is polynomial time reducible to $\mathrm{WCSP}^K(B)$ [$\mathrm{WCSP}^{KD}(B)$].*

**PROOF.** Analogous to the proof of Lemma 19. □

**Lemma 29 (Extended Substitution Lemma)** *Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ [with $m = n$]. For every $a \in \mathbb{Z}$, the problem $\mathrm{WCSP}^K(B(a))$ [$\mathrm{WCSP}^{KD}(B(a))$] is polynomial time reducible to $\mathrm{WCSP}^K(B(X))$ [$\mathrm{WCSP}^{KD}(B(X))$].*

**PROOF.** As Lemma 21, this is obvious. □

**Lemma 30 (Extended Renaming Lemma)** *Let $p \in \mathbb{Z}[X]$ and $B \in \mathbb{Z}[X]^{m\times n}$ [with $m = n$] be a $p$-matrix. Let $q \in \mathbb{Z}[X] \setminus \{0, 1, -1\}$, and let $C$ be the matrix obtained from $B$ by replacing powers of $p$ with corresponding powers of $q$.*

*Then $\mathrm{WCSP}^K(B)$ [$\mathrm{WCSP}^{KD}(B)$] is polynomial time reducible to $\mathrm{WCSP}^K(C)$ [$\mathrm{WCSP}^{KD}(C)$].*

**PROOF.** Analogously to the proof of the Renaming Lemma 27.

**Lemma 31 ([1])** *Let $\Gamma$ be a constraint language over a domain $D$. Then $\#\mathrm{CSP}(\Gamma \cup K(D))$ is polynomial time reducible to $\#\mathrm{CSP}(\Gamma)$.*

We write $P \leq P'$ to denote that problem $P$ is polynomial time reducible to problem $P'$.

**Lemma 32 (Constant Reduction Lemma)** *Let $p(X) \in \mathbb{Z}[X]$ be an irreducible polynomial and $B \in \mathbb{Z}[X]^{m\times n}$ a $p$-matrix. Then $\mathrm{WCSP}^K(B)$ is polynomial time reducible to $\mathrm{WCSP}(B)$.*

**PROOF.** By Lemma 20, the Substitution Lemma 21, the Renaming Lemma 27, the Extended Substitution Lemma 29, and the Extended Renaming Lemma 30, there is a non-negative matrix $C \in \mathbb{Z}^{m\times n}$, which is obtained from $B$ by substituting $X$ by a suitable integer $a$, such that both the problems $\mathrm{WCSP}(B)$, $\mathrm{WCSP}(C)$ and the problems $\mathrm{WCSP}^K(B)$, $\mathrm{WCSP}^K(C)$ are polynomial time equivalent. Let $C$ be such a matrix and let $(D_C, \{\alpha_C, \beta_C\})$ be the canonical template for $C$. By Corollary 8, $\mathrm{WCSP}(C)$ is polynomial time equivalent to

#CSP($\{\alpha_C, \beta_C\}$). By Lemma 31 of [1], the problem #CSP($\{\alpha_C, \beta_C\} \cup K(D_C)$) is polynomial time reducible to #CSP($\{\alpha_C, \beta_C\}$).

We shall prove that $\mathrm{WCSP}^K(C)$ is polynomial time reducible to #CSP($\{\alpha_C, \beta_C\} \cup K(D_C)$. Then the statement of the lemma follows by the following chain of reductions:

$$\begin{aligned}\mathrm{WCSP}^K(B) \leq \mathrm{WCSP}^K(C) &\leq \#\mathrm{CSP}(\{\alpha_C, \beta_C\} \cup K(D_C)) \\ &\leq \#\mathrm{CSP}(\{\alpha_C, \beta_C\}) \leq \mathrm{WCSP}(C) \leq \mathrm{WCSP}(B).\end{aligned}$$

It remains to reduce $\mathrm{WCSP}^K(C)$ to #CSP($\{\alpha_C, \beta_C\} \cup K(D_C)$. It is important for the following reduction to understand the construction of the canonical template and the canonical weighted template for $C$ and the difference between the two (see Subsection 4.1). The canonical weighted template has one domain element $(i, j)$ for each matrix entry $C_{ij}$, and the weight of $(i, j)$ is precisely $C_{ij}$. The canonical template has $C_{ij}$ domain elements for the matrix entry $C_{ij}$, and these elements form the intersection of the $i$the equivalence class of $\alpha_C$ and the $j$th class of $\beta_C$. Let $C_1, \ldots, C_m$ be the equivalence classes of $\alpha_C$ and $D_1, \ldots, D_n$ the equivalence classes of $\beta_C$, both enumerated in such a way that for $1 \leq i \leq m$ and $1 \leq j \leq n$ we have

$$C_{ij} = |C_i \cap D_j|.$$

Now let $\mathcal{P} = (V, D, \mathcal{C})$ be an instance of $\mathrm{WCSP}^K(C)$. We have to construct an instance $\mathcal{P}' = (V', D_C, \mathcal{C}')$ of #CSP($\{\alpha_C, \beta_C\} \cup K(D_C)$) such that $\mathcal{Z}_C^K(\mathcal{P})$ is the number of solutions of $\mathcal{P}'$. Note that $D$ is the domain of the canonical weighted template for $C$, that is, $D = \{1, \ldots, m\} \times \{1, \ldots, n\}$. If $\mathcal{C}$ contains a constraint $\langle v, \kappa_{(i,j)} \rangle$, then without loss of generality we may assume that $C_{ij} \neq 0$, because otherwise we have $\mathcal{Z}_C^K(\mathcal{P}) = 0$. If $C_{ij} \neq 0$ then $C_i \cap D_j \neq \emptyset$. Let us fix an (arbitrary) element $d_{ij} \in C_i \cap D_j$ for all $i, j$ with $C_{ij} \neq 0$. Now the idea is to replace the constraint $\langle v, \kappa_{(i,j)} \rangle$ by $\langle v, \kappa_{d_{ij}} \rangle$. However, this fixes $v$ in all solutions to be mapped to $d_{ij}$ and thus reduces the number of solutions too strongly — the correct number would be obtained if $v$ was allowed to be mapped to any element of $C_i \cap C_j$. Unfortunately, we cannot express this in our limited constraint language. Instead, we introduce an additional variable $v'$ that we fix to be mapped to $d_{ij}$, and we only require $v$ to be mapped to any element in the same $\alpha$-class and $\beta$-class as $v'$.

Let $F_{\mathcal{P}}$ be the set of all fixed values, that is, the set of all $v \in V$ such that there is a constraint of the form $\langle v, \kappa_{(i,j)} \rangle$ in $\mathcal{C}$. We define an instance $\mathcal{P}' = (V', D_C, \mathcal{C}')$ of #CSP($\{\alpha_C, \alpha_C\} \cup K(D_C)$) as follows:

- $V' = V \cup \{v' \mid v \in F_{\mathcal{P}}\}$.
- For every constraint of the form $\langle (u_1, u_2), \alpha \rangle, \langle (v_1, v_2), \beta \rangle \in \mathcal{C}$, we include the constraints $\langle (u_1, u_2), \alpha_C \rangle$, $\langle (v_1, v_2), \beta_C \rangle$ into $\mathcal{C}'$.

- For every $\langle (v),(i,j)\rangle \in \mathcal{C}$, we include $\langle (v,v'),\alpha_C\rangle$, $\langle (v,v'),\beta_C\rangle$ and $\langle (v'),\kappa_{d_{ij}}\rangle$ into $\mathcal{C}'$.

Using the same idea as in the proof of Lemma 7, it is easy to see that $Z_C^K(\mathcal{P})$ equals the number of solutions to $\mathcal{P}'$. □

In the next lemma, we turn to $\mathrm{WCSP}^{KD}(B)$. The lemma shows that we can work with symmetric matrices, and it also shows that we can get rid of the diagonal relation $\theta$ in the constraint language of $\mathrm{WCSP}^{KD}(B)$.

**Lemma 33 (Symmetrisation Lemma)** *Let $B \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix and $C = B \cdot B^\top$. Then*

*(1) $C$ is a symmetric non-negative matrix.*
*(2) $\mathsf{rank}(C) = \mathsf{rank}(B)$.*
*(3) If $B$ has a block of rank at least 2 then $C$ also has such a block.*
*(4) $\mathrm{WCSP}^{KD}(C)$ is polynomial time reducible to $\mathrm{WCSP}^{K}(B)$.*

**PROOF.** Since $C$ is the Gram matrix of the collection of rows of $B$, its rank equals $\mathsf{rank}(B)$. Notice that the same holds for every block of $C$. Therefore, for any block of $B$, the matrix $C$ has a block of the same rank. This proves (1)–(3).

To prove (4), let $\mathcal{D}_B = (D,\alpha,\beta,\omega)$ and $\mathcal{D}_C = (D',\alpha',\beta',\omega')$ be the canonical weighted templates for $B$ and $C$, respectively. Recall that $D = \{1,\dots,m\} \times \{1,\dots,n\}$ and $D' = \{1,\dots,m\}^2$. Also recall that the rows of $B$ [$C$] correspond to the equivalence classes of $\alpha$ [$\alpha'$, respectively] and the columns of $B$ [$C$] correspond to the equivalence classes of $\beta$ [$\beta'$, respectively].

Take an instance $\mathcal{P}' = (V',D',\mathcal{C}')$ of $\mathrm{WCSP}^{KD}(C)$. We define an instance $\mathcal{P} = (V,D,\mathcal{C})$ of $\mathrm{WCSP}^{K}(B)$ (=$\mathrm{WCSP}(\{\alpha,\beta\} \cup K(D))$) as follows:

(i) $V = \{v^1, v^2 \mid v \in V'\} \cup \{v^3, v^4 \mid \langle v, \kappa_{(i,j)}\rangle \in \mathcal{C}'\}$;
(ii) for every constraint $\langle (v,w),\alpha'\rangle \in \mathcal{C}'$, we include $\langle (v^1,w^1),\alpha\rangle$ into $\mathcal{C}$, and for every constraint $\langle (v,w),\beta'\rangle \in \mathcal{C}'$, we include $\langle (v^2,w^2),\alpha\rangle$ into $\mathcal{C}$;
(iii) for every $v \in V'$ we include $\langle (v^1,v^2),\beta\rangle$ into $\mathcal{C}$;
(iv) for every constraint $\langle (v),\kappa_{(i,j)}\rangle \in \mathcal{C}'$, we include the constraints $\langle (v^3),\kappa_{(i,1)}\}\rangle$, $\langle (v^4),\kappa_{(j,1)}\}\rangle$, $\langle (v^1,v^3),\alpha\rangle$, $\langle (v^2,v^4),\alpha\rangle$ into $\mathcal{C}$;
(v) for every constraint $\langle (v),\theta\rangle \in \mathcal{C}'$, we include $\langle (v^1,v^2),\alpha\rangle$ into $\mathcal{C}$.

To understand this, observe that the constraints in (ii) say that whenever $v,w \in V'$ are forced to be mapped to the same row of $C$, then $v^1$ and $w^1$ are forced to the same row of $B$. If $v$ and $w$ are forced to the same column of $C$, then $v^2$ and $w^2$ are forced to the same row(!) of $B$. The constraints in (iii) force $v^1$ and $v^2$ to the same column of $B$. The constraints in (iv) say that if $v \in V'$ is forced to the $(i,j)$-entry of $C$, then $v^3$ is forced to the first element of the $i$th row and $v^4$ is forced to the first element of the $j$th row. Finally, (v) says that if $v' \in V$ is forced onto the diagonal, then $v^1$ and $v^2$ are forced to the same row; since by (iii) they are also forced to the same column, this implies that they are forced to the same position.

Now let us try to understand how solutions for $\mathcal{P}'$ relate to solutions for $\mathcal{P}$. Observe first that the domain of $\varphi_{\mathcal{P}}$, that is, the variables that are fixed by constraints $\kappa_d$, is the set of all variables of the form $v^3, v^4$. Note that variables $v^3, v^4$ are only added for those $v \in V'$ that occur in some constraint $\langle v, \kappa_{(i,j)} \rangle$.

Let $\sigma'$ be a solution of $\mathcal{P}'$. Let $\Psi(\sigma')$ be the set of all solutions $\sigma$ for $\mathcal{P}$ such that for all $v \in V'$, if $\sigma'(v) = (i,j)$, then $\sigma(v^1) = (i,k)$ and $\sigma(v^2) = (j,k)$ for some $k \in \{1, \ldots, n\}$. Then the sets $\Psi(\sigma')$, where $\sigma'$ ranges over all solutions of $\mathcal{P}'$, form a partition of the space of solutions of $\mathcal{P}$. To see that every solution $\sigma$ of $\mathcal{P}$ belongs to some $\Psi(\sigma')$, just recall that by (iii), $v^1$ and $v^2$ must be mapped to the same column.

More formally, for every solution $\sigma$ of $\mathcal{P}$ we define a mapping $\sigma' : V' \to D'$ by letting, for every $v \in V'$,

$$\sigma'(v) = (i,j) \text{ if } \sigma(v^1) = (i,k) \text{ and } \sigma(v^2) = (j,k).$$

This mapping is well-defined on $V'$, because by the constraints (iii) there always exists a suitable $k$. Then, clearly, $\sigma \in \Psi(\sigma')$. To see that $\sigma'$ is a solution of $\mathcal{P}'$, note the following:

- For every constraint $\langle (u,v), \alpha' \rangle \in \mathcal{C}'$, the constraint $\langle (u^1, v^1), \alpha \rangle$ implies that $\sigma(u^1), \sigma(v^1)$ are in the same row, say, $i$. Hence, $(\sigma'(u), \sigma'(v)) = ((i,j),(i,k)) \in \alpha'$ for certain $1 \leq j,k \leq m$.
- For every constraint $\langle (u,v), \beta' \rangle \in \mathcal{C}'$, the constraint $\langle (u^2, v^2), \alpha \rangle$ implies that $\sigma(u^2), \sigma(v^2)$ are in the same row, say, $j$. Therefore, $(\sigma'(u), \sigma'(v)) = ((i,j),(k,j)) \in \beta'$ for certain $1 \leq i,k \leq m$.
- For every constraint $\langle v, \theta \rangle \in \mathcal{C}'$, the constraint $\langle (v^1, v^2), \alpha \rangle$ implies that $\sigma(v^1) = \sigma(v^2)$ are in the same row, say, $i$. Hence $\sigma'(v) = (i,i) \in \theta$.
- For every $v$ such that $\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'$, the constraints $\langle (v^3), \kappa_{(i,1)}\} \rangle$, $\langle (v^4), \kappa_{(j,1)}\} \rangle$ imply $\sigma(v^3) = (i,1)$ and $\sigma(v^4) = (j,1)$. Then the constraints $\langle (v^1, v^3), \alpha \rangle$, $\langle (v^2, v^4), \alpha \rangle$ imply that $\sigma(v^1)$ is in row $i$ and $\sigma(v^2)$ in row $j$. Thus $\sigma'(v) = (i,j) \in \kappa_{(i,j)}$.

Finally, we have

$$\begin{aligned}
\mathcal{Z}_B^K(\mathcal{P}) &= \left(\prod_{\langle (v),\kappa_{(i,j)}\rangle\in\mathcal{C}'} B_{i1}B_{j1}\right)\cdot\left(\sum_{\substack{\sigma' \text{ solution}\\ \text{to } \mathcal{P}'}}\ \sum_{\sigma\in\Psi(\sigma')}\ \prod_{\substack{v\in V'\\ \sigma'(v)=(i,j)\\ \sigma(v^1)=(i,k),\sigma(v^2)=(j,k)}} B_{ik}B_{jk}\right)\\
&= \left(\prod_{\langle (v),\kappa_{(i,j)}\rangle\in\mathcal{C}'} B_{i1}B_{j1}\right)\cdot\left(\sum_{\substack{\sigma' \text{ solution}\\ \text{to } \mathcal{P}'}}\ \prod_{\substack{v\in V'\\ \sigma'(v)=(i,j)}}\ \sum_{k=1}^{n} B_{ik}B_{jk}\right)\\
&= \left(\prod_{\langle (v),\kappa_{(i,j)}\rangle\in\mathcal{C}'} B_{i1}B_{j1}\right)\cdot\left(\sum_{\substack{\sigma' \text{ solution}\\ \text{to } \mathcal{P}'}}\ \prod_{\substack{v\in V'\\ \sigma'(v)=(i,j)}} C_{ij}\right)\\
&= \left(\prod_{\langle (v),\kappa_{(i,j)}\rangle\in\mathcal{C}'} B_{i1}B_{j1}\right)\cdot \mathcal{Z}_C^{KD}(\mathcal{P}').
\end{aligned}$$

Since the term $\prod_{\langle (v),\kappa_{(i,j)}\rangle\in\mathcal{C}'} B_{i1}B_{j1}$ can easily be computed in polynomial time, this yields the desired reduction. □

The last goal of this section is to prove the Extended X-Lemma 36, a version of the X-Lemma 17 for the extended language of $\mathrm{WCSP}^{KD}(B)$. To prove this lemma, we need to extend further results of the previous subsection.

By $\mathrm{COUNT}^{KD}(B)$ we denote the problem of finding the number $N_B^{KD}(\mathcal{P}, w)$ of solutions $\sigma$ of an instance $\mathcal{P}$ of $\mathrm{WCSP}^{KD}(B)$ such that $\omega_B(\sigma) = w$.

**Lemma 34** *Let $B \in \mathbb{Z}[X]^{m\times m}$. Then the problems* $\mathrm{WCSP}^{KD}(B)$ *and* $\mathrm{COUNT}^{KD}(B)$ *are polynomial time equivalent.*

**PROOF.** Analogous to the proof of Lemma 22.

**Lemma 35 (Extended Prime Filter Lemma)** *Let $B \in \mathbb{Z}[X]^{m\times m}$ be a non-negative matrix, and $p$ an irreducible polynomial. Then* $\mathrm{WCSP}^{KD}(B|_p)$ *is polynomial time reducible to* $\mathrm{WCSP}^{KD}(B)$*.*

**PROOF.** Analogous to the proof of the Prime Filter Lemma 25.

Finally, we have reached our goal:

**Lemma 36 (Extended X-Lemma)** *Let $B \in \mathbb{Z}[X]^{m\times m}$ be a symmetric non-negative matrix that has a block of row rank at least 2.*

*Then there exists a symmetric $X$-matrix $C \in \mathbb{Z}[X]^{m\times m}$ such that $C$ has a block of row rank at least 2 and $\mathrm{WCSP}^{KD}(C)$ is reducible to $\mathrm{WCSP}^{KD}(B)$.*

*Moreover, if $B$ is positive then $C$ can also be assumed to be positive.*

**PROOF.** By the Extended Prime Filter Lemma 35 and the Prime Rank Lemma 26, we may assume that $B$ is a $p$-matrix for some irreducible polynomial $p$. Now we can apply the Extended Renaming Lemma 30 with $q = X$.

### *6.4 Permutable Equivalence Relations*

For binary relations $\gamma$ and $\delta$, we let $\gamma \circ \delta$ be the relation consisting of all pairs $(x, y)$ such that there exists a $z$ with $(x, z) \in \alpha$ and $(z, y) \in \beta$. Two equivalence relations $\alpha, \beta$ are said to be *permutable* if

$$\alpha \circ \beta = \beta \circ \alpha.$$

As is easily seen, $\alpha, \beta$ are not permutable if and only if there are $1 \le i, j \le m$, $1 \le k, l \le n$ such that $B(\alpha, \beta)_{ik}, B(\alpha, \beta)_{il}, B(\alpha, \beta)_{jk} \neq 0$, but $B(\alpha, \beta)_{jl} = 0$.

**Lemma 37 ([1])** *If $\alpha, \beta$ are equivalence relations that are not permutable, then the problem $\#\mathrm{CSP}(\{\alpha, \beta\})$ is $\#\mathrm{P}$-hard.*

Our first result in this section is an extension of this lemma to the weighted problem:

**Lemma 38 (Non-Permutability Lemma)** *Let $B \in \mathbb{Z}[X]^{m\times n}$ be a nonnegative matrix such that there exists $1 \le i, k \le n$, $1 \le j, \ell \le n$ with $B_{ik}$, $B_{i\ell}$, $B_{jk} \neq 0$ and $B_{j\ell} = 0$. Then $\mathrm{WCSP}(B)$ is $\#\mathrm{P}$-hard.*

**PROOF.** Let $1 \le i, k \le n$, $1 \le j, \ell \le n$ such that $B_{ik}$, $B_{il}$, $B_{jk} \neq 0$ and $B_{jl} = 0$. Take an integer $a$ such that the matrix $B' = B(a)$ is non-negative and $B'_{ik}, B'_{il}, B'_{jk} \neq 0$. By the Substitution Lemma 21, the problem $\mathrm{WCSP}(B')$ is polynomial time reducible $\mathrm{WCSP}(B)$. Let $(D_{B'}, \{\alpha', \beta'\})$ be the canonical template for $B'$. By Corollary 8, $\#\mathrm{CSP}(\{\alpha_{B'}, \beta_{B'}\})$ is polynomial time reducible to $\mathrm{WCSP}(B')$. Furthermore, $\alpha'$ and $\beta'$ are not permutable, and thus by Lemma 37, $\#\mathrm{CSP}(\{\alpha', \beta'\})$ is $\#\mathrm{P}$-hard. $\square$

The goal of this section is to prove the following lemma:

**Lemma 39 (0-Elimination Lemma)** *Let $B \in \mathbb{Z}[X]^{m\times m}$ be a non-negative symmetric matrix that has a block of row rank at least 2.*

*Then there exists an $n \le m$ and a positive symmetric $X$-matrix $C \in \mathbb{Z}[X]^{n\times n}$ such that $C$ has a block of row rank at least 2 and $\mathrm{WCSP}^{KD}(C)$ is reducible to $\mathrm{WCSP}^{KD}(B)$.*

Since any $X$-matrix is non-negative, the lemma amounts to eliminating 0-entries. The basic idea is to let $C$ be a connected component of $B$, but this does not quite work, because we do not know how to filter out all those solutions for an instance of $\mathrm{WCSP}^{KD}(B)$ that map all variables to a fixed connected component of $B$. Lemma 41 gives a way to circumvent this problem.

Let us call a matrix *quasi-diagonal* if it is of the form displayed in Figure 1. Of course the blocks $B_1, \dots, B_k$ may be of different sizes.

**Lemma 40** *Let $B \in \mathbb{Z}[X]^{m\times m}$ be a symmetric matrix that has a block of row rank at least 2. Then there exists a permutation $\pi$ of $\{1, \dots, m\}$ such that the matrix $\pi B \in \mathbb{Z}[X]^{m\times m}$ with $(\pi B)_{ij} = B_{\pi(i)\pi(j)}$ is quasi-diagonal with blocks $B_1, \dots, B_r$, and $\mathsf{rank}(B_1) \ge 2$.*

*Furthermore, $\mathrm{WCSP}^{KD}(\pi B)$ and $\mathrm{WCSP}^{KD}(B)$ are polynomial time equivalent.*

**PROOF.** The proof is straightforward. For the equivalence of the two problems, recall Lemma 9. $\square$

$$\left(\begin{array}{c|c|c|c} B_1 & 0 & \cdots & 0 \\ \hline 0 & B_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & B_k \end{array}\right)$$

Fig. 1. A quasi-diagonal matrix $B$

**Lemma 41** *Let $B(X) \in \mathbb{Z}[X]^{m\times m}$ be a non-negative quasi-diagonal symmetric matrix with blocks $B_1, \dots, B_k$. Furthermore, assume that $B_1$ is positive and $\mathsf{rank}(B_1) \ge 2$.*

*Suppose that $B_1$ is an $n \times n$-matrix, and let $C \in \mathbb{Z}[X]^{n\times n}$ be the matrix with entries*

$$C_{ij} = B_{ij} \cdot B_{1i} \cdot B_{1j}.$$

*Then $C$ is positive, $\mathsf{rank}(C) \geq 2$, and $\mathrm{WCSP}^{KD}(C)$ is polynomial time reducible to $\mathrm{WCSP}^{KD}(B)$.*

**PROOF.** Clearly, $C$ is positive, as its entries are products of the (positive) entries of $B_1$.

To prove that $\mathsf{rank}(C) \geq 2$, consider a submatrix $B_{\{i,i'\}\{j,j'\}}$ of $B_1$ of row rank 2. Then the following determinant is non-zero:

$$\begin{vmatrix} B_{ij} & B_{ij'} \\ B_{i'j} & B_{i'j'} \end{vmatrix} \neq 0.$$

Then we have

$$\begin{vmatrix} C_{ij} & C_{ij'} \\ C_{i'j} & C_{i'j'} \end{vmatrix} = \begin{vmatrix} B_{ij}B_{1i}B_{1j} & B_{ij'}B_{1i}B_{1j'} \\ B_{i'j}B_{1i'}B_{1j} & B_{i'j'}B_{1i'}B_{1j'} \end{vmatrix} = B_{1i}B_{1i'}B_{1j}B_{1j'} \cdot \begin{vmatrix} B_{ij} & B_{ij'} \\ B_{i'j} & B_{i'j'} \end{vmatrix} \neq 0.$$

It remains to prove that $\mathrm{WCSP}^{KD}(C)$ is polynomial time reducible to $\mathrm{WCSP}^{KD}(B)$. Let $\mathcal{D} = (D, \alpha, \beta, \omega)$ and $\mathcal{D}' = (D', \alpha', \beta', \omega')$ be the canonical weighted template for $C$ and $B$, respectively. Note that

$$D = \{1, \dots, n\}^2 \subseteq \{1, \dots, m\}^2 = D'.$$

Furthermore, let $\theta$ be the diagonal of $D$ and $\theta'$ the diagonal of $D'$. The $\kappa_d$ for $d \in D$ are the same on both $D$ and $D'$.

Let $\mathcal{P} = (V, D, \mathcal{C})$ be an instance of $\mathrm{WCSP}^{KD}(C)$. We transform it to an instance $\mathcal{P}' = (V', D', \mathcal{C}')$ of $\mathrm{WCSP}^{KD}(B)$ as follows:

(i) $V' = V \cup \{v^1, v^2 \mid v \in V\} \cup \{x\}$.
(ii) For every constraint from $\mathcal{C}$ we include into $\mathcal{C}'$ the analogous constraint replacing $\alpha, \beta, \theta$ by $\alpha', \beta', \theta'$, respectively, and $\kappa_d$ by $\kappa_d$.
(iii) For every $v \in V$ we include into $\mathcal{C}'$ the constraints $\langle (v, v^1), \alpha\rangle$, $\langle (v, v^2), \beta\rangle$, $\langle (v^1, x), \beta\rangle$, $\langle (v^2, x), \alpha\rangle$.
(iv) We include the constraint $\langle x, \kappa_{(1,1)}\rangle$.

Note that the constraints (iii) and (iv) guarantee that $x$ is forced to $(1, 1)$, and for every $v \in V$, $v^1$ is forced into the same row as $v$ and column 1, and $v^2$ is forced into row 1 and the same column as $v$.

Now let $\sigma$ be a solution of $\mathrm{WCSP}^{KD}(C)$. Define

$$\sigma' : V \cup \{v^1, v^2 \mid v \in V\} \cup \{x\} \to D$$

as follows: For every $v \in V$, let $\sigma'(v) = \sigma(v)$. If $\sigma(v) = (i, j)$, let $\sigma'(v^1) = (i, 1)$ and $\sigma'(v^2) = (1, j)$. Finally, let $\sigma'(x) = (1, 1)$. Then $\sigma'$ is a solution of $\mathrm{WCSP}^{KD}(B)$.

Conversely, if $\sigma'$ is a solution of $\mathrm{WCSP}^{KD}(B)$ of non-zero weight $\omega'(\sigma')$, then $\sigma'(x) = (1, 1)$, and for all $v \in V$ with $\sigma'(v) = (i, j)$ we have $\sigma'(v^1) = (i, 1)$ and $\sigma'(v^2) = (1, j)$. Now $\omega'(\sigma') \neq 0$ implies that $(i, j) \in D$, because all entries of row 1 and column 1 of $B$ that are outside of $B_1$ are zero.

Thus there is a one-to-one correspondence between solutions $\sigma$ of $\mathcal{P}$ and solutions $\sigma'$ of $\mathcal{P}'$ of non-zero weight.

For every $\sigma : V \to D$, let $\sigma_1$ and $\sigma_2$ the projection of $\sigma$ on the first and second component, respectively, that is, if $\sigma(v) = (i, j)$, then $\sigma_1(v) = i$ and $\sigma_2(v) = j$. Then

$$\omega'(\sigma') = B_{11} \cdot \prod_{v \in V} B_{\sigma_1(v)\sigma_2(v)} \cdot B_{\sigma_1(v)1} \cdot B_{1\sigma_2(v)} = B_{11} \cdot \omega(\sigma).$$

Thus the mapping $\mathcal{P} \mapsto \mathcal{P}'$ yields the desired reduction from $\mathrm{WCSP}^{KD}(C)$ to $\mathrm{WCSP}^{KD}(B)$. $\square$

**Proof of the 0-Elimination Lemma 39.** Let $B \in \mathbb{Z}[X]^{m \times m}$ be a non-negative symmetric matrix that has a block of row rank at least 2. By Lemma 40, we may assume that $B$ is quasi-diagonal with blocks $B_1, \dots, B_k$ (as in Figure 1) and that $B_1$ has row rank at least 2. Suppose that $B_1$ is an $n \times n$-matrix.

If $B_1$ is positive, we can apply Lemma 41 and then the Extended X-Lemma 36 to the resulting $C$.

If $B_1$ is not positive, then there are $1 \leq i, j, k, \ell \leq n$ such that $B_{ik}$, $B_{i\ell}$, $B_{jk} \neq 0$, and $B_{j\ell} = 0$ (this follows from the fact that $B_1$ is a block and hence indecomposable). Then by the Non-Permutability Lemma 38, $\mathrm{WCSP}(B)$ and hence $\mathrm{WCSP}^{KD}(B)$ is #P-hard. In this case, we can simply let

$$C = \begin{pmatrix} 1 & X \\ X & 1 \end{pmatrix}$$

$\square$

The goal of this subsection is to prove that the 1-entries of our matrix can be arranged in square cells around diagonal. That is, we show that it will be sufficient to consider matrices of the form displayed in Figure 2 (on page 43), where the $*$-cells contain powers of $X$ greater than 1. This is the content of the General Conditioning Lemma 47.

The following lemma implies that we can always assume that our matrix contains 1-entries.

**Lemma 42** *Let $p$ be an irreducible polynomial and $B \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix such that such that every entry of $B$ is divisible by $p$. Then* $\mathrm{WCSP}^{KD}(\frac{1}{p}B)$ *is polynomial time reducible to* $\mathrm{WCSP}^{KD}(B)$.

**PROOF.** Take $\mathcal{P} = (V, D, \mathcal{C}) \in \mathrm{WCSP}^{KD}(\frac{1}{p}B)$. Then

$$\mathcal{Z}^{KD}_{\frac{1}{p}B}(\mathcal{P}) = \frac{1}{p^{|V|}}\mathcal{Z}^{KD}_{B}(\mathcal{P}).$$

□

Recall that a principal submatrix of an $(m \times m)$-matrix $B$ is a submatrix $B_K$ of $B$, for some $K \subseteq \{1, \dots, n\}$, that is obtained from $B$ by deleting all rows and columns whose index is not in $K$.

A row [column] of the matrix $B$ is called an *1-row* [*1-column*] if 1 occurs in it.

**Lemma 43 (1-Row Lemma)** *Let $B \in \mathbb{Z}[X]^{m\times m}$ be a symmetric positive $X$-matrix, and let $C$ be the principal submatrix of $B$ obtained by removing all non-1-rows and all non-1-columns and $A$ the submatrix obtained by removing all non-1-rows. Then*

*(1)* $\mathrm{WCSP}^{KD}(C)$ *is polynomial time reducible to* $\mathrm{WCSP}^{KD}(B)$*;*
*(2)* $\mathrm{WCSP}^{K}(A)$ *is polynomial time reducible to* $\mathrm{WCSP}^{K}(B)$.

**PROOF.** (1) Without loss of generality we may assume that the first $n$ rows of $B$ are the 1-rows and thus that the first $n$-columns are the 1-columns. Let $C'$ denote the $n \times n$-matrix with entries $C'_{ij} = \ell_i \cdot \ell_j \cdot C_{ij}$, where $\ell_i$ is the number of 1s in the $i$th row of $C$. We first show that $\mathrm{WCSP}^{KD}(C')$ is polynomial time reducible to $\mathrm{WCSP}^{KD}(B)$. Let $\mathcal{D} = (D, \alpha, \beta, \omega)$ and $\mathcal{D}' = (D', \alpha', \beta', \omega')$ be the canonical weighted templates of $C'$ and $B$, respectively. Then $D = \{1, \dots, n\}^2$ and $D' = \{1, \dots, m\}^2$. Let $\theta$, $\theta'$ be the diagonals of $D$ and $D'$.

We use $\Delta$ to denote the maximal degree of $X$ in $C'$. Let $\mathcal{P} = (V, D, \mathcal{C})$ be an instance of $\text{WCSP}^{KD}(C')$. We define an instance $\mathcal{P}' = (V', D', \mathcal{C}')$ of $\text{WCSP}^{KD}(B)$ as follows:

(i) Let $V' = V \cup \{v_1, \ldots, v_k, v^1, \ldots, v^k \mid v \in V\}$, where $k = |V| \cdot \Delta + 1$.
(ii) For every constraint $\langle (u, v), \alpha \rangle \in \mathcal{C}$, add the constraint $\langle (u, v), \alpha' \rangle$ to $\mathcal{C}'$. Similarly, for every constraint $\langle (u, v), \beta \rangle \in \mathcal{C}$, add the constraint $\langle (u, v), \beta' \rangle$ to $\mathcal{C}'$, and for every constraint $\langle (v), \theta \rangle \in \mathcal{C}$, add the constraint $\langle (v), \theta' \rangle$ to $\mathcal{C}'$.
(iii) For every constraint $\langle v, \kappa_d \rangle \in \mathcal{C}$, add the constraint $\langle v, \kappa_d \rangle$ to $\mathcal{C}'$.
(iv) For every $v \in V$ and $1 \leq i \leq k$, add the constraints $\langle (v, v_i), \alpha \rangle$ and $\langle (v, v^i), \beta \rangle$ to $\mathcal{C}'$.
(v) For every $v \in V$ and $1 \leq i < k$, add the constraints$\langle (v_i, v_{i+1}), \alpha \rangle$, $\langle (v_i, v_{i+1}), \beta \rangle$, $\langle (v^i, v^{i+1}), \alpha \rangle$, and $\langle (v^i, v^{i+1}), \beta \rangle$ to $\mathcal{C}'$.

The constraints (i)–(iii) just make sure that the restriction of a solution of $\mathcal{P}'$ to $V$ is a solution of $\mathcal{P}$. By (iv), for every $v \in V$, the variables $v_1, \ldots, v_k$ are forced to the same row as $v$ and the variables $v^1, \ldots, v^k$ to the same column. By the constraints in (v), for every $v \in V$ the variables $v_1, \ldots, v_k$ are forced to the same entry, and so are the vertices $v^1, \ldots, v^k$.

Let $\sigma' : V' \to D'$ be a solution of $\mathcal{P}'$. Then

$$\omega'(\sigma') = \prod_{v \in V} B_{\sigma'(v)} \cdot B^k_{\sigma'(v_1)} \cdot B^k_{\sigma'(v^1)}.$$

Observe that $\mathsf{deg}(\omega'(\sigma')) < k$ if and only if for every $v \in V$,

$$B_{\sigma'(v_1)} = B_{\sigma'(v^1)} = 1.$$

This is only possible if $\sigma'(v)$ is contained in a 1-row and in a 1-column, that is, in $D$. In this case, the restriction $\sigma$ of $\sigma'$ to $V$ is a solution of $\mathcal{P}$. Conversely, for every solution $\sigma$ of $\mathcal{P}$ there is an extension $\sigma'$ that is a solution of $\mathcal{P}'$ with $\mathsf{deg}(\omega'(\sigma')) < k$. As can be easily seen, there are

$$\ell_1^{|V_1|} \cdot \ldots \cdot \ell_n^{|V_n|} \cdot \ell_1^{|V^1|} \cdot \ldots \cdot \ell_n^{|V^n|}$$

such extensions where $V_i$ denotes the set of variables $v$ such that $\sigma(v) = (i, j)$ for a certain $j$, and $V^i$ denotes the set of variables $v$ such that $\sigma(v) = (j, i)$ for a certain $j$. Therefore,

$$\sum_{\substack{\sigma' \\ \sigma \text{ restriction of } \sigma'}} \omega'(\sigma') = \omega(\sigma).$$

Thus

$$\mathcal{Z}^{KD}_{C'} = \sum_{\sigma \text{ solution of } \mathcal{P}} \omega(\sigma) = \sum_{\substack{\sigma' \text{ solution of } \mathcal{P}' \\ \mathsf{deg}(\omega'(\sigma')) < k}} \omega(\sigma').$$

This yields a reduction from WCSP$^{KD}(C')$ to COUNT$^{KD}(B)$ and thus to WCSP$^{KD}(B)$ by Lemma 34.

Observe that $C = C'|_X$. By the Extended Prime Filter Lemma 35, WCSP$^{KD}(C'|_X)$ is polynomial time reducible to WCSP$^{KD}(C')$ and thus to WCSP$^{KD}(B)$.

(2) The proof in this case is similar. Let $A'$ denote the $n \times m$-matrix with entries $A'_{ij} = \ell_i \cdot A_{ij}$, where $\ell_i$ is the number of 1s in the $i$th row of $A$. We first show that WCSP$^K(A')$ is polynomial time reducible to WCSP$^K(B)$. Let $\mathcal{D}'' = (D'', \alpha, \beta, \omega)$ be the canonical weighted templates of $A'$. Then $D'' = \{1, \dots, m\} \times \{1, \dots, n\}$.

We use $\Delta$ to denote the maximal degree of $X$ in $A'$. Let $\mathcal{P} = (V, D'', \mathcal{C})$ be an instance of WCSP$^K(A')$. We define an instance $\mathcal{P}' = (V', D', \mathcal{C}')$ of WCSP$^K(B)$ as follows:

(i) Let $V' = V \cup \{v_1, \dots, v_k \mid v \in V\}$, where $k = |V| \cdot \Delta + 1$.
(ii) For every constraint $\langle (u,v), \alpha \rangle \in \mathcal{C}$, add the constraint $\langle (u,v), \alpha' \rangle$ to $\mathcal{C}'$. Similarly, for every constraint $\langle (u,v), \beta \rangle \in \mathcal{C}$, add the constraint $\langle (u,v), \beta' \rangle$ to $\mathcal{C}'$.
(iii) For every constraint $\langle v, \kappa_d \rangle \in \mathcal{C}$, add the constraint $\langle v, \kappa_d \rangle$ to $\mathcal{C}'$.
(iv) For every $v \in V$ and $1 \le i \le k$, add the constraints $\langle (v, v_i), \alpha \rangle$ to $\mathcal{C}'$.
(v) For every $v \in V$ and $1 \le i < k$, add the constraints$\langle (v_i, v_{i+1}), \alpha \rangle$, $\langle (v_i, v_{i+1}), \beta \rangle$, to $\mathcal{C}'$.

As well as in part (1), the constraints (i)–(iii) just make sure that the restriction of a solution of $\mathcal{P}'$ to $V$ is a solution of $\mathcal{P}$. By (iv), for every $v \in V$, the variables $v_1, \dots, v_k$ are forced to the same row as $v$. By the constraints in (v), for every $v \in V$ the variables $v_1, \dots, v_k$ are forced to the same entry.

Let $\sigma' : V' \to D'$ be a solution of $\mathcal{P}'$. Then

$$\omega'(\sigma') = \prod_{v \in V} B_{\sigma'(v)} \cdot B^k_{\sigma'(v_1)}.$$

Observe that $\mathsf{deg}(\omega'(\sigma')) < k$ if and only if for every $v \in V$,

$$B_{\sigma'(v_1)} = 1.$$

This is only possible if $\sigma'(v)$ is contained in a 1-row, that is, in $D''$. In this case, the restriction $\sigma$ of $\sigma'$ to $V$ is a solution of $\mathcal{P}$. Conversely, for every solution $\sigma$ of $\mathcal{P}$ there is an extension $\sigma'$ that is a solution of $\mathcal{P}'$ with $\mathsf{deg}(\omega'(\sigma')) < k$. As can be easily seen, there are

$$\ell_1^{|V_1|} \cdot \ldots \cdot \ell_n^{|V_n|}$$

such extensions, where $V_i$ denotes the set of variables $v$ such that $\sigma(v) = (i, j)$ for a certain $j$. Therefore,

$$\sum_{\substack{\sigma' \\ \sigma \text{ restriction of } \sigma'}} \omega'(\sigma') = \omega(\sigma).$$

Then we finish the proof as in the previous case. □

A *1-cell* in a matrix $B \in \mathbb{S}^{m\times n}$ is a submatrix $B_{KL}$ such that $B_{ij} = 1$ for all $i \in K, j \in L$ and $B_{ij} \neq 1$ for all $i \in K, j \in \{1, \ldots, n\} \setminus L$ and $i \in \{1, \ldots, m\} \setminus K, j \in L$.

**Lemma 44** *Let $B \in \mathbb{Z}[X]^{m\times n}$ be a symmetric matrix such that not all 1-entries of $B$ are contained in 1-cells. Then* WCSP$(B)$ *is* #P*-hard.*

**PROOF.** If not all 1-entries of $B$ are contained in 1-cells, then there are $i, j, k, \ell \in \{1, \ldots, m\}$ such that $B_{ik} = B_{i\ell} = B_{jk} = 1$ and $B_{j\ell} \neq 1$. Let $B'$ be the matrix obtained from $B$ by replacing all entries different from 1 by 0. By the Prime Elimination Lemma 24, WCSP$(B')$ is polynomial time reducible to WCSP$(B)$. By the Non-Permutability Lemma 38, WCSP$(B')$ is #P-hard.

**Lemma 45** *Let $B \in \mathbb{Z}[X]^{m\times n}$ be an $X$-matrix and let $C = (BB^\top)|_X$. Then for every 1-cell $B_{KL}$ of $B$, the principal submatrix $C_{KK}$ is a 1-cell of $C$.*

**PROOF.** Let $C = (BB^\top)|_X$ and note that $C_{ij} = 1$ if and only if $B_{ik} = B_{jk} = 1$ for some $k \in \{1, \ldots, n\}$. The claim follows. □

By the results we have proved so far, from now on we may assume that our matrix $B$ satisfies the following conditions. (This will be proved in the General Conditioning Lemma 47.)

**Conditions 46 (General Conditions)** *$B \in \mathbb{Z}[X]^{m\times m}$ such that:*

*(A)* $\mathsf{rank}(B) \geq 2$.
*(B) $B$ is symmetric.*
*(C) All entries of $B$ are powers of the indeterminate $X$.*
*For $1 \leq i, j \leq n$, let $\ell_{ij} = \mathsf{deg}(B_{ij})$ (so $B_{ij} = X^{\ell_{ij}}$).*
*(D) There are a $k \geq 2$ and $1 = m_0 < m_1 < \ldots < m_k = m + 1$ such that, for $1 \leq i \leq k - 1$, the principal submatrices $B_{\{m_{i-1},\ldots,m_i-1\}}$ are 1-cells of $B$, the principal submatrix $B_{\{m_{k-1},\ldots,m_k-1\}}$ may be a 1-cell (or may be not), and all 1-entries of $B$ are contained in one of these 1-cells.*

$$\left(\begin{array}{c|c|c|c|c}
\begin{matrix}1&\cdots&1\\ \vdots&\ddots&\vdots\\ 1&\cdots&1\end{matrix} & * & \cdots & * & * \\
\hline
* & \begin{matrix}1&\cdots&1\\ \vdots&\ddots&\vdots\\ 1&\cdots&1\end{matrix} & \cdots & * & * \\
\hline
\begin{matrix}\vdots\\ \vdots\end{matrix} & \begin{matrix}\vdots\\ \vdots\end{matrix} & \begin{matrix}\ddots & \\ & \ddots\end{matrix} & \begin{matrix}\vdots\\ \vdots\end{matrix} & \begin{matrix}\vdots\\ \vdots\end{matrix} \\
\hline
* & * & \cdots & \begin{matrix}1&\cdots&1\\ \vdots&\ddots&\vdots\\ 1&\cdots&1\end{matrix} & * \\
\hline
* & * & \cdots & * & */1
\end{array}\right)$$

Fig. 2. The cellular structure of the matrix $B$

Condition (D) means that $B$ has a cellular structure as indicated in Figure 2. The $*$-cells contain no 1-entries; the bottom right cell is either a 1-cell or a $*$-cell. The 1-cells are squares on the diagonal, but they may be of different sizes.

**Lemma 47 (General Conditioning Lemma)** *Let $B \in \mathbb{S}^{m\times n}$ a non-negative matrix that has a block of rank at least 2. Then there is $k \leq m$ and a $k \times k$-matrix $B'$ satisfying Conditions 46 such that* $\mathrm{WCSP}^{KD}(B')$ *is polynomial time reducible to* $\mathrm{WCSP}(B)$.

**PROOF.**

By the X-Lemma 17, there is an $X$-matrix $B_1 \in \mathbb{Z}[X]^{m\times n}$ of rank at least 2 such that $\mathrm{WCSP}(B_1) \leq \mathrm{WCSP}(B)$. By the Symmetrisation Lemma 33 the Constant Reduction Lemma 32, the Extended X-Lemma 36 and the Extended Renaming Lemma 30, there is a symmetric $X$-matrix $B_2 \in \mathbb{Z}[X]^{m\times m}$ that has a block of row rank at least 2 such that $\mathrm{WCSP}^{KD}(B_2) \leq \mathrm{WCSP}(B_1)$. By the 0-Elimination Lemma 39, there is a positive symmetric X-matrix $B_3 \in \mathbb{Z}[X]^{k\times k}$ of rank at least 2 such that $\mathrm{WCSP}^{KD}(B_3) \leq \mathrm{WCSP}^{KD}(B_2)$ (for some $k \leq m$).

Note that $B_3$ satisfies conditions (A)–(C). By Lemma 42, we may assume that $B_3$ contains at least one 1-entry. If $B_3$ contains 1-entries that are not contained in some 1-cell, then $\mathrm{WCSP}(B_3)$ is #P-hard by Lemma 44, and thus we can reduce $\mathrm{WCSP}^{KD}(B')$ for any matrix $B'$ to $\mathrm{WCSP}^{KD}(B_3)$. Thus we may assume that all 1-entries of $B_3$ are contained in 1-cells. If $B_3$ contains exactly one 1-cell, then this 1-cell must be a principal submatrix, because $B_3$ is symmetric. By permuting the rows and columns, we can bring $B_3$ into the desired form satisfying (D) with $k = 1$. If $B_3$ contains more than one 1-cell, then by Lemma 45, after suitably permuting rows and columns the matrix $B_4 = (B_3 \cdot B_3^\top)|_X$ satisfies (D). Since $B_4$ contains two 1-cells, its rank is at least 2, thus it also satisfies (A). It immediately follows from the definition of $B_4$ that it satisfies (B) and (C). Finally, $\mathrm{WCSP}^{KD}(B_4) \leq \mathrm{WCSP}^{KD}(B_3)$ by the Symmetrisation Lemma 33 and the Extended Prime Filter Lemma 35. $\square$

*Thus it remains to prove the* $\#\mathrm{P}$*-hardness of* $\mathrm{WCSP}^{KD}(B)$ *for all matrices* $B$ *satisfying the General Conditions 46.*

### *6.7 Matrices with at least two 1-cells*

In this section, we will take care of those matrices $B$ with at least two 1-cells. The main result of this section is the following lemma:

**Lemma 48 (Two 1-Cell Lemma)** *Let* $B \in \mathbb{Z}[X]^{m\times m}$ *be a positive symmetric matrix that has at least two 1-cells. Then* $\mathrm{WCSP}^{KD}(B)$ *is* $\#\mathrm{P}$*-hard.*

We first show that we may assume that a matrix with at least two 1-cells satisfies the General Conditions 46 and the following conditions:

**Conditions 49 (Two 1-Cell Conditions)** *(E) $B$ has at least two 1-cells. (F) All diagonal entries of $B$ are 1.*

**Lemma 50** *Let* $B \in \mathbb{Z}[X]^{m\times m}$ *be a positive symmetric matrix that has at least two 1-cells. Then there is a matrix* $B'$ *satisfying the General Conditions 46 and the Two 1-Cell Conditions 49 such that* $\mathrm{WCSP}^{KD}(B')$ *is polynomial time reducible to* $\mathrm{WCSP}^{KD}(B)$.

**PROOF.** Unfortunately, to prove this lemma we need to repeat some of the earlier proofs (specifically parts of the proof of the General Conditioning Lemma 47) and make sure that they preserve the property of having two 1-cells.

Let $B \in \mathbb{Z}[X]^{m\times m}$ be a positive symmetric matrix that has at least two 1-cells. Let $i, j, i', j'$ be indices such that $B_{ij} = B_{i'j'} = 1$ and $B_{i'j} \neq 1$. Let $p$ be an irreducible polynomial that divides $B_{i'j}$. Let $B_1$ be the matrix obtained from $B|_p$ by replacing all powers of $p$ by the corresponding powers of $X$. By the

Extended Prime Filter Lemma 35 and the Extended Renaming Lemma 30, $\mathrm{WCSP}^{KD}(B_1)$ is reducible to $\mathrm{WCSP}^{KD}(B)$.

$B_1$ satisfies conditions (A)–(C) and (E). We may further assume that all 1-entries of $B_1$ are contained in 1-cells, because otherwise $\mathrm{WCSP}(B_1)$ is #P-hard by Lemma 44. If all 1-cells of $B_1$ are on the diagonal, then we can satisfy (D) simply by permuting rows and columns. Otherwise, after suitably permuting rows and columns the matrix $B_2 = (B_1 \cdot B_1^\top)|_X$ satisfies (D), and it still satisfies (E). Arguing as in the proof of the General Conditioning Lemma 47, we can show that $B_2$ also satisfies (A)–(C) and that $\mathrm{WCSP}^{KD}(B_2)$ is reducible to $\mathrm{WCSP}^{KD}(B_1)$.

Condition 49 (F) can be achieved by the 1-Row Lemma 43. $\square$

Let $B$ be a matrix satisfying the General Conditions 46. The *cells* of $B$ are the submatrices $B_{IJ}$, where $I = \{m_{i-1}, \dots, m_i - 1\}$, $J = \{m_{j-1}, \dots, m_j - 1\}$ for some $1 \le i, j \le k$. These are precisely the "cells" of Figure 2 (on page 43). We call $B$ a *cell matrix* if for all cells $B_{IJ}$ all entries within the cell $B_{IJ}$ are equal, that is, for $i, i' \in I$ and $j, j' \in J$ we have $B_{ij} = B_{i'j'}$.

**Lemma 51** *Let $B \in \mathbb{Z}^{m \times m}[X]$ be a matrix satisfying the General Conditions 46 and the Two 1-Cell Conditions 49. Then there is a cell matrix $C \in \mathbb{Z}^{m \times m}[X]$ that still satisfies the General Conditions 46 and the Two 1-Cell Conditions 49, such that* $\mathrm{WCSP}^{KD}(C)$ *is polynomial time reducible to* $\mathrm{WCSP}^{KD}(B)$.

**PROOF.** Observe that for every matrix $B$ satisfying Conditions 46 and 49, the matrix $B' = (B \cdot B^\top)|_X$ also satisfies the conditions, and the problem $\mathrm{WCSP}^{KD}(B')$ is polynomial time reducible to $\mathrm{WCSP}^{KD}(B)$. (We have already used this in the proof of Lemma 50.)

Furthermore, for $1 \le i, j \le m$ we have $B'_{ij} = X^{n_{ij}}$, where $n_{ij} = \min_{1 \le k \le m}\{\mathsf{deg}(B_{ik}) + \mathsf{deg}(B_{jk})\}$. Since $B_{jj} = 1$ and thus $\mathsf{deg}(B_{jj}) = 0$,

$$\mathsf{deg}(B'_{ij}) \le \mathsf{deg}(B_{ij}).$$

Let $B_0 = B$ and, for $i \ge 0$, $B_{i+1} = (B_i \cdot B_i^\top)|_X$. Since the degrees of all entries are decreasing, there is a $k$ such that $B_{k+1} = B_k$. We shall prove that $C = B_k$ is a cell matrix.

Let $C_{IJ}$ be a cell of $C$ that is not a 1-cell, and let $i \in I, j \in J$ such that $\mathsf{deg}(C_{ij})$ is minimum among the degrees of all entries of the cell. Then, since $C = C' = (C \cdot C^\top)|_X$, for all $j' \in J$,

$$\mathsf{deg}(C_{ij'}) = \mathsf{deg}(((C \cdot C^\top)|_X)_{ij})$$

$$= \min_{1\le q\le n}\{\mathsf{deg}(C_{iq}) + \mathsf{deg}(C_{j'q})\}$$
$$\le \mathsf{deg}(C_{ij}),$$

because $C_{j'j} = 1$. Thus by the minimality of $\mathsf{deg}(C_{ij})$ we have

$$\mathsf{deg}(C_{ij}) = \mathsf{deg}(C_{ij'}).$$

Now for all $i' \in J$, analogously we get

$$\mathsf{deg}(C_{i'j'}) \le \mathsf{deg}(C_{ij'}),$$

which implies $\mathsf{deg}(C_{ij}) = \mathsf{deg}(C_{ij'}) = \mathsf{deg}(C_{i'j'})$. Thus $C_{ij} = C_{i'j'}$. $\square$

We now prove the #P-hardness of WCSP($B$) for cell-matrices $B$ satisfying Conditions 46 and 49 that have exactly two 1-cells.

**Lemma 52** *Let $B(X) \in \mathbb{Z}[X]^{m\times m}$ be of the form*

$$\left(\begin{array}{ccc|ccc} 1 & \cdots & 1 & X^\delta & \cdots & X^\delta \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & X^\delta & \cdots & X^\delta \\ \hline X^\delta & \cdots & X^\delta & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ X^\delta & \cdots & X^\delta & 1 & \cdots & 1 \end{array}\right).$$

*Then the problem* WCSP($B(X)$) *is* #P*-hard.*

**PROOF.** Let $k, \ell$ be the sizes of the two 1-cells in $B(X)$.

Since by Lemma 10, EVAL($(B(X) \cdot B(X)^\top$) is polynomial time reducible to WCSP($B(X)$), it suffices to prove that EVAL($(B(X) \cdot B(X)^\top$) is #P-hard.

Analogously to the proof of Lemma 22 (also see [5]) it can be shown that for every symmetric matrix $A \in \mathbb{S}^{n\times n}$ the following graph version GCOUNT($A$) of the problem COUNT($A$) can be reduced to EVAL(A):

| |
|---|
| *Input:* Graph $G = (V, E)$, $w \in \mathbb{S}$. |
| *Objective:* Compute $N_A(G, w)$, the number of mappings $\sigma : V \to \{1, \dots, k\}$ with $\omega_A(\sigma) = w$. |

This implies (as in the Prime Filter Lemma 25) that EVAL($(B(X)\cdot B(X)^\top)|_X$) is polynomial time reducible to EVAL($(B(X)\cdot B(X)^\top$) and thus to WCSP($B(X)$). Observe that $(B(X)\cdot B(X)^\top)|_X = B(X)$. Let $C = B(2)$; we shall actually prove that EVAL($C$) is #P-hard.

In [5], Dyer and Greenhill considered a generalised version of EVAL($A$), in which vertex-weights are also allowed. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $F \in \mathbb{R}^{n\times n}$ a diagonal matrix with positive diagonal entries (the idea is that the entry $F_{ii}$ denotes the weight assigned to $i$). For every graph $G = (V, E)$, let

$$Z_{A,F}(G) = \sum_{\sigma:V\to\{1,\dots,n\}} \prod_{\{u,v\}\in E} A_{\sigma(u)\sigma(v)} \prod_{v\in V} F_{\sigma(v)\sigma(v)}.$$

EVAL($A, F$) is the problem of computing $Z_{A,F}(G)$ for a given graph $\mathcal{G}$. Dyer and Greenhill [5] proved that EVAL($A$) is polynomial time reducible to EVAL($A, F$).

Let $G = (V, E)$ be a graph. For every partition $(V_1, V_2)$ of $V$, let $s(V_1, V_2) = |E \cap (V_1 \times V_2)|$ be the number of edges from $V_1$ to $V_2$. Observe that

$$\begin{aligned}
Z_C(G) &= \sum_{(V_1,V_2)\text{ Partition of }V} \; \sum_{\sigma_1:V_1\to\{1,\dots,\ell\}} \; \sum_{\sigma_2:V_2\to\{\ell+1,\dots,k\}} 2^{\delta\cdot s(V_1,V_2)} \\
&= \sum_{(V_1,V_2)\text{ Partition of }V} k^{|V_1|}\ell^{|V_2|}2^{\delta\cdot s(V_1,V_2)} \\
&= Z_{A,F}(G),
\end{aligned}$$

where

$$A = \begin{pmatrix} 1 & 2^\delta \\ 2^\delta & 1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix}.$$

Thus EVAL($A, F$) and therefore EVAL($A$) is reducible to EVAL($C$). It is easy to see that the #P-hard problem #MAX-CUT of counting the number of maximum cuts of a given graph is reducible to GCOUNT($A$) and hence to EVAL($A$). To see this, $G = (V, E)$ be a graph. Each mapping $\sigma : V \to \{1, 2\}$ gives rise to a cut $(\sigma^{-1}(1), \sigma^{-1}(2))$ of the graph, and the weight $\omega_A(\sigma)$ of the mapping is $2^{\delta\cdot k}$, where $k$ is the number of edges from $\sigma^{-1}(1)$ to $\sigma^{-1}(2)$). □

**Lemma 53** *Let $B \in \mathbb{Z}^{n\times n}[X]$ be a cell-matrix satisfying the General Conditions 46 and the Two 1-Cell Conditions 49. Then* WCSP$^{KD}(B)$ *is* #P*-hard.*

**PROOF.** Let $\delta = \min\{\mathsf{deg}(B_{ij}) \mid B_{ij} \neq 1\}$ and $\Delta = \max\{\mathsf{deg}(B_{ij}) \mid B_{ij} \neq 1\}$.

Let $B_{IJ}$ be a cell of $B$ whose entries are $X^\delta$. By symmetry and the definition of the cells, we have

$$C = B_{(I\cup J)\,(I\cup J)} = \left(\begin{array}{ccc|ccc} 1 & \dots & 1 & X^\delta & \cdots & X^\delta \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & X^\delta & \cdots & X^\delta \\ \hline X^\delta & \cdots & X^\delta & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ X^\delta & \cdots & X^\delta & 1 & \cdots & 1 \end{array}\right).$$

By Lemma 52, WCSP$(C)$ is #P-hard. We shall reduce WCSP$(C)$ to WCSP$^{KD}(B)$.

Let $\mathcal{D}' = (D', \alpha', \beta', \omega')$ be the canonical weighted template of $B$. Let $\theta'$ be the diagonal of $D'$ and, for $d \in D'$, $\kappa_d = \{d\}$. Recall that $D' = \{1, \dots, n\}^2$. Let $D = (I \cup J) \times (I \cup J) \subseteq D'$. $\alpha = \alpha' \cap D^2$, $\beta = \beta' \cap D^2$, and $\omega = \omega'|_D$. Observe that $\mathcal{D} = (D, \alpha, \beta, \omega)$ is isomorphic to the canonical weighted template for $C$. It will be more convenient to work with this template than with the canonical one.

Let $\mathcal{P} = (V, D, \mathcal{C})$ be an instance of WCSP$(C)$. We define an instance $\mathcal{P}' = (V', D', \mathcal{C}')$ of WCSP$^{KD}(B)$ (see Figure 3) as follows: Let $k = |V| \cdot \Delta + 1$, and let $i_0 \in I, j_0 \in J$.

(i) Let $V' = V \cup \{v^i_j \mid v \in V, 1 \le i \le 4, 1 \le j \le k\} \cup \{x, y\}$.
(ii) For every constraint $\langle (u, v), \alpha\rangle \in \mathcal{C}$, add the constraint $\langle (u, v), \alpha'\rangle$ to $\mathcal{C}'$. Similarly, for every constraint $\langle (u, v), \beta\rangle \in \mathcal{C}$, add the constraint $\langle (u, v), \beta'\rangle$ to $\mathcal{C}'$.
(iii) Add the constraints $\langle x, \kappa_{(i_0,j_0)}\rangle$ and $\langle y, \kappa_{(j_0,i_0)}\rangle$.
(iv) For every $v \in V$, $1 \le i \le 4$, and $1 \le j < k$, add the constraints $\langle (v^i_j, v^i_{j+1}), \alpha\rangle$ and $\langle (v^i_j, v^i_{j+1}), \beta\rangle$.
(v) For every $v \in V$, add the constraints $\langle (x, v^1_1), \beta\rangle$, $\langle (x, v^2_1), \alpha\rangle$, $\langle (y, v^3_1), \beta\rangle$, $\langle (y, v^4_1), \alpha\rangle$.
(vi) For every $v \in V$, add the constraints $\langle (v, v^1_1), \alpha\rangle$, $\langle (v, v^2_1), \beta\rangle$, $\langle (v, v^3_1), \alpha\rangle$, $\langle (v, v^4_1), \beta\rangle$.

The constraints in (ii) make sure that the restriction of a solution of $\mathcal{P}'$ to $V$ is a solution of $\mathcal{P}$, provided that the range of the solution of $\mathcal{P}'$ is contained in $D$. The constraints in (iii) guarantee that $x$ is mapped to $(i_0, j_0)$ and $y$ is mapped to $(j_0, i_0)$. The constraints in (iv) guarantee that $v^i_j$ and $v^i_{j'}$ get the same value for all $i, j, j'$. The constraints in (v) force $v^1_1$ into column $j_0$, $v^2_1$ into row $i_0$, $v^3_1$ into column $i_0$, and $v^4_1$ into row $j_0$. Finally, the constraints in (vi) force $v$ into the same row as $v^1_1$ and $v^3_1$, which also implies that $v^1_1$ and $v^3_1$ are forced in the same row. Moreover, they force $v$ into the same column as

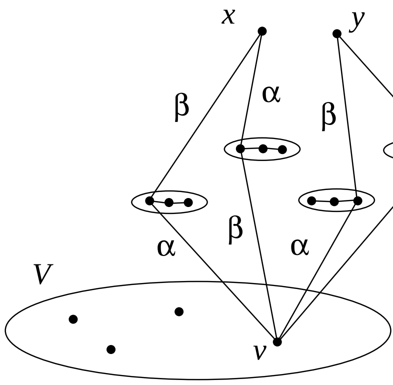


Fig. 3.

$v_1^2$ and $v_1^4$.

For every solution $\delta'$ of $\mathcal{P}'$, let $\sigma$ be the restriction of $\sigma'$ to $V$. Observe that

$$\omega'(\sigma') = X^{2\delta} \cdot X^{K(\sigma')} \cdot \prod_{v \in V} \sigma(v)$$

for some $K$ such that $0 \leq K(\sigma') \leq 4|V|\Delta$. The crucial observation is:

- If $\sigma'(v_1^i) \in D'$ for all $v \in V$ and $1 \leq i \leq 4$, then $K(\sigma') = 2k|V|\delta$.
- If $\sigma'(v_1^i) \in D \setminus D'$ for some $v \in V$ and $1 \leq i \leq 4$, then $K(\sigma') \geq 2k|V|\delta + k$.

Since $\mathsf{deg}(\prod_{v \in V} \sigma(v)) \leq |V| \cdot \Delta < k$ by the definition of $k$ and since $\sigma'(v) \in D' \iff \sigma'(v_1^1), \ldots, \sigma'(v_1^4) \in D'$ (by the constraints in (vi)), it follows that

$$\mathsf{deg}(\omega'(\sigma')) < 2\delta + 2k|V|\delta + k \iff \sigma'(V) \subseteq D'.$$

This yields a reduction from WCSP$(C)$ to COUNT$^{KD}(B)$ and thus to WCSP$^{KD}(B)$ by Lemma 34. $\square$

**Proof of the Two 1-Cell Lemma 48.** Follows immediately from Lemmas 50, 51, and 53. $\square$

### *6.8 Matrices with a single 1-cell*

In this section we consider the remaining case that $B \in \mathbb{Z}[X]^{m \times m}$ is a matrix satisfying the General Conditions 46 and only has one 1-cell. Our goal is to prove the following lemma:

**Lemma 54 (Single 1-Cell Lemma)** *Let $B \in \mathbb{Z}[X]^{m \times m}$ be a matrix that satisfies the General Conditions 46 and has exactly one 1-cell. Then* WCSP$^{KD}(B)$ *is* #P*-hard.*

We will complete the proof of the One 1-Cell Lemma at the end of the section and then summarise how the Main Hardness Theorem 16 can be obtained from our lemmas.

Before we get to the heart of the matter, we need two more simple reductions in the style of the previous sections.

**Lemma 55** *Let $B \in \mathbb{Z}^{m\times m}[X]$, $1 \leq s \leq m, 1 \leq t \leq m$, and let $C \in \mathbb{Z}^{m\times m}[X]$ the matrix with*

$$C_{ij} = B_{ij} \cdot (B_{ii})^s \cdot (B_{jj})^t$$

*for $1 \leq i \leq m, 1 \leq j \leq m$. Then $\mathrm{WCSP}^{KD}(C)$ is polynomial time reducible to $\mathrm{WCSP}^{KD}(B)$.*

**PROOF.** Note that the canonical templates for $B$ and $C$ only differ in their weight functions $\omega_B, \omega_C$. Let $\mathcal{P} = (V, D, \mathcal{C})$ be an instance of $\mathrm{WCSP}^{KD}(C)$. We construct an instance $\mathcal{P}' = (V', D, \mathcal{C}')$ of $\mathrm{WCSP}^{KD}(B)$ as follows:

(i) $V' = V \cup \{v_1^1, \ldots, v_s^1, v_1^2, \ldots, v_t^2 \mid v \in V\}$;
(ii) $\mathcal{C}' = \mathcal{C} \cup \mathcal{C}_1 \cup \mathcal{C}_2$ where:
- $\mathcal{C}_1 = \{\langle (v_1^1), \theta\rangle, \ldots, \langle (v_s^1), \theta\rangle, \langle (v_1^2), \theta\rangle, \ldots, \langle (v_t^2), \theta\rangle \mid v \in V\}$,
- $\mathcal{C}_2 = \{\langle (v, v_1^1), \alpha\rangle, \ldots, \langle (v, v_s^1), \alpha\rangle, \langle (v, v_1^2), \beta\rangle, \ldots, \langle (v, v_t^2), \beta\rangle \mid v \in V\}$.

Every solution $\sigma$ of $\mathcal{P}$ can be extended to a solution of $\mathcal{P}'$ in a unique way, because, for any element $d \in D$, there is only one 'diagonal' element from the $\alpha$-class and only one from $\beta$-class containing $d$. Conversely, the restriction of any solution of $\mathcal{P}'$ onto $V$ is a solution of $\mathcal{P}$. Finally, for a solution of $\mathcal{P}$ and the corresponding solution $\sigma'$ of $\mathcal{P}'$, we have

$$\begin{aligned}
\omega_B(\sigma') &= \prod_{v\in V} B_{\sigma_1(v)\sigma_2(v)} \cdot B^s_{\sigma_1(v)\sigma_1(v)} \cdot B^t_{\sigma_2(v)\sigma_2(v)} \\
&= \prod_{v\in V} C_{\sigma_1(v)\sigma_2(v)} \\
&= \omega_C(\sigma).
\end{aligned}$$

Here $\sigma(v) = (\sigma_1(v), \sigma_2(v))$ for all $v \in V$. □

We will use the previous lemma to show that we may assume that our matrix $B$ satisfies the following conditions (in addition to the General Conditions 46).

**Conditions 56 (Single 1-Cell Conditions)** *Let $B \in \mathbb{Z}[X]^{m\times m}$ be a matrix satisfying the General Conditions 46, and, for $1 \leq i, j \leq m$, let $\ell_{ij} = \mathsf{deg}(B_{ij})$. Let $r$ be the column of the first entry greater than 1 in row 1 of $B$, that is, $r = \min\{j \mid 1 \leq j \leq m, \ell_{1j} > 0\}$.*

*(G) $B$ has exactly one 1-cell.*
*(H) The first $(r-1)$ rows of $B$ are identical.*
*(I) For $1 \leq i < r$, $r \leq i' \leq m$, and $1 \leq j \leq m$,*

$$\ell_{ij} \leq \ell_{i'j}.$$

**Lemma 57** *Let $B \in \mathbb{Z}[X]^{m\times m}$ be a matrix that satisfies the General Conditions 46 and has precisely one 1-cell. Then there is a matrix $B'$ satisfying the General Conditions 46 and the One 1-Cell Conditions 56 such that* $\mathrm{WCSP}^{KD}(B')$ *is polynomial time reducible to* $\mathrm{WCSP}^{KD}(B)$*.*

**PROOF.** Observe that if a matrix $B$ satisfies (A)–(D), (G), and (H), then (I) can easily be satisfied by applying Lemma 55. Indeed, it is not hard to see that $C$ obtained from $B$ as in Lemma 55 for $s = t$ such that $\max\{\ell_{1r}, \ldots, \ell_{1m}\} < s \cdot \min\{\ell_{rr}, \ldots, \ell_{mm}\}$, satisfies (A)–(D), (G), (H) and (I). So we only have to worry about (A)–(D), (G), and (H).

Note that $B$ already satisfies conditions (A)–(D) and (G).

The proof is by induction on $m$: For $m = 2$, condition (H) is trivially satisfied. So let $m \geq r > 2$ and suppose that $B$ does not satisfy (H). Let $C$ be $(r-1)\times m$ matrix consisting of the first $(r-1)$-rows of $B$. By the 1-Row Lemma 43 (2), $\mathrm{WCSP}^{K}(C)$ is polynomial time reducible to $\mathrm{WCSP}^{K}(B)$. Since $C_{i1} = 1$ for $1 \leq i \leq r-1$, but the rows of $C$ are not identical, we have $\mathsf{rank}(C) \geq 2$. Let $D = C \cdot C^{\top}$. By the Symmetrisation Lemma 33, $\mathrm{WCSP}^{KD}(D)$ is polynomial time reducible to $\mathrm{WCSP}^{K}(C)$.

$D$ is an $(r-1) \times (r-1)$-matrix with $\mathsf{rank}(D) = \mathsf{rank}(C) > 1$. We apply the General Conditioning Lemma 47 to $D$ and obtain a $(k\times k)$-matrix $D'$, for some $k \leq r-1$, that satisfies the General Conditions 46 such that $\mathrm{WCSP}^{KD}(D')$ is polynomial time reducible to $\mathrm{WCSP}^{KD}(D)$. If $D'$ has at least two 1-cells, then by the Two 1-Cell Lemma 48, $\mathrm{WCSP}^{KD}(D')$ is #P-hard. Hence $\mathrm{WCSP}^{KD}(B)$ is #P-hard, and we can take $B'$ to be an arbitrary matrix satisfying the conditions.

If $D'$ has only one 1-cell, then by the induction hypothesis there is a matrix $D''$ satisfying Conditions (A)–(D), (G), (H) such that $\mathrm{WCSP}^{KD}(D'')$ is reducible to $\mathrm{WCSP}^{KD}(D')$ and hence to $\mathrm{WCSP}^{KD}(B)$. □

*For the rest of this section, we fix a matrix $B \in \mathbb{Z}[X]^{m\times m}$ that satisfies the General Conditions 46 and the Single 1-Cell Conditions 56. We also let $\ell_{ij} = \mathsf{deg}(B_{ij})$ for $1 \leq i, j \leq m$ and $r = \min\{j \mid 1 \leq j \leq m, \ell_{1j} > 0\}$.*

**Lemma 58** *Let $k$ be a natural number and $B^{[k]}$ the matrix with $B^{[k]}_{ij} = B_{ij} \cdot (B_{1j})^{k-1}$. Then* $\mathrm{WCSP}^{KD}(B^{[k]})$ *is polynomial time reducible to* $\mathrm{WCSP}^{KD}(B)$.

**PROOF.** Let $\mathcal{D} = (D, \alpha, \beta, \omega)$ be the canonical weighted template for $B$ and note that the canonical template for $B^{[k]}$ is the same except for its weight function, which we denote by $\omega_k$.

Let $\mathcal{P} = (V, D, \mathcal{C})$ be an instance of $\mathrm{WCSP}^{KD}(B^{[k]})$. We construct an instance $\mathcal{P}' = (V', D, \mathcal{C}')$ of $\mathrm{WCSP}^{KD}(B)$ as follows

(i) $V' = V \cup \{v_1, \dots, v_{k-1} \mid v \in V\} \cup \{x\}$.
(ii) Add all constraints in $\mathcal{C}$ to $\mathcal{C}'$.
(iii) Add a constraint $\langle x, \kappa_{(1,1)} \rangle$ to $\mathcal{C}'$.
(iv) For every $v \in V$, add the constraints $\langle (v_1, x), \alpha \rangle, \dots, \langle (v_{k-1}, x), \alpha \rangle$ to $\mathcal{C}$.
(v) For every $v \in V$, add the constraints $\langle (v, v_1), \beta \rangle, \dots, \langle (v, v_{k-1}), \beta \rangle$.

The constraints in (ii) guarantee that the restriction of every solution of $\mathcal{P}'$ to $V$ is a solution of $\mathcal{P}$. The constraint (iii) makes sure that $x$ is mapped to $(1, 1)$. Thus the constraints in (iv) guarantee that all $v_i$ are mapped to the first row. The constraints in (v) make sure that for every $v$ the $v_i$ are mapped to the same column as $v$. Thus (iv) and (v) together imply that if $v$ is mapped to $(i, j)$ by a solution, then $v_1, \dots, v_{k-1}$ are mapped to $(1, j)$.

Thus every solution $\sigma$ of $\mathcal{P}$ can be extended to a solution $\sigma'$ of $\mathcal{P}'$ in a unique way, and conversely, the restriction $\sigma$ of any solution $\sigma'$ of $\mathcal{P}'$ to $V$ is a solution of $\mathcal{P}$. Furthermore, for every solution $\sigma'$ of $\mathcal{P}'$,

$$\begin{aligned}\omega(\sigma') &= B_{11} \cdot \prod_{v \in V} B_{\sigma_1(v)\sigma_2(v)} \cdot B^{k-1}_{1\sigma_2(v)} \\ &= B_{11} \cdot \prod_{v \in V} C_{\sigma_1(v)\sigma_2(v)} \\ &= B_{11} \cdot \omega_k(\sigma).\end{aligned}$$

Here $\sigma(v) \in (\sigma_1(v), \sigma_2(v))$. Note that the factor $B_{11}$ is needed to account for the variable $x$ with $\sigma'(x) = (1, 1)$. $\square$

We need a few facts about polynomials. We consider polynomials over the field $\mathbb{Q}$ of rational numbers, which we view as a subfield of the complex numbers $\mathbb{C}$. Let $f \in \mathbb{Q}[X]$ and $\lambda \in \mathbb{C}$. Then $\mathsf{mult}(\lambda, f)$ denotes the *multiplicity* of $\lambda$ in $f$ if $\lambda$ is a root of $f$ and $\mathsf{mult}(\lambda, f) = 0$ otherwise. The *kth root* of a complex number $\lambda$ is the $k$-element set $\lambda^{1/k} = \{\mu \mid \mu^k = \lambda\}$. Slightly abusing notation we will denote $\lambda^{1/k}$ any element from this set. We shall use the following basic facts on polynomials, roots and their multiplicity.

**Lemma 59** *Let $f \in \mathbb{Q}[X]$ be a polynomial and $\lambda \in \mathbb{C}$ a complex number.*

*(1) There exists a unique (up to a scalar factor) irreducible polynomial $p_\lambda \in \mathbb{Q}[X]$ such that $\lambda$ is a root of $p_\lambda$. If $\lambda$ is a root of $f$ then $p_\lambda | f$.*
*(2) If $\mathsf{mult}(\lambda, f) = s$ then $f = p_\lambda^s \overline{f}$ for some $\overline{f} \in \mathbb{Q}[X]$ with $\overline{f}(\lambda) \neq 0$.*
*(3) For every root $\lambda$ of $f(X)$, $\lambda^{1/k}$ is a root of $f(X^k)$. Moreover,*

$$\mathsf{mult}(\lambda^{1/k}, f(X^k)) = \mathsf{mult}(\lambda, f(X)).$$

The following lemma is the technical core of the whole proof. It is very hard to motivate the particular construction or give simple intuitions as to why it works. The general idea is to construct a matrix $C$ with $\text{WCSP}^{KD}(C)$ being reducible to $\text{WCSP}^{KD}(B)$ such that $C$ has more than one 1-cell, so that we can apply the Two 1-Cell Lemma of the previous subsection. It seems a good strategy to generate an infinite family of matrices $C_k$ by some kind of uniform "powering" construction and hope that at least one of the $C_k$ works. The construction below is essentially the simplest we could come up with that does exactly this.

Recall that $B \in \mathbb{Z}[X]^{m\times m}$ is a matrix that satisfies the General Conditions 46 and the Single 1-Cell Conditions 56, and

$$\ell_{ij} = \mathsf{deg}(B_{ij}), \tag{5}$$
$$r = \min\{j \mid 1 \leq j \leq m, \ell_{1j} > 0\}. \tag{6}$$

In the following, for $k \geq 1$, we let

$$C^{[k]} = B^{[k]} \cdot (B^{[k]})^\top. \tag{7}$$

Observe that $C^{[k]}$ is a symmetric positive matrix in $\mathbb{Z}[X]^{m\times m}$.

For every root $\lambda$ of $C_{11}^{[1]}$, every $r \leq j \leq m$ and $k$, we denote the multiplicity of $\lambda^{1/k}$ in $C_{1j}^{[k]}$ by $m(\lambda, j, k)$, and we let

$$m(\lambda, j) = \min_{k\geq 1} m(\lambda, j, k).$$

**Lemma 60** *(1) For any root $\lambda$ of $C_{11}^{[1]}$, any $r \leq j \leq m$ and any positive integer $k$, $\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda, C_{11}^{[1]}) \geq m(\lambda, j)$ and $\mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}) \geq m(\lambda, j)$.*
*(2) For any root $\lambda$ of $C_{11}^{[1]}$, any $r \leq j \leq m$ such that the first and $j$th rows are linearly dependent and any positive integer $k$,*

$$\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}).$$

(3) *For any $r \leq j \leq m$ such that the first and the $j$th row are linearly independent, there is a root $\lambda$ of $C_{11}^{[1]}$ such that* $\mathsf{mult}(\lambda, C_{11}^{[1]}) > m(\lambda, j)$

**PROOF.** Let $j \in \{r, \ldots, m\}$. Let $b = \min\{\ell_{j1} - \ell_{11}, \ldots, \ell_{jm} - \ell_{1m}\}$. By the One 1-Cell Condition 56(I), $b \geq 0$.

To simplify the notation, let $a_1 = \ell_{11} = 0, \ldots, a_{r-1} = \ell_{1r-1} = 0$, $a_r = \ell_{1r}, \ldots, a_m = l_{1m}$, $b_1 = \ell_{j1} - b, \ldots, b_m = \ell_{jm} - b$ and $c_i = b_i - a_i$ for $1 \leq i \leq m$. Note that $c_i \geq 0$ for $1 \leq i \leq m$ and all the $c_i$ are equal to 0 if and only if the first and $j$th rows are linearly dependent. Note also that if the first and $j$th rows are linearly independent, then not all of the $c_i$ are equal.Then

$$B = \begin{pmatrix} X^{a_1} & X^{a_2} & X^{a_3} & \cdots & X^{a_m} \\ \vdots & \vdots & \vdots & & \vdots \\ X^{b+b_1} & X^{b+b_2} & X^{b+b_3} & \cdots & X^{b+b_m} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix},$$

$$B^{[k]} = \begin{pmatrix} X^{ka_1} & X^{ka_2} & X^{ka_3} & \cdots & X^{ka_m} \\ \vdots & \vdots & \vdots & & \vdots \\ X^{b+\overbrace{b_1-a_1}^{=c_1}+ka_1} & X^{b+\overbrace{b_2-a_2}^{=c_2}+ka_2} & X^{b+\overbrace{b_3-a_3}^{=c_3}+ka_3} & \cdots & X^{b+\overbrace{b_m-a_m}^{=c_m}+ka_m} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}.$$

For the matrix $C^{[k]}$ we have

$$C_{11}^{[k]} = X^{2ka_1} + X^{2ka_2} + \ldots + X^{2ka_m}, \tag{8}$$
$$C_{1j}^{[k]} = X^{b}(X^{c_1+2ka_1} + X^{c_2+2ka_2} + \ldots + X^{c_m+2ka_m}), \tag{9}$$
$$C_{jj}^{[k]} = X^{2b}(X^{2c_1+2ka_1} + X^{2c_2+2ka_2} + \ldots + X^{2c_m+2ka_m}). \tag{10}$$

Take a root $\lambda$ of $C_{11}^{[1]}$. Then $\lambda \neq 0$, because $a_1 = 0$. Let $k \geq 1$. Note first that (8) and Lemma 59(3) imply that $\lambda^{1/k}$ is a root of $C_{11}^{[k]}$ with

$$\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda, C_{11}^{[1]}). \tag{11}$$

If the first and $j$th rows are linearly dependent, then $c_1 = c_2 = \ldots = c_m = 0$. Thus by Equalities (8)–(10),

$$C_{1j}^{[k]} = X^{b} \cdot C_{11}^{[k]} \quad \text{and} \quad C_{jj}^{[k]} = X^{2b} \cdot C_{11}^{[k]}.$$

Since $\lambda \neq 0$, it follows that

$$\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}).$$

This proves Lemma 60(2).

In the following, we assume that rows 1 and $j$ are linearly independent. In particular, not all the $c_i$ are equal. If, for some $k$, $\lambda^{1/k}$ is not a root of $C_{1j}^{[k]}$, then $m(\lambda, j) = 0$, and Lemma 60(1) and (3) hold trivially for $\lambda$. In the following, we assume that $\lambda^{1/k}$ is a root of $C_{1j}^{[k]}$. Our first goal is to find $m(\lambda, j)$.

Let $\alpha \in \mathbb{C}$ such that $\lambda = e^{\alpha}$. Then for every $k \geq 1$,

$$C_{1j}^{[k]}(\lambda^{1/k}) = e^{\alpha b/k}\left(e^{\alpha(2a_1+\frac{c_1}{k})} + e^{\alpha(2a_2+\frac{c_2}{k})} + \ldots + e^{\alpha(2a_m+\frac{c_m}{k})}\right) = 0$$

(because $\lambda^{1/k}$ is a root of $C_{1j}^{[k]}$). Consider the function

$$f_\lambda(z) = e^{\alpha(2a_1+c_1 z)} + e^{\alpha(2a_2+c_2 z)} + \ldots + e^{\alpha(2a_m+c_m z)}.$$

For every $k \geq 1$ we have $f_\lambda(1/k) = 0$.

Claim 1. Suppose that $g(z) = u(z) + iv(z)$ is a function that is analytic in the real segment $[0,1]$ and that $\{r_n\}_{n\geq 1}$, $\{s_n\}_{n\geq 1}$ from the real segment $[0,1]$ such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = 0$ and $u(r_n) = v(s_n) = 0$ for all $n \geq 1$. Then

(a) $g(0) = 0$;
(b) there are sequences $\{r'_n\}_{n\geq 1}$, $\{s'_n\}_{n\geq 1}$ from the real segment $[0,1]$ such that

$$\lim_{n\to\infty} r'_n = \lim_{n\to\infty} s'_n = 0$$

and $u'(r'_n) = v'(s'_n) = 0$ for all $n \geq 1$, where $u', v'$ denote the derivatives of the corresponding functions.

**PROOF.** Without loss of generality we may assume that $\{r_n\}$, $\{s_n\}$ are monotone. Then, since $g$ is continuous,

$$g(0) = \lim_{z\to 0} g(z) = \lim_{z\to 0} u(z) + i \lim_{z\to 0} v(z) = \lim_{n\to\infty} u(r_n) + i \lim_{n\to\infty} v(s_n) = 0.$$

Furthermore, let $u_0, v_0$ denote the restrictions of $u, v$ onto the real interval $[0,1]$. Then $u_0, v_0$ are continuous and differentiable real functions. Therefore, for any $n$, there are $r'_n \in [r_{n+1}, r_n]$ and $s'_n \in [s_{n+1}, s_n]$ such that $u'_0(r'_n) = v'_0(s'_n) = 0$. Clearly, $\lim_{n\to\infty} r'_n = \lim_{n\to\infty} s'_n = 0$ and $u'(r'_n) = u'_0(r'_n) = 0$, $v'(s'_n) = v'_0(s'_n) = 0$.

This completes the proof of Claim 1.

The function $f_\lambda(z)$ is analytic everywhere including $[0,1]$. Moreover, for any $k$,

$$f_\lambda(1/k) = e^{\alpha(2a_1+\frac{c_1}{k})} + e^{\alpha(2a_2+\frac{c_2}{k})} + \ldots + e^{\alpha(2a_m+\frac{c_m}{k})} = \frac{C_{1j}^{[k]}(\lambda^{1/k})}{e^{b/k}} = 0.$$

Therefore, by Claim 1, for any $\ell \geq 1$, the $\ell$th derivative $f_\lambda^{(\ell)}(0) = 0$.

Computing the derivatives at 0 we get

$$f_\lambda^{(l)}(0) = (\alpha c_1)^l e^{2\alpha a_1} + (\alpha c_2)^l e^{2\alpha a_2} + \ldots + (\alpha c_m)^l e^{2\alpha a_m} = 0.$$

Observe that for $1 \leq i \leq r-1$ we have $c_i = c_1$ by the One 1-Cell Condition 56 (H). Without loss of generality we may assume that

$$c_1 = \ldots = c_{s_1}, \quad c_{s_1+1} = \ldots = c_{s_2}, \quad \ldots, \quad c_{s_{t-1}+1} = \ldots = c_{s_t} = 0,$$

where $s_0 = 0, s_t = m$, and that $c_{s_1}, \ldots, c_{s_t}$ are all different. (We therefore assume that $c_1 \neq 0$. It may well not be the case, but we use this assumption only once in the next paragraph, and it is easy to see that what we really need is $t > 1$.) Moreover, we have $t \geq 2$, because not all the $c_i$ are equal by our assumption that rows 1 and $j$ are linearly independent, and $s_1 \geq r-1$.

Denoting $Y_i = e^{2\alpha a_{s_{i-1}+1}} + \ldots + e^{2\alpha a_{s_i}}$, $1 \leq i \leq t-1$, we get a system of linear equations

$$\begin{cases} c_{s_1}Y_1 + c_{s_2}Y_2 + \ldots + c_{s_{t-1}}Y_{t-1} = 0 \\ (c_{s_1})^2 Y_1 + (c_{s_2})^2 Y_2 + \ldots + (c_{s_{t-1}})^2 Y_{t-1} = 0 \\ \vdots \\ (c_{s_1})^{t-1} Y_1 + (c_{s_2})^{t-1} Y_2 + \ldots + (c_{s_{t-1}})^{t-1} Y_{t-1} = 0 \end{cases}$$

The determinant of the system is Vandermonde. Therefore, $Y_1 = \ldots = Y_{t-1} = 0$. Denoting

$$\begin{aligned} g_1(X) &= X^{2a_1} + \ldots + X^{2a_{s_1}} \\ &\vdots \\ g_{t-1}(X) &= X^{2a_{s_{t-2}+1}} + \ldots + X^{2a_{s_{t-1}}} \\ g_t(X) &= X^{2a_{s_{t-1}+1}} + \ldots + X^{2a_{s_t}}, \end{aligned}$$

we have $g_1(\lambda) = \ldots = g_{t-1}(\lambda) = 0$ and, since $C_{11}^{[1]}(X) = g_1(X) + \ldots + g_{t-1}(X) + g_t(X)$ and $\lambda$ is a root of $C_{11}^{[1]}$, $g_t(\lambda) = 0$ as well. (If $c_1 = 0$ then we have $g_2(\lambda) = \ldots = g_t(\lambda) = 0$, from which we conclude $g_1(\lambda) = 0$.)

Everything we have done so far is independent of the specific root $\lambda$. Thus, for every irreducible polynomial $g$ with $g|C_{11}^{[1]}$, we have $g|g_1, \ldots, g_t$. Let $h_1, \ldots, h_q$ be the different irreducible divisors of $C_{11}^{[1]}$. Without loss of generality we may assume that the leading coefficients of the $h_i$ are positive. Then

$$\begin{aligned}
C_{11}^{[1]} &= \overline{g}_1(X)h_1^{r_{11}}(X)\ldots h_q^{r_{1q}}(X) + \ldots + \overline{g}_t(X)h_1^{r_{t1}}(X)\ldots h_q^{r_{tq}}(X)\\
&= h_1^{m_1}(X)\ldots h_q^{m_q}(X)\\
&\qquad \cdot\Big(\overline{g}_1(X)h_1^{r'_{11}}(X)\ldots h_q^{r'_{1q}}(X) + \ldots + \overline{g}_t(X)h_1^{r'_{t1}}(X)\ldots h_q^{r'_{tq}}(X)\Big),
\end{aligned}$$

for suitably chosen polynomials $\overline{g}_i(X)$ and non-negative integers $r_{ij}, r'_{ij}$, and $m_i = \min(r_{1i}, \ldots, r_{ti})$. To simplify the notation, we set

$$h(X) = h_1^{m_1}(X)\ldots h_q^{m_q}(X), \qquad f_i(X) = \overline{g}_i(X)h_1^{r'_{i1}}(X)\ldots h_q^{r'_{iq}}(X).$$

Then $g_i(X) = h(X)\cdot f_i(X)$ for $1 \le i \le t$ and

$$C_{11}^{[1]} = h(X)\cdot\Big(f_1(X) + \ldots + f_t(X)\Big).$$

Since $s_1 \ge r-1$, the polynomial $g_1$ is the only one with a non-zero constant term. Thus $g_1$ and $g_i$ for $2 \le i \le t$ differ by more than a constant factor. Since $h(X)$ is the greatest common divisor of $g_1, \ldots, g_t$, the degree of at least one of the polynomials $f_i$ is positive. Let $1 \le i \le t$. Since all coefficients of $g_i$ and the leading coefficient of $h(X)$ are positive, the leading coefficient of $f_i$ is positive. Thus

$$\mathsf{deg}(f_1(X) + \ldots + f_t(X)) > 0. \tag{12}$$

To simplify the notation in Claims 2 and 3, suppose now that $\lambda$ is a root of $h_1$.

Claim 2. $m(\lambda, j) = m_1$.

**PROOF.** We need to show that $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) \ge m_1$ for all $k$ and that there is a positive integer $k$ such that $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) = m_1$.

As is easily seen, for any $k$,

$$C_{1j}^{[k]} = X^b(X^{c_{s_1}}g_1(X^k) + \ldots + X^{c_{s_t}}g_t(X^k))$$

$$= X^b h(X^k)\Big(X^{c_{s_1}} f_1(X^k) + \ldots + X^{c_{s_t}} f_t(X^k)\Big). \tag{13}$$

Therefore $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) \geq \mathsf{mult}(\lambda, h) = m_1$.

Regroup the summands in $f_\lambda(z)$:

$$\begin{aligned} f_\lambda(z) &= e^{\alpha c_{s_1} z}(e^{2\alpha a_1} + \ldots + e^{2\alpha a_{s_1}}) + \ldots + e^{\alpha c_{s_t} z}(e^{2\alpha a_{s_{t-1}+1}} + \ldots + e^{2\alpha a_{s_t}}) \\ &= h(\lambda)\Big(\lambda^{c_{s_1} z} f_1(\lambda) + \ldots + \lambda^{c_{s_t} z} f_t(\lambda)\Big). \end{aligned}$$

Let

$$\bar{f}(X, z) = X^{c_{s_1} z} f_1(X) + \ldots + X^{c_{s_t} z} f_t(X).$$

Then $f_\lambda(z) = h(\lambda) \cdot \bar{f}(\lambda, z)$.

Let $\beta_1 = \lambda^{c_{s_1}/t!}, \ldots, \beta_t = \lambda^{c_{s_t}/t!}$. Then, for any $\ell \leq t$,

$$f_\lambda\left(\frac{\ell}{t!}\right) = h(\lambda) \cdot \bar{f}\left(\lambda, \frac{\ell}{t!}\right) = h(\lambda) \cdot \Big(\beta_1^\ell f_1(\lambda) + \ldots + \beta_t^\ell f_t(\lambda)\Big).$$

Suppose for contradiction that $\bar{f}(\lambda, \ell/t!) = 0$ for $l = 1, \ldots, t$. Consider the system

$$\begin{cases} \beta_1 f_1(\lambda) + \ldots + \beta_t f_t(\lambda) = 0 \\ (\beta_1)^2 f_1(\lambda) + \ldots + (\beta_t)^2 f_t(\lambda) = 0 \\ \vdots \\ (\beta_1)^t f_1(\lambda) + \ldots + (\beta_t)^t f_t(\lambda) = 0 \end{cases}$$

Since $\beta_i \neq \beta_{i'}$ whenever $i \neq i'$, we get $f_1(\lambda) = \ldots = f_t(\lambda) = 0$, which contradicts (12).

Thus for some $\ell \leq t$, $\bar{f}(\lambda, \ell/t!) \neq 0$. Pick such an $\ell$ and let $k = t!/\ell$. Note that

$$\bar{f}\left(X^k, 1/k\right) = X^{c_{s_1}} f_1(X^k) + \ldots + X^{c_{s_t}} f_t(X^k)$$

and recall (13). Since $\bar{f}(\lambda, 1/k) \neq 0$, $\lambda^{1/k}$ is not a root of the polynomial on the left hand side, and by (13) this implies $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) = m_1$.

This completes the proof of Claim 2.

Claim 3. For every every positive integer $k$, $\mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}) \geq m_1$.

**PROOF.** Let us consider $C_{jj}^{[k]}$:

$$\begin{aligned} C_{jj}^{[k]} &= X^{2b}(X^{2c_1+2ka_1} + \ldots + X^{2c_m+2ka_m}) \\ &= X^{2b}(X^{2c_1}(X^k)^{2a_1} + \ldots + X^{2c_m}(X^k)^{2a_m}) \\ &= X^{2b}h(X^k)(X^{2c_{s_1}}f_1(X^k) + \ldots + X^{2c_{s_t}}f_t(X^k)). \end{aligned}$$

Then

$$\mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}) \geq \mathsf{mult}(\lambda^{1/k}, h(X^k)) = \mathsf{mult}(\lambda, h(X)) = m_1.$$

This completes the proof of Claim 3.

Clearly, Lemma 60(1) follows from Claims 1 and 2. To prove (3), we recall that

$$C_{11}^{[1]} = h_1^{m_1}(X) \ldots h_q^{m_q}(X)\big(f_1(X) + \ldots + f_t(X)\big).$$

Thus every root $\lambda$ of $f_1(X) + \ldots + f_t(X)$ is also a root of $C_{11}^{[1]}$ and, therefore, it is a root of one of $h_1, \ldots, h_q$. Then $\mathsf{mult}(\lambda, C_{11}^{[1]}) > m(\lambda, j)$. Choose $k$ by Claim 2. Then

$$\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda, C_{11}^{[1]}) > m_i = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}).$$

Note that such a number $k$ exists for every root $\lambda$ such that $\mathsf{mult}(\lambda, C_{11}^{[1]}) > m(\lambda, j)$. $\square$

**Lemma 61** *There exist $j \in \{r, \ldots, m\}$, a root $\lambda$ of $C_{11}^{[1]}$ and a positive integer $k$ such that*

*(1)* $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) < \mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]})$*;*
*(2) for every $i \in \{r, \ldots, m\}$,* $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) \leq \mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]})$ .

**PROOF.** We choose $\lambda$ and $j \in \{r, \ldots, m\}$ such that the first and $j$th rows of $B$ are linearly independent and $m(\lambda, j)$ is the least number for all pairs $\lambda, j$ satisfying Lemma 60(3). By Lemma 60(1), $\mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}) \geq m(\lambda, j) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]})$ for a certain $k$. For any $i \neq j$, if $m(\lambda, i) \leq m(\lambda, j)$ and the first and $i$th rows are linearly independent, then the pair $\lambda, i$ satisfies Lemma 60(3), $m(\lambda, i) = m(\lambda, j)$ by the choice of $\lambda, j$ and $\mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]}) \geq m(\lambda, i) = m(\lambda, j) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]})$. If $m(\lambda, i) \geq m(\lambda, j)$ and the first and $i$th rows are linearly independent, then

$$\mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]}) \geq m(\lambda, i) \geq m(\lambda, j) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}).$$

Finally, if the first and $i$th rows are linearly dependent, then, by Lemma 60(2),

$$\mathsf{mult}(\lambda^{1/k}, C_{1i}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) > m(\lambda, j)$$

and

$$\mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]}) \geq \mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) > m(\lambda, j).$$

□

Finally, we are ready to put everything together.

**PROOF of the Single 1-Cell Lemma 54.** Let $B \in \mathbb{Z}[X]^{m \times m}$ be a matrix that satisfies the General Conditions 46 and has exactly one 1-cell. By Lemma 57, we may assume that $B$ satisfies the Single 1-Cell-Conditions 56.

We use the same notation as above; in particular, we define $r$ as in (6) on page 53 and $C^{[k]}$ as in (7) on page 53.

Choose $j, \lambda, k$ according to Lemma 61. Let

$$t = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}).$$

Then $\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) > t$ and $\mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]}) \geq t$ for $r \leq i \leq m$. Let $p_\lambda$ be an irreducible polynomial such that $\lambda$ is a root of $p_\lambda$ and let

$$C = (C^{[k]})|_{p_\lambda}.$$

By the Extended Prime Filter Lemma 35 $\text{WCSP}^{KD}(C)$ is polynomial time reducible to $\text{WCSP}^{KD}(C^{[k]})$ and hence to $\text{WCSP}^{KD}(B)$.

Case 1. For all $u, v$,

$$\mathsf{mult}(\lambda^{1/k}, C_{uv}^{[k]}) \geq t.$$

In this case, the matrix $C'$ obtained from $C$ by dividing by $p_\lambda^t$ is a positive symmetric matrix with at least two 1-cells, because $C_{1j} = C_{j1} = 1$, but $C_{11} \neq 1$. Then $\text{WCSP}^{KD}(C')$ is #P-hard by the Two 1-Cell Lemma 48. By Lemma 42, $\text{WCSP}^{KD}(C')$ is reducible to $\text{WCSP}^{KD}(C)$ and hence to $\text{WCSP}^{KD}(B)$.

Case 2. There are $u, v$ such that

$$\mathsf{mult}(\lambda^{1/k}, C_{uv}^{[k]}) < t.$$

In this case, let $s$ be the least multiplicity of $\lambda^{1/k}$ in the entries of $C^{[k]}$. Denote by $C'$ the matrix $C$ divided by $p_\lambda^s$. Let $u, v$ be indices with $\mathsf{mult}(\lambda^{1/k}, C_{uv}^{[k]}) < t$.

We claim that $u \geq r$ or $v \geq r$. To see this, recall that by the Single 1-Cell Condition 56 (H), the first $r-1$ rows of $B$ and hence of $B^{[k]}$ are identical. Since $C^{[k]} = B^{[k]} \cdot (B^{[k]})^\top$, this implies that for $u', v' \leq r-1$ we $C^{[k]}_{u'v'} = C^{[k]}_{11}$, and $\mathsf{mult}(\lambda^{1/k}, C^{[k]}_{11}) > t > s$. This proves our claim that $u, v \geq r$.

We have $C'_{uv} = C'_{vu} = 1$, and $C'_{uu} \neq 1$ or $C'_{vv} \neq 1$. Therefore, $C'$ has at least two 1-cells. Then $\mathrm{WCSP}^{KD}(C')$ is #P-hard by the Two 1-Cell Lemma 48. $\square$

**Proof of Theorem 16.** Let $B \in \mathbb{S}^{k \times \ell}$ be a non-negative matrix such that at least one block of $B$ has row rank at least 2. By the General Conditioning Lemma 47, without loss of generality, we may assume that the matrix $B$ satisfies the General Conditions 46. If $B$ has at least two 1-cells, then $\mathrm{WCSP}^{KD}(B)$ is #P-hard by the Two 1-Cell Lemma 48. If $B$ has just one 1-cell, then $\mathrm{WCSP}^{KD}(B)$ is #P-hard by the Single 1-Cell Lemma 54. $\square$

## 7 Conclusions

We give a complete complexity theoretic classification for the problem of evaluating the partition function of a symmetric non-negative matrix $A$, which may be viewed as the adjacency matrix of an undirected weighted graph $H$. Our proofs explore a correspondence between this evaluation problem and weighted constraint satisfaction problems for constraint languages with two equivalence relations.

Peculiarly, our proof does not go through for matrices with negative entries. Indeed, we do not know whether the evaluation problem for the matrix

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

is #P-hard. (Observe that the evaluation problem for this matrix is equivalent to the problem of counting induced subgraphs with an even number of edges.)

The more important open problem is to obtain a classification result for the evaluation problem for non-symmetric matrices, corresponding to directed graphs. We believe that with our results such a classification may now be within reach, in particular because our main hardness result goes through for directed graphs. The ultimate goal of this line of research is a classification of counting and weighted CSP for arbitrary constraint languages. Towards a solution of this problem, one may try to reduce the weighted CSP to evaluation problems for directed graphs. It is interesting to note that the known reduction between the corresponding decision problems does not give a reduction between the counting problems we are interested in here.

**Acknowledgement** We wish to thank Mark Jerrum for many useful discussions.

## References


[1] A. Bulatov and V. Dalmau. Towards a dichotomy theorem for the counting constraint satisfaction problem. In *Proceedings of the 44th IEEE Symposium on Foundations of Computer Science, FOCS'03*, pages 562–571, 2003.

[2] A.A. Bulatov. A dichotomy theorem for constraints on a three-element set. In *Proceedings of the 43rd IEEE Symposium on Foundations of Computer Science, FOCS'02*, pages 649–658, 2002.

[3] A.A. Bulatov. Tractable conservative constraint satisfaction problems. In *Proceedings of the 18th Annual IEEE Simposium on Logic in Computer Science*, pages 321–330, 2003.

[4] N. Creignou and M. Hermann. Complexity of generalized satisfiability counting problems. *Information and Computation*, 125(1):1–12, 1996.

[5] M. Dyer and C. Greenhill. The complexity of counting graph homomorphisms. *Random Structures and Algorithms*, 17:260–289, 2000.

[6] M.E. Dyer, L.A. Goldberg, and M. Jerrum. Counting and sampling $H$-colourings. In J.D.P. Rolim and S.P. Vadhan, editors, *Proceedings of the 6th International Workshop on Randomization and Approximation Techniques*, volume 2483 of *Lecture Notes in Computer Science*, pages 51–67. Springer-Verlag, 2002.

[7] T. Feder and M.Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory. *SIAM Journal of Computing*, 28:57–104, 1998.

[8] L.A. Goldberg, M. Jerrum, and M. Paterson. The computational complexity of two-state spin systems. *Random Structures and Algorithms*, 23:133–154, 2003.

[9] L.A. Goldberg, S. Kelk, and M. Paterson. The complexity of choosing an $H$-colouring (nearly) uniformly at random. In *Proceedings of the 34rd ACM Simposium on Theory of Computing*, pages 53–62, 2002.

[10] M. Grötschel, L. Lovasz, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimazation*. Springer-Verlag, 1993. 2nd edition.

[11] P. Hell and J. Nešetřil. On the complexity of $H$-coloring. *Journal of Combinatorial Theory, Ser.B*, 48:92–110, 1990.

[12] P. Hell, J. Nešetřil, and X. Zhu. Duality and polynomial testing of tree homomorphisms. *Trans. of the AMS*, 348(4):1281–1297, 1996.

[13] F. Jaeger, D.L. Vertigan, and D.J.A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Mathematical Proceedings of the Cambridge Philosophical Society*, 108:35–53, 1990.

[14] P.G. Jeavons, D.A. Cohen, and M. Gyssens. Closure properties of constraints. *Journal of the ACM*, 44(4):527–548, 1997.

[15] K. Ko. *Complexity Theory of Real Functions*. Birkhäuser, 1991.

[16] T.J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the 10th ACM Symposium on Theory of Computing*, pages 216–226, 1978.