BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 5, September 1973

THE COMPREHENSIVE FACTORIZATION OF A FUNCTOR

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Communicated by Saunders Mac Lane, March 12, 1973

In this article we show that every functor has a factorization into an initial functor followed by a discrete 0-fibration and that this factorization is functorial. Size considerations will be ignored but may be easily filled in; we assume the existence of a category of sets large enough to dwarf any given finite number of categories.

There is an analogy between the category **Set** of sets and the category **Cat** of categories which is partly explained by the observation that each is a category of types for a suitable hyperdoctrine. A hyperdoctrine (Lawvere [**4**]) consists of a category T of types and a functor $P: T^{\mathrm{op}} \to \mathbf{Cat}$ satisfying conditions. The comprehension schema (also see [**4**]) is expressed by a pair of adjoint functors

$$PX \leftrightarrows T/X$$

for each object X of T, where T/X is the category of objects over X. It is often the case that this structure arises from more usual structure on the category T; namely:

(1) a factorization system (E, M) on T;

(2) a category object Ω in T which "classifies the M-subobjects".

The sense in which (1) is intended is that of Freyd-Kelly [**2**]. We say that Ω classifies the M-subobjects when there is a natural equivalence of categories $T(X, \Omega) \approx M(X)$, where $M(X)$ is the full subcategory of T/X consisting of the arrows in M with target X. Then $PX = T(X, \Omega)$. From (1), the functor "take the (E, M)-image" is the left adjoint of the inclusion $M(X) \to T/X$; this adjunction combines with (2) to yield the comprehension schema.

The familiar example of a hyperdoctrine which arises in the above way is provided by the power-set functor $P: \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$. Here M consists of monomorphisms, E of epimorphisms, and Ω is a set with two elements. These considerations lie at the heart of the elementary theory of the category of sets in the new elegant form—elementary topos theory—due to Lawvere-Tierney (see Freyd [**1**]). A topos T is a finitely complete, cartesian closed category satisfying (2) where M consists of the mono-

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AMS (MOS) subject classifications (1970). Primary 18A20; Secondary 18A25, 18A40, 18D05, 18D30.

Key words and phrases. Factorization of functors, initial functors, discrete fibrations, comprehension schema, comma categories, Kan extensions.

morphisms. It can be proved that (E, M) is a factorization system on T where E consists of the epimorphisms.

It is our aim to show here that there is a factorization system on **Cat** which gives rise to the hyperdoctrine with $T = \mathbf{Cat}$ and PX equal to the category of set-valued functors on X. Moreover, we show that the classes E, M are already distinguished in the literature; namely, E consists of the initial functors, M consists of the discrete 0-fibrations. Forthcoming papers (see [6]) on the elementary theory of the 2-category of categories will consider the abstraction of the concept of discrete 0-fibration to an arbitrary 2-category and the corresponding condition (2).

For any category A, let $!: A \to \mathbf{1}$ denote the unique functor into the terminal object of **Cat**. Let $*: \mathbf{1} \to \mathbf{Set}$ denote the constant functor at the one point set. For functors $A \to^{r} D$, $B \to^{s} D$ with the same codomain, the *comma category* r/s is defined by the pullback

$$\begin{array}{ccc} r/s & \longrightarrow & D^2 \\ \binom{d_0}{d_1}\Big\downarrow & & \Big\downarrow\binom{d_0}{d_1} \\ A \times B & \xrightarrow[r \times s]{} & D \times D. \end{array}$$

The natural transformation

$$\begin{array}{ccc} r/s & \xrightarrow{d_1} & B \\ {\scriptstyle d_0}\Big\downarrow & \nearrow^{\lambda} & \Big\downarrow{\scriptstyle s} \\ A & \xrightarrow[r]{} & D \end{array}$$

corresponding to $r/s \to D^2$ is characterized by the universal property that composition with λ sets up a bijection

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle u}\swarrow & \Big\downarrow{\scriptstyle w} & \searrow{\scriptstyle v} & \\ A & & & & B \\ & {\scriptstyle d_0}\nwarrow & r/s & \nearrow{\scriptstyle d_1} & \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} X & \xrightarrow{v} & B \\ {\scriptstyle u}\Big\downarrow & \nearrow^{\sigma} & \Big\downarrow{\scriptstyle s} \\ A & \xrightarrow[r]{} & D \end{array}$$

between arrows of spans w and natural transformations σ. The particular case which is important here is the comma category $*/k$ for a functor $B \to^{k} \mathbf{Set}$; the objects are pairs (b, ξ) where b is an object of B and ξ is an element of kb, and the arrows $(b, \xi) \to^{\beta} (b', \xi')$ are arrows $b \to^{\beta} b'$ in B satisfying $(k\beta)\xi = \xi'$.

A natural transformation

$$\begin{array}{ccc} A & \xrightarrow{j} & B \\ & {\scriptstyle f}\searrow \;\curvearrowright\; \swarrow{\scriptstyle k} & \\ & \mathbf{Set} & \end{array}$$

is said to exhibit k as a *left extension* of f along j when it sets up a bijection between natural transformations

$$B \underset{g}{\overset{k}{\Downarrow}} \mathbf{Set}$$

and natural transformations

$$A \underset{gj}{\overset{f}{\Downarrow}} \mathbf{Set}.$$

Recall also (see Mac Lane [**5**, p. 236]) the formula

$$kb = \operatorname{colim}(j/b \xrightarrow{d_0} A \xrightarrow{f} \mathbf{Set}).$$

PROPOSITION 1. *For any functor $B \to^k$ **Set**, the natural transformation*

$$\begin{array}{ccc} */k & \xrightarrow{d_1} & B \\ {\scriptstyle ! = d_0}\downarrow & \lambda \nearrow & \downarrow{\scriptstyle k} \\ \mathbf{1} & \xrightarrow[*]{} & \mathbf{Set} \end{array}$$

exhibits k as a left extension of $*!$ along d_1. □

A functor $E \to^p B$ is called a *discrete* 0-*fibration* when there exists a functor $B \to^k$ **Set** and an isomorphism $E \cong */k$ such that

$$\begin{array}{ccc} E & \cong & */k \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle d_1} & \\ & B & \end{array}$$

commutes. By Proposition 1, k is uniquely determined up to isomorphism by p.

A functor $A \to^e E$ is called *initial* when the canonical natural transformation

$$\begin{array}{ccc} \mathbf{Set}^E & \xrightarrow{\mathbf{Set}^e} & \mathbf{Set}^A \\ & {\scriptstyle \lim}\searrow \ \xrightarrow{\text{res}_e} \ \swarrow{\scriptstyle \lim} & \\ & \mathbf{Set} & \end{array}$$

is an isomorphism.

These two definitions can be made independent of the choice of the category **Set**. A functor $E \to^p B$ is called a 0-*fibration* (Gray [**3**]) when the canonical functor $E^2 \to p/B$, induced by the natural transformation

$$\begin{array}{ccc} E^2 & \xrightarrow{pd_1} & B \\ {\scriptstyle d_0}\downarrow & p\lambda \nearrow & \downarrow{\scriptstyle 1} \\ E & \xrightarrow[p]{} & B, \end{array}$$

has a left adjoint with identity unit. The 0-fibration p is discrete if and only if, for each object b in B, the category E_b, defined by the pullback

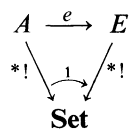
$$\begin{array}{ccc} E_b & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{1} & \underset{b}{\longrightarrow} & B \end{array}$$

is discrete. This is easily proved by defining $B \to^k \mathbf{Set}$ on objects by $kb = E_b$. That (a), (c) in the following proposition are equivalent is well known (see Mac Lane [**5** p. 213] for the dual); we give a different proof. Condition (b) is the important one for us.

PROPOSITION 2. *The following conditions are equivalent*:

(a) *the functor* $A \to^e E$ *is initial*;

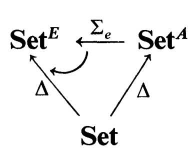
(b) *the identity natural transformation*,



exhibits $*!$ *as a left extension of* $*!$ *along* e;

(c) *for each object* x *in* E, *the comma category* e/x *is nonempty and pathwise connected.*

PROOF. Consider the natural transformation in the triangle



obtained from the triangle containing res_e by replacing each functor by its left adjoint. Then Δ takes a set to the constant functor at that set, and Σ_e takes a functor to its left extension along e. Also e is initial if and only if $\Sigma_e\Delta \to \Delta$ is an isomorphism. Every constant functor $A \to \mathbf{Set}$ is a coproduct of copies of the constant functor $*!$ and Σ_e, being a left adjoint, preserves coproducts. So $\Sigma_e\Delta \to \Delta$ is an isomorphism if and only if $\Sigma_e(*!) = *!$. This proves (a) $\Leftrightarrow$ (b). From the construction of colimits in **Set** it is clear that (c) says precisely that, for each object x in E,

$$\mathrm{colim}(e/x \xrightarrow{!} \mathbf{1} \xrightarrow{*} \mathbf{Set})$$

is the one point set. Using the formula for left extensions, we see that this is equivalent to (b). $\square$

THEOREM 3. *Any functor $A \to^f B$ admits a factorization*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle e}\searrow \quad \nearrow{\scriptstyle p} & \\ & E & \end{array}$$

where e is an initial functor and p is a discrete 0-fibration.

PROOF. Define $B \to^k \mathbf{Set}$ to be the left extension of $A \to^! \mathbf{1} \to^* \mathbf{Set}$ along f. Then define $E \to^p B$ to be $*/k \to^{d_1} B$. So p is a discrete 0-fibration. The natural transformation

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle !}\downarrow & \Rightarrow & \downarrow{\scriptstyle k} \\ \mathbf{1} & \xrightarrow[*]{} & \mathbf{Set} \end{array}$$

factors through the natural transformation

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ {\scriptstyle !}\downarrow & \Rightarrow & \downarrow{\scriptstyle k} \\ \mathbf{1} & \xrightarrow[*]{} & \mathbf{Set} \end{array}$$

by the universal property of the latter. So we have a functor e such that

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & & \searrow{\scriptstyle f} & \\ \mathbf{1} & & {\scriptstyle e}\downarrow & & B \\ & \nwarrow & & \nearrow{\scriptstyle p} & \\ & & E & & \end{array}$$

commutes. Consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{e} & E & \xrightarrow{p} & B \\ & {\scriptstyle *!}\searrow \;\Leftarrow & \downarrow{\scriptstyle *!} \;\Rightarrow & \swarrow{\scriptstyle k} & \\ & & \mathbf{Set} & & \end{array}$$

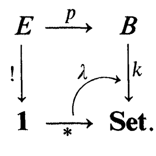
The outside triangle is a left extension by definition of k. The right triangle is a left extension by Proposition 1. So the left triangle is a left extension; e is initial. □

THEOREM 4. *Suppose, in the commutative diagram*

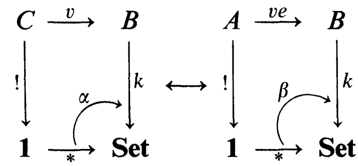
$$\begin{array}{ccc} A & \xrightarrow{e} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow[p]{} & B \end{array}$$

of functors, e is initial and p is a discrete 0-fibration. Then there exists a unique functor $C \to^{w} E$ such that $pw = v$ and $we = u$.

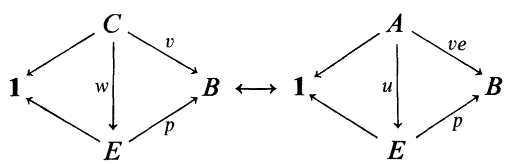
PROOF. Since p is a discrete 0-fibration there is a functor k and a universal natural transformation



Since e satisfies (b) of Proposition 2, composing with e sets up a bijection



between natural transformations α and natural transformations β. By the universal property of λ, this implies that composing with e sets up a bijection



between arrows of spans w and arrows of spans u. $\square$

It follows from Theorem 4 that the factorization of Theorem 3 is unique up to isomorphism and that the notion of image so obtained is functorial.

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