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# The Compressible Euler Equations in a Bounded Domain: Existence of Solutions and the Incompressible Limit

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**Abstract.** A short-time existence theorem is proven for the Euler equations for nonisentropic compressible fluid flow in a bounded domain, and solutions with low Mach number and almost incompressible initial data are shown to be close to corresponding solutions of the equations for incompressible flow.

## 1. Introduction

The Euler equations for compressible fluid flow are

$$\begin{aligned} \varrho_t+(u\cdot\nabla)\varrho+\varrho\nabla\cdot u&=0\,,\\ \varrho[u_t+(u\cdot\nabla)u]+\nabla P&=0\,,\\ S_t+(u\cdot\nabla)S&=0\,, \end{aligned}\tag{1.1}$$

to which must be added an equation of state prescribing the density $\varrho$ as a function of the pressure $P$ and entropy $S$. We will be considering a one-parameter family of equations of state $\varrho=f(P/\lambda^2,S)$; for example, for polytropic gases $\varrho=[(1+P/\lambda^2)e^{-A(S)}]^{1/\gamma}$. The equation of state allows (1.1) to be rewritten as

$$\begin{aligned} \frac{1}{\lambda^2\varrho}\frac{\partial\varrho}{\partial(P/\lambda^2)}(P_t+(u\cdot\nabla)P)+\nabla\cdot u&=0\,,\\ \varrho[u_t+(u\cdot\nabla)u]+\nabla P&=0\,,\\ S_t+(u\cdot\nabla)S&=0\,, \end{aligned}\tag{1.2}$$

a quasilinear symmetric hyperbolic system.

Formally letting $\lambda\to\infty$ in (1.2) produces the incompressibility condition $\nabla\cdot u=0$; the taking of this limit will be made rigorous by showing that, for suitable initial data, smooth solutions of the initial-boundary-value problem for (1.2) indeed converge, as $\lambda\to\infty$, to corresponding solutions of the equations for

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incompressible stratified fluids [17],

$$
\begin{aligned}
\nabla \cdot u &= 0\,, \\
\varrho[u_t + (u \cdot \nabla)u] + \nabla P &= 0\,, \\
\varrho_t + (u \cdot \nabla)\varrho &= 0\,.
\end{aligned}
\tag{1.3}
$$

[Note that the last equation in (1.3) follows from $S_t + (u \cdot \nabla)S = 0$, since $\varrho = \varrho(S)$ in the limit $\lambda \to \infty$.]

As noted in [10, 13], the one-parameter family of equations of state can be best interpreted physically as the effective equation of state of a fixed fluid whose equations of motion are written in nondimensionalized form, with $\lambda$ essentially the inverse of the initial Mach number of the fluid. The theorem to be proven therefore justifies the use (at least for small times) of the incompressible equations when the Mach number is small and the initial data is almost incompressible.

Before establishing this convergence result, solutions of the compressible Euler equations in a bounded domain must first be shown to exist, because a short-time existence theorem for initial-boundary-value problems for quasilinear symmetric hyperbolic systems has been developed [14, 16, Appendix A] only in the case when the boundary is noncharacteristic, and so does not apply to system (1.2) with the usual solid-wall boundary condition.

The existence theorem will be proven in Sects. 3 and 4, and the convergence theorem in Sect. 5. Analogous results have been proven for baratropic flow, in which $\varrho$ depends only on $P$, by Ebin [18] and by Klainerman and Majda [10] for space-periodic flow, and by Ebin [5, 6], Beirao Da Veiga [21, 22], and Agemi [1] for flow in a bounded domain; the proofs here are simpler than previous proofs for the bounded domain case.

Recently, Rauch and Nishida [15] have proven existence for baratropic flow in a bounded domain by methods similar to those used here. A general convergence theorem for singular limits of quasilinear symmetric hyperbolic systems with periodic boundary conditions has been proven by Browning and Kreiss [4] under more stringent assumptions about the initial data than those used in the special case here; the results of Barker [2] in effect, relax the assumptions on the initial data to those used here for the case of one space dimension, again assuming periodic boundary conditions. The convergence proof given here uses a modified version of the key idea in Barker's proof.

## 2. Preliminaries

$\Omega$ will denote a bounded region in $\mathbb{R}^n$ in which the flow takes place; we will consider only the case of greatest physical interest, i.e. $n=3$, although mathematically this is just a matter of convenience. $\nu$ will denote the outward normal on the boundary $\partial\Omega$.

$H^s$ denotes the usual Sobolev space of order $s$, which when $s$ is a nonnegative integer can be obtained by completing $C^\infty(\bar{\Omega})$ in the norm $\|\cdot\|_s \equiv \sum_{|\alpha| \leqq s} \|D^\alpha \cdot\|_{L^2}$.

Two basic facts about Sobolev spaces that we will need are that in three dimensions

$$\begin{aligned} &\text{i)} \qquad \|f\|_{C^0} \leqq k\|f\|_{H^s} \quad \text{for} \quad s > \tfrac{3}{2}, \\ &\text{ii)} \qquad \|fg\|_{H^r} \leqq k\|f\|_{H^{r+1}}\|g\|_{H^{r+1}}; \end{aligned} \tag{2.1}$$

the first of these is Sobolev's inequality, while the second follows from the Hölder and Gagliardo-Nirenberg inequalities.

The space $X_3 = \bigcap_{k=0}^{3} C^k([0, T]; H^{3-k})$ will be given the norm

$$|||w|||_{3,\lambda,T} \equiv \sup_{0 \leqq t \leqq T} |||w(t)|||_{3,\lambda} \equiv \|w(t)\|_3 + \|w_t(t)\|_2 + \frac{1}{\lambda}\|w_{tt}(t)\|_1 + \frac{1}{\lambda^2}\|w_{ttt}(t)\|_0;$$

similarly,

$$|||w(t)|||_{2,\lambda} \equiv \|w(t)\|_2 + \|w_t(t)\|_1 + \frac{1}{\lambda}\|w_{tt}(t)\|_0 .$$

The $\lambda$ dependence of these norms will be important only for the convergence theorem in Sect. 5.

The boundary condition for both (1.2) and (1.3) is

$$u \cdot v = 0 \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

The initial conditions for (1.2) are

$$\begin{aligned} P(0, x, \lambda) &= P_0(x, \lambda), \\ u(0, x, \lambda) &= u_0(x, \lambda), \\ S(0, x, \lambda) &= S_0(x, \lambda), \end{aligned} \tag{2.3}$$

while for (1.3) they are

$$\begin{aligned} u(0, x) &= \hat{u}_0(x), \\ \varrho(0, x) &= \varrho_0(x), \end{aligned} \tag{2.4}$$

where $\nabla \cdot \hat{u}_0 = 0$.

## 3. Existence Theorem for Fixed $\lambda$

**Theorem 1.** *System* (1.2), (2.2), (2.3) *has a classical solution* $(P, u, S)$ *on some time interval* $[0, T]$, *provided that*

i) $\Omega$ *is open and bounded in* $\mathbb{R}^3$ *and* $\partial\Omega$ *is in* $C^\infty$;

ii) $\exists N \subset \mathbb{R}^2$ *and a* $\delta_0 > 0$ *such that* $N_0 \equiv \{(P_0(x, \lambda), \ S_0(x, \lambda)) | x \in \Omega\} \subset\subset N$, $\varrho \in C^5(N)$, *and* $\varrho(P/\lambda^2, S) \geqq \delta_0$ *and* $\dfrac{\partial \varrho}{\partial(P/\lambda^2)} \geqq \delta_0$ *for* $(P, S) \in N$;

iii) $P_0(x, \lambda)$, $u_0(x, \lambda)$, $S_0(x, \lambda) \in H^3(\Omega)$;

iv) $v \cdot \partial_t^k u(0) = 0$ *on* $\partial\Omega$, $0 \leqq k \leqq 2$, *where* $\partial_t^k u(0)$ *is the* $k^{th}$ *time derivative at* $t = 0$ *of any solution of* (1.2) *and* (2.2), *as calculated from* (1.2) *to yield an expression in terms of* $P_0$, $u_0$, *and* $S_0$.

$T$ depends only on $\lambda$, $\Omega$, $\delta_0$, $\operatorname{dist}(N_0, \partial N)$, the $C^5$ norm of $\varrho$ and the $H^3$ norms of $P_0$, $u_0$, and $S_0$. $(P, u, S)$ are in $C^1([0,T]\times\bar{\Omega})\cap\left(\bigcap_{j=0}^{2} C^j([0,T]; H^{3-j-\delta_1}(\Omega))\right)$ for any $\delta_1>0$, and $\partial_t^j(P,u,S)$ are in $L^\infty([0,T]; H^{3-j}(\Omega))$ for $j=0,1,2,3$; furthermore their norms in these spaces can be bounded in terms of $\delta_1$ and the quantities $T$ depends on. Finally, the solution is unique in $C^1$.

*Remark.* If $\varrho$ and the initial data are smoother and more compatibility conditions like those in iv) are satisfied, then the solution will be correspondingly smoother; see [14] for a discussion of compatibility conditions.

*Proof.* The uniqueness part is standard; see e.g. Sect. 1 of [11]. The existence part consists of three steps: first, approximate Eqs. (1.2), (2.2), (2.3) by ones for which the boundary is noncharacteristic so that an existence theorem can be applied; second, establish uniform estimates for the solutions of the approximating equations; third, take a limit to obtain the solution to the original equations. The first and last of these steps will be performed in this section, while the middle step will be deferred to the next section.

In order for the solutions of the approximating equations defined below to be sufficiently smooth, it is necessary to approximate the initial data $f_0\equiv(P_0, u_0, S_0)$ in $H^3$ by functions $f_0^{(n)}$ in $H^5$ that satisfy the compatibility conditions through order three. Lemma 3.3 of [14] says that such approximations exist for linear equations with a nonsingular boundary matrix, and the proof given there can be extended to the case at hand, as follows:

As in the proof of Lemma 3.3 in [14], the approximants $f_0^{(n)}$ will be obtained by picking $g^{(n)}$ in $H^5$ such that $g^{(n)}\to f$ in $H^3$ and then picking $h^{(n)}$ in $H^5$ such that $h^{(n)}\to 0$ in $H^3$ and $f_0^{(n)}\equiv g^{(n)}-h^{(n)}$ obeys the desired compatibility conditions. In [14] it is shown that $h^{(n)}$ should obey on $\partial\Omega$ the conditions

$$M(A^\nu)^k\partial_\nu^k h^{(n)} = M(A^\nu)^k\partial_\nu^k g^{(n)} + M\cdot(\text{other terms}),\ 0\leqq k\leqq 3\,, \tag{3.1}$$

where $M$ is the matrix giving the boundary condition, i.e. for system (1.2), (2.2),

$$M=(0\quad \nu^T\quad 0)\,,$$

and $A^\nu$ is the boundary matrix of the system, defined by $A^\nu=\nu^j A^j$, where the differential equations can be written as $A^0 u_t + A^j u_{x_j} = F$; for system (1.2),

$$(A^\nu)^k=\begin{cases}\begin{pmatrix}0 & \nu^T & 0\\ \nu & 0 & 0\\ 0 & 0 & 0\end{pmatrix} & \text{if } k \text{ is odd},\\[2ex] \begin{pmatrix}1 & 0 & 0\\ 0 & \nu\otimes\nu & 0\\ 0 & 0 & 0\end{pmatrix} & \text{if } k \text{ is even}.\end{cases}$$

Also, the "other terms" in (3.1) contain fewer than $k$ normal derivatives, so that even though these terms now depend nonlinearly on $h^{(n)}$ they are known quantities and have the appropriate amount of smoothness by finite induction. The fact that

$A^\nu$ is singular is overcome by noting that $\mathrm{Range}[M(A^\nu)^k]=\mathrm{Range}[M]$, so (3.1) can be solved for $\partial_\nu^k h^{(n)}$. The rest of the construction now proceeds as in the proof of Lemma 3.3 in [14].

Thus we obtain $f_0^{(n)}=(P_0^{(n)}, u_0^{(n)}, S_0^{(n)})$ in $H^5$ obeying the compatibility conditions for system (1.2), (2.2) through order three, and converging in $H^3$ to $f_0=(P_0, u_0, S_0)$ as $n\to\infty$. In particular, $\partial_t^k u^{(n)}(0), \partial_t^k S^{(n)}(0)\in H^{5-k}(\Omega), 0\leqq k\leqq 5$. Pick some $T_0>0$; by Theorem 2.5.7 of [8] there exist $\tilde{u}^{(n)}, \tilde{S}^{(n)}\in H^5([0,T_0]\times\Omega)$ satisfying $\partial_t^k\tilde{u}^{(n)}(0)=\partial_t^k u^{(n)}(0)$, $\partial_t^k\tilde{S}^{(n)}(0)=\partial_t^k S^{(n)}(0)$, $0\leqq k\leqq 4$. [Recall that $\partial_t^k u^{(n)}(0)$, $\partial_t^k S^{(n)}(0)$ are to be defined by plugging $f_0^{(n)}$ into (1.2).] Now extend $v$ to be in $C^\infty(\bar{\Omega})$ and define the approximating equations, for sufficiently small positive $\varepsilon$:

$$\begin{aligned} &\frac{1}{\lambda^2\varrho}\frac{\partial\varrho}{\partial(P/\lambda^2)}[P_t+(u\cdot\nabla)P]+\nabla\cdot u=0,\\ &\varrho[u_t+(u\cdot\nabla)u]+\varepsilon(v\cdot\nabla)u+\nabla P=\varepsilon(v\cdot\nabla)\tilde{u}^{(n)},\\ &S_t+(u\cdot\nabla)S+\varepsilon(v\cdot\nabla)S=\varepsilon(v\cdot\nabla)\tilde{S}^{(n)}, \end{aligned}\tag{3.2}$$

together with boundary condition (2.2) and initial condition

$$(P(0),u(0),S(0))=(P_0^{(n)},u_0^{(n)},S_0^{(n)}).\tag{3.3}$$

The boundary is noncharacteristic for this system, the boundary condition is maximally nonnegative, and the compatibility conditions are satisfied through order three, so system (3.2), (2.2), (3.3) has a solution

$$P(t,x,\varepsilon,n), u(t,x,\varepsilon,n), S(t,x,\varepsilon,n)\in\bigcap_{j=0}^{4}C^j([0,T(\varepsilon,n)],H^{4-j}(\Omega)).$$

Because only a sketch of the existence theorem just used has appeared previously [14], I have included a precise statement and a proof in Appendix A.

This completes step one of the existence proof; now suppose we can show that $T(\varepsilon,n)$ can be taken to be some fixed $T>0$ provided $\varepsilon\leqq\tilde{\varepsilon}(n)$, and can show further that for such $\varepsilon$

$$|||P(\varepsilon,n),u(\varepsilon,n),S(\varepsilon,n)|||_{3,\lambda,T}\leqq C\tag{3.4}$$

for some $C$ independent of $\varepsilon$ and $n$. Then, letting $\varepsilon(n)=\min\left(\tilde{\varepsilon}(n),\frac{1}{n}\right)$, defining $v(n)=(P(\varepsilon(n),n),u(\varepsilon(n),n),S(\varepsilon(n),n))$, and abbreviating (3.2) as

$$L_n(v(n))v(n)=\varepsilon(n)F^{(n)},$$

we have

$$L_{n_1}(v(n_1))[v(n_1)-v(n_2)]=[L_{n_2}(v(n_2))-L_{n_1}(v(n_1))]v(n_2)+\varepsilon(n_1)F^{(n_1)}-\varepsilon(n_2)F^{(n_2)}.\tag{3.5}$$

Multiplying (3.5) by $v(n_1)-v(n_2)$, integrating over $\Omega$, and using Sobolev's inequality, (2.1), and (3.4) then gives

$$\frac{d}{dt}\|v(n_1)-v(n_2)\|_{L^2}^2\leqq\mathrm{const}\cdot\left[\|v(n_1)-v(n_2)\|_{L^2}^2+\frac{1}{n_1^2}+\frac{1}{n_2^2}\right].\tag{3.6}$$

Since $\|v(n_1)-v(n_2)\|_{L^2|t=0}\to 0$ as $n_1, n_2\to\infty$, applying Gronwall's inequality to (3.6) shows that as $n\to\infty$ $P(\varepsilon(n),n)$, $u(\varepsilon(n),n)$, $S(\varepsilon(n),n)$ converge in $C^0([0,T];L^2(\Omega))$ to some $P, u, S$. An interpolation argument (see e.g. pp. 39–40 of [13]) then shows that convergence takes place in $\bigcap_{j=0}^{2} C^j([0,T];H^{3-j-\delta})\supset C^1([0,T]\times\bar{\Omega})$, so $P, u, S$ satisfy (1.2), (2.2), (2.3) classically. Also, since a ball in $H^S$ is closed under $L^2$ convergence, (3.4) and (3.2) imply that

$$\sup_{t\in[0,T]} \|\partial_t^k(P,u,S)\|_{H^{3-k}}\leqq C_1$$

for some $C_1$ depending on the quantities stated in the theorem, including $\lambda$.

Furthermore, in order to show that $(P(\varepsilon,n), u(\varepsilon,n), S(\varepsilon,n))$ continue to exist and satisfy (3.4), it suffices to show merely that they satisfy (3.4) under the assumption that the solution exists and is in $X_4$, since a continuation principle says that the solution will exist and be in $X_4$ as long as it remains bounded in $X_3$. A proof of this continuation principle is given in Appendix A; [13] gives a sharper version for the initial-value problem.

Let $c(n)=|||(v\cdot\nabla)\tilde{u}^{(n)}|||_{3,\lambda,T_0}+|||(v\cdot\nabla)\tilde{S}^{(n)}|||_{3,\lambda,T_0}$, and define $\tilde{\varepsilon}(n)=1/c(n)$ so that the right side of (3.2) is bounded in $X_3$ if $\varepsilon\leqq\tilde{\varepsilon}(n)$. Given that the right side of (3.2) is so bounded, we will show in Sect. 4 that the solution of (3.2), (2.2), (3.3) satisfies the estimates

$$\frac{d}{dt}|||P(t),u(t),S(t)|||_{E_1}^2\leqq F_{1,\lambda}(|||P(t),u(t),S(t)|||_{3,\lambda}) \tag{3.7}$$

and

$$|||P(t),u(t),S(t)|||_{E_2}\leqq F_{2,\lambda}(|||P(t),u(t),S(t)|||_{E_1}), \tag{3.8}$$

where

$$k_1(\lambda)\,[|||\cdot|||_{E_1}+|||\cdot|||_{E_2}]\leqq|||\cdot|||_{3,\lambda}\leqq k_2(\lambda)\,[|||\cdot|||_{E_1}+|||\cdot|||_{E_2}], \tag{3.9}$$

and $F_{1,\lambda}$ is a continuous nondecreasing function on $(0,\infty)$, $F_{2,\lambda}$ is a continuous nondecreasing function on $[0,2|||P_0,u_0,S_0|||_{E_1}]$, and $F_{1,\lambda}$ and $F_{2,\lambda}$ are independent of $\varepsilon$ for $0<\varepsilon\leqq\varepsilon_0$. Now $P_0^{(n)}, u_0^{(n)}, S_0^{(n)}\to P_0, u_0, S_0$ in $H^3$, and (3.2) and the first half of (3.9) show that convergence also holds in the $|||\ \ |||_{E_1}$ norm, so $|||P_0^{(n)}, u_0^{(n)}, S_0^{(n)}|||_{E_1}\leqq\frac{3}{2}|||P_0,u_0,S_0|||_{E_1}$ for $n$ sufficiently large. Hence substituting (3.8) into the second half of (3.9) and plugging the result into (3.7) yields a differential inequality that shows $|||P(\varepsilon,n), u(\varepsilon,n), S(\varepsilon,n)|||_{E_1}$ is bounded on some fixed time interval $[0,T]$ provided $n$ is sufficiently large and $\varepsilon\leqq\tilde{\varepsilon}(n)$. Using (3.8) and the second half of (3.9) once more then yields (3.4).

## 4. Estimates

In the standard estimates for symmetric hyperbolic systems (see [9, 13, 14, 16, Appendix A]), only estimates involving normal derivatives near the boundary require inverting the boundary matrix $A^\nu$, which for (3.2) would yield estimates of order $\frac{1}{\varepsilon}$. Thus,

$$\frac{d}{dt}|||P(t,\varepsilon),u(t,\varepsilon),S(t,\varepsilon)|||^2_{3,\lambda,\tan}\leqq F_{3,\lambda}(|||P(t,\varepsilon),u(t,\varepsilon),S(t,\varepsilon)|||_{3,\lambda}), \tag{4.1}$$

where the $|||\ \ |||_{3,\lambda,\tan}$ norm is a sum of norms of localizations and includes no normal derivatives in a neighborhood of $\partial\Omega$. Define $\Omega_\delta\equiv\{x\in\bar{\Omega}|\mathrm{dist}(x,\partial\Omega)\leqq\delta\}$; given any $\delta$ we can pick the localizations so that $|||\ \ |||_{3,\lambda,\tan}$ is equivalent to $|||\ \ |||_{3,\lambda}$ on $\Omega\backslash\Omega_\delta$.

Next, the estimate

$$\frac{d}{dt}|||P(t,\varepsilon),u(t,\varepsilon),S(t,\varepsilon)|||^2_{2,\lambda}\leqq F_{4,\lambda}(|||P(t,\varepsilon),u(t,\varepsilon),S(t,\varepsilon)|||_{3,\lambda}) \tag{4.2}$$

can be obtained in standard fashion by simply not integrating the spatial derivatives by parts.

Furthermore, since $A^\nu$ for (1.1) had rank two we can solve for the normal derivative of two components of $(P,u,S)$ in a neighborhood of $\partial\Omega$ without obtaining terms of order $\frac{1}{\varepsilon}$ as follows: Let $\partial\Omega$ be given locally as the solution set of

$$0=\phi(x_1,x_2,x_3)\equiv x_3-\psi(x_1,x_2), \tag{4.3}$$

and make the standard change of variables $y_1=x_1$, $y_2=x_2$, $y_3=\phi(x_1,x_2,x_3)$. Then the equations from (3.2) for $w\equiv(u\cdot\nabla\phi,P)$ can be written as

$$\tilde{A}^\nu\partial_{y_3}w\equiv\begin{pmatrix}1 & \dfrac{1}{\lambda^2\varrho}\dfrac{\partial\varrho}{\partial(P/\lambda^2)}(u\cdot\nabla\phi)\\ \varepsilon|\nabla\phi|+\varrho(u\cdot\nabla\phi) & 1+[\psi_{y_1}(y_1,y_2)]^2+[\psi_{y_2}(y_1,y_2)]^2\end{pmatrix}\begin{pmatrix}\partial_{y_3}u\cdot\nabla\phi\\ \partial_{y_3}P\end{pmatrix}$$
$$=\text{sum of terms with only time and tangential derivatives of }(P,u). \tag{4.4}$$

For $\varepsilon\leqq$some $\varepsilon_0$, det $\tilde{A}$ will be bounded away from zero provided $|P,S|\leqq$some constant $k$ and $|u\cdot\nabla\phi|\leqq$some small constant $\varepsilon_1$ depending on $\varepsilon_0$, $k$, etc.

Fix $\varepsilon_0$, and fix $k$ such that $\|P,S\|_2\leqq 2\|P_0,S_0\|_2$ implies $|P,S|\leqq k$, thereby fixing $\varepsilon_1$. Because the boundary condition (2.2) says precisely that $u\cdot\nabla\phi=0$ on $\partial\Omega$, and $u_0^{(n)}$ is bounded in $H^3$ independent of $n$ and satisfies this boundary condition by assumption, $|u_0^{(n)}\cdot\nabla\phi|\leqq\frac{1}{2}\varepsilon_1$ on $\Omega_{\delta^*}$ for some sufficiently small $\delta^*$ independent of $n$. For later use, pick this $\delta^*$ small enough so that $\Omega_{\delta^*}$ can be covered by sets in which $\partial\Omega$ is defined by an equation like (4.3), and fix the $|||\ \ |||_{3,\lambda,\tan}$ norm so that it is equivalent to $|||\ \ |||_{3,\lambda}$ on $\Omega\backslash\Omega_{\frac{\delta^*}{2}}$. If $\mathbf{x}$ is in $\Omega_{\delta^*}$, then

$$|u(t_1,\mathbf{x})\cdot\nabla\phi|\leqq|u_0^{(n)}\cdot(\mathbf{x})\cdot\nabla\phi|+t_1\left[\sup_{0\leqq t\leqq t_1}|u_t(t)|\right]|\nabla\phi|\leqq\tfrac{1}{2}\varepsilon_1+C_1t_1|||u|||_{3,\lambda,t_1}\leqq\varepsilon_1 \tag{4.5}$$

for $t_1$ sufficiently small if $|||u|||_{3,\lambda,T}$ is finite for some $T>0$. Hence whatever estimate for $|||P,u,S|||_{3,\lambda,T_1}$ is eventually derived by assuming $\det\tilde{A}^\nu\geqq\frac{1}{2}$ in $\Omega_{\delta^*}$, (4.2) and (4.5) show that $\det \tilde{A}^{\nu}$ will indeed be $\geqq \frac{1}{2}$ there for $t \leqq$ some $T_2$. The estimate derived for $|||P, u, S|||_{3,\lambda,T_1}$ is therefore valid at least for $|||P, u, S|||_{3,\lambda,T}$, where $T = \min(T_1, T_2)$.

Now multiply (4.4) by $(\tilde{A}^{\nu})^{-1}$, take two time derivatives or one time and one normal derivative or two normal derivatives, and integrate over a set $G \subset \Omega_{\delta^*}$ in which $\partial\Omega$ is given by (4.3). Using Sobolev's inequality, estimate (2.1), the estimate

$$\|\partial_{y_1}^{k_1}\partial_{y_2}^{k_2}\partial_{y_3}^{k_3} f\|_{L^2} \leqq \varepsilon_2 \|\partial_{y_3}^{k_1+k_2+k_3} f\|_{L^2} + c(\varepsilon_2)\left[\|f\|_{k_1+k_2+k_3,\tan} + \|f\|_{k_1+k_2+k_3-1}\right], \tag{4.6}$$

and the estimate(s) already obtained from the case(s), if any, when fewer normal derivatives were taken, we obtain the estimate

$$\begin{aligned} |||\partial_{y_3} P, \partial_{y_3}(u \cdot \nabla\phi)|||_{2,\lambda,G} &\leqq F_{5,\lambda}(|||P, u, S|||_{2,\lambda}) \cdot [c(\varepsilon_2)(1 + |||P, u, S|||_{3,\lambda,\tan}) \\ &\quad + \varepsilon_2(|||\partial_{y_3}, P, \partial_{y_3}(u \cdot \nabla\phi)|||_{2,\lambda,G} + |||\partial_{y_3} u^1, \partial_{y_3} u^2|||_{2,\lambda,G})] \,. \end{aligned} \tag{4.7}$$

Estimate (4.6) which we just used can be proven for a whole space by using the Fourier transform and then extended to the case of a bounded domain by injecting $H^{k_1+k_2+k_3}(\Omega)$ boundedly into $H_0^{k_1+k_2+k_3}(\Omega') \subset H^{k_1+k_2+k_3}(\mathbb{R}^n)$ for some $\Omega' \supset \Omega$ (see pp. 274–276 of [7] for this bounded injection map).

Estimates (4.1) and (4.2), and essentially (4.7) also, are standard estimates for symmetric hyperbolic systems. The remaining estimates will make use of the special structure of (3.2). Taking the curl of the equation for $u$ in (3.2) and multiplying by $\frac{1}{\varrho}$ gives

$$[(\nabla \times u)_t + (u \cdot \nabla)\nabla \times u] + \frac{\varepsilon}{\varrho}(v \cdot \nabla)\nabla \times u + \text{lower order terms} = \frac{\varepsilon}{\varrho}\nabla \times (v \cdot \nabla)\tilde{u}^{(n)}, \tag{4.8}$$

because curl grad $\equiv 0$. Taking $D^{\alpha}$ of (4.8), multiplying by $2D^{\alpha}(\nabla \times u)$, summing over all $\alpha$ with $|\alpha| \leqq 2$ (including both time and space derivatives), and using standard estimates gives

$$\frac{d}{dt}|||\nabla \times u|||_{2,\lambda}^2 \leqq F_{\lambda}(|||P, u, S|||_{3,\lambda}) - \varepsilon \sum_{|\alpha| \leqq 2} \int_{\partial\Omega} |D^{\alpha}\nabla \times u|^2 \leqq F_{\lambda}(|||P, u, S|||_{3,\lambda}) \,. \tag{4.9}$$

The points are that i) given $P$ and $S$, the equation for $\nabla \times u$ is itself a symmetric hyperbolic system, and ii) the boundary terms in the estimates for this system always have a helpful sign so that the estimates proceed exactly as in the whole-space case. Analogous points hold for the equation for $S$ in (3.2) considered by itself, so

$$\frac{d}{dt}|||S|||_{3,\lambda,T}^2 \leqq F(|||P, u, S|||_{3,\lambda,T}) \,. \tag{4.10}$$

We are now ready to define the $E_1$ and $E_2$ norms and show that (3.7)–(3.9) hold. Define

$$|||P, u, S|||_{E_1} = (|||P, u, S|||_{3,\lambda,\tan}^2 + |||P, u, S|||_{2,\lambda}^2 + |||\nabla \times u|||_{2,\lambda}^2 + |||S|||_{3,\lambda}^2)^{1/2} \,,$$

and

$$|||P,u,S|||_{E_2}=\sum_G |||\partial_{y_3}P,\partial_{y_3}(u\cdot \nabla\phi)|||_{2,\lambda}, \tag{4.11}$$

where the sum is taken over a finite set of $G$s that cover $\Omega_{\frac{\delta^*}{2}}$, are contained in $\Omega_\delta$, and are such that in each of them $\partial\Omega$ can be represented as in (4.3). Since $|||\ \ |||_{3,\lambda,\tan}$ is equivalent to $|||\ \ |||_{3,\lambda}$ in $\Omega\backslash\Omega_{\frac{\delta^*}{2}}$, in order to show that (3.9) holds it suffices to show that it holds for functions supported in one of the $G$s. Within a $G$, $\nabla\times u$ can be written in the $y$-coordinates defined after (4.3) as $(\partial_{y_2}u^3-\psi_{y_2}\partial_{y_3}u^3-\partial_{y_3}u^2, \partial_{y_3}u^1-\partial_{y_1}u^3-\psi_{y_1}\partial_{y_3}u^3, \partial_{y_2}u^1-\psi_{y_2}\partial_{y_3}u^1-\partial_{y_1}u^2+\psi_{y_1}\partial_{y_3}u^2)$. Since the tangential derivatives $\partial_{y_1}$ and $\partial_{y_2}$ are included in the $|||\ \ |||_{3,\lambda,\tan}$ norm,

$$|||\partial_{y_3}(u^2+\psi_{y_2}u^3),\partial_{y_3}(u^1+\psi_{y_1}u^3)|||_{2,\lambda,G}\leqq c(|||u|||_{3,\lambda,\tan}+|||\nabla\times u|||_{2,\lambda}).$$

On the other hand $|||\ \ |||_{E_2}$ includes $\|\partial_{y_3}(u^3-\psi_{y_1}u^1-\psi_{y_2}u^3)\|_{2,\lambda,G}$. Since

$$\det\begin{pmatrix}1 & 0 & \psi_{y_1}\\ 0 & 1 & \psi_{y_2}\\ -\psi_{y_1} & -\psi_{y_2} & 1\end{pmatrix}=1+\psi_{y_1}^2+\psi_{y_2}^2\neq 0,$$

$$|||\partial_{y_3}u|||_{2,\lambda,G}\leqq k(|||P,u,S|||_{E_1}+|||P,u,S|||_{E_2}). \tag{4.12}$$

No calculation is even needed to show that (4.12) holds when $u$ on the left side is replaced by $P$ or $S$, so (3.9) holds.

Finally, (4.12) and the definition of the $E_1$ norm allow us to rewrite (4.7) as

$$\begin{aligned}|||\partial_{y_3}P,\partial_{y_3}u\cdot\nabla\phi|||_{2,\lambda,G}&\leqq F_{6,\lambda}(|||P,u,S|||_{E_1})\\ &\cdot[c(\varepsilon_2)(1+|||P,u,S|||_{E_1})+\varepsilon_2|||\partial_{y_3}P,\partial_{y_3}u\cdot\nabla\phi|||_{2,\lambda,G}],\end{aligned}$$

which, upon choosing $\varepsilon_2=\dfrac{1}{F_{6,\lambda}(3|||P(0),u(0),S(0)|||_{E_1})}$, can be solved for $|||\partial_{y_3}P,\partial_{y_3}u\cdot\nabla\phi|||_{2,\lambda,G}$ to obtain

$$|||\partial_{y_3}P,\partial_{y_3}u\cdot\nabla\phi|||_{2,\lambda,G}\leqq\frac{F_{7,\lambda}(|||P,u,S|||_{E_1})}{1-\dfrac{F_{6,\lambda}(|||P,u,S|||_{E_1})}{F_{6,\lambda}(3|||P(0),u(0),S(0)|||_{E_1})}}. \tag{4.13}$$

Adding (4.13) for all $G$s that occur in the sum in (4.11) gives us (3.8), while adding (4.1), (4.2), (4.9), and (4.10) yields (3.7).

## 5. Convergence as $\lambda\to\infty$

**Theorem 2.** *Assume*

i) *the hypotheses of Theorem 1 are satisfied for $\lambda\geqq$ some $\lambda_0$, with $\delta_0$ and* dist$(N_0,\partial N)\geqq$ *some constant independent of* $\lambda$;

ii) $\|P_0(x,\lambda)\|_3+\lambda\|\nabla\cdot u_0(x,\lambda)\|_2\leqq$ *constant*;

iii) $\exists\hat{u}_0,\hat{S}_0\in H^3$ *such that*

$$\lim_{\lambda\to\infty}\|u_0(x,\lambda)-\hat{u}_0(x)\|_3+\|S_0(x,\lambda)-\hat{S}_0(x)\|_3=0.$$

*Then*

a) *the solution* $(P(\lambda), u(\lambda), S(\lambda))$ *of* (1.2), (2.2), (2.3) *(shown to exist in Theorem 1) exists for a time T independent of* $\lambda$ *and satisfies*

$$\left|\left|\left|\frac{P(\lambda)}{\lambda}, u(\lambda), S(\lambda)\right|\right|\right|_{3,\lambda,T} + |||\nabla P(\lambda)|||_{2,\lambda,T} + \lambda|||\nabla \cdot u(\lambda)|||_{2,\lambda,T} \leqq c \tag{5.1}$$

*for some c independent of* $\lambda$;

b) *there exists* $\hat{u}, \hat{\varrho}$ *in* $C([0,T]; H^{3-\delta}) \cap C^1([0,T]; H^{2-\delta}) \cap C^1([0,T] \times \bar{\Omega})$ *for any* $\delta > 0$, *such that as* $\lambda \to \infty$ $\left(u(\lambda), \varrho\left(\frac{P(\lambda)}{\lambda^2}, S(\lambda)\right)\right) \to (\hat{u}, \hat{\varrho})$ *in* $C([0,T]; H^{3-\delta})$;

c) *there exists* $\hat{P}$ *in* $C^0([0,T]; H^3)$ *such that as* $\lambda \to \infty$ $\nabla P(\lambda) \to \nabla \hat{P}$ *weak-* in* $L^\infty([0,T]; H^2)$;

d) $(\hat{P}, \hat{u}, \hat{\varrho})$ *satisfy* (1.3), (2.2), (2.4) *with* $P, u, \varrho$ *replaced by* $\hat{P}, \hat{u}, \hat{\varrho}$, *and* $\varrho_0$ *defined by* $\varrho_0 \equiv \varrho(0, \hat{S}_0)$.

*Remark.* The scaling used here differs from [10, 13, 16] in that what is denoted $P$ here is usually called $\lambda^2 P$ there.

*Proof.* Once assertion a) of the theorem is established, assertions b)–d) follow as in Theorem 2.4 of [13] or Theorem 1 of [10], provided i) the test function we integrate against to establish that $(\hat{P}, \hat{u}, \hat{\varrho})$ is a weak solution is chosen to have compact support in $\Omega$, ii) we note that $S$ converges in $C^1([0,T]; H^{2-\delta})$ since there are no $0(\lambda)$ terms in its equation, iii) we observe that $u \to \hat{u}$ in $C^0([0,T]; H^{2-\delta}) \supset C^0([0,T] \times \bar{\Omega})$ implies $\hat{u}$ satisfies the boundary condition since $u$ does, and iv) we prove uniqueness, by standard arguments, for classical solutions of (1.3), (2.2), (2.4).

Furthermore, the initial data could be approximated by smoother data obeying more compatibility conditions, the resulting solutions would exist and remain in $X_4$ as long as their $X_3$ norm $|||\ |||_{3,\lambda,T}$ remained bounded, and any estimates for derivatives through order three derived for the approximating solutions will hold for the original solution by a limit argument, so in order to prove a) it suffices to show that (5.1) holds under the assumption that the solution exists and is in $X_4$ on $[0,T]$. Finally, it suffices to prove (5.1) for $\lambda$ sufficiently large.

Define $r = P/\lambda$ and $a(r/\lambda, S) = \dfrac{1}{\varrho(r/\lambda, S)} \dfrac{\partial \varrho(r/\lambda, S)}{\partial(r/\lambda)}$, so that we can rewrite (1.2) as

$$\begin{aligned} a[r_t + (u \cdot \nabla) r] + \lambda \nabla \cdot u &= 0, \\ \varrho[u_t + (u \cdot \nabla) u] + \lambda \nabla r &= 0, \\ S_t + (u \cdot \nabla) S &= 0, \end{aligned} \tag{5.2}$$

assumption ii) of the theorem as

$$\lambda(\|r(0,x,\lambda)\|_3 + \|\nabla \cdot u(0,x,\lambda)\|_2) \leqq \text{constant}, \tag{5.3}$$

and (5.1) as

$$|||r(\lambda), u(\lambda), S(\lambda)|||_{3,\lambda,T} + \lambda|||\nabla r, \nabla \cdot u|||_{2,\lambda,T} \leqq \text{constant}. \tag{5.4}$$

Note that (5.3) implies that

$$|||r(\lambda,0),u(\lambda,0),S(\lambda,0)|||_{3,\lambda}+\lambda|||\nabla r(\lambda,0),\nabla\cdot u(\lambda,0)|||_{2,\lambda}\leqq \text{constant}, \qquad (5.5)$$

i.e. that the desired estimate holds at time zero.

As in Sect. 4, we seek norms $|||\ |||_{E_1}$ and $|||\ |||_{E_2}$ satisfying

$$k_1(|||\cdot|||_{E_1}+|||\cdot|||_{E_2})\leqq|||\cdot|||_{3,\lambda}\leqq k_2(|||\cdot|||_{E_1}+|||\cdot|||_{E_2}), \qquad (5.6)$$

for which we can show that

$$\frac{d}{dt}|||(r,u,S)|||_{E_1}^2<F_1(|||r,u,S|||_{3,\lambda}) \qquad (5.7)$$

and

$$|||(r,u,S)|||_{E_2}\leqq F_2(|||r,u,S|||_{E_1}), \qquad (5.8)$$

but now the $k_i$ and $F_i$ are to be independent of $\lambda$. The $|||\ |||_{E_1}$ and $|||\ |||_{E_2}$ to be used here differ from those in Sect. 4; specifically, defining $V=(r,u,S)$,

$$|||V|||_{E_1}^2=(V,A^0V)+(V_t,A^0V_t)+\left(\frac{1}{\lambda}V_{tt},A^0\frac{1}{\lambda}V_{tt}\right)$$
$$+\left(\frac{1}{\lambda^2}V_{ttt},A^0\frac{1}{\lambda^2}V_{ttt}\right)+|||S|||_{3,\lambda}^2+\|\nabla\times u\|_2^2, \qquad (5.9)$$

$$|||V|||_{E_2}\equiv\|\nabla u\|_2+\|\nabla u_t\|_1+\left\|\frac{1}{\lambda}\nabla u_{tt}\right\|_0$$
$$+\|\nabla r\|_2+\|\nabla r_t\|_1\left\|\frac{1}{\lambda}\nabla r_{tt}\right\|_0, \qquad (5.10)$$

where $A^0=\operatorname{diag}(a,\varrho,\varrho,\varrho,1)$ and $(\cdot,\cdot)$ is the $L^2$ inner product on $\Omega$. Since $a$ and $\varrho$ are positive initially and the estimates to be proven show they remain so for a time independent of $\lambda$, (5.6) holds. However, we will use the estimate [3]

$$\|u\|_s\leqq c(\|\nabla\times u\|_{s-1}+\|\nabla\cdot u\|_{s-1}+\|u\|_0+\|u\cdot v\|_{s-\frac{1}{2},\partial\Omega}) \qquad (5.11)$$

to replace $|||V|||_{E_2}$ by

$$|||V|||_{E_3}\equiv\|\nabla\cdot u\|_2+\|\nabla\cdot u_t\|_1+\left\|\frac{1}{\lambda}\nabla\cdot u_{tt}\right\|_0$$
$$+\|\nabla\times u_t\|_1+\left\|\frac{1}{\lambda}\nabla\times u_{tt}\right\|_0+\|\nabla r\|_2+\|\nabla r_t\|_1+\left\|\frac{1}{\lambda}\nabla r_{tt}\right\|_0. \qquad (5.12)$$

That is, (5.11) implies that (5.6) still holds when $|||\ |||_{E_2}$ is replaced by $|||\ |||_{E_3}$, because $u\cdot v=0$ on $\partial\Omega$. Hence, in order to show that $|||r,u,S|||_{3,\lambda,T}\leqq c$ it suffices to show (5.7) and

$$|||r,u,S|||_{E_3}\leqq F_3(|||r,u,S|||_{E_1}). \qquad (5.13)$$

We need to derive several estimates in order to establish (5.7) and (5.13). First, taking $\partial_t^k$ of (5.2), $0 \leqq k \leqq 3$, multiplying by $2\partial_t^k(r, u, S)$ times the appropriate power of $\frac{1}{\lambda}$, integrating over $\Omega$, and integrating by parts as usual in the terms containing derivatives of order $k+1$ gives

$$\frac{d}{dt}\left[(V, A^0 V)+(V_t, A^0 V_t)+\left(\frac{1}{\lambda} V_{tt}, A^0 \frac{1}{\lambda} V_{tt}\right)+\left(\frac{1}{\lambda^2} V_{ttt}, A^0 \frac{1}{\lambda^2} V_{ttt}\right)\right] \leqq F_4(|||V|||_{3,\lambda}), \tag{5.14}$$

because the boundary terms always vanish by (2.2) and the factors of $\frac{1}{\lambda}$ work out correctly. Estimates for $S$ like those in Sect. 4 also have the correct $\lambda$-dependence, i.e.

$$\frac{d}{dt} |||S|||_{3,\lambda}^2 \leqq F_5 |||V|||_{3,\lambda}). \tag{5.15}$$

The same is true for spatial-dervative estimates for $\nabla \times u$ from Sect. 4, i.e.

$$\frac{d}{dt} \|\nabla \times u\|_2^2 \leqq F_6(|||V|||_{3,\lambda}). \tag{5.16}$$

Adding (5.14)–(5.16) yields (5.7).

Estimates time derivatives of $\nabla \times u$ like those in Sect. 4 would not have the correct $\lambda$-dependence because of the presence of $\frac{1}{\varrho} \nabla \varrho \times u_t$ as one of the "lower order terms" in (4.8); we therefore will estimate such terms as follows: Take the curl of the equation for $u$ in (5.2) and divide by $\varrho$ to obtain

$$\nabla \times u_t = -\frac{1}{\varrho} \nabla \varrho \times u_t - \frac{1}{\varrho} \nabla \times (\varrho(u \cdot \nabla) u). \tag{5.17}$$

Taking the $L^2$ norm of both sides and using calculus inequalities (2.1) to estimate the right side gives

$$\|\nabla \times u_t\|_0 \leqq F_7(\|V\|_3)(1+\|u_t\|_0). \tag{5.18}$$

Taking up to one spatial derivative of (5.17), taking the $L^2$ norm and using calculus inequalities gives

$$\begin{aligned} \|\nabla \times u_t\|_1 &\leqq F_7(\|V\|_3)(1+\|u_t\|_0+\|\nabla u_t\|_0) \\ &\leqq c F_7(\|V\|_3)(1+\|u_t\|_0+\|\nabla \cdot u_t\|_0+\|\nabla \times u_t\|_0). \end{aligned} \tag{5.19}$$

Finally, taking two time derivatives of (5.17) yields

$$\left\|\frac{1}{\lambda} \nabla \times u_{tt}\right\|_0 \leqq F_8(\|V\|_3)\|u_{tt}\|_0+\frac{1}{\lambda} F_9(|||V|||_{3,\lambda}). \tag{5.20}$$

The remaining basic estimates will be derived from the equations

$$\nabla \cdot u = -\frac{1}{\lambda} a[r_t + (u \cdot \nabla) r],$$
$$\nabla r = -\frac{1}{\lambda} \varrho [u_t + (u \cdot \nabla) u], \tag{5.21}$$

which are just (5.2) rewritten.

Taking two time derivatives of (5.21) and multiplying by $\frac{1}{\lambda}$, or one time derivative, or one time derivative and up to one spatial derivative, or up to two spatial derivatives, and taking the $L^2$ norm of both sides of the result yields

$$\left\|\frac{1}{\lambda} \nabla \cdot u_{tt}\right\|_0 + \left\|\frac{1}{\lambda} \nabla r_{tt}\right\|_0 \leqq F_{10}(\|V\|_3) \left\|\frac{1}{\lambda^2} V_{ttt}\right\|_0 + \frac{1}{\lambda} F_{11}(|||V|||_{3,\lambda}), \tag{5.22}$$

$$\|\nabla \cdot u_t\|_0 + \|\nabla r_t\|_0 \leqq F_{12}(\|V\|_3) \left\|\frac{1}{\lambda} V_{tt}\right\|_0 + \frac{1}{\lambda} F_{13}(|||V|||_{3,\lambda}), \tag{5.23}$$

$$\begin{aligned} \|\nabla \cdot u_t\|_1 + \|\nabla r_t\|_1 &\leqq F_{14}(\|V\|_3) \left\|\frac{1}{\lambda} V_{tt}\right\|_1 + \frac{1}{\lambda} F_{15}(|||V|||_{3,\lambda}) \\ &\leqq c F_{14}(\|V\|_3) \left[\left\|\frac{1}{\lambda} V_{tt}\right\|_0 + \left\|\frac{1}{\lambda} \nabla S_{tt}\right\|_0 + \left\|\frac{1}{\lambda} \nabla r_{tt}\right\|_0 \right. \\ &\quad \left. + \left\|\frac{1}{\lambda} \nabla \cdot u_{tt}\right\|_0 + \left\|\frac{1}{\lambda} \nabla \times u_{tt}\right\|_0 \right] \\ &\quad + \frac{1}{\lambda} F_{15}(|||V|||_{3,\lambda}), \end{aligned} \tag{5.24}$$

$$\|\nabla \cdot u\|_2 + \|\nabla r\|_2 \leqq \frac{1}{\lambda} F_{16}(|||V|||_{3,\lambda}). \tag{5.25}$$

Now,

$$\begin{aligned} \|V\|_3 &\leqq c(\|V\|_0 + \|\nabla r\|_2 + \|\nabla u\|_2 + \|\nabla S\|_2) \\ &\leqq c(\|V\|_0 + \|\nabla S\|_2 + \|\nabla \times u\|_2) + c(\|\nabla \cdot u\|_2 + \|\nabla r\|_2) \\ &\leqq c\left(|||V|||_{E_1} + \frac{1}{\lambda} |||V|||_{3,\lambda}\right) \end{aligned} \tag{5.26}$$

by (5.11) and (5.25). Substituting (5.26) into (5.18)–(5.20) and (5.22)–(5.25), substituting (5.18) and (5.23) into (5.19), substituting (5.20) and (5.22) into (5.24), and adding the resulting (5.19), (5.20), (5.22), (5.24), and (5.25) yields

$$|||V|||_{E_3} \leqq F_{16}(|||V|||_{E_1}) + \frac{1}{\lambda} F_{17}(|||V|||_{3,\lambda}) \leqq F_{16}(|||V|||_{E_1}) + \frac{1}{\lambda} F_{18}(|||V|||_{E_1}, |||V|||_{E_3}). \tag{5.27}$$

For $\lambda$ sufficiently large (5.27) can be solved for $|||V|||_{E_3}$ in terms of $|||V|||_{E_1}$, yielding (5.13).

Thus, $|||r, u, S|||_{3,\lambda,T} \leqq c$ for some $T$ and $c$ independent of $\lambda$, and (5.25) then implies (5.4), so the theorem is established.

## Appendix A

The purpose of this appendix is to state and prove the existence theorem for quasilinear symmetric hyperbolic systems with noncharacteristic boundary and maximally nonnegative boundary conditions that was used in Sect. 3. A sketch of a more general version of this theorem was presented in [14]; it should also be noted that an existence theorem for more complicated boundary conditions was proven in [20].

Because a solution to the quasilinear equation $L(u)u = F$ will be obtained by finding a fixed point of the map $v \to u$ given by $L(v)u = F$, we need a linear regularity theorem that bounds some norm of the solution $u$ of $Lu = F$ in terms of the same norm of the coefficients of $L$.

Before stating this slightly more precise version of the linear regularity theorem from [14], let us recall a few definitions.

First,

$$X_k([0,T];\Omega) \equiv \bigcap_{j=0}^{k} C^j([0,T]; H^{k-j}(\Omega))$$

has the norm

$$|||\cdot|||_{k,T} \equiv \sup_{0 \leqq t \leqq T} |||\cdot(t)|||_k \equiv \left[\sum_{j=0}^{k} \|\partial_t^j \cdot (t)\|^2_{H^{k-j}(\Omega)}\right]^{1/2}.$$

Note that

$$\|\cdot\|^2_{H^k([0,T]\times\Omega)} = \int_0^T |||\cdot(t)|||_k^2 .$$

A $|||\cdot|||_{k,\tan}$ norm will be defined similarly except that no normal derivatives are to be included in a neighborhood of the boundary $\partial\Omega$; that is a $|||\cdot|||_{k,\tan}$ norm is a sum of norms of localizations, and in the patches intersecting $\partial\Omega$ a coordinate system is used in which $\partial\Omega$ is mapped into a portion of the hyperplane $\{x_n = 0\}$ (cf. [9]), and no $\partial_{x_n}$ derivatives are included. The $||| \;\; |||_{k,\tan}$ norm therefore depends on the choice of patches and coordinates.

Next, the boundary matrix $A^\nu$ of a system $A^0 u_t + A^j u_{x_j} + Bu = F$ is defined by $A^\nu = \nu^j A^j$, where $\nu$ is the outer normal on $\partial\Omega$. The boundary condition $Mu = 0$ is called maximally nonnegative if $A^\nu$ is positive semidefinite on the null space $N$ of $M$ but not on any space containing $N$ as a proper subspace. For later use, $\nu$ can be extended to be in $C^\infty(\bar{\Omega})$ (assuming $\partial\Omega$ is smooth), and then $A^\nu$ is also defined for $x \in \Omega$, but the above condition is applied only on $\partial\Omega$.

Given the system $A^0 u_t + A^j u_{x_j} + Bu = F$ and initial data $u(0) = f$, "$\partial_t^i u(0)$" is defined by formally taking $i-1$ time derivatives of the system, solving for $\partial_t^i u$ and evaluating at time $t=0$; e.g. "$u(0)$" $= f$, "$\partial_t^1 u(0)$" $= (A^0(0,x))^{-1}(F(0,x) - A^j(0,x) f_{x_j} - B(0,x) f)$. The quotation marks remind us that "$\partial_t u(0)$" is not the derivative of a known function but rather the value that the derivative of the sought-for function $u$ will have provided $u$ exists. Finally, for $A$ a matrix, $|A|$ denotes its operator norm, i.e. $|A|=\sup|Av|/|v|$.

**Theorem A1.** *The system*

$$Lu \equiv A^0 u_t + A^j u_{x_j} + Bu = F \quad in \quad [0,T]\times\Omega, \tag{A1}$$

$$u(0,x)=f(x) \quad in \quad \Omega, \tag{A2}$$

$$M(x)v=0 \quad on \quad [0,T]\times\partial\Omega \tag{A3}$$

*has a unique solution in* $X_m([0,T];\Omega)$, $m\geqq 1$, *provided*

i) $\Omega$ *is open and bounded in* $R^n$ *and* $\partial\Omega$ *is smooth;*

ii) $A^0$, *the* $A_j$, *and* $B$ *are in* $X_s$, *where* $s=\max\left(m,\left[\frac{n}{2}\right]+2\right)$;

iii) $A^0$ *and the* $A^j$ *are symmetric and* $A^0\geqq$ *some* $c_1>0$;

iv) $M(x)$ *is in* $C^\infty(\bar\Omega)$;

v) $A^\nu$ *is nonsingular on* $[0,T]\times\partial\Omega$ *and* $|(A^\nu)^{-1}|\leqq$ *some* $c_2$ *there;*

vi) *the boundary condition is maximally nonnegative;*

vii) $F$ *is in* $H^m([0,T]\times\Omega)$ *and* $f$ *is in* $H^m(\Omega)$;

viii) $M''\partial_t^i u(0)''=0$ *on* $\partial\Omega$, $0\leqq i\leqq m-1$.

The solution obeys the estimates

$$|||u(t)|||_{m-1}^2+|||u(t)|||_{m,\tan}^2 \leqq F_1(1/c_1, |||A^0,A^j|||_{s-1,T})e^{F_2(|||A^0,A^j,B|||_{s,T},1/c_1,c_2)t} \cdot\left\{|||u(0)|||_{m-1}^2+|||u(0)|||_{m,\tan}^2+F_3(|||A^0,A^j,B|||_{s,T},c_2)\cdot\int_0^t e^{-F_2t'}|||F(t')|||_m^2dt'\right\}, \tag{A4$_m$}$$

$$|||u(t)|||_m\leqq F_4(|||A^0,A^j,B|||_{s-1,T},c_2)(1+|||A^0,A^j|||_{s,T}^a)\cdot\{|||u(t)|||_{m-1}+|||u(t)|||_{m,\tan}+|||F(t)|||_{m-1}\}, \tag{A5$_m$}$$

where $a$ is a constant $<1$ that depends only on $m$ and the dimension $n$, and the $F_j$ are positive, continuous, and nondecreasing. (The $F_j$ also depend on $m$, $\Omega$, and $M$ but we will consider these fixed.) Finally, if $m>\left[\frac{n}{2}\right]+2$ then

$$|||u(t)|||_{m,\tan}^2\leqq F_1(1/c_1,|||A^0,A^j|||_{m-1,T})e^{F_5(|||A^0,A^j,B|||_{m-1,T},1/c_1,c_2)t} \cdot\left\{|||u(0)|||_{m,\tan}^2+F_6(|||A^0,A^j,B|||_{m-1,T},c_2)\left[|||u|||_{m-1,T}^2\int_0^t|||A^0,A^j,B|||_m^2 +\int_0^t e^{-F_5t'}|||F(t')|||_m^2dt'\right]\right\}, \tag{A6$_m$}$$

$$|||u(t)|||_m\leqq F_7(|||A^0,A^j,B|||_{m-1,T},c_2)\{|||u(t)|||_{m-1}+|||u(t)|||_{m,\tan}+|||F(t)|||_{m-1}\}. \tag{A7$_m$}$$

*Proof.* We will first prove the theorem in the case when $L$ is smooth, and then show by an approximation argument that it holds generally. For smooth $L$, the existence of a unique solution $u$ in $X_m$ follows easily from Theorem 3.1 of [14]. To show that (A4)–(A7) hold we will follow the derivation of similar estimates in [14, 9] while keeping track of how these estimates depend on the coefficients of $L$.

Using Lemma 3.3 of [14] we can approximate $F$ and $f$ in $H^m$ by $F_k$ and $f_k$ in $H^{m+\left[\frac{n}{2}\right]+2}$ such that $M''\partial_t^i u^k(0)''=0$ on $\partial\Omega$, $0\leqq i\leqq m+\left[\frac{n}{2}\right]+1$. Now if $(A4)$–$(A7)_m$ hold for the resulting solutions $u^k$ then straightforward limit arguments show that $u^k\to u$ in $X_m$ and $u$ obeys $(A4)$–$(A7)_m$ as well, so we can assume that $u$, $f$, and $F$ are in $C^{m+1}$, which ensures that all of the calculations to be made later are legitimate.

As in [9] and Lemma 3.2 of [14], we will use a partition of unity $\{\phi^i\}$ and changes of dependent and independent variables to reduce to the case when either $\Omega=\{x|x_n>0, |x|<1\}$, $\operatorname{supp} u\subset\{|x|<\frac{1}{2}\}$, $Mu=0$ on $\{x_n=0\}$ with $M$ a constant matrix, and $|(A^n)^{-1}|\leqq 2c_2$, or else $\Omega=\mathbb{R}^n$ and $\operatorname{supp} u\subset\{|x|<1\}$. Note that $|A^\nu(y)-A^\nu(x)|\leqq c(\Omega)\,\|A^\nu\|_{C^\alpha}|x-y|^\alpha\leqq c(\Omega)\,|||A^j|||_{\left[\frac{n}{2}\right]+1,T}|y-x|^\alpha$ by standard inbedding lemmas when $\alpha=1/8$ say, so that

$$|A^\nu(y)^{-1}|=|A^\nu(x)^{-1}(I-[A^\nu(x)-A^\nu(y)]A^\nu(x)^{-1})^{-1}|$$
$$\leqq c_2/(1-c(\Omega)\,|||A^j|||_{s-1,T}|y-x|^\alpha c_2)\leqq 2c_2$$

if

$$x\in\partial\Omega \quad\text{and}\quad |y-x|^{-\alpha}<[2c(\Omega)c_2|||A^j|||_{s-1,T}]\,.$$

Hence the choice of the open cover, and so of the partition of unity subordinate to it, depends on $L$ only through $c_2$ and $|||A^j|||_{s-1,T}$, while the changes of dependent and independent variables depend only on $\Omega$ and $M$, respectively, not on $L$.

Let $(\ ,\ )$ and $\|\ \ \|$ denote the spatial $L^2$ inner product and norm, respectively. In $R^n$ $|||\ \ |||_{k,\tan}\equiv|||\ \ |||_k$, while in $\Omega=\{x|x_n>0,\ |x|<1\}$ $|||\cdot|||_{k,\tan}\equiv\sum_{\substack{|\alpha|\leqq m\\ \alpha_n=0}}\|D^\alpha\cdot\|$; here $\alpha=(\alpha_0,\alpha_1,\dots,\alpha_n)$ and $D^\alpha=\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}$. The $|||\ \ |||_{k,\tan}$ norm for the original domain is defined by $|||\cdot|||_{k,\tan}\equiv\sum_i|||\phi^i\cdot|||_{k,\tan}$, where the norms on the right side are evaluated in the transformed domains.

Then, since $L(\phi^i u)=\phi^i F+(\phi_t^i A^0+\phi_{x_j}^i A^j)u$, estimates like (A4)–(A7) for the $\phi^i u$ imply that $u$ satisfies (A4)–(A7) with $|||F|||_k$ replaced by $|||F|||_k+|||(A^0,A^j)u|||_k$; in particular, $F_1$, $F_4$, etc. still depend only on the $X_{s-1}$ norm of the coefficients. The calculus inequalities

$$|||fg|||_k\leqq c|||f|||_{\left[\frac{n}{2}\right]+1}|||g|||_k\,,\qquad k\leqq\left[\frac{n}{2}\right]+1\,, \tag{A8}$$

$$|||fg|||_k\leqq c(|||f|||_k\,|||g|||_{k-1}+|||f|||_{k-1}\,|||g|||_k)\,,\qquad k\geqq\left[\frac{n}{2}\right]+2\,, \tag{A9}$$

which will be proven in Appendix B, then show that $u$ satisfies the original (A5) and (A7), and satisfies (A4) and (A6) with $|||F|||_k$ replaced by $|||F|||_k+|||u|||_k$. Substituting (A5) and (A7) into the new (A4) and (A6), respectively, allow us to apply Gronwall's inequality to show that the original (A4) and (A6) hold, so it suffices to prove the estimates for the localizations.

Since the estimates for $R^n$ are similar, but simpler and better known, we will consider only the boundary patches.

*Claim.* In order to show that (A4)–(A7) hold, it suffices to show that for any positive $\varepsilon$,

$$\frac{d}{dt}\sum_{\substack{|\alpha|\leqq m\\ |\alpha|\leqq m-1 \text{ or } \alpha_n=0}}(D^\alpha u, A^0 D^\alpha u)\leqq F(|||A^0, A^j, B|||_{s,T})|||u|||_m^2+c|||F|||_m^2, \quad \text{(A10)}$$

$$\begin{aligned}\|\partial_t^i\partial_n^{m-i}u\| &\leqq F(|||A^0, A^j, B|||_{s-1,T}, c_2)\,\|\partial_t^{i+1}\partial_n^{m-i-1}u\|+\varepsilon\sum_{j=0}^{m-1}\|\partial_t^j\partial_n^{m-j}u\|\\ &+F(|||A^0, A^j, B|||_{s-1,T}, c_2, 1/\varepsilon)\,(1+|||A^0, A^j|||_{s,T}^a)\,(|||u|||_{m,\tan}+|||u|||_{m-1}\\ &+|||F|||_{m-1}), \quad 0\leqq 1\leqq m-1, \qquad \text{(A11)}\end{aligned}$$

and that if $m>\left[\frac{n}{2}\right]+2$, then the term $|||A^0, A^j|||_{s,T}^a$ can be omitted from (A11) and the right side of (A10) can be replaced by $F(|||A^0, A^j, B|||_{m-1,T})$ $(|||A^0, A^j, B|||_{m,T}^2|||u|||_{m-1}^2+|||u|||_m^2)+c|||F|||_m^2$.

*Proof of the Claim.* After choosing $\varepsilon$ sufficiently small, the estimates in (A11) can be solved for the $\|\partial_t^i\partial_n^{m-i}u\|$, showing that these quantities are bounded by the right side of (A5). The remaining terms on the left side of (A5) have the form $\|D^\alpha u\|$ with $|\alpha|\leqq m$ and $\alpha_0+\alpha_n\leqq m-1$, so the calculus inequality,

$$\sum_{\substack{|\alpha|\leqq m\\ \alpha_0+\alpha_n\leqq m-1}}\|D^\alpha f\|\leqq\varepsilon\sum_{i=0}^{m-1}\|\partial_t^i\partial_n^{m-i}f\|+F(1/\varepsilon,\Omega)\,(|||f|||_{m,\tan}+|||f|||_{m-1}), \quad \text{(A12)}$$

then shows that (A5) holds. [Inequality (A12) is a straightforward consequence of the calculus inequality

$$\sum_{\substack{|\alpha|\leqq k\\ \alpha_0=0,\,\alpha_n\leqq k-1}}\|D^\alpha f\|\leqq\varepsilon\|\partial_n^k f\|+c(1/\varepsilon,\Omega)\left(\sum_{\substack{|\beta|=k\\ \beta_0=0=\beta_n}}\|D^\beta f\|+\|f\|_{H^{k-1}}\right),$$

which can be proven for $R^n$ by Fourier transform and then extended to bounded domains.] Next, substituting (A5) into the right side of (A10) and using $\|\cdot\|^2\leqq\frac{1}{c_1}(\cdot, A^0\cdot)$ allows us to apply Gronwalls inequality to the transformed (A10); using

$$c_1\|\cdot\|^2\leqq(\cdot, A^0\cdot)\leqq|||A^0|||_{s-1,T}\|\cdot\|^2$$

once more then yields (A4). Finally, a similar argument shows that the alternate forms of (A10–A11) imply (A7) and (A6).

To prove (A10), let $\alpha_n=0$ and $|\alpha|\leqq m$, take $D^\alpha$ of $Lu=F$, and define $u^\alpha\equiv D^\alpha u$ to obtain

$$A^0u_t^\alpha+A^ju_{x_j}^\alpha+Bu^\alpha=D^\alpha F+[A^0, D^\alpha]u_t+[A^j, D^\alpha]u_{x_j}+[B, D^\alpha]u. \quad \text{(A13)}$$

Multiply (A 13) by $2u^\alpha$, integrate over the spatial variables, and integrate by parts in the terms on the left side to get

$$\begin{aligned}\frac{d}{dt}(u^\alpha, A^0u^\alpha) &= \left(\int_{\{x_n=0\}}[-(u^\alpha)^T A^n u^\alpha]\right) + (u^\alpha, (A_t^0 + A_{x_j}^j - 2B)u^\alpha) \\ &\quad + (2u^\alpha, D^\alpha F) + (2u^\alpha, [A^0, D^\alpha]u_t + [A^j, D^\alpha]u_{x_j} + [B, D^\alpha]u) \\ &\leqq \left(\int_{\{x_n=0\}}[-(u^\alpha)^T A^n u^\alpha]\right) + (\|A_t^0 + A_{x_j}^j - 2B\|_{C^0} + c)\|u^\alpha\|^2 \\ &\quad + \|D^\alpha F\|^2 + \|[A^0, D^\alpha]u_t\|^2 + \|[A^j, D^\alpha]u_{x_j}\|^2 + \|[B, D^\alpha]u\|^2 .\end{aligned} \tag{A 14}$$

Now $MD^\alpha u = D^\alpha M u = 0$ because the transformed $M$ is constant and $\alpha_n = 0$, and $A^n$ is a scalar multiple of the original $A^\nu$ so $-(u^\alpha)^T A^n u^\alpha \leqq 0$ on $\{x_n = 0\}$ because the boundary condition is nonnegative. Also,

$$\|A_t^0 + A_{x_j}^j - 2B\|_{C^0} \leqq c|||A^0, A^j, B|||_{\left[\frac{n}{2}\right]+2, T} .$$

The last three terms in (A 14) themselves consist of terms of the form $\|(D^\alpha A)(D^\beta u)\|^2$, with $|\alpha| + |\beta| \leqq m+1$; $|\alpha|, |\beta| \leqq m$; and $A = A^0$, $A^j$, or $B$. Now if $|\beta| = m$ then

$$\|(D^\alpha A)(D^\beta u)\|^2 \leqq \|D^\alpha A\|_{C^0}^2 \|D^\beta u\|^2 \leqq |||A^0, A^j, B|||^2_{\left[\frac{n}{2}\right]+2} \cdot |||u|||_m^2 ,$$

while if $s = m = |\alpha|$ then

$$\|(D^\alpha A)(D^\beta u)\|^2 \leqq \|D^\alpha A\|^2 \|D^\beta u\|_{C^0}^2 \leqq |||A^0, A^j, B|||_m^2 |||u|||^2_{\left[\frac{n}{2}\right]+2} .$$

In the remaining cases $|\alpha| \leqq s-1$ and $|\beta| \leqq m-1$, so the calculus inequality

$$\begin{aligned}\|(D^\alpha f)(D^\beta g)\| &\leqq c|||f|||_{s_1}^{a_1} |||f|||_{s_1-1}^{1-a_1} |||g|||_{s_2}^{a_2} |||g|||_{s_2-1}^{1-a_2} \\ &\leqq c|||f|||_{s_1} |||g|||_{s_2}, \qquad |\alpha| < s_1, \qquad |\beta| < s_2, \\ & |\alpha| + |\beta| + \frac{n}{2} < s_1 + s_2 ,\end{aligned} \tag{A 15}$$

where $0 < a_i < 1$, shows that $\|(D^\alpha A)(D^\beta u)\|^2 \leqq c|||A^0, A^j, B|||_s^2 |||u|||_m^2$. Also if $m \geqq \left[\frac{n}{2}\right] + 3$ and $n \geqq 2$ or if $m \geqq \left[\frac{n}{2}\right] + 4$ and $n = 1$, then by the second part of (A 15)

$$\|(D^\alpha A)(D^\beta u)\|^2 \leqq c(|||A^0, A^j, B|||_m^2 |||u|||_{m-1}^2 + |||A^0, A^j, B|||_{m-1}^2 |||u|||_m^2) ,$$

while if $n = 1$ and $m = 3$, then with each $s_i = m$ each $a_i$ in the first part of (A 15) is $< \frac{1}{2}$ so the same conclusion holds. [Inequality (A 15) will also be proven in Appendix B.]

Similar estimates are obtained for the case $|\alpha| \leqq m-1$ by following the same procedure but omitting the integration by parts in the spatial derivative terms to avoid boundary terms. Adding these estimates for all $|\alpha| \leqq m$ with $|\alpha| \leqq m-1$ or $\alpha_n = 0$ yields (A 10), including the special form when $m > \left[\frac{n}{2}\right] + 2$.

To prove (A 11), solve $Lu=F$ for $u_{x_n}$, and take $\partial_t^i\partial_n^{m-i-1}$ of the result to get

$$\|\partial_t^i\partial_n^{m-i}u\| \leqq \left\|\partial_t^i\partial_n^{m-i-1}\left[(A^n)^{-1}\left(F-A^0u_t-\sum_{j=1}^{n-1}A^ju_{x_j}-Bu\right)\right]\right\|. \tag{A 16}$$

Using Sobolev's inequality ($\|\cdot\|_{C^0}\leqq\|\cdot\|_{\left[\frac{n}{2}\right]+1,T}$), (A 8–A 9), (A 12), (A 15), the elementary estimates $c_1^a c_2^{1-a}\leqq\varepsilon c_1+c(1/\varepsilon)c_2$ and $\|A\|_{C^0}\leqq c|A|$ for matrices $A$, and the fact that $|(A^n)^{-1}|\leqq 2c_2$ to estimate the right side of (A 16) yields (A 11), with the term $|||A^0, A^j|||_s^a$ present only if $m\leqq\left[\frac{n}{2}\right]+2$, thereby completing the proof for the case when $L$ is smooth.

Now consider the case when $L$ is not smooth. By the proof of Lemma 3.3 of [14], there exist sequences $\tilde{F}_k$ and $f_k$ in $H^{m+1}$ such that $\tilde{F}_k\to F$ in $H^m([0,T]x\Omega)$, $f_k\to f$ in $H^m(\Omega)$, and $M$"$\partial_t^i u^k(0)$"$=0$ on $\partial\Omega$, $0\leqq i\leqq m-1$, where "$\partial_t^i u^k(0)$" is defined like "$\partial_t^i u(0)$" but using $\tilde{F}_k$ and $f_k$ in place of $F$ and $f$. (Note that this much weaker version of the lemma holds even though $L$ is not smooth.) By repeating elements of the sequence if necessary, we can arrange that $\|\tilde{F}_k\|_{H^{m+1}}$, $\|f_k\|_{H^{m+1}}\leqq ck$. Then, since "$\partial_t^i u^k(0)$" is in $H^{m-i+1/2}(\Omega)$, Theorem 2.5.7 of [8] says there exists $U^k$ in $H^{m+1}([0,T]x\Omega)$ such that $\partial_t^i U^k(0)=$"$\partial_t^i u^k(0)$", $0\leqq i\leqq m$, and $\|U^k\|_{H^{m+1}}\leqq ck$.

Now pick $B_k$ and symmetric $A_k^0$ and $A_k^j$ in $C^\infty$ such that

$$|||A_k^0-A^0, A_k^j-A^j, B_k-B|||_s\leqq\frac{1}{k^2}, \qquad |A_k^\nu-A^\nu|\leqq\frac{1}{k^2},$$

and consider the system

$$\begin{aligned} L_k u^k \equiv A_k^0 u_t^k + \left(A_k^j+\frac{1}{k^2}\nu^j I\right)u_{x_j}+B_k u = F_k \equiv \tilde{F}_k + (A_k^0-A^0)U_t^k \\ + \left(A_k^j+\frac{1}{k^2}\nu^j I - A^j\right)U_{x_j}+(B_k-B)U\,, \end{aligned} \tag{A 17}$$

$$u^k(0)=f_k\,, \tag{A 18}$$

$$Mu^k=0 \quad \text{on} \quad \partial\Omega\,. \tag{A 19}$$

For $k$ sufficiently large, conditions i–v and vii of Theorem A1 are satisfied for (A 17)–(A 19), and since "$\partial_t^i u^k(0)$", $0\leqq i\leqq m$, is the same for $L_k u^k=F_k$, $u^k(0)=f_k$ as for $Lu^k=\tilde{F}_k$, $u^k(0)=f_k$, condition viii also holds, by construction. Next, the boundary matrix for (A 17) is $A_k^\nu+\frac{1}{k^2}I$, and if $Mv=0$, then

$$v^T\left(A_k^\nu+\frac{1}{k^2}I\right)v\geqq v^TA^\nu v-|v^T(A_k^\nu-A^\nu)v|+\frac{1}{k^2}|v|^2\geqq 0\,.$$

Also, if $w\in N^\perp$ and $|w|=1$, then picking $z\in N$ to maximize $c$ in $(z+w)^TA^\nu(z+w) < -c|z+w|^2$ gives a $c(w)$ that is continuous and hence bounded away from zero.

Hence for $k$ sufficiently large

$$(z+w)^T\left(A_k^\nu+\frac{1}{k^2}I\right)(z+w)<0,$$

so the boundary condition is still maximally nonnegative.

Therefore, since $L_k$ is smooth, (A17)–(A19) has a solution $u^k$ in $X_m$ obeying (A4)–(A7)$_m$. Since $\|F_k-\tilde{F}_k\|_{H^m}\leqq c/k$ by (A8), $F_k\to F$ in $H^m$. Also, $|||u^k|||_{m,T}\leqq c$ since the $\|F_k\|_{H^m}$ etc. are bounded. Next,

$$L(u^k-u^l)=F_k-F_l+(L-L_k)u^k-(L-L_l)u^l,$$

and multiplying this by $u^k-u^l$, integrating over $\Omega$, integrating by parts, and applying Gronwall's inequality shows that $\{u^k\}$ is a Cauchy sequence in $C^0([0,T];L^2)$. Since $Lu^k=F_k+(L-L_k)u^k\to F$ in $L^2$, the limit $u$ of the sequence $\{u^k\}$ is a strong solution of (A1)–(A3).

Next, since $|||u^k|||_m$ is bounded and a ball in $H^m$ is closed under $L^2$-convergence, $u$ is in $L^\infty([0,T];H^m)$. Also, since

$$\|u^k-u^l\|_{H^{m-\delta}}\leqq c\|u^k-u^l\|_{L^2}^{\delta/m}\|u^k-u^l\|_{H^m}^{1-\delta/m}$$

by the Sobolev interpolation inequalities, $u^k\to u$ in $C^0(0,T];H^{m-\delta}(\Omega))$ for any $\delta>0$. Solving for $u_t^k$ and $u_t$ in the equations $L_ku^k=F_k$ and $Lu=F$ now allows us to conclude that $u^k\to u$ in $\bigcap_{j=0}^{m-1} C^j([0,T];H^{m-j-\delta})$ and $\partial_t^i u\in L^\infty([0,T];H^{m-i})$. Arguments like those on pages 40 [replace $H^{-s}$ by $(H^s)'$] and 44–46 [replace $A^0(u^k)$ by $A_{k+1}^0$ and note that $D^\alpha u^k(0)$ still $\to D^\alpha u(0)$ by construction] of [13] then prove that $u(t)$ is also continuous in the $|||\ |||_{m,\tan}$ norm, and using $Lu=F$ to express normal derivatives in terms of tangential ones then shows that $u$ is in $X_m$.

In order to verify that $u$ satisfies (A4)–(A7), it suffices to show that

$$\|D^\alpha u(t)\|_{L^2(\Omega)}\leqq\overline{\lim}\|D^\alpha u^k(t)\|_{L^2(\Omega)},\qquad |\alpha|\leqq m,\qquad 0\leqq t\leqq T,\tag{A20}$$

since taking the limit of the (A4)–(A7) obeyed by $u^k$ then yields (A4)–(A7) for $u$. If $|\alpha|\leqq m-1$ then $D^\alpha u^k(t)\to D^\alpha u(t)$ in $L^2$, so (A20) certainly holds. Also, if $D^\alpha$ includes any spatial derivative, say $D^\alpha=\partial_{x_j}D^\beta$, then for all $\phi\in C_0^\infty(\Omega)$,

$$\begin{aligned}(\phi,D^\alpha u^k(t))&=(\phi,\partial_{x_j}D^\beta u^k(t))=(-\partial_{x_j}\phi,D^\beta u^k(t))\\&\to(-\partial_{x_j}\phi,D^\beta u(t))=(\phi,D^\alpha u(t)),\end{aligned}$$

i.e. $D^\alpha u^k(t)$ converges weakly in $L^2$ to $D^\alpha u(t)$ and again (A20) holds. The only remaining possibility is $D^\alpha=\partial_t^m$, so

$$\begin{aligned}(\phi,\partial_t^m u^k(t))&=(\phi,\partial_t^{m-1}[(A_k^0)^{-1}(F_k-A_k^j u_{x_j}^k-B_k u^k)])\\&\to(\phi,\partial_t^{m-1}[(A^0)^{-1}(F-A^j u_{x_j}-Bu)])=(\phi,\partial_t^m u(t)),\end{aligned}$$

and hence (A20) holds in all cases.

With the linear regularity theorem in hand, we are now ready for the quasilinear existence theorem:

**Theorem A2.** *The system*

$$A^0(t,x,u)u_t + A^j(t,x,u)u_{x_j} + B(t,x,u)u = F(t,x) \quad in \quad [0,t]\times\Omega, \tag{A21}$$

$$u(0,x) = f(x) \quad in\ \Omega, \tag{A22}$$

$$M(x)u = 0 \quad on \quad [0,T]\times\partial\Omega \tag{A23}$$

*has a unique classical solution if T is sufficiently small, provided*

i) $\Omega$ *is open and bounded in* $R^n$ *and* $\partial\Omega$ *is in* $C^\infty$

ii) *for some* $\varepsilon_0$ *and* $T_0$, $A^0(t,x,v)$, $A^j(t,x,v)$, *and* $B(t,x,v)$ *are in* $C^m([0,T_0]\times N_0)$ *with* $m \geqq \left[\frac{n}{2}\right]+2$, *where* $N_0 \equiv \{(x,v)|x\in\bar{\Omega},\ |v-f(x)|\leqq\varepsilon_0\}$;

iii) $A^0$ *and the* $A^j$ *are symmetric and* $A^0 \geqq$ *some* $c_1 > 0$, *for* $t,x,v$ $[0,T_0]\times N_0$;

iv) $M(x)$ *is in* $C^\infty(\bar{\Omega})$;

v) $A^\nu(t,x,v) \equiv \nu^j(x)A^j(t,x,v)$ *is nonsingular and* $|(A^\nu)^{-1}| \leqq$ *some* $c_2$, *for all*

$$t,x,v \in [0,T_0]\times N_0$$

*such that* $x\in\partial\Omega$ *and* $M(x)v=0$;

vi) *the boundary condition* $M(x)u=0$ *is maximally nonnegative for* $A^\nu(t,x,v)$, *for all* $t,x,v$ *as in* v).

vii) $F$ *is in* $H^m([0,T]x\Omega)$ *and* $f$ *is in* $H^m(\Omega)$;

viii) $M$"$\partial_t^i u(0)$"$=0$ *on* $\partial\Omega$, $0\leqq i\leqq m-1$.

The time $T$ up to which the solution is shown to exist depends only on $\Omega$, $M$, $n$, $m$, $\varepsilon_0$, $T_0$, $c_1$, $c_2$, $F$, the $C^m$ norms of $A^0$, the $A^j$, and $B$, and the $H^m$ norm of $f$. (If $F\in X_m$ $T$ depends on $F$ only via $|||F|||_{m,T_0}$.) The solution is in $X_m([0,T];\Omega)$ $\left[\text{which is contained in } C^{m-\left[\frac{n}{2}\right]-1}([0,T]x\Omega)\right]$, and its norm in this space is bounded in terms of $T$ and the quantities $T$ depends on.

Theorem (A2) follows from the lemmas (which have conditions i–viii as hypotheses):

**Lemma A3.** *The set* $X=\{U\in X_m([0,T_0];\Omega)|MU=0$ *on* $\partial\Omega$, $\partial_t^i U(0)=$"$\partial_t^i u(0)$", $0\leqq i\leqq m\}$ *is nonempty.*

**Lemma A4.** *There is a function* $g(R)$ *with* $\lim_{R\to\infty} g(R)=\infty$ *and a* $|||\ \ |||_{m,\tan}$ *norm depending only on* $R$, $\Omega$, *and* $M$ *such that if* $T_1$ *is sufficiently small and* $R$ *is sufficiently large then the set* $X_{(R,T_1)}\equiv\{U\in X_m([0,T_1];\Omega)|MU=0$ *on* $\partial\Omega$; $\partial_t^i U(0)$ $=$"$\partial_t^i u(0)$", $0\leqq i\leqq m$; $\sup_{0\leqq t\leqq T} |||U|||_{m-1}^2+|||U|||_{m,\tan}^2\leqq R$, $|||U|||_{m,T}\leqq g(R)\}$ *is mapped into itself by the map* $v\to w$ *given by* $L(v)w=F$, $w(0)=f$, $Mw=0$ *on* $\partial\Omega$.

**Lemma A5.** *If* $T_1$ *is sufficiently small the map in Lemma A4 is a contraction in the* $X_0$ *norm.*

*Proof of Theorem A2, Given the Lemmas.* Pick $u^0$ in $X$ and choose $R_1$ large enough so that $u^0$ is in $X_{(R_1,T_0)}$. Pick $R\geqq R_1$ sufficiently large and $T$ sufficiently small so that the conclusions of Lemmas A4 and A5 hold with $T_1 = T$. Then the sequence $\{u^n\}$ defined by $L(u^n)u^{n+1} = F$, $u^{n+1}(0) = f$, $Mu^{n+1} = 0$ on $\partial\Omega$ converges in $X_0([0, T]; \Omega)$ to some $u$. Since the $u^k$ are bounded in $X_m$, an interpolation argument and the fact that

$$u_t^{k+1} = [A^0(u^k)]^{-1}(F - A^j(u^k)u_{x_j}^{k+1} - B(u^k)u^{k+1})$$

show that $u^k \to u$ in $\bigcap_{j=0}^{m-1} C^j([0, T]; H^{m-j-\delta})$ and $\partial_t^j u \in L^\infty([0, T]; H^{m-j})$. In particular $u^k \to u$ in $C^1$ so $u$ satisfies (A21)–(A23). Finally, the argument in [13] can be adapted in similar fashion as for Theorem A1 to show that $u \in X_m$. $\Big($For this, note that by the proof of Theorem A1, if $|||\cdot|||_{m-1}^2 + |||\cdot|||_{m,\tan}^2$ is replaced by

$$\sum_{\substack{|\alpha| \leq m \\ |\alpha| \leq m-1 \text{ or } \alpha_n = 0}} \sum_i (D^\alpha \phi^i \cdot, A^0(u^k) D^\alpha \phi^i \cdot)$$

then $u^{k+1}$ satisfies (A4) with the factor $F_1$ omitted.$\Big)$

*Remark.* The $u^0$ used to start the above iteration is not the one suggested in [14], but is analogous to the ones used in [1, 5], which deal with one particular system.

*Proof of Lemma A3.* We will let $U$ be the solution of $L(f)U = G$, $U(0) = f$, $MU = 0$ on $\partial\Omega$, where $G$ is to be chosen so that $G \in H^m$ and $\partial_t^i U(0) = $ "$\partial_t^i u(0)$", $0 \leq i \leq m$. Assuming by induction that "$\partial_t^i U(0)$" = "$\partial_t^i u(0)$" for $0 \leq i \leq j$ (which clearly is true when $j = 0$), we find that it will also hold for $i = j+1$ provided

$$\partial_t^j G(0) = \text{“}\partial_t^j F - [\partial_t^j, A^0(u)]\partial_t u - [\partial_t^j, A^j(u)]u_{x_j} - [\partial_t^j, B]u|_{t=0}\text{”}$$

(interpreted to be an expression in terms of $f$ and $F$). Theorem 2.5.7 of [8] guarantees the existence of a $G$ in $H^m$ having these derivatives of $t = 0$ provided $\partial_t^j G(0) \in H^{m-j-1/2}(\Omega)$, which can be shown to hold by using the assumptions on $F$ and $f$, (A8), and the fact (to be proven in Appendix B) that

for $m \geq \left[\frac{n}{2}\right] + 1$, if $A \in C^m$ and $h \in X_m$, then $A(h) \in X_m$ and

$$|||A(h)|||_m \leq C \|A\|_{C^m(|\cdot| \leq c_1 |||h|||_{[\frac{n}{2}]+1})} (1 + |||h|||_m^m); \tag{A24}$$

if $m \geq \left[\frac{n}{2}\right] + 2$, then $(1 + |||h|||_m^m)$ may be replaced by $(1 + |||h|||_{m-1}^{m-1})(1 + |||h|||_m)$.

*Proof of Lemma A4.* Let $k_1 = |||$"$u(0)$"$|||_{m-1}^2 + |||$"$u(0)$"$|||_{m,\tan}^2$. Define "$\partial_t^i A(t=0)$", where $A = A^0$, $A^j$, or $B$ by formally taking $\partial_t^i$ of $A(t, x, u)$ and setting $\partial_t^j u$ equal to "$\partial_t^j u(0)$" $0 \leq j \leq i$, and let

$$k_2 = |||A^0(t=0), A^j(t=0), B(t=0)|||_{m-1}.$$

Let $k_3$ be a constant such that $\|\cdot\|_{C^0} \leq k_3 \|\cdot\|_{H^{[\frac{n}{2}]+1}}$ and $k_4$ a constant such that

$$|||\cdot(t)|||_{m-1} \leq k_4 |||\cdot|||_{H^m([0, T_0] \times \Omega)}.$$

Let $G_1(\cdot)$ be such that (A24) implies

$$|||A^0(\cdot), A^j(\cdot), B(\cdot)|||_k \leqq G_1(|||\cdot|||_k), \qquad k=m-1, m;$$

let $G_2(\cdot)$ be such that (A24) implies

$$|||A^0(\cdot), A^j(\cdot), B(\cdot)|||_m \leqq G_2(|||\cdot|||_{m-1})(1+|||\cdot|||_m).$$

Now define $R_0 = 2F_1(1/c_1, 2k_2)(k_1+1)$, where $F_1$ is the $F_1$ in (A4), and for $R \geqq R_0$ define

$$g(R) = 1 + G_2(\sqrt{R})^a + [2F_4(2k_2, c_2)G_2(\sqrt{R})^a(\sqrt{2R} + K_4\|F\|_{H^m([0,T_0]\times\Omega)})]^{\frac{1}{1-a}},$$

where $a$ is the $a$ in (A5). Also for $R \geqq R_0$, pick $T_1 = T_1(R)$ such that $T_1 \leqq \varepsilon_0/k_3 g(R)$, $T \leqq k_2/G_1(g(R))$, $T_1 \leqq \ln 2/F_2(G_1(g(R)), 1/c_1, c_2)$, and
$F_3(G_1(g(R)), c_2) \int_0^{T_1} |||F(t)|||_m^2 \leqq 1$.

Suppose $v \in X_{(R, T_1)}$, where $R$ and $T_1$ satisfy the above conditions. Then for $0 \leqq t \leqq T_1$,

$$|v(t,x) - f(x)| \leqq t \sup_{0 \leqq t \leqq T_1} |v_t| \leqq k_3 T_1 |||v|||_{\left[\frac{n}{2}\right]+2, T_1} \leqq \varepsilon_0.$$

Hence, since $A^0(t,x,v)$, etc. are in $X_m$ by (A24), Theorem A1 shows that there is a unique $w$ in $X_m([0, T_1]; \Omega)$ satisfying $L(v)w = F$, $w(0) = f$, $Mw = 0$ on $\partial\Omega$. These equations together with the fact that $\partial_t^i v(0) = $ "$\partial_t^i u(0)$", $0 \leqq i \leqq m$, imply that $\partial_t^i w(0) = $ "$\partial_t^i u(0)$", $0 \leqq i \leqq m$, as well. Now

$$|||A^0(v), A^j(v), B(v)|||_{m-1, T_1} \leqq |||A^0(t=0), A^j(t=0), B(t=0)|||_{m-1}$$
$$+ T_1 |||A^0(v), A^j(v), B(v)|||_{m, T_1} \leqq k_2 + T_1 G_1(g(R)) \leqq 2k_2,$$
$$e^{F_2(|||A^0, A^j, B|||_{m, T_1}, 1/c_1, c/2)t} \leqq e^{F_2(G_1(g(R)), 1/c_1, c_2)T_1} \leqq 2,$$

and

$$F_3(|||A^0, A^j, B|||_{m, T_1}, c_2) \int_0^t e^{-F_2 t'} |||F(t')|||_m^2 \leqq 1 \quad \text{for} \quad 0 \leqq t \leqq T.$$

Hence by (A4),

$$|||u(t)|||_{m-1}^2 + |||u(t)|||_{m,\tan}^2 \leqq 2F_1(1/c_1, 2k_2)(k_1+1) = R_0 \leqq R.$$

Similarly, (A5) shows that $|||u|||_{m, T_1} \leqq g(R)$.

*Proof of Lemma A5.* A Standard $L^2$ Energy Estimate.

The following continuation principle was also used in Sect. 3.

**Theorem A6.** *Suppose the hypotheses of Theorem A2 are satisfied with* $m > \left[\frac{n}{2}\right] + 2$, *that* (A21)–(A23) *has a solution* $u$ *in* $X_{\left[\frac{n}{2}\right]+2}([0, T]; \Omega)$ *with* $T \leqq T_0$, *and that* $(x, u(t,x)) \in N_0$ *for* $x \in \bar{\Omega}$, $0 \leqq t \leqq T$. *Then* $u \in X_m([0, T]; \Omega)$.

*Proof.* A standard continuation principle says that $u$ can be continued in $X_m$ as long as $x, u(t,x) \in N_0$ for all $x \in \bar{\Omega}$ and $|||u|||_m$ remains finite. (By Theorem A2, if

$|||u(t_0)|||_m \leqq c$ then $u$ has a solution for a time interval $[t_0, t_0+\varepsilon]$ of fixed length, so gluing together these solutions on the interval $[0,\varepsilon]$, $[\varepsilon, 2\varepsilon]$, etc. produces an $X_m$ solution for as long as $|||u|||_m \leqq c$, since the hyperplane $t=$constant is noncharacteristic.) Also, by induction it suffices to prove $|||u|||_{m,T}<\infty$ under the assumption that $|||u|||_{m-1,T}\leqq c$.

Now $u$ satisfies the linear equation $Lu=F$, where $L\equiv L(u)$, so $u$ obeys (A 6), (A 7) with $A^0\equiv A^0(u)$, etc. The point is that (A 6), (A 7), and (A 24) are all essentially linear in the highest norm, so

$$|||A^0, A^j, B|||_m \leqq c(|||u|||_{m-1})(|||u|||_m+1)\leqq c(|||u|||_{m-1})(|||u|||_{m,\tan}+1)$$

by (A 24) and (A 7), and substituting this into (A 6) shows

$$|||u(t)|||^2_{m,\tan}\leqq c+c\int_0^t |||u|||^2_{m,\tan},$$

and hence by Gronwall's inequality

$$|||u|||^2_{m,\tan}\leqq c(1+|||u(0)|||^2_{m,\tan})\leqq c$$

on $[0,T]$. Substituting this into (A 7) shows $|||u|||_m\leqq c$.

*Remark.* Theorem (A 6) remains valid when the boundary is characteristic if (A 4)–(A 7) hold with $|||\ \ |||_{m,\tan}$ is replaced by some $|||\ \ |||_{m,E_1}$ norm; in particular, it is valid for system (5.2), (2.2), (2.3).

## Appendix B

We will prove the following calculus inequalities used in Appendix A:

$$|||fg|||_k\leqq c|||f|||_{\left[\frac{n}{2}\right]+1}|||g|||_k, \quad k\leqq\left[\frac{n}{2}\right]+1, \tag{B 1}$$

$$|||fg|||_k\leqq c(|||f|||_k|||g|||_{k-1}+|||f|||_{k-1}|||g|||_k), \quad k\geqq\left[\frac{n}{2}\right]+2, \tag{B 2}$$

$$\|(D^\alpha f)(D^\beta g)\|_{L^2}\leqq c|||f|||_{s_1}^{a_1}|||f|||_{s_1-1}^{1-a_1}|||g|||_{s_2}^{a_2}|||g|||_{s_2-1}^{1-a_2} \tag{B 3}$$

with $0<a_i<1$, provided $|\alpha|\leqq s_1-1$, $|\beta|\leqq s_2-1$, $|\alpha|+|\beta|+\frac{n}{2}<s_1+s_2$,

$$|||A(h)|||_m\leqq c\|A\|_{C^m(|\cdot|\leqq c_1|||h|||_{[\frac{n}{2}]+1})}(1+|||h|||_m^m), \quad m\geqq\left[\frac{n}{2}\right]+1; \tag{B 4}$$

if $m\geqq\left[\frac{n}{2}\right]+2$, then $(1+|||h|||_m^m)$ may be replaced by $(1+|||h|||_{m-1}^{m-1})(1+|||h|||_m)$.

*Remark.* Except for the inclusion of time derivatives, these inequalities are well-known; see e.g. the appendix of [10].

*Proof of* (B 1).

$$|||fg|||_k \leqq c \sum_{|\alpha|+|\beta|\leqq k} \|(D^\alpha f)(D^\beta g)\|_{L^2},$$

where $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_n)$ and $D^\alpha=\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\ldots\partial_{x_n}^{\partial_n}$, etc. Any term with $|\beta|=k$ is

$$\leqq c\|f\|_{C^0}|||g|||_k \leqq c|||f|||_{\left[\frac{n}{2}\right]+1}|||g|||_k$$

by Sobolev's inequality, and if $k=\left[\frac{n}{2}\right]+1=|\alpha|$ just switch $f$ with $g$. Otherwise (B 3) applies. (B 2) is proven similarly.

*Proof of* (B 3). We will use Holder's inequality $\|fg\|_{L^2}\leqq\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$, where

$$\frac{1}{p_1}+\frac{1}{p_2}=\tfrac{1}{2},$$

and the Gagliardo-Nirenberg inequalities ([19], p. 27) $\|D^j h\|_{L^p}\leqq c\|h\|_{H^m}^a\|h\|_{L^2}^{1-a}$, where $j<m$, $j/m<a<1$, $\frac{1}{p}=\frac{1}{2}+\frac{j-am}{n}$, $2<p<\infty$, and $D^j$ includes only spatial derivatives. It suffices to find estimates of the form

$$\|(D^{|\alpha|-\alpha_0}\partial_t^{\alpha_0}f)(D^{|\beta|-\beta_0}\partial_t^{\beta_0}g)\|_{L^2}\leqq\|D^{|\alpha|-\alpha_0}\partial_t^{\alpha_0}f\|_{L^{p_1}}\|D^{|\beta|-\beta_0}\partial_t^{\beta_0}g\|_{L^{p_2}},$$
$$\|D^{|\alpha|-\alpha_0}\partial_t^{\alpha_0}f\|_{L^{p_1}}\leqq c\|\partial_t^{\alpha_0}f\|_{H^{s_1-\alpha}}^{a_1}\|\partial_t^{\alpha_0}f\|_{L^2}^{1-a_1},$$
$$\|D^{|\beta|-\beta_0}\partial_t^{\beta_0}g\|_{L^{p_2}}\leqq c\|\partial_t^{\beta_0}g\|_{H^{s_2-\beta_0}}^{a_2}\|\partial_t^{\beta_0}g\|_{L^2}^{1-a_2}.$$

Given $a_1$ and $a_2$, $p_1$, and $p_2$ are defined by $\frac{1}{p_1}=\frac{1}{2}+\frac{|\alpha|-\alpha_0-a_1(s_1-\alpha_0)}{n}$ and

$$\frac{1}{p_2}=\tfrac{1}{2}+\frac{|\beta|-\beta_0-a_2(s_2-\beta_0)}{n},$$

so we just need to satisfy the conditions

i) $$-\tfrac{1}{2}<\frac{|\alpha|-\alpha_0-a_1(s_1-\alpha_0)}{n}<0,$$

ii) $$\frac{|\alpha|-\alpha_0}{s_1-\alpha_0}<a_1<1,$$

iii) $$-\tfrac{1}{2}<\frac{|\beta|-\beta_0-a_2(s_2-\beta_0)}{n}<0,$$

iv) $$\frac{|\beta|-\beta_0}{s_2-\alpha_0}<a_1<1,$$

v) $$-\tfrac{1}{2}=\frac{|\alpha|-\alpha_0-a_1(s_1-\alpha_0)+|\beta|-\beta_0-a_2(s_2-\beta_0)}{n}.$$

Now if $a_1$ and $a_2$ were set equal to one, the right side of v) would equal $[|\alpha|+|\beta|-(s_1+s_2)]/n$ which is $<-\frac{1}{2}$ by assumption. On the other hand if $a_1$ and $a_2$ were set to $\frac{|\alpha|-\alpha_0}{s_1-\alpha_0}$ and $\frac{|\beta|-\beta_0}{s_2-\beta_0}$, respectively, the right side of v) would equal zero. Therefore since $s_1-\alpha_0$ and $s_2-\beta_0$ are both positive, there exist $a_1$ and $a_2$ satisfying ii) and iv) that make v) hold. Then $\frac{|\alpha|-\alpha_0-a_1(s_1-\alpha_0)}{n}$ and $\frac{|\beta|-\beta_0-a_2(s_2-\alpha_0)}{n}$ are both negative and their sum $=-\frac{1}{2}$, so i) and iii) are satisfied.

*Proof of* (B4). We allow $A$ to have explicit time and space dependence also. Now $|||A(u)|||_m$ is a sum of terms of the form

$$\|(D^{\alpha}_{t,x}D^{p}_{u}A)D^{\beta^{(1)}}u\ldots D^{\beta^{(p)}}u\|_{L^2},$$

with $0\leqq p\leqq m$ and $|\alpha|+\sum|\beta^{(i)}|\leqq m$, and each term is

$$\leqq c\|A\|_{C^m(|\cdot|\leqq\|u\|_{C^0})}|||u^p|||_m.$$

By induction starting from $(B_1)$ and $(B_2)$ we can show that $|||u^p|||_m\leqq c|||u|||^p_m$ for $p\geqq 1$ when $m\geqq\left[\frac{n}{2}\right]+1$ and $|||u^p|||_m\leqq c|||u|||^{p-1}_{m-1}|||u|||_m$ for $p\geqq 1$ when $m\geqq\left[\frac{n}{2}\right]+2$, and then (B 4) follows easily.

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