INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 20, 255–261 (1995)

# THE COMPRESSIBLE NEWTONIAN EXTRUDATE SWELL PROBLEM

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SUMMARY

We solve the compressible Newtonian extrudate swell problem in order to investigate the effect of compressibility on the shape of the extrudate. We employ a first-order equation of state relating the density to the pressure and use finite elements for the numerical solution of the problem. Our results show that the shape of the extrudate and the final extrudate swell ratio are not significantly affected even at high compressibility values.

KEY WORDS Finite elements Compressible Newtonian flow Extrudate swell

## 1. INTRODUCTION

The extrudate swell problem has been one of the most extensively studied problems in the area of polymer-processing simulations. Since the first solution of the two-dimensional Newtonian problem by Nickell *et al.*,[1] who used a Picard iteration, finite element scheme, considerable progress has been made in extrusion modelling. Ruschak[2] proposed the full Newton iteration method in which the position of the free surface is computed simultaneously with the velocity and pressure fields. Omodei[3,4] presented complete series of results at various Reynolds and capillary numbers for both the planar and axisymmetric Newtonian extrudate swell problems. Crochet and Keunings[5] solved the viscoelastic extrudate swell problem using differential constitutive models and Papanastasiou *et al.*[6] developed the streamlined finite element method for integral constitutive models. Georgiou *et al.*[7,8] developed singular finite elements for the solution of the Newtonian problem. More recent progress includes the development of new techniques for integral models,[9,10] three-dimensional extrusion simulations[11–13] and die design.[13–15]

In this paper we consider the axisymmetric Newtonian extrudate swell problem and examine the effect of compressibility on the shape of the free surface and the final extrudate swell ratio. A simple first-order equation of state is used to express the density of the fluid as a function of pressure. The density is eliminated and a standard finite element method is employed for the numerical solution of this free surface problem. The numerical results show that the free surface profile and the final extrudate swell ratio are not significantly affected even when the fluid is highly compressible. It should be added, however, that compressibility combined with slip at the wall can alter dramatically the dynamics of the flow. This has been demonstrated in Reference 16 for time-dependent compressible Poiseuille flow.

In Section 2 we present the governing equations and boundary conditions. In Section 3 we give a brief description of the finite element formulation. The numerical results are presented and discussed in Section 4.

CCC 0271–2091/95/030255–07


*Received 4 January 1994*
*Revised 27 June 1994*

## 2. GOVERNING EQUATIONS

We consider the steady compressible axisymmetric extrudate swell problem. The geometry is shown in Figure 1. If $\rho$, $p$, $\mathbf{v}$ and $\sigma$ are the density, pressure, velocity vector and stress tensor respectively, the continuity and momentum equations in the absence of body forces read

$$\nabla \cdot \rho \mathbf{v} = 0, \tag{1}$$

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla \cdot \sigma = \mathbf{0}. \tag{2}$$

For the bulk viscosity we make the usual yet arbitrary assumption that it is zero.[17] The stress tensor for compressible Newtonian flow is then given by

$$\sigma = -p(\rho)\mathbf{I} + \eta[(\nabla \mathbf{v}) + (\nabla \mathbf{v})^T] - \tfrac{2}{3}\eta \mathbf{I} \nabla \cdot \mathbf{v}, \tag{3}$$

where $\mathbf{I}$ is the unit tensor, $\eta$ is the viscosity and the superscript $T$ denotes the transpose. The viscosity is assumed to be independent of the pressure.

The above equations are completed by a thermodynamic equation of state. As in Reference 16, we use the first-order expansion

$$\rho = \rho_0[1 + \beta(p - p_0)], \tag{4}$$

where

$$\beta = -\frac{1}{V_0}\left(\frac{\partial V}{\partial p}\right)_{p_0, T}$$

is the isothermal compressibility, $\rho_0$ and $V_0$ are the density and specific volume at the reference pressure $p_0$ respectively and $T$ is the temperature.

It is convenient to work with dimensionless equations. For that purpose we scale the lengths by the radius $R$ and the velocity components by $\dot{M}/\rho_0 R^2$, where $\dot{M}$ is the mass flow rate. The pressure $p - p_0$ and stress components are scaled by $\eta \dot{M}/\rho_0 R^3$ and the density by $\rho_0$. Two dimensionless numbers are obtained, namely the Reynolds number $Re$ and a compressibility number $B$:

$$Re \equiv \frac{\dot{M}}{\eta R}, \qquad B \equiv \frac{\beta \eta \dot{M}}{\rho_0 R^3}. \tag{5}$$

We thus obtain the set of dimensionless equations

$$\nabla^* \cdot \rho^* \mathbf{v}^* = 0, \tag{6}$$

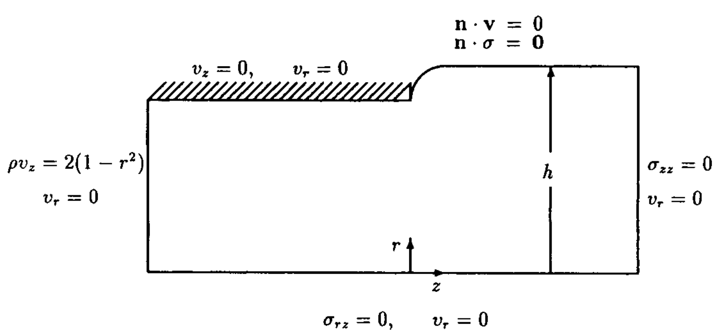


Figure 1. Boundary conditions for extrudate swell problem

$$Re\,\rho^*\mathbf{v}^*\cdot\nabla^*\mathbf{v}^* - \nabla^*\cdot\sigma^* = \mathbf{0}, \tag{7}$$

$$\sigma^* = -p^*\mathbf{I} + [(\nabla^*\mathbf{v}^*) + (\nabla^*\mathbf{v}^*)^T] - \tfrac{2}{3}\mathbf{I}\nabla^*\cdot\mathbf{v}^*, \tag{8}$$

$$\rho^* = 1 + Bp^*. \tag{9}$$

The superscript asterisks denote non-dimensional variables and will be dropped hereafter without any risk of confusion.

The boundary conditions are shown in Figure 1. Along the axis of symmetry we have the usual symmetry conditions and along the wall both velocity components vanish. At the exit plane the flow is considered fully translational. Surface tension is neglected and therefore the normal and tangential stress components vanish on the free surface. Another condition at the free surface is the kinematic one

$$\mathbf{n}\cdot\mathbf{v} = 0, \tag{10}$$

where $\mathbf{n}$ is the outward unit vector normal to the free surface. The kinematic condition provides the additional equation used for the calculation of the free surface location. Finally, at the inlet plane we assume that the density is constant, the axial velocity is parabolic and the mass flow rate is $\pi$. Therefore we have

$$\rho v_z = 2(1 - r^2). \tag{11}$$

The above assumptions are consistent with the analytical solution for compressible Poiseuille flow.[16]

## 3. FINITE ELEMENT FORMULATION

The full Newton iteration method is employed for this free surface problem. In other words, the unknown position of the free surface $h$ is calculated simultaneously with the velocity and pressure fields.[2] The density is eliminated by means of the equation of state (9). We use the standard biquadratic velocity ($P^2$–$C^0$) and bilinear pressure ($P^1$–$C^0$) elements with a quadratic expansion for $h$.

Let $\Psi^j$, $\Phi^j$ and $\chi^j$ denote bilinear, biquadratic and quadratic shape functions respectively. The unknowns $p$, $\mathbf{v}$ and $h$ are expanded as

$$p = \sum_j^{N_p} p^j\Psi^j, \qquad \mathbf{v} = \sum_j^{N_p} \mathbf{v}^j\Phi^j, \qquad h = \sum_j^{N_h} h^j\chi^j. \tag{12}$$

Here $p^j$, $\mathbf{v}^j$ and $h^j$ are the values of the unknowns at the $j$th node and $N_p$, $N_v$ and $N_h$ are the numbers of pressure, velocity and free-surface nodes respectively. For steady flow the discretized Galerkin equations in non-dimensional form read

$$\int_\Omega \nabla\cdot(1 + Bp)\mathbf{v}\Psi^i \,\mathrm{d}\Omega = 0, \quad i = 1, 2, \ldots, N_p, \tag{13}$$

$$\int_\Omega [Re(1 + Bp)\mathbf{v}\cdot\nabla\mathbf{v}\Phi^i + \sigma\cdot\nabla\Phi^i]\,\mathrm{d}\Omega - \int_{\partial\Omega} \mathbf{n}\cdot\sigma\Phi^i \,\mathrm{d}s = \mathbf{0}, \quad i = 1, 2, \ldots, N_v, \tag{14}$$

$$\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{v}\{X^i\}\,\mathrm{d}s = 0, \quad i = 1, 2, \ldots, N_h, \tag{15}$$

where $\Omega$ and $\partial\Omega$ denote the domain and its boundary respectively.

The discretized $z$-momentum equations along the inlet are replaced by

$$\int_{\text{inlet}} [\rho v_z - 2(1 - r^2)]\Phi^i r \mathrm{d}r = 0. \tag{16}$$

It should be added that in the earlier stages of this work the density was not eliminated but was approximated using the same low-order basis functions as for the pressure. Such a mixed finite element should satisfy the inf-sup condition for stability purposes. According to Fortin and Soulaimani,[18] this element gives good results at low Mach and Reynolds numbers. For the values of $Re$ and $B$ considered here, the results obtained with the above element are essentially the same as those of the present formulation.

## 4. RESULTS

Results have been obtained for various values of $Re$ and $B$. The convergence of the numerical scheme has been checked by employing meshes of various lengths and refinements. All the results presented in the sequence have been produced with a mesh extending 5 radii upstream and 5 radii downstream and consisting of $72 \times 10$ elements. In Figure 2 we show the free surface profiles computed at $Re = 0$ and various values of the compressibility number $B$. For low values of $B$, swelling is reduced as $B$ is increased, and this agrees with the results of Beverly and Tanner;[19] above a critical value of $B$ (about 0·06), however, swelling is enhanced with compressibility. The results for non-zero $Re$ are similar.

The final extrudate swell ratio $h_f$ is plotted as a function of $B$ in figure 3, where we provide the results for $Re = 0$ and 1. The changes caused by compressibility are not larger than 1·5% despite the fact that the values of $B$ considered here are rather high and correspond to large density changes uncommon to fluids typically used in extrusion. (A typical value for the compressibility number in polymer extrusion is $B = 0{\cdot}0003$.[16]) As an example, for $B = 0{\cdot}1$ and $Re = 0$ the density varies from 0·24 (near the lip) to 3·05 (at the inlet). The density contours for this flow are shown in Figure 4.

One would intuitively expect that swelling should be enhanced with compressibility, since the density of the fluid becomes lower outside the tube. As indicated in Figure 4, however, the fluid is

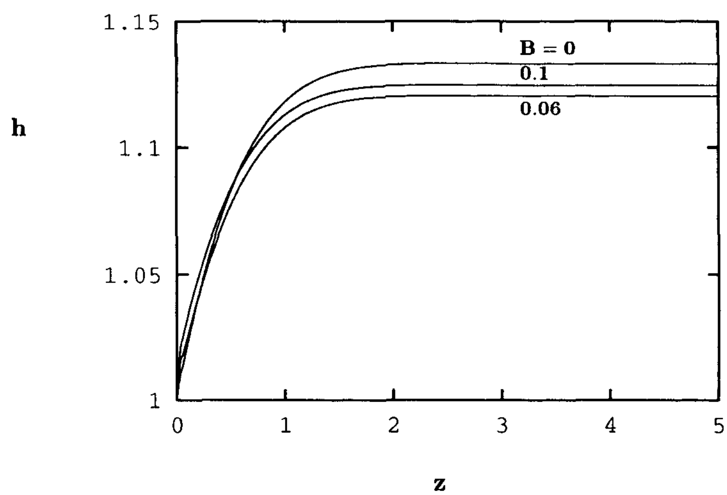


Figure 2. Free surface profiles for creeping compressible flow

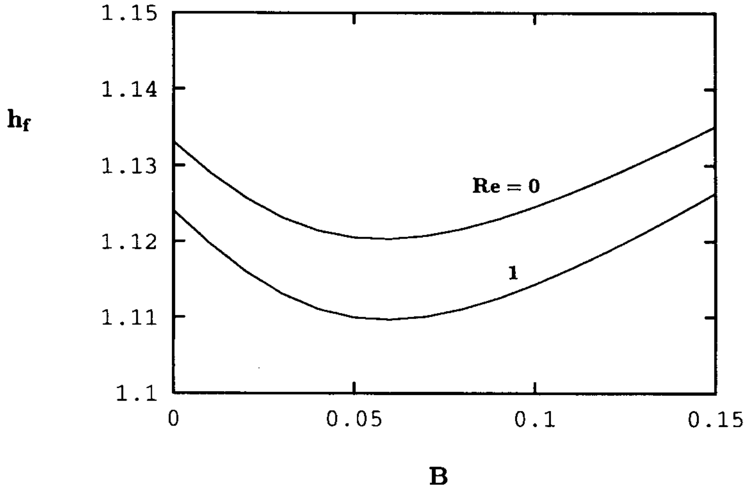


Figure 3. Final extrudate swell ratio for $Re = 0$ and 1

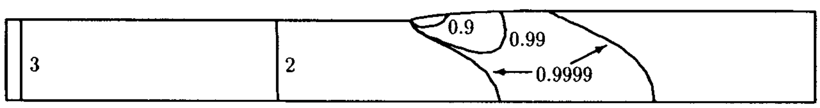


Figure 4. Density contours for $B = 0.1$ and $Re = 0$

slightly compressed in the extrudate region. Because of the presence of the singularity at the exit of the tube, there exists a negative pressure regime in the extrudate region, around the singularity and just after the exit. Therefore the fluid is compressed as it moves downstream where the pressure is zero. When the compressibility number is low, this compression is more significant than the decompression of the fluid at the exit, which explains the minimum we observe in Figure 3.

Finally, in Figure 5 we plot the calculated pressures and volumetric flow rates at the inlet. The inlet plane is located 5 radii upstream of the exit. The pressure drop decreases considerably with the compressibility number and so does the volumetric flow rate, in close agreement with the analytical solution for compressible Poiseuille flow.[16]

## 5. CONCLUSIONS

We have modelled the compressible extrudate swell problem using finite elements and a first-order equation of state. The numerical results show that the shape of the extrudate and the final extrudate swell ratio are not significantly affected even when the compressibility is high.


### ACKNOWLEDGEMENTS

The suggestion of Professor Marcel Crochet are gratefully acknowledged. The results presented in this paper have been obtained within the framework of Interuniversity Attraction Poles initiated by the Belgian State, Prime Minister's Office, Science Policy Programming.

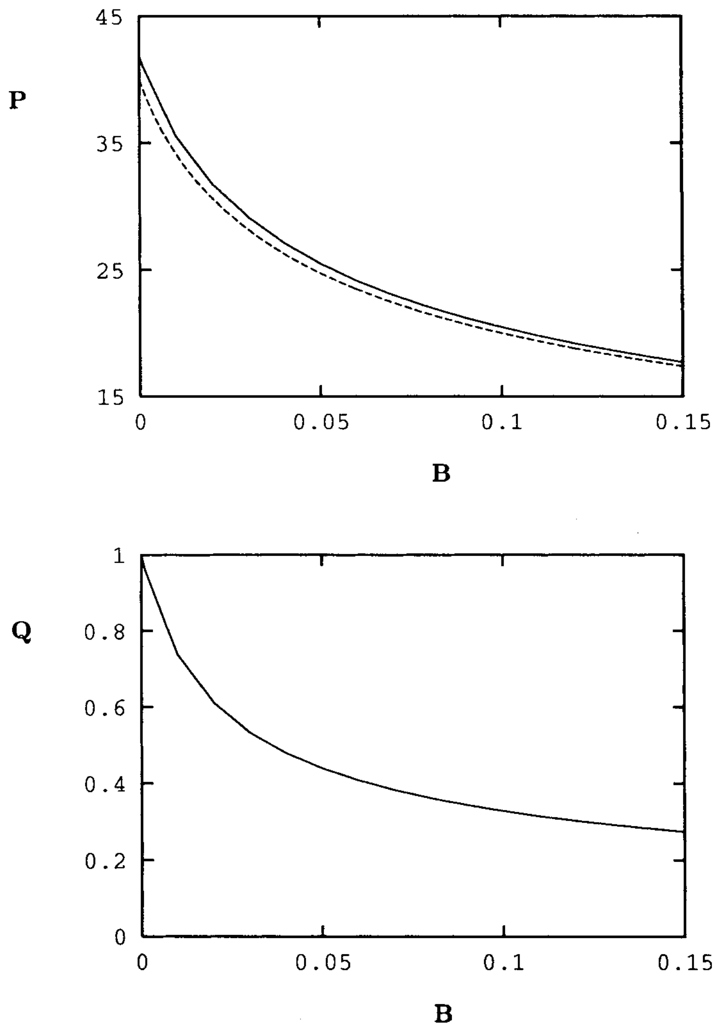


Figure 5. (a) Pressure and (b) volumetric flow rate at the inflow plane located 5 radii upstream of the exit. The broken line is the analytical solution for compressible Poiseuille flow

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