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THE CONCAVITY OF THE GAUSSIAN CURVATURE OF THE CONVEX LEVEL SETS OF MINIMAL SURFACES WITH RESPECT TO THE HEIGHT

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For the minimal graph with strictly convex level sets, we find an auxiliary function to study the Gaussian curvature of the level sets. We prove that this curvature function is a concave function with respect to the height of the minimal surface while this auxiliary function is almost sharp when the minimal surface is the catenoid.

1. Introduction

Consider a function whose graph is minimal and whose level sets are strictly convex. Extending work of Longinetti [1987], we explore the relation between the Gaussian curvature of the level sets and the height.

The nature of the level sets of the solutions of elliptic partial differential equations is a subject with a long history, going back to results of Shiffman in the 1950s for minimal surfaces. The curvature of such level sets has also been studied for several decades. Some key contributions to these problems are listed in the introduction of [Chen and Shi 2011]. Here we just mention some recent developments directly relevant to our problem.

Jost, Ma, and Ou [Jost et al. 2012] and Ma, Ye, and Ye [Ma et al. 2011] proved that the Gaussian and principal curvatures of convex level sets of three-dimensional harmonic functions attain their minima on the boundary. Ma, Ou, and Zhang [2010] gave estimates of the Gaussian curvature of convex level sets of higher-dimensional harmonic functions based on the Gaussian curvature of the boundary and the norm of the gradient on the boundary. Wang and Zhang [2012] have given estimates for the Gaussian curvature of convex level sets of minimal surfaces, Poisson equations, and a class of semilinear elliptic partial differential equations studied by Caffarelli and Spruck [1982].

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In this paper we use the support function of strictly convex level sets and the maximum principle to obtain the concavity of the Gaussian curvature of convex level sets of minimal graphs with respect to the height:

Theorem 1.1. *Let* Ω *be a bounded smooth domain in* $\mathbb{R}^n$, $n \geq 2$, *and let*

$$u \in C^4(\Omega) \cap C^2(\overline{\Omega}), \quad t_0 \leq u(x) \leq t_1$$

be a minimal graph in Ω, *that is, one such that*

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \textit{in } \Omega. \tag{1-1}$$

Assume $|\nabla u| \neq 0$ *in* $\overline{\Omega}$. *Let*

$$\Gamma_t = \{x \in \Omega : u(x) = t\} \quad \textit{for } t_0 < t < t_1$$

be the level sets of u *and let* K *be their Gaussian curvature function. For*

$$f(t) = \min\left\{\left[\left(\frac{|\nabla u|^2}{1+|\nabla u|^2}\right)^{\frac{n-3}{2}} K\right]^{\frac{1}{n-1}}(x) : x \in \Gamma_t\right\},$$

if the level sets of u *are strictly convex with respect to the normal* ∇u, *we have the differential inequality*

$$D^2 f(t) \leq 0 \quad \textit{in } (t_0, t_1).$$

Under the same assumption as in Theorem 1.1, Wang and Zhang [2012] proved the following statement: *for* $n \geq 2$, *the function* $(|\nabla u|^2/(1+|\nabla u|^2))^\theta K$ *attains its minimum on the boundary, where* $\theta = -\frac{1}{2}$ *or* $\theta \geq \frac{1}{2}(n-3)$. From this fact they got the lower bound estimates for the Gaussian curvature of the level sets.

Corollary 1.2. *Let* u *satisfy*

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 & \textit{in } \Omega = \Omega_0 \backslash \overline{\Omega}_1, \\ u = 0 & \textit{on } \partial\Omega_0, \\ u = 1 & \textit{on } \partial\Omega_1, \end{cases} \tag{1-2}$$

where Ω_0 *and* Ω_1 *are bounded smooth convex domains in* $\mathbb{R}^n$, $n \geq 2$, $\overline{\Omega}_1 \subset \Omega_0$. *Assume* $|\nabla u| \neq 0$ *in* $\overline{\Omega}$ *and the level sets of* u *are strictly convex with respect to normal* ∇u. *Let* K *be the Gaussian curvature of the level sets. For any point* $x \in \Gamma_t$, $0 < t < 1$, *we have the following estimates.*

- *For* $n = 3$, *we have*

$$K(x)^{1/2} \geq (1-t)(\min_{\partial\Omega_0} K)^{1/2} + t(\min_{\partial\Omega_1} K)^{1/2}. \tag{1-3}$$

- *For $n \neq 3$, we have*

$$\left[\left(\frac{|\nabla u|^2}{1+|\nabla u|^2}\right)^{\frac{n-3}{2}} K\right]^{\frac{1}{n-1}}(x) \geq (1-t)\min_{\partial\Omega_0}\left[\left(\frac{|\nabla u|^2}{1+|\nabla u|^2}\right)^{\frac{n-3}{2}} K\right]^{\frac{1}{n-1}} + t\min_{\partial\Omega_1}\left[\left(\frac{|\nabla u|^2}{1+|\nabla u|^2}\right)^{\frac{n-3}{2}} K\right]^{\frac{1}{n-1}}. \tag{1-4}$$

Remark 1.3. The following example shows that our estimates are almost sharp in a sense. Let $u(r,\theta)$, $r > 2$, be the n-dimensional catenoid:

$$u(r,\theta) = \int_{-r}^{-2} \frac{1}{\sqrt{s^{2(n-1)}-1}}\,ds. \tag{1-5}$$

Then

$$|\nabla u| = \frac{1}{\sqrt{r^{2(n-1)}-1}}, \tag{1-6}$$

and the Gaussian curvature of the level set at x is $K(x) = r^{1-n}$. Hence,

$$f(t) = \left[\left(\frac{|\nabla u|^2}{1+|\nabla u|^2}\right)^{\frac{n-3}{2}} K\right]^{\frac{1}{n-1}} = r^{2-n}. \tag{1-7}$$

For $n = 2$, $f(t)$ becomes a constant function, which shows that our estimate of its concavity is sharp. Now we turn to the case $n > 2$.

Set

$$R = \int_{-\infty}^{-2} \frac{1}{\sqrt{s^{2(n-1)}-1}}\,ds.$$

Then we have

$$\begin{aligned} -u + R &= \int_{-\infty}^{-r} \frac{1}{s^{n-1}}\,ds - \int_{-\infty}^{-r} \frac{1}{s^{n-1}}\left[1 - \frac{1}{\sqrt{1-s^{-2(n-1)}}}\right] ds \\ &= \frac{(-1)^n}{2-n} r^{2-n} + \mathbb{O}(r^{4-3n}). \end{aligned} \tag{1-8}$$

This means that

$$f(t) = (-1)^n(2-n)(R-t) + \mathbb{O}(r^{4-3n}), \tag{1-9}$$

which shows the "almost sharpness" of our estimate in higher dimensions.

To prove these theorems, let K be the Gaussian curvature of the convex level sets, and let $\varphi = \log K(x) + \rho(|\nabla u|^2)$. For suitable choices of ρ and β, we shall show the elliptic differential inequality

$$L(e^{\beta\varphi}) \leq 0 \quad \text{mod } \nabla_\theta \varphi \quad \text{in } \Omega, \tag{1-10}$$

where L is the elliptic operator associated with the equation we discussed and here we have suppressed the terms involving $\nabla_\theta \varphi$ (see the notations below) with locally bounded coefficients. Then we apply the strong minimum principle to obtain the main results.

In Section 2, we first give brief definitions on the support function of the level sets, and then we obtain the equation of the minimal graph in terms of the support function. We prove Theorem 1.1 in Section 3 by formal calculations. The main technique in the proof consists of rearranging the second and third derivative terms using the equation and the first derivative condition for φ. The key idea is Pogorelov's method in a priori estimates for fully nonlinear elliptic equations.

2. Notations and preliminaries

Let Ω_0 and Ω_1 be bounded smooth open convex subsets of $\mathbb{R}^n$ such that $\overline{\Omega}_1 \subset \Omega_0$, and let $\Omega = \Omega_0 \backslash \overline{\Omega}_1$. Let $u : \overline{\Omega} \to \mathbb{R}$ be a smooth function with $|Du| > 0$ in Ω and let its level sets be strictly convex with respect to the normal direction Du.

For simplicity, we will assume that

$$\begin{aligned} u &= 0 \quad \text{on } \partial\Omega_0, \\ u &= 1 \quad \text{on } \partial\Omega_1, \end{aligned}$$

and we extend u to Ω_1 with the value 1. For $0 \le t \le 1$, we set

$$\overline{\Omega}_t = \{x \in \overline{\Omega}_0 : u \ge t\};$$

Then every $x \in \Omega$ belongs to the boundary of $\overline{\Omega}_{u(x)}$.

Next we define the *support function* of u, denoted by

$$H : \mathbb{R}^n \times [0, 1] \to \mathbb{R}$$

as follows: for each $t \in [0, 1]$, $H(\cdot, t)$ is the support function of the convex body $\overline{\Omega}_t$, that is,

$$H(X, t) = H_{\overline{\Omega}_t}(X) \quad \text{for all } X \in \mathbb{R}^n,\ t \in [0, 1].$$

For details, see [Colesanti and Salani 2003; Longinetti and Salani 2007].

The rest of this section is devoted to deriving the minimal graph by means of the support function. For this we need a reformulation of the first and second derivatives of u in terms of the support function h_{Ω_t}, which is the restriction of $H(\cdot, t)$ to the unit sphere $\mathbb{S}^{n-1}$; see [Chiti and Longinetti 1992; Longinetti and Salani 2007]. For the convenience of the reader, we report the main steps here.

Recall that h is the restriction of H to $\mathbb{S}^{n-1} \times [0, 1]$, so $h(\theta, t) = H(Y(\theta), t) = h_{\overline{\Omega}_t}(Y(\theta))$ where $t \in [0, 1]$ and $Y(\theta) \in \mathbb{S}^n$ is a unit vector with coordinate θ. Since the level sets of u are strictly convex and $h(\theta, t)$ is well defined, the map

$$x(X, t) = x_{\overline{\Omega}_t}(X),$$

which assigns to every $(X, t) \in \mathbb{R}^n \backslash \{0\} \times (0, 1)$ the unique point $x \in \Omega$ on the level surface $\{u = t\}$, where the gradient of u is parallel to X (and orientation reversed).

Let

$$T_i = \frac{\partial Y}{\partial \theta_i},$$

so that $\{T_1, \dots, T_{n-1}\}$ is a tangent frame field on $\mathbb{S}^{n-1}$, and let

$$x(\theta, t) = x_{\overline{\Omega}_t}(Y(\theta));$$

we denote its inverse map by

$$\nu : (x_1, \dots, x_n) \to (\theta_1, \dots, \theta_{n-1}, t).$$

Notice that all these maps (h, x, and ν) depend on the considered function u (like H), even if we do not adopt any explicit notation to stress this fact.

For $h(\theta, t) = \langle x(\theta, t), Y(\theta)\rangle$, since Y is orthogonal to $\partial\overline{\Omega}_t$ at $x(\theta, t)$, deriving the previous equation, we obtain

$$h_i = \langle x, T_i\rangle.$$

In order to simplify some computations, we can also assume that $\theta_1, \dots, \theta_{n-1}, Y$ is an orthonormal frame positively oriented. Hence, from the previous two equalities, we have

$$x = hY + \sum_i h_i T_i$$

and

$$\frac{\partial T_i}{\partial \theta_j} = -\delta_{ij} Y \quad \text{at } x,$$

where the summation index runs from 1 to $n - 1$ if no extra explanation is given, and δ_{ij} is the standard Kronecker symbol. Following [Chiti and Longinetti 1992], we obtain, at the point x under consideration,

$$\frac{\partial x}{\partial t} = h_t Y + \sum_i h_{ti} T_i,$$
$$\frac{\partial x}{\partial \theta_j} = hT_j + \sum_i h_{ij} T_i, \quad j = 1, \dots, n-1.$$

The inverse of the above Jacobian matrix is

$$\text{(2-1)} \qquad \begin{aligned} \frac{\partial t}{\partial x_\alpha} &= h_t^{-1}[Y]_\alpha, && \alpha = 1, \dots, n, \\ \frac{\partial \theta_i}{\partial x_\alpha} &= \sum_j b^{ij}[T_j - h_t^{-1} h_{tj} Y]_\alpha, && \alpha = 1, \dots, n, \end{aligned}$$

where $[\,\cdot\,]_i$ denotes the i-coordinate of the vector in the bracket and

$$b_{ij} = \left\langle \frac{\partial x}{\partial \theta_i}, \frac{\partial Y}{\partial \theta_j} \right\rangle = h\delta_{ij} + h_{ij} \tag{2-2}$$

denotes the inverse tensor of the second fundamental form of the level surface $\partial\overline{\Omega}_t$ at $x(\theta, t)$. The eigenvalues of the tensor b^{ij} are the principal curvatures $\kappa_1, \dots, \kappa_{n-1}$ of $\partial\overline{\Omega}_t$ at $x(\theta, t)$; see [Schneider 1993].

The first equation of (2-1) can be rewritten as

$$Du = \frac{Y}{h_t},$$

where the left hand side is computed at $x(\theta, t)$, while the right hand side is computed at (θ, t). It follows that

$$|Du| = -\frac{1}{h_t}.$$

By the chain rule and (2-1), the second derivatives of u in terms of h can be computed as

$$u_{\alpha\beta} = \sum_{i,j} [-h_t^{-2} h_{ti} Y + h_t^{-1} T_i]_\alpha b^{ij} [T_j - h_t^{-1} h_{tj} Y]_\beta - h_t^{-3} h_{tt} [Y]_\alpha [Y]_\beta \tag{2-3}$$

for $\alpha, \beta = 1, \dots, n$.

In these new coordinates, the minimal graph equation, $\operatorname{div} \dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0$, reads

$$h_{tt} = \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_{ti} h_{tj}] b^{ij}, \tag{2-4}$$

and the associated linear elliptic operator is

$$L = \sum_{i,j,p,q} [(1 + h_t^2)\delta_{pq} + h_{tp} h_{tq}] b^{ip} b^{jq} \frac{\partial^2}{\partial \theta_i \, \partial \theta_j} - 2 \sum_{i,j} h_{tj} b^{ij} \frac{\partial^2}{\partial \theta_i \, \partial t} + \frac{\partial^2}{\partial t^2}. \tag{2-5}$$

Now we recall the well-known commutation formulas for the covariant derivatives of a smooth function $u \in C^4(S^n)$.

$$u_{ijk} - u_{ikj} = -u_k \delta_{ij} + u_j \delta_{ik}, \tag{2-6}$$

$$u_{ijkl} - u_{ijlk} = u_{ik} \delta_{jl} - u_{il} \delta_{jk} + u_{kj} \delta_{il} - u_{lj} \delta_{ik}. \tag{2-7}$$

They will be used during the calculations in the next section. By the definition of b_{ij} and the above commutation formulas, we easily get the following Codazzi-type formula:

$$b_{ij,k} = b_{ik,j}. \tag{2-8}$$

3. Gauss curvature of the level sets of minimal graph

In this section we prove Theorem 1.1. We state a technical lemma.

Lemma 3.1 [Ma et al. 2010]. *Let* $\lambda \ge 0$, $\mu \in \mathbb{R}$, $b_k > 0$, *and* $c_k \in \mathbb{R}$ *for* $2 \le k \le n-1$. *Define the quadratic polynomial*

$$Q(X_2, \dots, X_{n-1}) = -\sum_{2\le k\le n-1} b_k X_k^2 - \lambda\Bigl(\sum_{2\le k\le n-1} X_k\Bigr)^2 + 4\mu \sum_{2\le k\le n-1} c_k X_k.$$

Then we have

$$Q(X_2, \dots, X_{n-1}) \le 4\mu^2\Gamma,$$

where

$$\Gamma = \sum_{2\le k\le n-1} \frac{c_k^2}{b_k} - \lambda\Bigl(1 + \lambda \sum_{2\le k\le n-1} \frac{1}{b_k}\Bigr)^{-1}\Bigl(\sum_{2\le k\le n-1} \frac{c_k}{b_k}\Bigr)^2.$$

For a continuous function $f(t)$ on $[0, 1]$, we define its *generalized second-order derivative* at any point t in $(0, 1)$ as

$$D^2 f(t) = \limsup_{h\to 0} \frac{f(t+h) + f(t-h) - 2f(t)}{h^2}.$$

Let B be the quotient set $B \equiv \mathbb{R}^n/2\pi\mathbb{Z}^n$ and let $Q \equiv B \times (0, 1)$. Let $G(\theta, t)$ be a regular function in Q such that $\mathscr{L}(G(\theta, t)) \ge 0$ for $(\theta, t) \in Q$, where $\mathscr{L}$ is an elliptic operator of the form

$$\mathscr{L} = \sum_{i,j} a^{ij} \frac{\partial^2}{\partial\theta_i\,\partial\theta_j} + \sum_i b^i \frac{\partial^2}{\partial\theta_i\,\partial t} + \frac{\partial^2}{\partial t^2} + \sum_i c^i \frac{\partial}{\partial\theta_i}$$

with regular coefficients a^{ij}, b^i, c^i.

Lemma 3.2 [Longinetti 1987]. *The function* $\phi(t) = \max\{G(\theta, t) : \theta \in B\}$ *satisfies the differential inequality*

$$D^2\phi(t) \ge 0.$$

Moreover, $\phi(t)$ *is a convex function with respect to* t.

The lemma is proved only in dimension $n = 2$ in [Longinetti 1987], but it is easy to see that it is valid for the general case $n \ge 2$.

Since the level sets of u are strictly convex with respect to the normal Du, the matrix of second fundamental form (b_{ij}) is positive definite in Ω. Set

$$\varphi = \rho(h_t^2) - \log K(x),$$

where $K = \det(b^{ij})$ is the Gaussian curvature of the level sets and $\rho(t)$ is a smooth function defined on $(0, +\infty)$. For suitable choices of ρ and β, we will derive the differential inequality

$$L(e^{\beta\varphi}) \le 0 \quad \text{mod } \nabla_\theta\varphi \quad \text{in } \Omega, \tag{3-1}$$

where the elliptic operator L is given in (2-5) and we have modified the terms involving $\nabla_\theta\varphi$ with locally bounded coefficients. Then, by applying a maximum principle argument in Lemma 3.2, we can obtain the desired result.

In order to prove (3-1) at an arbitrary point $x_0 \in \Omega$, we may assume the matrix $(b_{ij}(x_0))$ is diagonal by rotating the coordinate system suitably. From now on, all the calculations will be done at the fixed point x_0.

Proof of Theorem 1.1. We shall prove the theorem in three steps.

Step 1: *computation* $L(\varphi)$. Taking the first derivative of φ, we get

$$\frac{\partial\varphi}{\partial\theta_j} = 2\rho' h_t h_{tj} + \sum_{k,l} b^{kl} b_{kl,j}, \tag{3-2}$$

$$\frac{\partial\varphi}{\partial t} = 2\rho' h_t h_{tt} + \sum_{k,l} b^{kl} b_{kl,t}. \tag{3-3}$$

Taking the derivative of (3-2) and (3-3) once more, we have

$$\frac{\partial^2\varphi}{\partial\theta_i\,\partial\theta_j} = (2\rho' + 4\rho'' h_t^2) h_{ti} h_{tj} + 2\rho' h_t h_{tji} - \sum_{k,l,r,s} b^{kr} b_{rs,i} b^{sl} b_{kl,j} + \sum_{k,l} b^{kl} b_{kl,ji},$$

$$\frac{\partial^2\varphi}{\partial\theta_i\,\partial t} = (2\rho' + 4\rho'' h_t^2) h_{ti} h_{tt} + 2\rho' h_t h_{tti} - \sum_{k,l,r,s} b^{kr} b_{rs,i} b^{sl} b_{kl,t} + \sum_{k,l} b^{kl} b_{kl,ti},$$

$$\frac{\partial^2\varphi}{\partial t^2} = (2\rho' + 4\rho'' h_t^2) h_{tt}^2 + 2\rho' h_t h_{ttt} - \sum_{k,l,r,s} b^{kr} b_{rs,t} b^{sl} b_{kl,t} + \sum_{k,l} b^{kl} b_{kl,tt}.$$

So we can wrtie

$$L(\varphi) = I_1 + I_2 + I_3 + I_4, \tag{3-4}$$

with

$$I_1 = (2\rho' + 4\rho'' h_t^2)\Big[\sum_{i,j}[(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}] b^{ii} b^{jj} h_{ti} h_{tj} - 2\sum_i h_{ti}^2 b^{ii} h_{tt} + h_{tt}^2\Big],$$

$$I_2 = 2\rho' h_t\Big[\sum_{i,j}[(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}] b^{ii} b^{jj} h_{tji} - 2\sum_i h_{ti} b^{ii} h_{tti} + h_{ttt}\Big],$$

$$I_3 = -\sum_{k,l} b^{kk} b^{ll}\Big[\sum_{i,j}[(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}] b^{ii} b^{jj} b_{kl,i} b_{kl,j},\ -2\sum_i h_{ti} b^{ii} b_{kl,i} b_{kl,t} + b_{kl,t}^2\Big]$$

$$I_4 = \sum_k b^{kk} L(b_{kk}).$$

In the rest of this section, we will deal with the four terms above respectively. For the term I_1, by recalling our equation, that is,

$$h_{tt} = \sum_{i,j}[(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}]b^{ij}, \tag{3-5}$$

we have, by recalling that (b^{ij}) is diagonal at x_0,

(3-6)
$$\begin{aligned} I_1 &= (2\rho' + 4\rho'' h_t^2)\Big[\sum_{i,j}[(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}]b^{ii}b^{jj}h_{ti}h_{tj} - 2\sum_i h_{ti}^2 b^{ii} h_{tt} + h_{tt}^2\Big] \\ &= (2\rho' + 4\rho'' h_t^2)\Big[(1+h_t^2)\sum_i (h_{ti}b^{ii})^2 + \Big(\sum_i h_{ti}^2 b^{ii} - h_{tt}\Big)^2\Big] \\ &= (2\rho' + 4\rho'' h_t^2)(1+h_t^2)\sum_i (h_{ti}b^{ii})^2 + (2\rho' + 4\rho'' h_t^2)(1+h_t^2)^2\sigma_1^2, \end{aligned}$$

where $\sigma_1 = \sum_i b^{ii}$ is the mean curvature.

Now we treat the term I_2. Differentiating (3-5) with respect to t, we have

$$h_{ttt} = 2h_t h_{tt}\sigma_1 + 2\sum_{i,j} h_{tti}h_{tj}b^{ij} - \sum_{i,j}[(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}]b^{ii}b^{jj}b_{ij,t}. \tag{3-7}$$

By inserting (3-7) into I_2, we can get

$$\begin{aligned} I_2 &= 2\rho' h_t\Big[\sum_{i,j}[(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}]b^{ii}b^{jj}h_{tji} - 2\sum_i h_{ti}b^{ii}h_{tti} + h_{ttt}\Big] \\ &= 2\rho' h_t\Big[\sum_{i,j}[(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}]b^{ii}b^{jj}(h_{tji} - b_{ij,t}) + 2h_t h_{tt}\sigma_1\Big]. \end{aligned}$$

Recalling the definition of the second fundamental form, that is, (2-2), together with (3-5), we obtain

(3-8)
$$\begin{aligned} I_2 &= 2\rho' h_t\Big[\sum_{i,j}[(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}]b^{ii}b^{jj}(-h_t\delta_{ij}) + 2h_t h_{tt}\sigma_1\Big] \\ &= -2\rho' h_t^2(1+h_t^2)\sum_i (b^{ii})^2 - 2\rho' h_t^2\sum_i (h_{ti}b^{ii})^2 + 4\rho' h_t^2(1+h_t^2)\sigma_1^2 \\ &\qquad + 4\rho' h_t^2\sigma_1\sum_i h_{ti}^2 b^{ii}. \end{aligned}$$

Combining (3-6) and (3-8),

(3-9)
$$\begin{aligned} &I_1 + I_2 \\ &= 4\rho' h_t^2\sigma_1\sum_i h_{ti}^2 b^{ii} + [4\rho' h_t^2(1+h_t^2) + (2\rho' + 4\rho'' h_t^2)(1+h_t^2)^2]\sigma_1^2 \\ &\quad + [(2\rho' + 4\rho'' h_t^2)(1+h_t^2) - 2\rho' h_t^2]\sum_i (h_{ti}b^{ii})^2 - 2\rho' h_t^2(1+h_t^2)\sum_i (b^{ii})^2. \end{aligned}$$

In order to deal with the last two terms, we shall compute $L(b_{kk})$ in advance. In this process, the index k is not summed. By differentiating (3-5) twice with respect to θ_k, we have

$$h_{ttkk} = J_1 + J_2 + J_3 + J_4, \tag{3-10}$$

with

$$\begin{aligned}
J_1 &= \sum_{i,j} [(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}]_{kk} b^{ij},\\
J_2 &= 2\sum_{ij,p,q} [(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}]_k (-b^{ip} b_{pq,k} b^{qj}),\\
J_3 &= \sum_{ij,p,q,r,s} [(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}](2b^{ir} b_{rs,k} b^{sp} b_{pq,k} b^{qj}),\\
J_4 &= \sum_{ij,p,q} [(1+h_t^2)\delta_{ij} + h_{ti}h_{tj}](-b^{ip} b_{pq,kk} b^{qj}).
\end{aligned}$$

For the term J_1, we have

$$\begin{aligned}
J_1 &= \sum_{i,j} (2h_t h_{tk}\delta_{ij} + h_{tik}h_{tj} + h_{ti}h_{tjk})_k b^{ij}\\
&= 2h_{tk}^2\sigma_1 + 2h_t h_{tkk}\sigma_1 + 2\sum_i h_{tikk}h_{ti}b^{ii} + 2\sum_i h_{tik}^2 b^{ii}.
\end{aligned}$$

Noticing that

$$\begin{aligned}
h_{tik} &= h_{kit} = b_{ki,t} - h_t\delta_{ki},\\
h_{tikk} &= h_{ikkt} = b_{ik,kt} - h_{kt}\delta_{ik} = b_{kk,it} - h_{kt}\delta_{ik},
\end{aligned}$$

we obtain

$$\begin{aligned}
J_1 = 2h_{tk}^2\sigma_1 + 2h_t b_{kk,t}\sigma_1 - 2h_t^2\sigma_1 + 2\sum_i b_{kk,it}h_{ti}b^{ii}&\\
- 2h_{tk}^2 b^{kk} + 2\sum_l b_{kl,t}^2 b^{ll} - 4h_t b_{kk,t}b^{kk} + 2h_t^2 b^{kk}&.
\end{aligned} \tag{3-11}$$

For the term J_2, we have

$$\begin{aligned}
J_2 &= 2\sum_{i,j} (2h_t h_{tk}\delta_{ij} + h_{tik}h_{tj} + h_{ti}h_{tjk})(-b^{ii} b_{ij,k} b^{jj})\\
&= -4h_t h_{tk}\sum_i (b^{ii})^2 b_{ii,k} - 4\sum_{i,j} h_{tik}h_{tj} b^{ii} b^{jj} b_{ij,k}\\
&= -4h_t h_{tk}\sum_i (b^{ii})^2 b_{ii,k} - 4\sum_{i,l} h_{ti} b^{ii} b^{ll} b_{kl,i} b_{kl,t}\\
&\qquad + 4h_t\sum_j h_{tj} b^{kk} b^{jj} b_{kk,j}.
\end{aligned} \tag{3-12}$$

Note that we have changed the lower index during the above calculations and this will happen frequently in the following procedure.

Also we have

$$J_3 = 2\sum_{i,j,l}[(1+h_t^2)\delta_{ij}+h_{ti}h_{tj}]b^{ii}b^{jj}b^{ll}b_{kl,i}b_{kl,j}. \tag{3-13}$$

Applying the commutation rule $b_{ij,kl}-b_{ij,lk}=b_{jk}\delta_{il}-b_{jl}\delta_{ik}+b_{ik}\delta_{jl}-b_{il}\delta_{jk}$, for the term J_4, we have

$$\begin{aligned} J_4 &= -\sum_{i,j}[(1+h_t^2)\delta_{ij}+h_{ti}h_{tj}]b^{ii}b^{jj}b_{ij,kk} \\ &= -\sum_{i,j}[(1+h_t^2)\delta_{ij}+h_{ti}h_{tj}]b^{ii}b^{jj}(b_{kk,ij}+b_{ij}-b_{kk}\delta_{ij}). \end{aligned} \tag{3-14}$$

On the other hand,

$$h_{ttkk}=h_{kktt}=b_{kk,tt}-h_{tt}=b_{kk,tt}-\sum_{i,j}[(1+h_t^2)\delta_{ij}+h_{ti}h_{tj}]b^{ij}. \tag{3-15}$$

By putting (3-11)–(3-15) into (3-10), recalling the definition of the operator L, we obtain

$$\begin{aligned} L(b_{kk}) = &\sum_{i,j}[(1+h_t^2)\delta_{ij}+h_{ti}h_{tj}]b^{ij}+2h_{tk}^2\sigma_1+2h_tb_{kk,t}\sigma_1-2h_t^2\sigma_1 \\ &-2h_{tk}^2b^{kk}+2\sum_l b_{kl,t}^2b^{ll}-4h_tb^{kk}b_{kk,t}+2h_t^2b^{kk}-4h_th_{tk}\sum_i(b^{ii})^2b_{ii,k} \\ &-4\sum_{i,l}h_{ti}b^{ii}b^{ll}b_{kl,i}b_{kl,t}+2\sum_{i,j,l}[(1+h_t^2)\delta_{ij}+h_{ti}h_{tj}]b^{ii}b^{jj}b^{ll}b_{kl,i}b_{kl,j} \\ &+4h_t\sum_i h_{ti}b^{kk}b^{ii}b_{kk,i}-\sum_{i,j}[(1+h_t^2)\delta_{ij}+h_{ti}h_{tj}]b^{ii}b^{jj}(b_{ij}-b_{kk}\delta_{ij}). \end{aligned}$$

Therefore,

$$\begin{aligned} I_4 = &\,2\sum_{i,j,k,l}[(1+h_t^2)\delta_{ij}+h_{ti}h_{tj}]b^{ii}b^{jj}b^{kk}b^{ll}b_{kl,i}b_{kl,j}-4\sum_{i,k,l}h_{ti}b^{ii}b^{kk}b^{ll}b_{kl,i}b_{kl,t} \\ &+2h_t\sigma_1\sum_k b^{kk}b_{kk,t}-4h_t\sum_k(b^{kk})^2b_{kk,t}-2h_t^2\sigma_1^2+2\sum_{k,l}b^{kk}b^{ll}b_{kl,t}^2 \\ &+[(n-1)(1+h_t^2)+2h_t^2]\sum_i(b^{ii})^2+2\sigma_1\sum_i h_{ti}^2b^{ii} \\ &+(n-3)\sum_i(h_{ti}b^{ii})^2. \end{aligned} \tag{3-16}$$

By substituting (3-9) and (3-16) in (3-4), we obtain

(3-17)
$$L(\varphi)=\sum_{i,j,k,l}[(1+h_t^2)\delta_{ij}+h_{ti}h_{tj}]b^{ii}b^{jj}b^{kk}b^{ll}b_{kl,i}b_{kl,j}-2\sum_{i,k,l}h_{ti}b^{ii}b^{kk}b^{ll}b_{kl,i}b_{kl,t}$$
$$+\sum_{k,l}b^{kk}b^{ll}b_{kl,t}^2+2h_t\sigma_1\sum_k b^{kk}b_{kk,t}-4h_t\sum_k(b^{kk})^2b_{kk,t}$$
$$+(2+4\rho'h_t^2)\sigma_1\sum_i h_{ti}^2b^{ii}+[(n-1)(1+h_t^2)+2h_t^2-2\rho'h_t^2(1+h_t^2)]\sum_i(b^{ii})^2$$
$$+[4\rho'h_t^2(1+h_t^2)+(2\rho'+4\rho''h_t^2)(1+h_t^2)^2-2h_t^2]\sigma_1^2$$
$$+[(2\rho'+4\rho''h_t^2)(1+h_t^2)-2\rho'h_t^2+(n-3)]\sum_i(h_{ti}b^{ii})^2.$$

Step 2: *calculation of $L(e^{\beta\varphi})$ and estimation of the third-order derivatives involving $b_{kk,t}$.* Notice that

$$L(e^{\beta\varphi})=\beta e^{\beta\varphi}\{L(\varphi)+\beta\varphi_t^2\}+\beta^2e^{\beta\varphi}\sum_{i,j,p,q}[(1+h_t^2)\delta_{pq}+h_{tp}h_{tq}]b^{ip}b^{jq}\frac{\partial\varphi}{\partial\theta_i}\frac{\partial\varphi}{\partial\theta_j}$$
$$-2\beta^2e^{\beta\varphi}\sum_{i,j}h_{tj}b^{ij}\frac{\partial\varphi}{\partial\theta_i}\frac{\partial\varphi}{\partial t}.$$

To reach (3-1), we only need to prove that, for some constant $\beta<0$,

$$L(\varphi)+\beta\varphi_t^2\geq 0 \mod \nabla_\theta\varphi.$$

We now compute $\beta\varphi_t^2$.

By (3-3), we have

$$\varphi_t^2=4(\rho')^2h_t^2h_{tt}^2+4\rho'h_th_{tt}\sum_k b^{kk}b_{kk,t}+\left(\sum_k b^{kk}b_{kk,t}\right)^2 \tag{3-18}$$
$$=4(\rho')^2h_t^2(1+h_t^2)^2\sigma_1^2+8(\rho')^2h_t^2(1+h_t^2)\sigma_1\sum_i h_{ti}^2b^{ii}$$
$$+4(\rho')^2h_t^2\left(\sum_i h_{ti}^2b^{ii}\right)^2+4\rho'h_t(1+h_t^2)\sigma_1\sum_k b^{kk}b_{kk,t}$$
$$+4\rho'h_t\left(\sum_i h_{ti}^2b^{ii}\right)\left(\sum_k b^{kk}b_{kk,t}\right)+\left(\sum_k b^{kk}b_{kk,t}\right)^2.$$

Joining (3-17) with (3-18), we regroup the terms in $L(\varphi)+\beta\varphi_t^2$ as follows:

$$L(\varphi)+\beta\varphi_t^2=P_1+P_2+P_3,$$

where

$$P_1=\sum_{k\neq l}\biggl(\sum_{i,j}h_{ti}h_{tj}b^{ii}b^{jj}b^{kk}b^{ll}b_{kl,i}b_{kl,j}-2\sum_i h_{ti}b^{ii}b^{kk}b^{ll}b_{kl,i}b_{kl,t}+b^{kk}b^{ll}b^2_{kl,t}\biggr),$$

$$\begin{aligned}P_2&=\sum_k(b^{kk}b_{kk,t})^2+\beta\Bigl(\sum_k b^{kk}b_{kk,t}\Bigr)^2\\&\quad+2\sum_k\biggl[[1+2\beta\rho'(1+h_t^2)]h_t\sigma_1+2\beta\rho'h_t\Bigl(\sum_i h_{ti}^2b^{ii}\Bigr)\\&\qquad-\sum_i h_{ti}b^{ii}b^{kk}b_{kk,i}-2h_tb^{kk}\biggr]\cdot(b^{kk}b_{kk,t}),\end{aligned}$$

$$\begin{aligned}P_3&=(1+h_t^2)\sum_{i,k,l}(b^{ii})^2b^{kk}b^{ll}b^2_{kl,i}+\sum_{i,j,k}h_{ti}h_{tj}b^{ii}b^{jj}b^{kk}b_{kk,i}b^{kk}b_{kk,j}\\&\quad+[2+4\rho'h_t^2+8\beta(\rho')^2h_t^2(1+h_t^2)]\sigma_1\sum_i h_{ti}^2b^{ii}\\&\quad+[(n-1)(1+h_t^2)+2h_t^2-2\rho'h_t^2(1+h_t^2)]\sum_i(b^{ii})^2\\&\quad+[4\rho'h_t^2(1+h_t^2)+(2\rho'+4\rho''h_t^2)(1+h_t^2)^2-2h_t^2+4\beta(\rho')^2h_t^2(1+h_t^2)^2]\sigma_1^2\\&\quad+[(2\rho'+4\rho''h_t^2)(1+h_t^2)-2\rho'h_t^2+(n-3)]\sum_i(h_{ti}b^{ii})^2\\&\quad+4\beta(\rho')^2h_t^2\Bigl(\sum_i h_{ti}^2b^{ii}\Bigr)^2.\end{aligned}$$

In the rest of this step, we will deal with the term P_2. Let $X_k=b^{kk}b_{kk,t}(k=1,2,\dots,n-1)$. Then P_2 can be rewritten as

$$P_2(X_1,X_2,\dots,X_{n-1})=\sum_k X_k^2+\beta\Bigl(\sum_k X_k\Bigr)^2+2\sum_k c_kX_k,$$

where

$$c_k=[1+2\beta\rho'(1+h_t^2)]h_t\sigma_1+2\beta\rho'h_t\Bigl(\sum_i h_{ti}^2b^{ii}\Bigr)-\sum_i h_{ti}b^{ii}b^{kk}b_{kk,i}-2h_tb^{kk}.$$

Denote by $\mathcal{P}_2$ the matrix

$$\begin{pmatrix}1+\beta&\beta&\cdots&\beta\\\beta&1+\beta&\cdots&\beta\\\vdots&\vdots&\ddots&\cdots\\\beta&\beta&\cdots&1+\beta\end{pmatrix}.$$

In a word, we want to bound $P_2(X_1, X_2, \dots, X_{n-1})$ from below. Thus the nonnegativity of $\mathcal{P}_2$ is necessary, and this requires

$$\beta \geq -\frac{1}{n-1}.$$

For convenience, Let us choose the degenerate case, that is, $\beta = -1/(n-1)$. By setting $\tau = (1, 1, \dots, 1)$, the null eigenvector of the matrix $\mathcal{P}_2$, we then have, by (3-2),

$$(\star) \qquad P_2(1, 1, \dots, 1) = 2\sum_k c_k = 2[n-3-2\rho'(1+h_t^2)]h_t\sigma_1 - 2\sum_i h_{ti} b^{ii} \frac{\partial \varphi}{\partial \theta_i},$$

which suggests that the simplest selection should be $\rho(t) = ((n-3)/2)\log(1+t)$.

From now on, let us fix $\rho(t) = ((n-3)/2)\log(1+t)$ and $\beta = -1/(n-1)$. But, for simplicity, we do not always substitute for the values of ρ and β.

By straightforward computation and $(\star)$, we have

$$\sum_k \left(X_k + \beta \sum_i X_i + c_k \right)^2 = P_2(X_1, X_2, \dots, X_{n-1}) + \sum_k c_k^2 + P_2(\nabla_\theta \varphi),$$

where

$$P_2(\nabla_\theta \varphi) = 2\beta \left(\sum_i X_i \right) \sum_k c_k = 2\beta \left(\sum_j X_j \right) \sum_i h_{ti} b^{ii} \frac{\partial \varphi}{\partial \theta_i}.$$

Putting ρ and β into some terms in c_k, we derive that

$$c_k = \frac{2}{n-1} h_t \sigma_1 - \frac{2}{n-1} \rho' h_t \left(\sum_i h_{ti}^2 b^{ii} \right) - \sum_i h_{ti} b^{ii} b^{kk} b_{kk,i} - 2h_t b^{kk}.$$

Therefore, together with (3-2), we get

$$\begin{aligned} P_2(X_1, X_2, \dots, X_{n-1}) &\\ \geq -\sum_k c_k^2 - P_2(\nabla_\theta \varphi) &\\ = -\sum_{i,j,k} h_{ti} h_{tj} b^{ii} b^{jj} b^{kk} b_{kk,i} b^{kk} b_{kk,j} - 4h_t \sum_{i,k} h_{ti} b^{ii} (b^{kk})^2 b_{kk,i} &\\ - 4h_t^2 \sum_k (b^{kk})^2 + \frac{4}{n-1} h_t^2 \sigma_1^2 - \frac{8}{n-1} \rho' h_t^2 \sigma_1 \sum_i h_{ti}^2 b^{ii} &\\ + \frac{4}{n-1} h_t^2 (\rho')^2 \left(\sum_i h_{ti}^2 b^{ii} \right)^2 + \widetilde{P}_2(\nabla_\theta \varphi),& \end{aligned}$$

where

$$\widetilde{P}_2(\nabla_\theta \varphi) = -P_2(\nabla_\theta \varphi) - \frac{4}{n-1} h_t \left[\sigma_1 - \rho' \sum_j h_{tj}^2 b^{jj} \right] \sum_i h_{ti} b^{ii} \frac{\partial \varphi}{\partial \theta_i}.$$

Observing that $P_1 \geq 0$,

$$L(\varphi)+\beta\varphi_t^2 \geq (1+h_t^2)\sum_{i,k,l}(b^{ii})^2 b^{kk}b^{ll}b_{kl,i}^2 - 4h_t\sum_{i,k}h_{ti}b^{ii}(b^{kk})^2 b_{kk,i} + \Big[2+4\rho' h_t^2+8\beta(\rho')^2h_t^2(1+h_t^2)-\frac{8}{n-1}\rho' h_t^2\Big]\sigma_1\sum_i h_{ti}^2 b^{ii} + [(n-1)(1+h_t^2)-2h_t^2-2\rho' h_t^2(1+h_t^2)]\sum_i (b^{ii})^2 + \Big[4\rho' h_t^2(1+h_t^2)+[(2\rho'+4\rho'' h_t^2)+4\beta(\rho')^2h_t^2](1+h_t^2)^2-\frac{2n-6}{n-1}h_t^2\Big]\sigma_1^2 + [(2\rho'+4\rho'' h_t^2)(1+h_t^2)-2\rho' h_t^2+(n-3)]\sum_i (h_{ti}b^{ii})^2+\widetilde{P}_2(\nabla_\theta\varphi). \tag{3-19}$$

In the next step we will concentrate on the following two terms:

$$R=(1+h_t^2)\sum_{i,k,l}(b^{ii})^2 b^{kk}b^{ll}b_{kl,i}^2 - 4h_t\sum_{i,k}h_{ti}b^{ii}(b^{kk})^2 b_{kk,i}.$$

Step 3: *conclusion of the proof of* (3-1). Recalling our first-order condition (3-2), we have

$$b^{11}b_{11,j}=\frac{\partial\varphi}{\partial\theta_j}-\sum_{k\geq 2}b^{kk}b_{kk,j}-2\rho' h_t h_{tj} \quad \text{for } j=1,2,\dots,n-1. \tag{3-20}$$

For the term R, we have

$$\begin{aligned} R &= (1+h_t^2)\Big[\sum_i\sum_{k\neq l}(b^{ii})^2 b^{kk}b^{ll}b_{kl,i}^2+\sum_{i,k}(b^{ii})^2(b^{kk}b_{kk,i})^2\Big] - 4\sum_{i,k}h_t h_{ti}b^{ii}(b^{kk})^2 b_{kk,i} \\ &= (1+h_t^2)\Big[2\sum_{k\geq 2}(b^{11})^2 b^{kk}b^{11}b_{k1,1}^2+2\sum_{i,k\geq 2}(b^{ii})^2 b^{kk}b^{11}b_{k1,i}^2 + \sum_i\sum_{\substack{k,l\geq 2\\ k\neq l}}(b^{ii})^2 b^{kk}b^{ll}b_{kl,i}^2+\sum_i (b^{ii})^2(b^{11}b_{11,i})^2 + \sum_i\sum_{k\geq 2}(b^{ii})^2(b^{kk}b_{kk,i})^2\Big] \\ &\quad - 4\sum_i h_t h_{ti}b^{ii}(b^{11})^2 b_{11,i} - 4\sum_i\sum_{k\geq 2}h_t h_{ti}b^{ii}(b^{kk})^2 b_{kk,i} \\ &= R_1+R_2+R_3, \end{aligned}$$

where

$$R_1=(1+h_t^2)\Big[2\sum_{k\ge 2}(b^{11})^2b^{kk}b^{11}b_{k1,1}^2+\sum_i(b^{ii})^2(b^{11}b_{11,i})^2\Big] -4\sum_i h_th_{ti}b^{ii}(b^{11})^2b_{11,i},$$

$$R_2=2\sum_{i,k\ge 2}(1+h_t^2)(b^{ii})^2b^{kk}b^{11}b_{k1,i}^2+\sum_i\sum_{\substack{k,l\ge 2\\ k\ne l}}(1+h_t^2)(b^{ii})^2b^{kk}b^{ll}b_{kl,i}^2,$$

$$R_3=\sum_i\sum_{k\ge 2}(1+h_t^2)(b^{ii})^2(b^{kk}b_{kk,i})^2-4\sum_i\sum_{k\ge 2}h_th_{ti}b^{ii}(b^{kk})^2b_{kk,i}.$$

By (3-20), one has

$$\begin{aligned}R_1=(1+h_t^2)\Big[&2b^{11}\sum_{i,k,l\ge 2}b^{ii}b^{kk}b^{ll}b_{kk,i}b_{ll,i}+8\rho' h_tb^{11}\sum_{i,k\ge 2}h_{ti}b^{ii}b^{kk}b_{kk,i}\\ &+8(\rho')^2h_t^2b^{11}\sum_{i\ge 2}h_{ti}^2b^{ii}+\sum_i\sum_{k,l\ge 2}(b^{ii})^2b^{kk}b^{ll}b_{kk,i}b_{ll,i}\\ &+4\rho' h_t\sum_i\sum_{k\ge 2}h_{ti}(b^{ii})^2b^{kk}b_{kk,i}+4(\rho')^2h_t^2\sum_i(h_{ti}b^{ii})^2\Big]\\ &+4h_t\sum_i\sum_{k\ge 2}h_{ti}b^{ii}b^{11}b^{kk}b_{kk,i}+8\rho' h_t^2b^{11}\sum_i h_{ti}^2b^{ii}+R(\nabla_\theta\varphi),\end{aligned}$$

where

$$\begin{aligned}R(\nabla_\theta\varphi)=(1+h_t^2)\Big[&2b^{11}\sum_{k\ge 2}b^{kk}\Big(\frac{\partial\varphi}{\partial\theta_k}\Big)^2-4b^{11}\sum_{k,l\ge 2}b^{kk}b^{ll}b_{ll,k}\frac{\partial\varphi}{\partial\theta_k}\\ &-8\rho' h_tb^{11}\sum_{k\ge 2}b^{kk}h_{tk}\frac{\partial\varphi}{\partial\theta_k}+\sum_i(b^{ii})^2\Big(\frac{\partial\varphi}{\partial\theta_i}\Big)^2\\ &-2\sum_i\sum_{k\ge 2}(b^{ii})^2b^{kk}b_{kk,i}\frac{\partial\varphi}{\partial\theta_i}-4\rho' h_t\sum_i(b^{ii})^2h_{ti}\frac{\partial\varphi}{\partial\theta_i}\Big]\\ &-4h_tb^{11}\sum_i b^{ii}h_{ti}\frac{\partial\varphi}{\partial\theta_i}.\end{aligned}$$

On the other hand,

$$\begin{aligned}R_2=(1+h_t^2)\Big[&2b^{11}\sum_{k\ge 2}(b^{kk})^3b_{kk,1}^2+2\sum_{\substack{i,k\ge 2\\ i\ne k}}(b^{ii})^2b^{kk}b^{11}b_{k1,i}^2\\ &+2\sum_{\substack{i,k\ge 2\\ i\ne k}}b^{ii}(b^{kk})^3b_{kk,i}^2+\sum_i\sum_{\substack{k,l\ge 2\\ k\ne l,k\ne i,l\ne i}}(b^{ii})^2b^{kk}b^{ll}b_{kl,i}^2\Big].\end{aligned}$$

Recall that $2\rho'(1+h_t^2)=n-3$, which will be denoted by α for simplicity in the following calculations. Now we are at a stage where we can rewrite the terms in R in a natural way: we denote by T_1 the terms involving $b_{kk,1}(k\geq 2)$, by T_2 the terms involving $b_{kk,i}(k,i\geq 2)$, and by T_3 all of the rest of the terms. More precisely,

$$T_1=\sum_{k\geq 2}(1+2b_{11}b^{kk})\cdot((1+h_t^2)^{1/2}b^{11}b^{kk}b_{kk,1})^2+\Big(\sum_{k\geq 2}(1+h_t^2)^{1/2}b^{11}b^{kk}b_{kk,1}\Big)^2 + 4h_th_{t1}b^{11}(1+h_t^2)^{-1/2}\sum_{k\geq 2}\Big(1+\frac{\alpha}{2}-b_{11}b^{kk}\Big)\cdot((1+h_t^2)^{1/2}b^{11}b^{kk}b_{kk,1})$$

and

$$T_2=(1+h_t^2)\sum_{i\geq 2}\bigg\{(1+2b_{ii}b^{11})\cdot\Big(\sum_{k\geq 2}b^{ii}b^{kk}b_{kk,i}\Big)^2+\sum_{\substack{k\geq 2\\ k\neq i}}2b_{ii}b^{kk}\cdot(b^{ii}b^{kk}b_{kk,i})^2 + \sum_{k\geq 2}(b^{ii}b^{kk}b_{kk,i})^2+4h_th_{ti}b^{ii}(1+h_t^2)^{-1} \times\sum_{k\geq 2}\Big[-b_{ii}b^{kk}+\frac{\alpha}{2}+(1+\alpha)b_{ii}b^{11}\Big]\cdot(b^{ii}b^{kk}b_{kk,i})\bigg\};$$

the rest of the terms are

$$T_3=h_t^2(1+h_t^2)^{-1}\Big[2\alpha^2b^{11}\sum_{i\geq 2}h_{ti}^2b^{ii}+\alpha^2\sum_i(h_{ti}b^{ii})^2+4\alpha b^{11}\sum_i h_{ti}^2b^{ii}\Big] + (1+h_t^2)\Big[2\sum_{\substack{i,k\geq 2\\ i\neq k}}(b^{ii})^2b^{kk}b^{11}b_{k1,i}^2+\sum_i\sum_{\substack{k,l\geq 2\\ k\neq l,k\neq i,l\neq i}}(b^{ii})^2b^{kk}b^{ll}b_{kl,i}^2\Big] + R(\nabla_\theta\varphi). \tag{3-21}$$

We shall minimize the terms T_1 and T_2 via Lemma 3.1 for different choices of parameters.

At first, let us examine the term T_1. set $X_k=(1+h_t^2)^{1/2}b^{11}b^{kk}b_{kk,1}$, $\lambda=1$, $\mu=h_{t1}b^{11}h_t(1+h_t^2)^{-1/2}$, $b_k=1+2b_{11}b^{kk}$, and $c_k=b_{11}b^{kk}-(1+\alpha/2)$, where $k\geq 2$. By Lemma 3.1, we have

$$T_1\geq -4h_t^2(1+h_t^2)^{-1}(h_{t1}b^{11})^2\Gamma_1,$$

where

$$\Gamma_1=\sum_{k\geq 2}\frac{c_k^2}{b_k}-\Big(1+\sum_{k\geq 2}\frac{1}{b_k}\Big)^{-1}\Big(\sum_{k\geq 2}\frac{c_k}{b_k}\Big)^2.$$

Next we shall simplify Γ_1. By denoting

$$\beta_k=\frac{1}{b_k},$$

we have

$$b_{11}b^{kk}=\frac{1}{2\beta_k}-\tfrac{1}{2},\qquad c_k=\frac{1}{2\beta_k}-\frac{3+\alpha}{2}.$$

Hence

$$\begin{aligned}\Gamma_1&=\sum_{k\geq 2}\beta_k\left(\frac{1}{2\beta_k}-\frac{3+\alpha}{2}\right)^2-\left(1+\sum_{k\geq 2}\beta_k\right)^{-1}\left[\sum_{k\geq 2}\beta_k\left(\frac{1}{2\beta_k}-\frac{3+\alpha}{2}\right)\right]^2\\&=\frac{1}{4}\sum_{k\geq 2}\frac{1}{\beta_k}-\left(1+\sum_{k\geq 2}\beta_k\right)^{-1}\frac{(n+1+\alpha)^2}{4}+\frac{(3+\alpha)^2}{4}.\end{aligned}$$

Since

$$1\leq 1+\sum_{k\geq 2}\beta_k\leq n-1,$$

it follows that

$$\begin{aligned}\Gamma_1&\leq\frac{1}{4}\sum_{k\geq 2}\frac{1}{\beta_k}-\frac{(n+1+\alpha)^2}{4(n-1)}+\frac{(3+\alpha)^2}{4}\\&=\frac{n-2}{4(n-1)}(2+\alpha)^2+\tfrac{1}{4}(2\sigma_1 b_{11}-2).\end{aligned}$$

Therefore,

$$T_1\geq-\left[\frac{(n-2)}{n-1}(2+\alpha)^2+2\sigma_1 b_{11}-2\right]h_t^2(1+h_t^2)^{-1}(h_{t1}b^{11})^2. \tag{3-22}$$

Now we will deal with T_2. For every $i\geq 2$ fixed, set $X_k=b^{ii}b^{kk}b_{kk,i}$, $\lambda=1+2b_{ii}b^{11}$, $\mu=-h_{ti}b^{ii}h_t(1+h_t^2)^{-1}$, $b_k=1+2b_{ii}b^{kk}(k\neq i)$, $b_i=1$, and $c_k=b_{ii}b^{kk}-\frac{1}{2}\alpha-(1+\alpha)b_{ii}b^{11}$. By Lemma 3.1, we have

$$T_2\geq -4(1+h_t^2)\sum_{i\geq 2}(h_{ti}b^{ii})^2\Gamma_i,$$

where

$$\Gamma_i=c_i^2+\sum_{\substack{k\geq 2\\ k\neq i}}\frac{c_k^2}{b_k}-\left(\frac{1}{\lambda}+1+\sum_{\substack{k\geq 2\\ k\neq i}}\frac{1}{b_k}\right)^{-1}\left(c_i+\sum_{\substack{k\geq 2\\ k\neq i}}\frac{c_k}{b_k}\right)^2.$$

For $k\neq i$, denoting

$$\beta_k=\frac{1}{b_k},$$

we have

$$b_{ii}b^{kk}=\frac{1}{2\beta_k}-\tfrac{1}{2},\quad c_k=\frac{1}{2\beta_k}-\delta,$$

where

$$\delta=\frac{1+\alpha}{2}+(1+\alpha)b_{ii}b^{11}.$$

Noticing that

$$c_i = \tfrac{3}{2} - \delta, \qquad \frac{\delta}{\lambda} = \frac{1+\alpha}{2},$$

we obtain

$$\begin{aligned}
\Gamma_i &= c_i^2 + \sum_{\substack{k\ge 2\\ k\ne i}} \beta_k\Big(\frac{1}{2\beta_k} - \delta\Big)^2 - \Big(\frac{1}{\lambda} + 1 + \sum_{\substack{k\ge 2\\ k\ne i}} \beta_k\Big)^{-1} \Big[c_i + \sum_{\substack{k\ge 2\\ k\ne i}} \beta_k\Big(\frac{1}{2\beta_k} - \delta\Big)\Big]^2 \\
&= \frac{1}{4}\sum_{\substack{k\ge 2\\ k\ne i}} \frac{1}{\beta_k} - \Big(\frac{1}{\lambda} + 1 + \sum_{\substack{k\ge 2\\ k\ne i}} \beta_k\Big)^{-1} \Big(\frac{n}{2} + \frac{\delta}{\lambda}\Big)^2 + \frac{9}{4} + \frac{\delta^2}{\lambda} \\
&= \frac{1}{4}\sum_{\substack{k\ge 2\\ k\ne i}} \frac{1}{\beta_k} - \Big(\frac{1}{\lambda} + 1 + \sum_{\substack{k\ge 2\\ k\ne i}} \beta_k\Big)^{-1} \frac{(n+1+\alpha)^2}{4} + \frac{9}{4} + \frac{1+\alpha}{2}\delta.
\end{aligned}$$

Obviously,

$$1 \le \frac{1}{\lambda} + 1 + \sum_{\substack{k\ge 2\\ k\ne i}} \beta_k \le n-1,$$

hence

$$\begin{aligned}
\Gamma_i &\le \frac{1}{4}\sum_{\substack{k\ge 2\\ k\ne i}} \frac{1}{\beta_k} - \frac{(n+1+\alpha)^2}{4(n-1)} + \frac{9}{4} + \frac{1+\alpha}{2}\delta \\
&= \frac{n-2}{4(n-1)}\alpha^2 - \frac{1}{n-1}\alpha + \frac{n-3}{2(n-1)} + \tfrac{1}{2}\sigma_1 b_{ii} + \tfrac{1}{2}\alpha^2 b_{ii} b^{11} + \alpha b_{ii} b^{11}.
\end{aligned}$$

Therefore, we have

$$\text{(3-23)} \quad T_2 \ge -\frac{h_t^2}{1+h_t^2} \sum_{i\ge 2} \Big(\frac{n-2}{n-1}\alpha^2 - \frac{4}{n-1}\alpha + \frac{2n-6}{n-1} + 2\sigma_1 b_{ii} + 2\alpha^2 b_{ii} b^{11} + 4\alpha b_{ii} b^{11}\Big)(h_{ti} b^{ii})^2.$$

Now, combining (3-21) , (3-22), and (3-23), we obtain

$$\text{(3-24)} \quad R \ge \frac{h_t^2}{1+h_t^2} \sum_i \Big(\frac{1}{n-1}\alpha^2 + \frac{4}{n-1}\alpha - \frac{2n-6}{n-1} - 2\sigma_1 b_{ii}\Big)(h_{ti} b^{ii})^2 + R(\nabla_\theta \varphi).$$

For choices of ρ and β, by (3-19) and (3-24), we have, for $n \ge 2$,

$$\begin{aligned}
L(\varphi) - \frac{1}{n-1}\varphi_t^2 &\ge \frac{2\sigma_1}{1+h_t^2} \sum_i h_{ti}^2 b^{ii} + (n-1)\sum_i (b^{ii})^2 + (n-3)\sigma_1^2 \\
&\qquad + \frac{2(n-3)}{1+h_t^2} \sum_i (h_{ti} b^{ii})^2 + \widetilde{P}_2(\nabla_\theta \varphi) + R(\nabla_\theta \varphi) \\
&\ge 0 \mod \nabla_\theta \varphi.
\end{aligned}$$

The proof of (3-1) is completed. □

Now we give a remark on Theorem 1.1.

Remark 3.3. In the proof of Theorem 1.1, if we restrict to the case $n = 2$ and just set $\rho = 0$, then (3-2) shows that

$$b_{11,1} = 0 \mod \nabla_\theta \varphi.$$

Applying this to the expression of $L(\varphi)$ in (3-17) will give

$$\begin{aligned} L(\varphi) &= (b^{11}b_{11,t})^2 - 2h_t(b^{11})^2 b_{11,t} + (b^{11})^2 h_{t1}^2 + (1+h_t^2)(b^{11})^2 \\ &= [b^{11}b_{11,t} - h_t b^{11}]^2 + (b^{11})^2 h_{t1}^2 + (b^{11})^2 \geq 0 \mod \nabla_\theta \varphi, \end{aligned}$$

and this means that, for any point $x \in \Gamma_t$, $0 < t < 1$,

$$\log K(x) \geq (1-t) \min_{\partial\Omega_0} \log K + t \min_{\partial\Omega_1} \log K,$$

which has already been proved by Longinetti [1987]. Also, by Remark 1.3 we know that this estimate is not sharp in the two-dimensional case.

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