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The Concept of Filament Strength and the Weibull Modulus

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ABSTRACT: The strength of high-performance filaments is a complex parameter which can not be fully described with a single value. The Weibull model is used to describe the intrinsic statistical nature of the fracture strength. Possibilities and limitations of the Weibull model are illustrated. The relationship between material properties and the parameters in the Weibull model is discussed.

KEY WORDS: high-performance filaments, strength, Weibull theory, Weibull modulus, fracture toughness, theoretical strength

The increase in performance of high-modulus, high-strength yarns not only puts stronger demands on the yarn manufacturing process but also on the accurate description of the structure and properties of the filaments composing the yarn. Notwithstanding their simple geomefrical form, filaments can be characterized by a large number of structural, mechanical, physical, and chemical parameters. One of the most important filament properties is the filament strength. Unlike physical parameters such as elastic modulus and thermal expansion coefficient, filament strength is a statistical parameter which can not be fully described by a single value. The statistical distribution of filament strengths at the monofilament level can usually be described by the so-called Weibull model. In this paper some aspects of this Weibullian description of the filament strength are discussed.

The most important mathematical equations of the Weibull model are summarized and illustrated. It is shown that the Weibull parameters are not material constants. The relation between fracture toughness of the filament material and the Weibull modulus is discussed. Finally, the relation between the Weibull modulus and the distribution of the size of the crack-nucleating defects is described.

Mathematical Formulation

The statistical description of the filament strength started with the work of Weibull *[1,2]*. His model and later modifications of this model are statistical models that are not related to the physical nature of the fracture process. Implicit in the model in its simplest form, as described here, is the assumption that failure is due to sudden catastrophic growth of pre-existing defects. Each defect corresponds to a certain local failure stress. Failure at the most serious defect (i.e., the defect with the lowest fracture stress) leads to the immediate failure of the entire sample. The fracture process is therefore of a perfectly brittle nature and hence independent of time and environment. It is assumed that the defects are homogeneously distributed throughout the sample or, in the case of filaments, along the filament. Furthermore, it is assumed that the strength distribution along a filament is of the same form as between individual filaments.

In this model a filament is regarded as a single chain of imaginary units of length L_0, each having a certain failure stress σ_{fi} (Fig. 1). Let $P_1(\sigma)$ be the probability of failure due to a stress σ for one unit. Then the probability of survival of that unit equals $1 - P_1(\sigma)$. For the entire serial chain of N units the cumulative failure probability equals P_N with

$$P_N(\sigma) = 1 - [1 - P_1(\sigma)]^N \tag{1}$$

or for very large N:

$$P_N(\sigma) = 1 - \exp[-N \cdot P_1(\sigma)] \tag{2}$$

Since N is proportional to the length, L, of the filament, Eq 2 can be rewritten as

$$P = 1 - \exp\left[\frac{L}{L_0}\,\phi(\sigma)\right] \tag{3}$$

where $\phi(\sigma)$ is an unknown function. On empirical grounds Weibull assumed a power law relation for $\phi(\sigma)$:

$$\phi(\sigma) = (\sigma/\sigma')^m \tag{4}$$

where σ is the applied stress and σ' a constant scale parameter with the same dimensions as used for the stress. The parameter m is dimensionless and is called the *Weibull modulus.* Combining Eqs 3 and 4 yields the following equation for the cumulative failure probability function, P (i.e., the fraction of units or samples which fail at or below a stress σ):

$$P = 1 - \exp\left[-\frac{L}{L_0}\cdot\left(\frac{\sigma}{\sigma'}\right)^m\right]$$

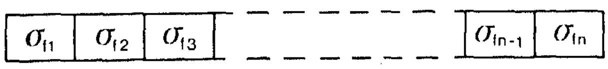

FIG. 1—*Schematic diagram of a fiber consisting of* n *units with fracture stresses* σ_{fi}.

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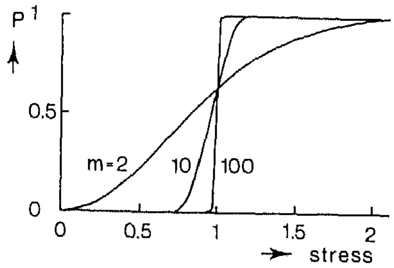

FIG. 2—*Cumulative failure probability versus applied stress for various values of* m (σ_0 = 1, L = 1).

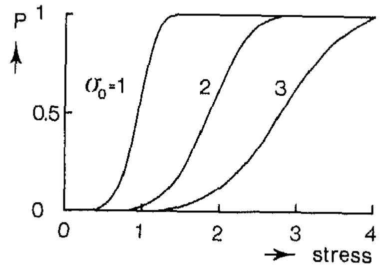

FIG. 3—*Cumulative failure probability versus applied stress for various values of* σ_0 (m = 5, L = 1).

which is usually written as

$$P = 1 - \exp\left[-L \cdot \left(\frac{\sigma}{\sigma_0}\right)^m\right] \tag{5}$$

where L is the gage length in meters and σ_0 no longer has simple units of stress.

Figures 2 and 3 show the cumulative failure probability versus the applied stress for various values of m and σ_0 respectively. From these figures it is evident that a high value for m corresponds to a narrow fracture stress distribution function.

In practice, rather than showing P versus σ it is advisable to plot $\ln(-\ln(1 - P)) - \ln(L)$ versus $\ln(\sigma)$, since this yields a linear dependence with slope m; rearrangement of Eq 5 gives

$$\ln(-\ln(1 - P)) - \ln(L) = m \cdot \ln(\sigma) - m \cdot \ln(\sigma_0) \tag{6}$$

In Figs. 4 and 5 the curves of Figs. 2 and 3 are presented in such a Weibull plot. They show that the shape of the fracture stress distribution is only determined by the magnitude of m and is independent of σ_0.

A very useful feature of the Weibull plot is the possibility of plotting fracture strength data obtained at various gage lengths on the same curve. Figure 6 shows the calculated failure probability versus the applied stress for three gage lengths (L = 1, L = 10, and L = 100 respectively, while m and σ_0 are kept constant). In the calculations 20 strength measurements per gage length are assumed. In this linear plot of P versus σ the three sets of data form three separate curves. When the data of Fig. 6 are presented in a Weibull plot (Fig. 7), the data form a single straight line in accordance with Eq 6.

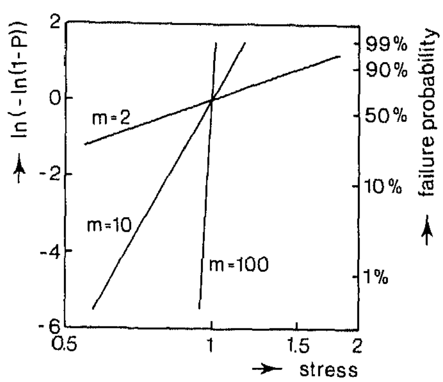

FIG. 4—*Curves of Fig. 2 replotted in a Weibull plot.*

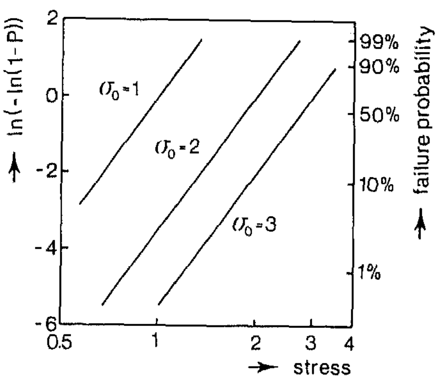

FIG. 5—*Curves of Fig. 3 replotted in a Weibull plot.*

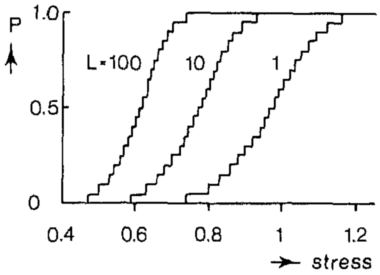

FIG. 6—*Calculated cumulative failure probability curves versus the applied stress for three gage lengths (20 samples per gage length;* m = 10, σ_0 = 1).

The cumulative failure probability P_i at a particular stress σ in these Weibull plots is approximated by

$$P_i \approx n_i/(1 + n) \tag{7}$$

where n_i is the number of filaments that have fractured at or below a stress σ and n is the total number of filaments tested. This procedure allows the plotting of all experimental data points on a Weibull plot. It has been shown [3,4] that $n_i/(1 + n)$ is an acceptable but conservative estimator for the fracture probability.

The failure probability density function $f(\sigma)$ (i.e., the normal-

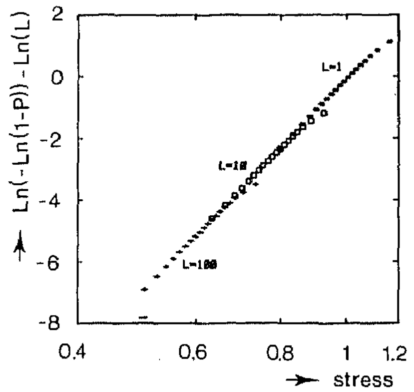

FIG. 7—*Data of Fig. 6 replotted in a Weibull plot.*

ized probability of a failure stress between σ and $\sigma + d\sigma$) can be obtained by differentiating Eq 5 and is given by

$$f(\sigma) = L\frac{m}{\sigma_0}\left(\frac{\sigma}{\sigma_0}\right)^{m-1} \exp\left\{-L\cdot\left(\frac{\sigma}{\sigma_0}\right)^m\right\} \tag{8}$$

The probability density functions corresponding to the curves in Figs. 2 and 3 are shown in Figs. 8 and 9.

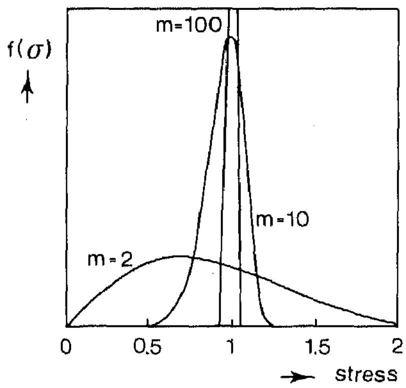

FIG. 8—*Failure probability density functions corresponding to the curves in Fig. 2.*

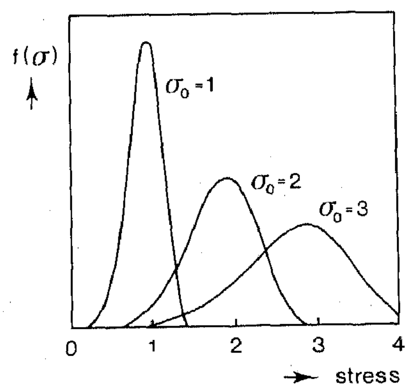

FIG. 9—*Failure probability density functions corresponding to the curves in Fig. 3.*

The average fracture stress of filaments with gage length L is given by

$$<\sigma> = \sigma_0 L^{-1/m}\,\Gamma(1 + 1/m) \tag{9}$$

where Γ is the gamma function. For the usual m values ($m \approx 5$ to 30) obtained for high-performance filaments: $\Gamma\,(1 + 1/m) \approx 0.95 \pm 0.03$. Equation 9 shows that the average fracture stress is not a constant but depends on the gage length. This fact has been realized (but not understood) for a very long time. (Cf. Aristotle (c. 350 B.C.) on the length dependence of the strength of wooden sticks and Leonardo da Vinci (c. A.D. 1500) on the strength of iron wires [5]). The change in average fracture stress with gage length follows directly from Eq 9 and is given by

$$<\sigma_1>/<\sigma_2> = (L_2/L_1)^{1/m} \tag{10}$$

Such a length dependence does not only apply to the average fracture stress but also to stresses at any failure probability level. The decrease in average fracture stress with increasing filament length is plotted in Fig. 10 for various values of m.

Equation 9 is only applicable over a limited length range. For very long gage lengths (> 10 m) the chance of a low fracture stress due to accidental damage (i.e., due to rare defects which are not part of the normal defect population) becomes significant. Furthermore, for very short gage lengths the predicted average fracture strength goes to infinity. Clearly this is impossible due to the finite value of the theoretical strength of all materials. Furthermore, deformation and fracture in or near the clamps during testing also put a much lower experimental limit on the maximum measurable fracture stress [6].

The Weibull modulus is usually determined from a Weibull plot as mentioned before. However, the Weibull modulus can also be estimated from the variation coefficient of the fracture strength (the standard deviation s divided by the average value), since

$$\text{var. coeff.} = \frac{s}{<\sigma>} = \frac{[\Gamma(1 + 2/m) - \Gamma^2(1 + 1/m)]}{\Gamma(1 + 1/m)} \tag{11}$$

The variation coefficient is independent of the gage length L and of σ_0. Figure 11 plots the Weibull modulus versus the variation coefficient in accordance with Eq 11. In the same figure the following simple approximation of Eq 11 is indicated by the dashed line:

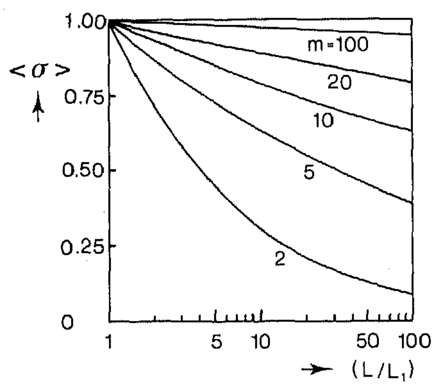

FIG. 10—*Decrease of average filament fracture stress with increasing gage length for various values of* m ($<\sigma> = 1$ at $L = L_1$).

$$m \approx 1.2/(s/<\sigma>) \qquad (12)$$

This simple procedure of determining the Weibull modulus from the variation coefficient yields a reasonably good estimate for m.

Weibull statistics can also be used to calculate the strength of an ideal untwisted bundle in which there is no interfilament interaction (the number of filaments in the bundle should be large, i.e., ≥ 100). Assuming a linear elastic behavior up to the point of failure and realizing that at the maximum bundle stress $dF/dl = 0$, the maximum bundle stress can be calculated [7] and given by

$$\sigma_{\text{max bundle}} = \sigma_0 \, (e\, m)^{-1/m} \cdot L^{-1/m} \qquad (13)$$

where e is the base of the natural logarithm. Combining Eqs 13 and 9 yields the following equation for the bundle efficiency, ϵ (i.e., the ratio of the maximum bundle stress and the average filament strength, both having the same length):

$$\epsilon = (e\, m)^{-1/m}/\Gamma(1 + 1/m) \qquad (14)$$

Equation 14 shows that the bundle efficiency is independent of L and σ_0 and a function of the Weibull modulus only. The bundle strength is always lower than the average filament strength. The bundle efficiency is plotted as a function of m in Fig. 12. Figure 13 shows the calculated bundle stress-strain curve for a material with a Weibull modulus $m = 10$; the fraction of broken filaments is also shown (dashed line). The maximum bundle stress is obtained when only 10% of the filaments have failed.

Finally, the following additional remarks should be made concerning the mathematics of the model:

- In the principal Weibull equation described here (Eq 5) there is a finite (although small) probability of extremely low or extremely high fracture stresses. The occurrence of fracture at $\sigma \approx 0$ can be prevented mathematically by incorporating a minimum strength level σ_u in Eq 5, yielding the so-called three-parameter Weibull equation:

$$P = 1 - \exp\left\{-L\left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m\right\} \quad \text{for} \quad \sigma > \sigma_u \qquad (15a)$$

and

$$P = 0 \quad \text{for} \quad \sigma \leq \sigma_u \qquad (15b)$$

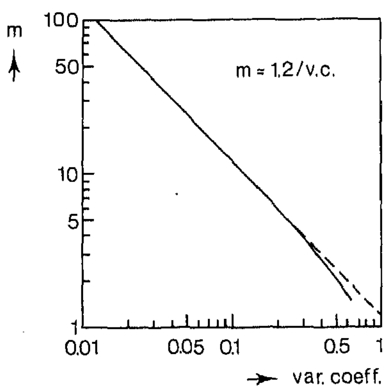

FIG. 11—*Weibull modulus versus the variation coefficient of the fracture stress data. Solid line is the exact relationship (Eq 11). Dashed line is the approximation (Eq 12).*

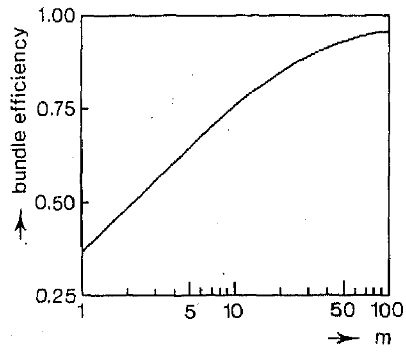

FIG. 12—*Bundle efficiency versus Weibull modulus.*

The incorporation of the term σ_u affects the results of the Weibull analysis only to a limited extent. Experimentally, σ_u is very hard to determine with any statistical significance, since its effect is only noticeable on the lower strength tail of the strength distribution. In this region the statistical uncertainty of the data is large. In practice σ_u is usually taken as $\sigma_u = 0$.

- The maximum fracture stress of any filament is limited by the theoretical strength of the filament material. In this case the probability density function is truncated and the Weibull theory is no longer applicable in its present form. It will be shown that if the average filament strength approaches the theoretical strength of the material, then the Weibull modulus approaches infinity.
- The theoretical limits to the value of the Weibull modulus are 0 for the lower limit and ∞ for the higher limit. A negative value for the Weibull modulus leads to an increase in average fracture stress with increasing filament length (or "a chain becomes stronger than its weakest link"). This is impossible within the concepts of brittle fracture. The maximum Weibull modulus value that can be measured experimentally is determined by the accuracy of the load at which the filament fails, the accuracy of the diameter of the filament, and in particular the variability of the filament diameter along a filament length [8]. These factors limit the maximum experimental value of the Weibull modulus which can be determined with statistical significance to about a value on the order of 100.
- The number of samples tested has a significant effect on the accuracy of the Weibull modulus derived from the Weibull plot. To illustrate this effect a computer simulation was performed in which 1000 filament strength values were generated which had a Weibullian distribution of known Weibull modulus. Of these 1000

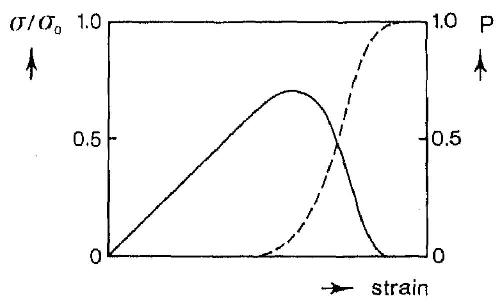

FIG. 13—*Calculated bundel stress-strain curve for a linear elastic material* (solid line) *and the cumulative failure probability* (dashed line) *during testing of the bundle* (m = 10).

filament strength values subsets of various sizes (n) were selected at random and the apparent Weibull modulus was determined for each subset. The results are shown in Figs. 14a, 14b, and 14c for $m = 5$, $m = 10$, and $m = 15$ respectively. The figures show that in particular for small subset sizes there is a rather wide range of "subset Weibull modulus" values. For example, take the case in which the "true" Weibull modulus for the total distribution is 5. Figure 14a shows that for a subset of 20 strength measurements there is about 5% chance of determining a Weibull modulus less than 3.7, 50% chance of determining a Weibull modulus less than 4.6, and about 5% chance of determining a Weibull modulus in excess of 6.8. More detailed analyses of the effects of sample size on the accuracy of m and σ_0 can be found elsewhere [*4,28-31*].

Is the Weibull Modulus a Material Constant?

From the foregoing discussion it is clear that the Weibull modulus is an important parameter as far as the description of filament strength is concerned. It is demonstrated here that m is not a material constant.

This follows directly from Fig. 15, which shows the Weibull curves for three grades of optical silica filaments (data from [*9*]). Fiber composition and testing conditions were identical; the filaments only differed in the production route followed. Figure 15 shows that m can vary for a particular material from ~3 to ~100. The data suggest that a high Weibull modulus corresponds to a high σ_0 value. Although this is not necessarily the case, it has often been observed for brittle fibres [*9-13*]. A high Weibull modulus indicates limited scatter in the fracture stress data. In general this means that weak filaments are absent. Consequently the average fracture stress increases and hence so does σ_0. It follows that if the average fracture stress approaches the theoretical strength, then the Weibull modulus should become very high (~100). For an isotropic material the theoretical strength is about 10% of Young's modulus. This yields a theoretical strength for silica filaments of 6.3 GN m^{-2}. The experimental average fracture stress of 6.0 GN m^{-2} for the filaments with $m = 98$ supports the suggestion that these filaments have indeed reached the maximum attainable strength for silica filaments.

A second example is shown in Fig. 16, which shows the Weibull plots for two rayon-based filaments. Once again it is clear that the Weibull modulus is not a material constant. Figure 16 also shows that a high Weibull modulus does not automatically mean that the

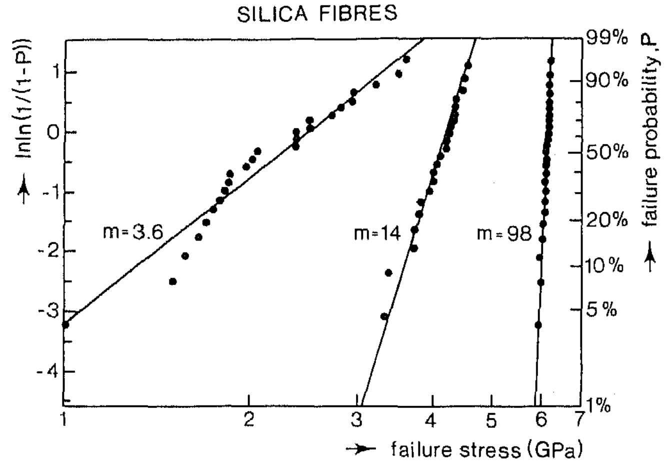

FIG. 15—*Weibull plot of fracture stress data for three grades of silica fibers; gage length is 25 mm* [9].

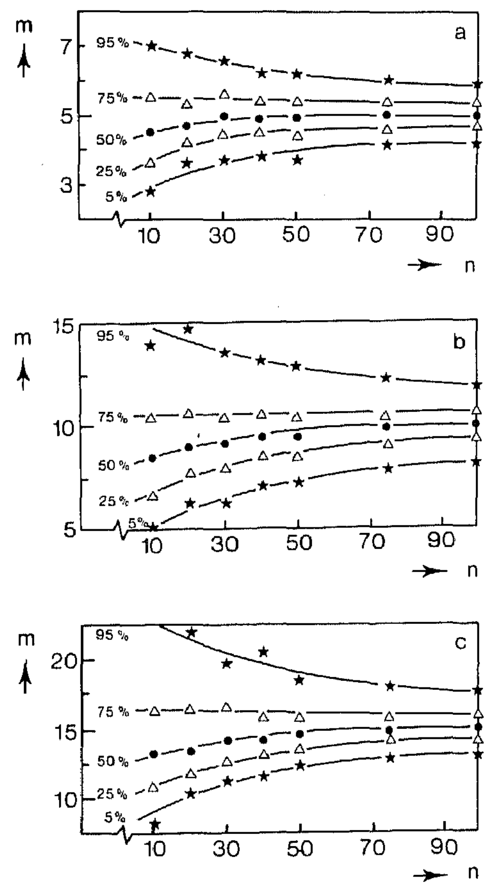

FIG. 14—*Weibull modulus versus number of samples tested* (a: m_{true} = 5; b: m_{true} = 10; c: m_{true} = 15).

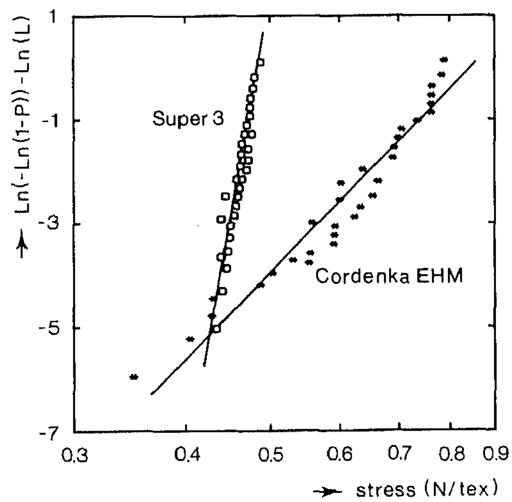

FIG. 16—*Weibull plot of fracture stress data for two types of cellulose fibers. Super 3 is a low-modulus fiber. Cordenka EHM is a high-modulus fiber.*

theoretical, and hence maximum, strength of a material is reached. The high Weibull modulus for Rayon Super 3 filaments is due to the rather plastic mode of deformation at high strains which yields $d\sigma/d\epsilon \approx 0$ for $\epsilon > 10\%$. In this rather plastic material the local defects in the filament give rise to a range of fracture strains rather than stresses. The fracture stress is more or less constant and equal to the yield stress of the material.

In the case of filaments having the theoretical strength both the Weibull moduli for fracture stress and for fracture strain should be high (≥ 100). Just using a criterion of a high Weibull modulus for the strength data [*14,15*] is clearly insufficient.

It should be stressed again that the Weibull theory remains a statistical theory, which is not based on detailed physical models for the fracture process. Therefore great care should be taken in using Weibull plots to draw conclusions on the physics or mechanics of the fracture process. Only if additional information is available can one use Weibull plots to support (not prove) new fracture models.

Weibull Modulus and Fracture Toughness

Several publications have appeared in which a relation between the Weibull modulus and the fracture toughness of a material was suggested [*16,17*]. The general notion is that structural reliability increases with increasing toughness. It has been argued by Kendall et al. [*18*] that such a relation between m and K_{Ic} is fortuitous: If a material behaves in a brittle, Griffith fashion then

$$\sigma_f = K_{Ic}/Y\sqrt{c} \qquad (16)$$

where σ_f is the fracture stress, K_{Ic} is the critical stress intensity factor, Y is a geometrical parameter, and c is the characteristic dimension of the crack-initiating defect. The maximum fracture strength in a set of fracture strength data therefore corresponds to a minimum defect size c_{min} and vice versa. Now imagine that the toughness K_{Ic} is increased while the material still follows the Griffith equation (16). If the flaw sizes remain the same, then the measured strengths σ_{max} and σ_{min} for the same series of tests will both increase by a factor K_{Ic2}/K_{Ic1}. Since for a normal set of Weibull data from q samples (from Eqs 6 and 7):

$$m = \frac{\ln(\ln(q+1)) - \ln(\ln(1+1/q))}{\ln(\sigma_{max}/\sigma_{min})} \qquad (17)$$

it is clear that m is independent of K_{Ic}.

This prediction has been confirmed experimentally [*18*] from strength data on ceramic "green bodies" (i.e., unfired but dried specimens) and on similar specimens after firing. Notwithstanding a 50-fold increase in K_{Ic} the Weibull modulus remained constant at $m = 7$. The change in K_{Ic} results in a horizontal shift of the line in the Weibull plot. The magnitude of the shift is $\ln(K_{Ic2}/K_{Ic1})$.

The above argument of a K_{Ic}-independent Weibull modulus is only valid if the tougher material still fails in a truly brittle manner. If, on the other hand, the increase in material toughness is obtained by incorporating a (localized) plasticizing process, the fracture toughness is no longer constant but depends on the crack length. In this case defects with dimensions between c_{min} and c_{max} cause smaller differences in fracture stress than in a Griffith-type material (Fig. 17). Such a change to a more Dugdale type of behavior is even more advantageous than a straightforward increase in

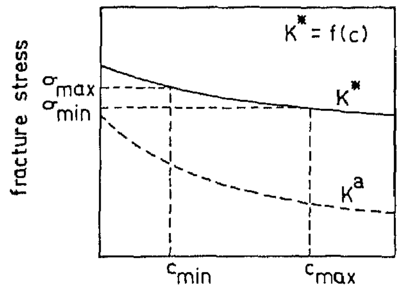

FIG. 17—*Effect of (localized) plasticity on the variation in fracture stress for a given range of defects (Curve K*); Curve K^a is for a brittle material.*

K_{Ic}. Not only does this increase the average fracture stress but the Weibull modulus is also higher. This mechanism was shown to operate in zirconia-toughened alumina [*19*]. In this material local phase transformation of the undercooled zirconia phase around the crack tip leads to crack arrest phenomena and quasi-ductile behavior. This gives a material with a high Weibull modulus ($m \approx 20$). Above the equilibrium phase transition temperature the toughening mechanism does not operate and the Weibull modulus drops to the value for the alumina reference specimens ($m \approx 10$). However, such a mechanism might be hard to realize in a filament because of its small diameter. On the other hand, the variation of m with testing conditions (such as loading rate, temperature, and environment) might give some qualitative information on changes in the fracture process.

Weibull Modulus and Defect Size Distribution

In the previous section it was shown that the variability in fracture stress is due to the distribution of defect sizes. In this respect it should be pointed out that the defect size should be regarded as an effective or equivalent size rather than an absolute dimension. Detailed TEM studies on carbon filaments [*20*] have shown that the orientation of the defect with respect to the graphite planes affects the effective size rather strongly. For other filament materials the position of the defect within the filament might also be important, with defects in the center being less damaging than surface defects.

The relation between the fracture strength distribution and the defect size distribution for isotropic brittle materials has been studied by several authors [*21–26*]. An experimentally observed defect size distribution function is given by [*27*]

$$f(c) = \frac{b^{n-1}}{(n-2)!} c^{-n} e^{-b/c} \qquad (18)$$

where c is the semi-crack size, b is a scaling parameter, and n is a parameter determining the shape of the distribution. The relation between the shape parameter n and the Weibull modulus m then becomes [*21*]

$$m \approx 2n - 2 \qquad (19)$$

Since the failure criteria in highly anisotropic materials are not yet well-formulated, the above relations can not be applied in their present form to the strength data on polymeric filaments. Studies on this problem are currently in progress.

Conclusions

It is shown that the Weibull theory provides a useful description of the intrinsic statistical variation in the fracture stress of filaments from high-performance materials. The Weibull modulus in this theory is an important parameter which controls among others the length dependence of the average fracture stress and the bundle efficiency. The Weibull modulus is not a material constant but reflects the shape of the defect population present in the material. Some relations between the Weibull modulus and the fracture toughness as well as the theoretical strength are discussed.

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