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The conformal transformation on a space with parallel Ricci tensor.

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Introduction

It is known that the conformal transformation group G of a space M with a Riemannian metric g coincides with the isometry group of M with some Riemannian metric g' conformally related to g, provided that the Weyl conformal tensor never vanishes on M [**4**]. Some of studies of conformal transformations, e.g., study of the group structure of conformal transformation groups, are therefore reduced to studies of isometries. So topics on conformal transformations may be limited to the case where M is conformally flat or to the relation between conformal transformations and properties of M which are not conformally invariant, e.g., the property to be symmetric, to which E. Cartan refered in his very first paper on "a remarkable class of Riemannian spaces". (It was proved in [**7**], [**11**] that a locally symmetric space does not admit an infinitesimal non-homothetic conformal transformation unless it is conformally flat.) In this paper we shall examine the relation between conformal transformations and the property that the Ricci tensor of M is parallel, and establish:

THEOREM. *Let g and g' be two complete Riemannian metrics on a manifold M ($2<\dim M=n$), such that the Ricci tensor of each of them is parallel. If g and g' are conformally related, they are homothetically related or (some connected component of) M with the metric g (and with g' also) is isometric to the sphere.*

Two Riemannian metrics g and g' on the same manifold are by definition *conformally* [*homothetically*] *related* if there exists a scalar ϕ on M such that $g'=\phi g$ [and ϕ is a constant]. ϕ is called the *associated function.*

COROLLARY 1. *Let M, $2<n$, be a complete connected Riemannian space whose Ricci tensor is parallel. Then, if M admits a conformal transformation, one of the three cases occurs:* 1) *it is an isometry,* 2) *it is homothetic and M is isometric to the euclidean space,* 3) *M is isometric to the sphere.*

COROLLARY 2. *A connected symmetric space does not admit a non-homothetic conformal transformation if it is not isometric to the sphere.*

As special cases of the theorem we already know the following three theorems which are necessary for the proof of Theorem and Corollary 1.

THEOREM Y. *Let M $(2<\dim M)$ be a complete connected Einstein space. If a conformal transformation of M belonging to the identity component of the conformal transformation group is not homothetic, M is then isometric to the sphere* (Yano-Nagano [**12**]).

THEOREM T. *Under the hypothesis of Theorem, assume that the Ricci tensor R of g satisfies the condition:* (S) *R has two eigenvalues e and f of multiplicity m and $n-m$ respectively with $f\leqq e$, $0\leqq m<n$ and $(n-m-1)e+(m-1)f=0$, and that an analogous condition for g' is satisfied except that the non-negative eigenvalue may be of multiplicity n. (The eigenvalues and the multiplicities may be different from those of R.) Then g and g' are homothetically related* (Tanaka [**7**]).

An eigenvalue e of multiplicity m of R is an eigenvalue of the (1,1)-type tensor $g^{ij}R_{jk}(p)$ whose eigenspace is of dimension m, p being a point of M. Since R is parallel, e and m are independent of p. It will hardly be necessary to explain the meaning of eigenvectors of R, the equality $m=0$, etc., only it should be understood that f is negative in case $m=0$ and that $f=e$ means $R=0$. The sphere corresponds to $m=n$, $0<e$.

THEOREM K. *If a complete connected Riemannian space admits a non-isometric homothetic transformation, it is isometric to the euclidean space* (Kobayashi [**4**] and Yano-Nagano [**13**]).

The hypothesis of Theorem will be preserved throughout this paper, M being of class C^∞. M will be assumed to be connected. The proof will be accomplished separately in three cases.

1. The first case.

In this paragraph we shall establish the theorem in case the Ricci tensor of any of the given metrics g and g' is not a positive number times the metric tensor. To that end we shall show that the Ricci tensors satisfy the condition (S) stated in Theorem T.

Since the Ricci tensor R is parallel we have [**6**, (6.7), p. 313]

$$\nabla_h C^h{}_{ijk}=0\,,$$

C being Weyl's conformal tensor. An analogous equation for g' holds good also. From these equations one can easily deduce

(1.1) $$(n-3)\,C^a{}_{ijk}\nabla_a\rho=0\,,$$

where ρ is defined by $\exp(2\rho)=\phi=$the associated function, using the relation [**6**, (5.2), p. 304] between the Levi-Civita connections of g and g' and the well known identities [**6**, (5.2) p. 306] satisfied by C. Since C is zero in case $n=3$, (1.1) always implies

(1.2) $$C^a{}_{ijk}\nabla_a\rho=0\,.$$

When M with the metric g is Einsteinian, the condition (S) is clearly satisfied. We should remember that the Ricci tensor R has been assumed not to be positive definite. Thus we assume that R has at least two different eigenvalues. Given an eigenvalue a of R, a distribution α is defined so that, $\alpha(p)$ is the eigenspace of $R(p)$ corresponding to a, p being an arbitrary point of M. Since R is parallel, α is well defined and $\alpha(p)$ is invariant under the homogeneous holonomy group. Therefore, given any two eigenvalues x and y of R, $x \neq y$, and two corresponding eigenvectors X and Y at a point, we find that

$$K^h{}_{ijk} X^i Y^k = 0\,, \tag{1.3}$$

where K is the Riemannian curvature tensor. Substituting (1.3) and the definition of C [**7**, (5.17), p. 306] into $C^a{}_{ijk} \nabla_a \rho \cdot X^i Y^k = 0$ derived from (1.2), we get $((n-1)(x+y)-r)(Y^h \nabla_h \rho) = 0$ at p, where r is the scalar curvature. When ρ is not constant on M, without loss of generality we may assume that $\nabla\rho(p) \neq 0$ at a point p and $Y^h \nabla_h \rho \neq 0$ at p, because the tangent space at p is spanned by the eigenvectors of the linear transformation $g^{ij}(p) R_{jk}(p)$. Then we obtain

$$(n-1)(x+y) = r\,. \tag{1.4}$$

In particular we find that R has just two different eigenvalues. Denoting by μ the multiplicity of x we have $\mu x + (n-\mu) y = r$. From (1.4) it follows that $(n-\mu-1)x + (\mu-1)y = 0$. Putting $e = \max(x, y)$ and $f = \min(x, y)$, one will readily find (S) satisfied by R, and in the same way by R'. Thus Theorem T can be applied and allows us to conclude that g and g' are homothetically related.

2. The second case.

This paragraph is devoted to the demonstration of the theorem in case the Ricci tensor of g is a positive multiple of g and M with g' is also Einsteinian. M is then compact by Myers' and Ambrose's theorem [**1**]. Putting the associated function $\phi = \exp(2\rho)$, we have [**6**, (5.8), p. 305]

$$R'_{ij} = R_{ij} - (n-2) S_{ij} - \lambda g_{ij}\,, \tag{2.1}$$

where

$$S_{ij} = \nabla_i \nabla_j \rho - \nabla_i \rho \nabla_j \rho \tag{2.2}$$

and

$$\lambda = \Delta\rho + g^{ab} \nabla_a \rho \nabla_b \rho\,, \tag{2.3}$$

Δ being the Laplacian, i. e.

$$\Delta\rho = g^{ab} \nabla_a \nabla_b \rho\,.$$

R and R' being scalar multiples of the metric tensor g, so is S. Hence the vector field u defined by $u_i = \nabla_i \exp(-\rho)$ is an infinitesimal conformal transformation [**13**] on M. Since M is compact, u generates a one-parameter group of conformal transformations. (Brinkmann seems to know this fact; he proved [**2**] that g' is obtained from g by some conformal transformation of M.) It cannot contain a non-isometric homothetic transformation by Theorem K. Hence M is isometric to the sphere by Theorem Y, unless u is a Killing vector. If u is a Killing vector it is parallel by the definition of u. Since the Ricci tensor is definite, u must then vanish, whence ρ is a constant. In other words g and g' are homothetically related.

3. The third case.

Now the demonstration of our main theorem is complete. But we shall here give another proof, which may be interesting, for the case where one of the Ricci tensors, say R' of g', is a positive multiple of g' and M with g is not Einsteinian. It suffices to show that g and g' are homothetically related under the assumption of simple-connectivity of M. M is compact, and is the Riemannian product of two Einstein spaces by de Rham's theorem [**5**] and the arguments in **2**. Let N be one of them whose scalar curvature is non-positive. We consider N as a subspace of M. In particular N is compact and simply connected. The injection of N into M induces a 1-form w from the 1-form$=d(\exp(-\rho))$ on M, and a Riemannian metric from g, which we shall denote by g also. Let v be the dual vector of w, i. e. $v^\alpha = g^{\alpha\beta} w_\beta$. As in **2**, v is easily seen to be an infinitesimal conformal transformation:

$$\pounds_v g = 2\nu g,$$

ν being a scalar. Since the scalar curvature s of N is constant, we have [**9**, (3.5), p. 160]

$$(\delta-1)\, g^{\alpha\beta} \nabla_\alpha \nabla_\beta \nu = -\nu s, \qquad \delta = \dim N.$$

δ is greater than 1, because N is compact and simply connected. Therefore we find

$$\Delta \nu^2 = 2g^{\alpha\beta}(\nu_\alpha \nu_\beta - s\nu^2/(\delta-1)), \qquad (\nu_\alpha = \nabla_\alpha \nu),$$

is non-negative. By Bochner's lemma [**10**], ν^2 is constant, which means v is homothetic and so isometric. (Kurita and Yano [**9**, p. 279] found this fact for the case $2<\delta$.) Since s is non-positive, v must be parallel by Bochner's theorem [**10**]. Therefore v is a harmonic vector field, which must thus vanish because N is simply connected. From (2.1), we thus obtain

$$R'_{ij} X^i X^j = R_{ij} X^i X^j - \lambda g_{ij} X^i X^j$$

for any vector X parallel to N; i. e., given any curve r joining the origin of X to a point of N, the parallelism of X along r carries X to a vector tangent to N. If $X \neq 0$, the left hand side is positive, while the first them in the right hand side is negative. So λ must be negative on M. Hence, by (2.3),

$\Delta\rho$ is negative on M, from which we conclude that ρ is constant on M by means of Bochner's lemma.

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Bibliography

[1] W. Ambrose, A theorem of Myers, Duke Math. J., **24** (1957), 345-351.
[2] H. W. Brinkmann, Einstein spaces which are mapped conformally on each other, Math. Ann., **94** (1925), 119-145.
[3] S. Kobayashi, A theorem on the affine transformation group of a Riemannian manifold, Nagoya Math. J., **9** (1955), 39-41.
[4] T. Nagano, On conformal transformations of Riemannian spaces, J. Math. Soc. Japan, **10** (1958), 79-93.
[5] G. de Rham, Sur la réductibilité d'un espace de Riemann, Comment. Math. Helv., **26** (1952), 328-344.
[6] J. A. Schouten, Ricci-Calculus, Springer, 1954.
[7] T. Sumitomo, On some transformations of Riemannian spaces, Tensor, **6** (1956), 136-140.
[8] T. Tanaka, Conformal connections and conformal transformations, to appear in Trans. Amer. Math. Soc.
[9] K. Yano, The theory of Lie derivatives and its applications, North-Holland Publ. Co., 1957.
[10] K. Yano and S. Bochner, Curvature and Betti numbers, Ann. of Math. Studies, **32** (1953).
[11] K. Yano and T. Nagano, Some theorems on projective and conformal transformations, Indag. Math., **19** (1957), 452-458.
[12] K. Yano and T. Nagano, Einstein spaces admitting a one-parameter group of conformal transformations, to appear in Ann. of Math.
[13] K. Yano and T. Nagano, The de Rham decomposition, isometries and affine transformations in Riemannian spaces, to appear.