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THE CONJUGACY PROBLEM FOR GRAPH PRODUCTS WITH CENTRAL CYCLIC EDGE GROUPS

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ABSTRACT. A graph product is the fundamental group of a graph of groups. Amongst the simplest examples are HNN extensions and free products with amalgamation.

Graph products with cyclic edge groups inherit a solvable conjugacy problem from their vertex groups under certain conditions, the most important of which imposed here is that all the edge group generators in each vertex group are powers of a common central element. Under these conditions the conjugacy problem is solvable for any two elements not both of zero reduced length in the graph product, and for arbitrary pairs of elements in HNN extensions, tree products and many graph products over finite-leaf roses. The conjugacy problem is not solvable in general for elements of zero reduced length in graph products over graphs with infinitely many circuits.

1. Introduction. A solvable conjugacy problem (S.C.P.) is generally not inherited by graph products of groups with S.C.P. (see Miller [**8**]). If attention is restricted to graph products with cyclic edge groups, more may be said. It is unlikely that this restriction can be lifted (cf. [**6**, p. 387; **7**, p. 114]). Finite groups, finitely generated free groups, finitely generated nilpotent groups, one-relator groups with torsion or nontrivial centre and certain small cancellation groups all have S.C.P., so there is a wealth of potential vertex groups which may be used in constructing such graph products.

In [**4**] the author shows that a recursively presented graph product with cyclic edge groups over a finite graph inherits S.C.P. from its vertex groups if the sets of cyclic generators in them are "semicritical", thus generalising [**5**, **7**] for HNN extensions and free products with amalgamation. However, semicriticality is a very restrictive condition, which does not hold if all the cyclic generators in a vertex group are powers of a common element. Such cases occur often enough: the celebrated Baumslag-Solitar non-Hopfian groups fall in this category.

Here this complementary case is considered. Not surprisingly, a further condition is imposed on the sets of cyclic generators: that they are central in their respective vertex groups. This is suggested by the direct proof that the Baumslag-Solitar groups have S.C.P., and by [**2**, §3]. It is comparatively straightforward to show that the conjugacy problem is solvable for any two elements in the graph product of which at least one has nonzero reduced length (Theorem 3.1). For elements of zero reduced length the problem is much more difficult, reducing to the question of whether a specific recursively enumerable (r.e.) set is recursive (Theorem 3.3).

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This condition holds for graph products over trees and for many over finite-leaf roses (Corollary 3.6), but fails in general for graphs with infinitely many circuits.

The reader is referred to [**9**] for the theory of computability and to [**10**] for the theory of graph products.

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2. Graph products. The following notation is based on [**10**, §5], but graphs with single, rather than double, edges are used. A *graph of groups* (G, D) consists of a nonempty connected directed graph $D = (E, V)$ where the edge set E and vertex set V are disjoint, with source and terminus maps $s, t\colon E \to V$, respectively; a vertex group G_v for every v in V; an edge group G_e for every e in E; and a pair of group monomorphisms $A_e\colon G_e \rightarrowtail G_{te}$ (denoted $g \mapsto g^e$ and $A_{\bar{e}}\colon G_e \rightarrowtail G_{se}$ (denoted $g \mapsto g^{\bar{e}}$). The fundamental groupoid of (G, D) will be denoted $\mathcal{F}$ and the graph product (fundamental group) will be denoted G^*.

The conjugacy problem for G^* is considered here in terms of the more general conjugacy problem for $\mathcal{F}$. For a discussion of the word and conjugacy problems for groupoids see [**3**, **4**]. Note that there the graph product is defined in terms of a slightly different graph, but after accounting for this the results of [**3**, §3; **4**, §2] translate directly.

DEFINITION 2.1. (a) The graph D is *recursive* if

(i) E and V are recursive sets.

(ii) $s, t\colon E \to V$ are partially recursive (with domain E).

(b) The graph of groups (G, D) is *recursively presented* if

(i) D is recursive.

(ii) G_v has a recursive presentation $\forall v$, uniformly given from V.

(iii) G_e has a recursive presentation $\forall e$, uniformly given from E.

(iv) A_e and $A_{\bar{e}}$ are partially recursive $\forall e$, uniformly given from E. □

If (G, D) is recursively presented it follows that $\mathcal{F}$ is a recursively presented groupoid [**3**, 3.5] and there is an algorithm to decide whether or not an arbitrary element of $\mathcal{F}$ is a loop. The conjugacy problem for $\mathcal{F}$ then reduces to the question of whether there is an algorithm to determine whether or not an arbitrary pair of loops in $\mathcal{F}$ are conjugate in $\mathcal{F}$.

Hereafter it will be assumed that $\mathcal{F}$ is (presented as) the fundamental groupoid of a recursively presented graph of groups (G, D) such that each edge group G_e is presented as a cyclic group on a single generator k_e.

DEFINITION 2.2. For each v in V, define $H_v \subseteq G_v$ to be

$$H_v = \{k_e^e \colon te = v\} \cup \{k_e^{\bar{e}} \colon se = v\}. \quad \square$$

The following conditions will also be imposed on $\mathcal{F}$.

CONDITIONS 2.3. (i) H_v is recursive $\forall v$, uniformly given from V.

(ii) G_v has S.C.P. $\forall v$, uniformly given from V.

(iii) There exists (known) $c_v \forall v$, uniformly given from V, such that $H_v \subseteq \langle c_v \rangle \subseteq \varsigma(G_v)$, the centre of G_v.

(iv) $\langle c_v \rangle$ has solvable extended word problem (S.E.W.P.) in $G_v \forall v$, uniformly given from V. □

These conditions imply that the order $O(c_v)$ of c_v for each v, and the powers k_e^e is of c_{te} and $k_e^{\bar{e}}$ is of c_{se} can all be calculated.

DEFINITION 2.4. If $e \in E$, let $\bar{e}$ represent the directed edge e traversed in the opposite direction (so $\bar{\bar{e}} = e$) and define $p_e = q_{\bar{e}}$ and $q_e = p_{\bar{e}}$ by $k_e^{\bar{e}} = c_{se}^{p_e}$ and $k_e^{\bar{e}} = c_{te}^{q_e}$. (If $e = e_i$ for some index i then p_e will be written p_i, and so on.) □

Under 2.3, the E.W.P. for $\langle k_e^e \rangle$ in G_{te} and $\langle k_e^{\bar{e}} \rangle$ in G_{se} is uniformly solvable from E, so the word problem for $\mathcal{F}$ is solvable [**3**, 3.6], and the process of finding a cyclically reduced loop conjugate to a given loop in $\mathcal{F}$ is algorithmic.

3. Conjugacy in fundamental groupoids.

THEOREM 3.1. *Under* 2.3 *the conjugacy problem is solvable for any two elements of* $\mathcal{F}$ *(and hence of* G^**) of which at least one has nonzero reduced length.*

PROOF. By [**3**, 2.6] it is necessary only to consider distinct pairs of nontrivial cyclically reduced loops $g = a_1 e_1 \cdots a_n e_n$ and $h = b_1 e_1 \cdots b_n e_n$ where $n \geq 1$, $e_i \in \{e, e^{-1}: e \in E\}$ and a_i and b_i lie in the same vertex group G_i (say). Let k be the minimum positive integer such that $n = qk$ and the sequence $e_1, \ldots, e_n$ is the sequence $e_1, \ldots, e_k$ repeated q times. Then, because the H_i are central, $g \sim_{\mathcal{F}} h$ if and only if there exist j, $1 \leq j \leq q$, and an integral solution $(r_1, \ldots, r_n)$ to the system of equations

$$(*) \qquad b_{l+i} a_i^{-1} = y_{i-1}^{r_{i-1}} x_i^{-r_i} \quad \text{in } G_i, \ 1 \leq i \leq n,$$

where $l = jk$, the subscripts are taken modulo n, and $x_i = k_e^{\bar{e}}(k_e^e)$ and $y_i = k_e^e(k_e^{\bar{e}})$ if $e_i = e(e^{-1})$. Each left-hand term in $(*)$ may be tested by 2.3(iv) in turn to see if it lies in the required cyclic group; if one of them does not, $g \not\sim_{\mathcal{F}} h$. Otherwise, calculate successively the integers t_i with $|t_i|$ a minimum, for which the ith left-hand term equals $c_i^{t_i}$. Set $s_i = p_i(q_i)$ if $x_i = k_e^{\bar{e}}(k_e^e)$ and $u_i = q_i(p_i)$ if $y_i = k_e^e(k_e^{\bar{e}})$. Then $g \sim_{\mathcal{F}} h$ if and only if the set of simultaneous linear congruences

$$\begin{aligned} t_1 &\equiv u_n r_n - s_1 r_1 \mod O(c_1), \\ t_i &\equiv u_{i-1} r_{i-1} - s_i r_i \mod O(c_i), \ 2 \leq i \leq n \end{aligned}$$

(setting $O(c_i) = 0$ if c_i has infinite order) has an integral solution $(r_1, \ldots, r_n)$. This is decidable. □

The conjugacy problem for elements of zero reduced length is not so tractable. Suppose g in G_v and h in G_w are of zero reduced length and, if $v = w$, $g \not\sim_{G_v} h$. Then $g \sim_{\mathcal{F}} h$ if and only if there exist an edge-sequence $\hat{e} = e_1, \ldots, e_n$ in D with $e_i \in \{e, \bar{e}: e \in E\}$, $te_i = se_{i+1} = v_{i+1}$ (say), $1 \leq i \leq n$, and integers $(r_1, \ldots, r_n)$ such that $g = x_1^{r_1}$, $y_{i-1}^{r_{i-1}} = x_i^{r_i}$, $2 \leq i \leq n$, and $y_n^{r_n} = h$, where $x_i = k_e^{\bar{e}}(k_e^e)$ and $y_i = k_e^e(k_e^{\bar{e}})$ if $e_i = e(\bar{e})$. If $g \notin \langle c_1 \rangle$ or $h \notin \langle c_{n+1} \rangle$ then $g \not\sim_{\mathcal{F}} h$. Therefore, suppose $g = c_1^{\alpha}$ and $h = c_{n+1}^{\beta}$ for known α and β. Thus $g \sim_{\mathcal{F}} h$ if and only if there exist an edge-sequence $\hat{e}$ and an integral solution $(r_1, \ldots, r_n)$ to the simultaneous linear congruences

$$(**) \qquad \begin{cases} \alpha \equiv p_1 r_1 \mod O(c_1), \\ q_{i-1} r_{i-1} \equiv p_i r_i \mod O(c_i), 2 \leq i \leq n, \\ q_n r_n \equiv \beta \mod O(c_{n+1}), \end{cases}$$

where $O(c_i^{p_i}) = O(c_{i+1}^{q_i})$ always. Such an edge-sequence may be assumed irreducible ($e_{i+1} \neq \bar{e}_i$, $1 \leq i \leq n-1$) without loss of generality.

Investigation of $(**)$ requires further notation. The greatest common divisor of integers $n_1, \ldots, n_k$ will be denoted $[n_1, \ldots, n_k]$.

DEFINITION 3.2. Let $\mathcal{U}(v,w)$ be the set of irreducible undirected edge-sequences from v to w in D. Let $\mathbf{Z}$ denote the integers and $\mathbf{Z}_m$ the integers modulo m. Define $\mathcal{P}(v,w)$ on $\hat{e} = e_1, \ldots, e_n$ in $\mathcal{U}(v,w)$ to be

(i) $\mathcal{P}(v,w)(\hat{e}) = (p_1 G_1(\hat{e})/G(\hat{e}),\ q_n G_n(\hat{e})/G(\hat{e})) \in \mathbf{Z} \times \mathbf{Z}$ where $G_i(\hat{e}) = \prod_{j=2}^{i} q_{j-1} \prod_{j=i+1}^{n} p_j,\ 1 \le i \le n$, and $G(\hat{e}) = [G_1(\hat{e}), \ldots, G_n(\hat{e})]$, if $O(k_i) = 0,\ 1 \le i \le n$,

(ii) $\mathcal{P}(v,w)(\hat{e}) = \prod_{j=1}^{n} q_j^* x_j \bmod m \in Z_m$ where $m = [m_1, \ldots, m_n]$, $p_i = p_i^* O(c_i)/m_i$, $q_i = q_i^* O(c_{i+1})/m_i$, $[p_i^*, m_i] = [q_i^*, m_i] = 1$ and x_i is a fixed solution of $p_i^* x_i \equiv 1 \bmod m_i,\ 1 \le i \le n$, if $O(k_i) = m_i \ge 2,\ 1 \le i \le n$,

(iii) $\mathcal{P}(v,w)(\hat{e}) = (0,0) \in \mathbf{Z}_{O(c_1)} \times \mathbf{Z}_{O(c_{n+1})}$, otherwise. □

Since E is recursive, so is $\mathcal{U}(v,w)$ for each pair (v,w), and $\mathcal{P}(v,w)$ is partially recursive. Hence $\operatorname{Im} \mathcal{P}(v,w)$ is r.e. but not necessarily recursive.

THEOREM 3.3. *Under* 2.3 *the conjugacy problem for* $\mathcal{F}$ (*and hence* G^*) *is solvable if and only if* $\operatorname{Im} \mathcal{P}(v,w)$ *is uniformly recursive for all* v, w *in* V.

PROOF. If for some $1 \le i \le n$, $p_i \equiv 0 \bmod O(c_i)$, the general solution to $(**)$ is given by $\alpha \equiv 0 \bmod O(c_1)$, $\beta \equiv 0 \bmod O(c_{n+1})$, and the identities of G_1 and G_{n+1} are conjugate by *any* edge-sequence joining v_1 to v_{n+1}. If $O(k_j) = 0$ and $O(k_l) \ge 2$ for some e_j, e_l in $\hat{e}$ then $O(k_i) = 1$ for some e_i in $\hat{e}$, hence $p_i \equiv 0 \bmod O(c_i)$.

If $O(k_i) = 0$ so $p_i \ne 0,\ 1 \le i \le n$, the general solution to equations $(2-n)$ in $(**)$ is given by

$$r_i = tG_i(\hat{e})/G(\hat{e}), \qquad 1 \le i \le n,\ t \in \mathbf{Z},$$

so for $g = c_v^\alpha$ and $h = c_w^\beta$ it follows that $g \sim_{\mathcal{F}} h$ if and only if there exists some $\hat{e}$ in $\mathcal{U}(v,w)$ such that $(\alpha, \beta) = (tp_1 G_1(\hat{e})/G(\hat{e}), tq_n G_n(\hat{e})/G(\hat{e}))$ for some t in $\mathbf{Z}$; that is, if and only if there exists a divisor t of $[\alpha, \beta]$ such that $(\alpha/t, \beta/t) \in \operatorname{Im} \mathcal{P}(v,w)$.

If $O(k_i) = m_i \ge 2$ so $p_i \not\equiv 0 \bmod O(c_i),\ 1 \le i \le n$, the general solution to $(**)$ is given by

$$r_i \equiv t(m_i/m) \left(\prod_{j=2}^{i} q_{j-1}^* x_j \right) x_1 \mod m_i, \qquad 1 \le i \le n,$$

where $\alpha = tO(c_1)/m$, $\beta = sO(c_{n+1})/m$ and $s \equiv t(\prod_{j=1}^{n} q_j^* x_j) \bmod m$. Thus $g \sim_{\mathcal{F}} h$ if and only if there exists some $\hat{e}$ in $\mathcal{U}(v,w)$ such that $\alpha = tO(c_v)/m$, $\beta = sO(c_w)/m$ and $s \equiv tx \bmod m$, where $x \in \mathcal{P}(v,w)(\hat{e})$. □

Whether $\operatorname{Im} \mathcal{P}(v,w)$ is recursive or not depends on D but more importantly on the exponent set $\{p_e, q_e \colon e \in E\}$. The following example shows that for every D there exists at least one class of graph products over D satisfying 2.3 with S.C.P.

COROLLARY 3.4. *Let* D *be a recursive graph, let* $G_e = G_v = \mathbf{Z}$ *for all* e *in* E *and* v *in* V, *and let* A_e *and* $A_{\bar{e}}$ *be multiplication by a fixed integer* $m \ne 0$ *for every* e. *Then* $\mathcal{F}$ (*and hence* G^*) *has S.C.P.*

PROOF. $\operatorname{Im} \mathcal{P}(v,w) = \{(m,m)\}$ for every (v,w) so is uniformly recursive. □

However, the more that is known about the edge-sequences of D, the more likely it is that a decision on the recursiveness of $\operatorname{Im} \mathcal{P}(v,w)$ can be made.

COROLLARY 3.5. *If* D *is a tree and* 2.3 *is satisfied then* $\mathcal{F}$ (*and hence* G^*) *has S.C.P.*

PROOF. For any vertices v, w in D, $\mathcal{U}(v,w)$ consists of the unique undirected arc from v to w in D, and since E is recursive, this arc may be found uniformly from v and w. $\square$

COROLLARY 3.6. *Under* 2.3 *if D is a finite-leaf rose (with $V = \{0\}$, say), and provided that $O(c_0) = 0$ implies $[p_j q_j, p_k q_k] = 1$ for every pair of edges $e_j \neq e_k$ in E, then $\mathcal{F}$ (and hence G^*) has S.C.P.*

PROOF. If either $O(c_0) \geq 2$ or $O(c_0) = 0$ and $[p_j q_j, p_k q_k] = 1$ for every pair of edges $e_j \neq e_k$ in E, then any edge-sequence $\hat{e}$ in $\mathcal{U}(v,w)$ may be replaced in (**) by the minimal edge-sequence with the same exponent sum on each edge-pair $\{e, \bar{e}\}$ appearing in $\hat{e}$. If D has edges $e_1, \ldots, e_n$ and $O(c_0) = 0$, then from 3.3 $g \sim_{\mathcal{F}} h$ if and only if $(\alpha/[\alpha,\beta], \beta/[\alpha,\beta]) = (\prod_{i=1}^n P_i^{\sigma_i}, \prod_{i=1}^n Q_i^{\sigma_i})$, where $\sigma_i \geq 0$ and $P_i = p_i/[p_i, q_i]$ $(q_i/[p_i, q_i])$ when $Q_i = q_i/[p_i, q_i]$ $(p_i/[p_i, q_i])$. If $O(c_0) \geq 2$ then from 3.3, $g \sim_{\mathcal{F}} h$ if and only if $\alpha = tO(c_0)/m$, $\beta = sO(c_0)/m$ and $s \equiv t \prod_{j=1}^l (R_{i_j} Z_{i_j})^{\sigma_j} \bmod m$, where $\sigma_j \geq 0$, for some $m = [m_{i_1}, \ldots, m_{i_l}]$ and $R_{i_j} = p_k^*$ (q_k^*) when $Z_{i_j} = x_k$ (y_k) and $p_k^* x_k \equiv 1 \bmod m_k$ $(q_k^* y_k \equiv 1 \bmod m_k)$ if $e_{i_j} = e_k$ $(\bar{e}_k)$. Both these conditions are decidable for known finite n. $\square$

The case of the finite-leaf rose under differing conditions has been investigated by Anshel [**1**].

If D is a one-leaf rose (i.e. G^* is the HNN extension $\langle G, e \colon \operatorname{rel} G,\ e^{-1} c^p e = c^q \rangle$ where G is a recursively presented group with S.C.P., $c \in \varsigma(G)$ and $\langle c \rangle$ has S.E.W.P. in G) then the relative primitivity condition of 3.6 is vacuous and G^* has S.C.P. These HNN extensions include the Baumslag-Solitar non-Hopfian groups, so they do not, in general, inherit conjugacy separability from G. They should be compared with other HNN extensions with S.C.P. (e.g. in [**2**, **5**, **6**]).

Corollary 3.5 may not be extended to graphs in general.

COROLLARY 3.7. *There exists an infinite graph satisfying the conditions of* 2.3 *which has unsolvable C.P. for elements of zero reduced length.*

PROOF. Let D be the graph with $E = \mathbf{Z}^+ = \{n \in \mathbf{Z} \colon n \geq 1\}; V = \mathbf{Z}^+ \cup \{0\}$; $t(n) = 0, \forall n \geq 1; s(2n) = s(2n-1) = n, \forall n \geq 1$. Set $G_e = G_v = \mathbf{Z}, \forall v \in v$, $e \in E$; let $\pi \colon \mathbf{Z} \to \mathbf{Z}$ be the recursive function $\pi(i) = p_i$, the ith prime, and let $\psi \colon \mathbf{Z} \to \mathbf{Z}$ be a one-to-one recursive function such that $\operatorname{Im} \psi$ is r.e. but not recursive [**9**, 5.2.V(a)]. Define $A_n(1) = 1$, $A_{\overline{n}}(1) = \pi \circ \psi(n)$, $\forall n \geq 1$ and $c_v = 1\ \forall v$. From 3.3, for g and h in G_0, $g \sim_{\mathcal{F}} h$ iff there exist distinct integers $i_1, \ldots, i_n$ such that $(\alpha/[\alpha,\beta], \beta/[\alpha,\beta]) = (\prod_{j=1}^n p_{\psi(k_j)}^{\sigma_j}, \prod_{j=1}^n p_{\psi(k_j^*)}^{\sigma_j})$ where $k_j = 2i_j\{2i_j - 1\}$ when $k_j^* = (2i_j - 1)\{2i_j\}$, and $\sigma_j > 0$, $1 \leq j \leq n$. If the prime decomposition of $\alpha/[\alpha,\beta]$ is $p_{l_1}^{t_1} \ldots p_{l_k}^{t_k}$ then $g \sim_{\mathcal{F}} h$ only if $\{p_{l_1}, \ldots, p_{l_k}\} \subseteq \operatorname{Im} \pi \circ \psi$; i.e. only if $\{l_1, \ldots, l_k\} \subseteq \operatorname{Im} \psi$, which is undecidable. $\square$

By 3.4 it is clear that the existence of countably many distinct circuits in a graph is not sufficient to prevent the graph product having S.C.P., though in view of 3.7 the C.P. is generally unsolvable for elements of zero reduced length. Each case should be tested separately. Finite graphs may be more amenable.

CONJECTURE. Under 2.3 if D is finite, $\mathcal{F}$ has S.C.P.

REFERENCES

1. M. Anshel, *Conjugate powers in HNN groups*, Proc. Amer. Math. Soc. **54** (1976), 19–23.
2. M. Anshel and P. Stebe, *The solvability of the conjugacy problem for certain HNN groups*, Bull. Amer. Math. Soc. **80** (1974), 266–270.
3. K. J. Horadam, *The word problem and related results for graph product groups*, Proc. Amer. Math. Soc. **82** (1981), 157–164.
4. ——, *The conjugacy problem for graph products with cyclic edge groups*, Proc. Amer. Math. Soc. **87** (1983), 379–385.
5. R. D. Hurwitz, *On cyclic subgroups and the conjugacy problem*, Proc. Amer. Math. Soc. **79** (1980), 1–8.
6. L. Larsen, *The conjugacy problem and cyclic HNN constructions*, J. Austral. Math. Soc. Ser. A **23** (1977), 385–401.
7. S. Lipschutz, *The conjugacy problem and cyclic amalgamations*, Bull. Amer. Math. Soc. **81** (1975), 114–116.
8. C. F. Miller III, *On group-theoretic decision problems and their classification*, Ann. of Math. Studies, no. 68, Princeton Univ. Press, Princeton, N.J., 1971.
9. H. Rogers, Jr. *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1968.
10. J.-P. Serre, *Trees* (translated by J. Stillwell), Springer-Verlag, Berlin, 1980.

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