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The connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$

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Abstract

We provide the first mathematical proof that the connective constant of the hexagonal lattice is equal to $\sqrt{2+\sqrt{2}}$. This value has been derived nonrigorously by B. Nienhuis in 1982, using Coulomb gas approach from theoretical physics. Our proof uses a parafermionic observable for the self-avoiding walk, which satisfies a half of the discrete Cauchy-Riemann relations. Establishing the other half of the relations (which conjecturally holds in the scaling limit) would also imply convergence of the self-avoiding walk to SLE(8/3).

1. Introduction

A famous chemist P. Flory [3] proposed to consider self-avoiding (i.e., visiting every vertex at most once) walks on a lattice as a model for spatial position of polymer chains. Self-avoiding walks turned out to be a very interesting mathematical object, leading to rich theories and challenging questions; see [9].

Denote by c_n the number of n-step self-avoiding walks on the hexagonal lattice $\mathbb{H}$ started from some fixed vertex, e.g., the origin. Elementary bounds on c_n (for instance $\sqrt{2}^n \le c_n \le 3\cdot 2^{n-1}$) guarantee that c_n grows exponentially fast. Since an $(n+m)$-step self-avoiding walk can be uniquely cut into an n-step self-avoiding walk and a parallel translation of an m-step self-avoiding walk, we infer that

$$c_{n+m} \le c_n c_m,$$

from which it follows that there exists $\mu \in (0,+\infty)$ such that

$$\mu := \lim_{n\to\infty} c_n^{\frac{1}{n}}.$$

The positive real number μ is called the *connective constant* of the hexagonal lattice.

Using Coulomb gas formalism, B. Nienhuis [10], [11] proposed physical arguments for μ to have the value $\sqrt{2+\sqrt{2}}$. We rigorously prove this statement.

While our methods are different from those applied by Nienhuis, they are similarly motivated by considerations of vertex operators in the $O(n)$ model.

THEOREM 1. *For the hexagonal lattice*,

$$\mu = \sqrt{2+\sqrt{2}}.$$

It will be convenient to consider walks between *mid-edges* of $\mathbb{H}$, i.e., centers of edges of $\mathbb{H}$ (the set of mid-edges will be denoted by H). We will write $\gamma : a \to E$ if a walk γ starts at a and ends at some mid-edge in $E \subset H$. In the case $E = \{b\}$, we simply write $\gamma : a \to b$. The *length* $\ell(\gamma)$ of the walk is the number of vertices visited by γ.

We will work with the partition function

$$Z(x) = \sum_{\gamma\,:\,a\to H} x^{\ell(\gamma)} \in (0, +\infty].$$

This sum does not depend on the choice of a, and is increasing in x. Establishing the identity $\mu = \sqrt{2+\sqrt{2}}$ is equivalent to showing that $Z(x) = +\infty$ for $x > 1/\sqrt{2+\sqrt{2}}$ and $Z(x) < +\infty$ for $x < 1/\sqrt{2+\sqrt{2}}$. To this end, we analyze walks restricted to bounded domains and weighted depending on their winding. The modified sum can be defined as a *parafermionic observable* arising from a disorder operator. Such observables exist for other models; see [5], [2], [13].

Let us mention that after this paper was circulated, its methods were used by Beaton, de Gier and Guttmann [1] to show that the critical fugacity for surface absorption of the self-avoiding walk on the hexagonal lattice is equal to $1+\sqrt{2}$. Their argument uses a generalization of our identity (2).

The paper is organized as follows. In Section 2, the parafermionic observable is introduced and its key property is derived. Section 3 contains the proof of Theorem 1. Section 4 discusses conformal invariance conjectures for self-avoiding walks. To simplify formulae, below we set $x_c := 1/\sqrt{2+\sqrt{2}}$ and $j = \mathrm{e}^{\mathrm{i}2\pi/3}$.

2. **Parafermionic observable**

A (hexagonal lattice) *domain* $\Omega \subset H$ is a union of all mid-edges emanating from a given collection of vertices $V(\Omega)$ (see Figure 1). A mid-edge z belongs to Ω if at least one end-point of its associated edge is in Ω; it belongs to $\partial\Omega$ if only one of them is in Ω. We further assume Ω to be simply connected, i.e., having a connected complement.

For a self-avoiding walk γ between mid-edges a and b (not necessarily the start and the end), we define its *winding* $\mathrm{W}_\gamma(a, b)$ as the total rotation of the direction in radians when γ is traversed from a to b; see Figure 1.

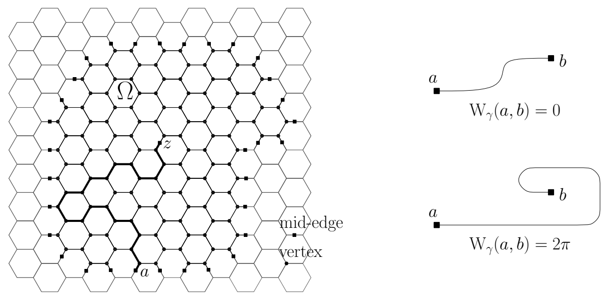

Figure 1. *Left.* A domain Ω with boundary mid-edges labeled by small black squares, and vertices of $V(\Omega)$ labeled by circles. *Right.* Winding of a curve γ.

Our main tool is given by the following

Definition 1. The *parafermionic observable* for $a \in \partial\Omega$, $z \in \Omega$, is defined by

$$F(z) = F_\Omega(a, z, x, \sigma) = \sum_{\gamma \subset \Omega:\ a \to z} \mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)} x^{\ell(\gamma)}.$$

LEMMA 1. *If $x = x_c$ and $\sigma = \frac{5}{8}$, then F satisfies the following relation for every vertex $v \in V(\Omega)$:*

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0, \tag{1}$$

where p, q, r are the mid-edges of the three edges adjacent to v.

Note that with $\sigma = 5/8$, the complex weight $\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)}$ can be interpreted as a product of terms λ or $\bar{\lambda}$ per left or right turn of γ drawn from a to z, with

$$\lambda = \exp\left(-\mathrm{i}\frac{5}{8} \cdot \frac{\pi}{3}\right) = \exp\left(-\mathrm{i}\frac{5\pi}{24}\right).$$

Proof. We start by choosing notation so that p, q and r follow counterclockwise around v. Note that the left-hand side of (1) can be expanded into the sum of contributions $c(\gamma)$ of all possible walks γ finishing at p, q or r. For instance, if a walk ends at the mid-edge p, its contribution will be given by

$$c(\gamma) = (p - v) \cdot \mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,p)}\ x_c^{\ell(\gamma)}.$$

One can partition the set of walks γ finishing at p, q or r into pairs and triplets of walks in the following way (see Figure 2):

- If a walk γ_1 visits all three mid-edges p, q, r, it means that the edges belonging to γ_1 form a disjoint self-avoiding path plus (up to a half-edge) a self-avoiding loop from v to v. One can associate to γ_1 the walk passing through the same edges, but exploring the loop from v to v in the other direction. Hence, walks visiting the three mid-edges can be grouped in pairs.

- If a walk γ_1 visits only one mid-edge, it can be associated to two walks γ_2 and γ_3 that visit exactly two mid-edges by prolonging the walk one step further. (There are two possible choices.) The reverse is true: a walk visiting exactly two mid-edges is naturally associated to a walk visiting only one mid-edge by erasing the last step. Hence, walks visiting one or two mid-edges can be grouped in triplets.

If one can prove that the sum of contributions to (1) of each pair or triplet vanishes, then their total sum is zero and (1) holds.

Let γ_1 and γ_2 be two associated walks as in the first case. Without loss of generality, we may assume that γ_1 ends at q and γ_2 ends at r. Note that γ_1 and γ_2 coincide up to the mid-edge p and then follow an almost complete loop in two opposite directions. It follows that

$$\ell(\gamma_1)=\ell(\gamma_2) \quad \text{and} \quad \begin{cases} \mathrm{W}_{\gamma_1}(a,q)=\mathrm{W}_{\gamma_1}(a,p)+\mathrm{W}_{\gamma_1}(p,q)=\mathrm{W}_{\gamma_1}(a,p)-\frac{4\pi}{3}, \\ \mathrm{W}_{\gamma_2}(a,r)=\mathrm{W}_{\gamma_2}(a,p)+\mathrm{W}_{\gamma_2}(p,r)=\mathrm{W}_{\gamma_1}(a,p)+\frac{4\pi}{3}. \end{cases}$$

In order to evaluate the winding of γ_1 between p and q above, we used the fact that a is on the boundary and Ω is simply connected. We conclude that

$$\begin{aligned} c(\gamma_1)+c(\gamma_2) &= (q-v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,q)} x_c^{\ell(\gamma_1)} + (r-v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_2}(a,r)} x_c^{\ell(\gamma_2)} \\ &= (p-v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,p)} x_c^{\ell(\gamma_1)} \left(j\bar{\lambda}^4 + \bar{j}\lambda^4 \right) = 0, \end{aligned}$$

where the last equality holds since $j\bar{\lambda}^4 = -i$ by our choice of $\lambda = \exp(-\mathrm{i}5\pi/24)$.

Let $\gamma_1, \gamma_2, \gamma_3$ be three walks matched as in the second case. Without loss of generality, we assume that γ_1 ends at p and that γ_2 and γ_3 extend γ_1 to q and r respectively. As before, we easily find that

$$\ell(\gamma_2)=\ell(\gamma_3)=\ell(\gamma_1)+1 \quad \text{and} \quad \begin{cases} \mathrm{W}_{\gamma_2}(a,r)=\mathrm{W}_{\gamma_2}(a,p)+\mathrm{W}_{\gamma_2}(p,q)=\mathrm{W}_{\gamma_1}(a,p)-\frac{\pi}{3}, \\ \mathrm{W}_{\gamma_3}(a,r)=\mathrm{W}_{\gamma_3}(a,p)+\mathrm{W}_{\gamma_3}(p,r)=\mathrm{W}_{\gamma_1}(a,p)+\frac{\pi}{3}. \end{cases}$$

Plugging these values into the respective contributions, we obtain

$$c(\gamma_1)+c(\gamma_2)+c(\gamma_3) = (p-v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,p)} x_c^{\ell(\gamma_1)} \left(1 + x_c j\bar{\lambda} + x_c\bar{j}\lambda\right) = 0.$$

Above is the *only* place where we use that x takes its critical value, i.e. $x_c^{-1} = \sqrt{2+\sqrt{2}} = 2\cos\frac{\pi}{8}$.

The claim of the lemma follows readily by summing over all pairs and triplets. □

Remark 1. Coefficients in (1) are three cube roots of unity multiplied by $p-v$, so its left-hand side can be seen as a discrete dz-integral along an elementary contour on the dual lattice. The fact that the integral of the parafermionic observable along discrete contours vanishes suggests that it is discrete holomorphic and that self-avoiding walks have a conformally invariant scaling limit; see Section 4.

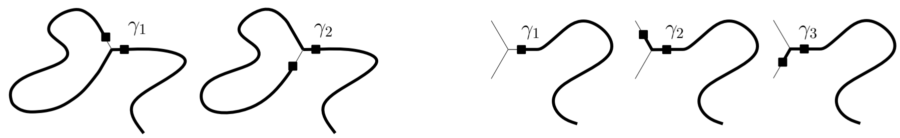

Figure 2. *Left*: a pair of walks visiting all the three mid-edges emanating from v and differing by rearranged connections at v. *Right*: a triplet of walks, one visiting one mid-edge, the two others visiting two mid-edges, and obtained by prolonging the first one through v.

3. **Proof of Theorem 1**

Counting argument in a strip domain. We consider a vertical strip domain S_T composed of T strips of hexagons and its finite version $S_{T,L}$ cut at heights $\pm L$ at angles $\pm\pi/3$; see Figure 3. Namely, position a hexagonal lattice $\mathbb{H}$ of meshsize 1 in $\mathbb{C}$ so that there exists a horizontal edge e with mid-edge a being 0. Then

$$V(S_T) = \left\{z \in V(\mathbb{H}) : 0 \le \mathrm{Re}(z) \le \frac{3T+1}{2}\right\},$$
$$V(S_{T,L}) = \{z \in V(S_T) : |\sqrt{3}\,\mathrm{Im}(z) - \mathrm{Re}(z)| \le 3L\}.$$

Denote by α the left boundary of S_T, by β the right one. Symbols ε and $\bar\varepsilon$ denote the top and bottom boundaries of $S_{T,L}$. Introduce the following (positive) partition functions:

$$A^x_{T,L} := \sum_{\gamma\subset S_{T,L}:\ a\to\alpha\setminus\{a\}} x^{\ell(\gamma)}, \qquad B^x_{T,L} := \sum_{\gamma\subset S_{T,L}:\ a\to\beta} x^{\ell(\gamma)},$$
$$E^x_{T,L} := \sum_{\gamma\subset S_{T,L}:\ a\to\varepsilon\cup\bar\varepsilon} x^{\ell(\gamma)}.$$

In the next lemma, we deduce from relation (1) a global identity without the complex weights.

Lemma 2. *For critical $x = x_c$, the following identity holds*:

$$1 = c_\alpha A^{x_c}_{T,L} + B^{x_c}_{T,L} + c_\varepsilon E^{x_c}_{T,L}, \tag{2}$$

with positive coefficients $c_\alpha = \cos\left(\frac{3\pi}{8}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}$ *and* $c_\varepsilon = \cos\left(\frac{\pi}{4}\right) = 1/\sqrt{2}$.

Proof. Sum the relation (1) over all vertices in $V(S_{T,L})$. Values at interior mid-edges cancel out, and we arrive at the identity

$$0 = -\sum_{z\in\alpha} F(z) + \sum_{z\in\beta} F(z) + j\sum_{z\in\varepsilon} F(z) + \bar{j}\sum_{z\in\bar\varepsilon} F(z). \tag{3}$$

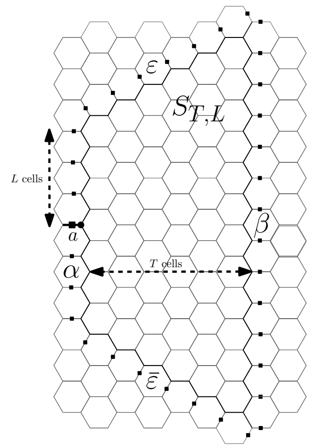

Figure 3. Domain $S_{T,L}$ and boundary intervals α, β, ε and $\bar{\varepsilon}$.

The symmetry of our domain implies that $F(\bar{z}) = \bar{F}(z)$, where $\bar{x}$ denotes the complex conjugate of x. Observe that the winding of any self-avoiding walk from a to the bottom part of α is $-\pi$ while the winding to the top part is π. Thus

$$\sum_{z\in\alpha} F(z) = F(a) + \sum_{z\in\alpha\setminus\{a\}} F(z) = F(a) + \frac{1}{2}\sum_{z\in\alpha\setminus\{a\}} (F(z) + F(\bar{z}))$$
$$= 1 + \frac{\mathrm{e}^{-\mathrm{i}\sigma\pi} + \mathrm{e}^{\mathrm{i}\sigma\pi}}{2} A_{T,L}^x = 1 - \cos\left(\frac{3\pi}{8}\right)\, A_{T,L}^x = 1 - c_\alpha A_{T,L}^x.$$

Above we have used the fact that the only walk from a to a is a trivial one of length 0, and so $F(a) = 1$. Similarly, the winding from a to any half-edge in β (resp. ε and $\bar{\varepsilon}$) is 0 (resp. $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$), therefore

$$\sum_{z\in\beta} F(z) = B_{T,L}^x \quad \text{and} \quad j\sum_{z\in\varepsilon} F(z) + \bar{j}\sum_{z\in\bar{\varepsilon}} F(z) = \cos\left(\frac{\pi}{4}\right)\, E_{T,L}^x = c_\varepsilon E_{T,L}^x.$$

The lemma follows readily by plugging the last three formulæ into (3). $\square$

Observe that sequences $(A_{T,L}^x)_{L>0}$ and $(B_{T,L}^x)_{L>0}$ are increasing in L and are bounded for $x \le x_c$ thanks to (2) and their monotonicity in x. Thus they have limits

$$A_T^x := \lim_{L\to\infty} A_{T,L}^x = \sum_{\gamma\subset S_T\colon\ a\to\alpha\setminus\{a\}} x^{\ell(\gamma)},\ \ B_T^x := \lim_{L\to\infty} B_{T,L}^x = \sum_{\gamma\subset S_T\colon\ a\to\beta} x^{\ell(\gamma)}.$$

Identity (2) then implies that $(E_{T,L}^{x_c})_{L>0}$ decreases and converges to a limit $E_T^{x_c} = \lim_{L\to\infty} E_{T,L}^{x_c}$. Passing to a limit in (2), we arrive at

$$1 = c_\alpha A_T^{x_c} + B_T^{x_c} + c_\varepsilon E_T^{x_c}. \tag{4}$$

Proof of Theorem 1. We start by proving that $Z(x_c) = +\infty$, and hence $\mu \geq \sqrt{2+\sqrt{2}}$. Suppose that for some T, $E_T^{x_c} > 0$. Actually, this quantity vanishes, but rather than proving this, it is simpler to work out the case of it being positive. As noted before, $E_{T,L}^{x_c}$ decreases in L and so

$$Z(x_c) \geq \sum_{L>0} E_{T,L}^{x_c} \geq \sum_{L>0} E_T^{x_c} = +\infty,$$

which completes the proof.

Assuming on the contrary that $E_T^{x_c} = 0$ for all T, we simplify (4) to

$$1 = c_\alpha A_T^{x_c} + B_T^{x_c}. \tag{5}$$

Observe that a walk γ entering into the count of $A_{T+1}^{x_c}$ and not into $A_T^{x_c}$ has to visit some vertex adjacent to the right edge of S_{T+1}. Cutting γ at the first such point (and adding half-edges to the two halves), we uniquely decompose it into two walks crossing S_{T+1} (these walks are usually called bridges), which together are one step longer than γ. We conclude that

$$A_{T+1}^{x_c} - A_T^{x_c} \leq x_c \left(B_{T+1}^{x_c}\right)^2. \tag{6}$$

Combining (5) for two consecutive values of T with (6), we can write

$$\begin{aligned} 0 = 1 - 1 &= (c_\alpha A_{T+1}^{x_c} + B_{T+1}^{x_c}) - (c_\alpha A_T^{x_c} + B_T^{x_c}) \\ &= c_\alpha (A_{T+1}^{x_c} - A_T^{x_c}) + B_{T+1}^{x_c} - B_T^{x_c} \leq c_\alpha x_c \left(B_{T+1}^{x_c}\right)^2 + B_{T+1}^{x_c} - B_T^{x_c}, \end{aligned}$$

and so

$$c_\alpha x_c \left(B_{T+1}^{x_c}\right)^2 + B_{T+1}^{x_c} \geq B_T^{x_c}.$$

It follows easily by induction that

$$B_T^{x_c} \geq \min[B_1^{x_c}, 1/(c_\alpha x_c)] \;/\; T$$

for every $T \geq 1$, and therefore

$$Z(x_c) \geq \sum_{T>0} B_T^{x_c} = +\infty.$$

This completes the proof of the estimate $\mu \geq x_c^{-1} = \sqrt{2+\sqrt{2}}$.

It remains to prove the opposite inequality $\mu \leq x_c^{-1}$. To estimate the partition function from above, we will decompose self-avoiding walks into bridges. A *bridge* of width T is a self-avoiding walk in S_T from one side to the opposite side, defined up to vertical translation. The partition function of bridges of width T is B_T^x, which is at most 1 by (4). Noting that a bridge of width T has length at least T, we obtain for $x < x_c$

$$B_T^x \leq \left(\frac{x}{x_c}\right)^T B_T^{x_c} \leq \left(\frac{x}{x_c}\right)^T.$$

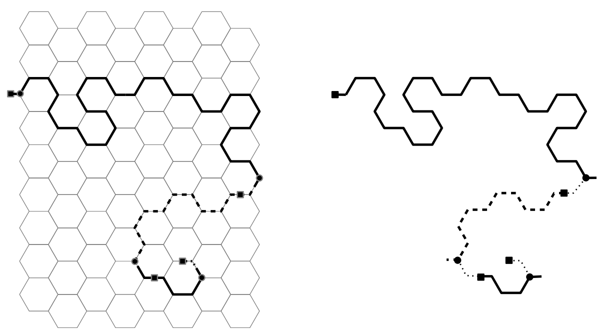

Figure 4. *Left*: Decomposition of a half-plane walk into four bridges with widths $8 > 3 > 1 > 0$. The first bridge corresponds to the maximal bridge containing the origin. Note that the decomposition contains one bridge of width 0. *Right*: The reverse procedure. If the starting mid-edge and the first vertex are fixed, the decomposition is unambiguous.

Thus, for $x < x_c$, the series $\sum_{T>0} B_T^x$ converges and so does the product $\prod_{T>0}(1 + B_T^x)$. Let us assume for the moment the following fact: *any self-avoiding walk can be canonically decomposed into a sequence of bridges of widths $T_{-i} < \cdots < T_{-1}$ and $T_0 > \cdots > T_j$, and, if one fixes the starting mid-edge and the first vertex visited, the decomposition uniquely determines the walk*. Applying the decomposition to walks starting at a (the first visited vertex is 0 or -1), we can estimate

$$Z(x) \leq 2 \sum_{\substack{T_{-i}<\cdots<T_{-1} \\ T_j<\cdots<T_0}} \left(\prod_{k=-i}^{j} B_{T_k}^x \right) = 2 \prod_{T>0} (1 + B_T^x)^2 < \infty.$$

The factor 2 is due to the fact that there are two possibilities for the first vertex once we fix the starting mid-edge. Therefore, $Z(x) < +\infty$ whenever $x < x_c$ and $\mu \leq x_c^{-1} = \sqrt{2+\sqrt{2}}$. To complete the proof of the theorem it only remains to prove that such a decomposition into bridges does exist. Such decomposition was first introduced by Hammersley and Welsh in [4]. (For a modern treatment, see Section 3.1 of [9].) We include the proof for completeness.

First assume that $\tilde{\gamma}$ is a half-plane self-avoiding walk, meaning that the start of $\tilde{\gamma}$ has extremal real part. We prove by induction on the width T_0 that the walk admits a canonical decomposition into bridges of widths $T_0 > \cdots > T_j$. Without loss of generality, we assume that the start has minimal real part. Out of the vertices having the maximal real part, choose the one visited last, say after n steps. The n first vertices of the walk form a bridge $\tilde{\gamma}_1$ of width T_0, which is the first bridge of our decomposition when prolonged to the mid-edge on the right of the last vertex. We forget about the $(n+1)$-th vertex, since there is no ambiguity in its position. The consequent steps form a half-plane walk $\tilde{\gamma}_2$ of width $T_1 < T_0$. Using the induction hypothesis, we know that $\tilde{\gamma}_2$ admits a decomposition into bridges of widths $T_1 > \cdots > T_j$. The decomposition of $\tilde{\gamma}$ is created by adding $\tilde{\gamma}_1$ before the decomposition of $\tilde{\gamma}_2$.

If the walk is a reverse half-plane self-avoiding walk, meaning that the end has extremal real part, we set the decomposition to be the decomposition of the reverse walk in the reverse order. If γ is a self-avoiding walk in the plane, one can cut the trajectory into two pieces γ_1 and γ_2: the vertices of γ up to the first vertex of maximal real part, and the remaining vertices. The decomposition of γ is given by the decomposition of γ_1 (with widths $T_{-i} < \cdots < T_{-1}$) plus the decomposition of γ_2 (with widths $T_0 > \cdots > T_j$).

Once the starting mid-edge and the first vertex are given, it is easy to check that the decomposition uniquely determines the walk by exhibiting the reverse procedure; see Figure 4 for the case of half-plane walks. □

4. **Remarks and conjectures**

Partition function of self-avoiding bridges. Our proof provides bounds for the partition function for bridges from a to the right side of the strip of width T, namely,

$$\frac{c}{T} \le B_T^{x_c} \le 1.$$

In [8, §§3.3.1 and 3.4.3], precise behaviors are conjectured for the number of self-avoiding walks between two points on the boundary of a domain, which yields the following (conjectured) estimate:

$$\sum_{\gamma \subset S_T : 0 \to T + \mathrm{i}yT} x_c^{\ell(\gamma)} \approx T^{-5/4} H(0, 1+\mathrm{i}y)^{5/4},$$

where H is the boundary derivative of the Poisson kernel. Integrating with respect to y, we obtain the following conjecture: *$B_T^{x_c}$ decays as $T^{-1/4}$, when $T \to \infty$.* Similar estimates are conjectured for walks in S_T from 0 to $\mathrm{i}yT$.

Counting self-avoiding walks. In [10], [11], Nienhuis proposed a more precise asymptotical behavior for the number of self-avoiding walks:

$$c_n \;\sim\; A\, n^{\gamma-1} \sqrt{2+\sqrt{2}}^{\,n}, \tag{7}$$

with $\gamma = 43/32$. Here the symbol $\sim$ means that the ratio of two sides is of the order $n^{o(1)}$, or perhaps even tends to a constant. Self-avoiding bridges and loops are expected to have $\gamma = 9/16$ and $\gamma = -1/2$, correspondingly, and the same connective constant μ.

Moreover, Nienhuis gave arguments in support of Flory's prediction that the mean-square displacement $\langle|\gamma(n)|^2\rangle$ satisfies

$$\langle|\gamma(n)|^2\rangle \;=\; \frac{1}{c_n}\sum_{\gamma\ n-\text{step SAW}}|\gamma(n)|^2 \;=\; n^{2\nu+o(1)}\ , \tag{8}$$

with $\nu = 3/4$. Despite the very precise statement of the predictions (7) and (8), the best rigorously known bounds are very far apart and almost 50 years old (see [9] for an exposition). The derivation of these exponents seems to be one of the most challenging problems in probability.

Conformally invariant scaling limit. In [8, Predictions 2 and 5, §4], it was shown by G. Lawler, O. Schramm and W. Werner that γ and ν could be proven to have the predicted values if the self-avoiding walk would posses a conformally invariant scaling limit. More precisely, let $\Omega \neq \mathbb{C}$ be a simply connected domain in the complex plane $\mathbb{C}$ with two points a and b on the boundary. For $\delta > 0$, we consider the discrete approximation given by the largest finite domain Ω_δ of $\delta\mathbb{H}$ included in Ω, and a_δ and b_δ to be the vertices of Ω_δ closest to a and b respectively. A probability measure $\mathbb{P}_{x,\delta}$ is defined on the set of self-avoiding trajectories γ joining a_δ and b_δ inside Ω_δ, by assigning to γ a weight proportional to $x^{\ell(\gamma)}$. We obtain a random curve denoted γ_δ. Conjectured conformal invariance of self-avoiding walks can be stated as follows; see [8, Prediction 1,§4].

CONJECTURE 1. *Let Ω be a simply connected domain (not equal to $\mathbb{C}$) with two distinct points a, b on its boundary. For $x = x_c$, the law of γ_δ in $(\Omega_\delta, a_\delta, b_\delta)$ converges when $\delta \to 0$ to a (chordal) Schramm-Loewner Evolution with parameter $\kappa = 8/3$ in Ω from a to b.*

As discussed in [7], [12], to prove convergence of a random curve to SLE it is sufficient to find a discrete observable with a conformally covariant scaling limit.

Thus it would suffice to show that a normalized version of F_δ has a conformally invariant scaling limit, which can be achieved by showing that it is holomorphic and has prescribed boundary values.

The winding of an interface leading to a boundary edge z is uniquely determined, and coincides with the winding of the boundary itself, as pointed out in [13]. Thus one can say that F_δ satisfies a discrete version of the following *Riemann boundary value problem* (a homogeneous version of the Riemann-Hilbert-Privalov BVP):

$$\text{Im}\left(F(z)\cdot(\text{tangent to }\partial\Omega)^{5/8}\right) \;=\; 0\ ,\ z\in\partial\Omega\ , \tag{9}$$

with a singularity at a. Note that the problem above has conformally covariant solutions (as $(dz)^{5/8}$-forms) and so is well defined even in domains with fractal boundaries.

As noted in Remark 1, relation (1) amounts to saying that discrete contour integrals of F_δ vanish. So any (subsequential) scaling limit of F_δ would have to be holomorphic. Unfortunately, relation (1) alone is unsufficient to deduce the existence of such a limit, unlike in the Ising case [2]. The reason is that for a domain with E edges, (1) imposes $\approx \frac{2}{3}E$ relations (one per vertice) for E values of F_δ, making it impossible to reconstruct F_δ from its boundary values. So F_δ is not exactly holomorphic; it can be rather thought of as a divergence-free vector field, which seems to have nontrivial curl. However, we expect that in the limit the curl vanishes, which is equivalent to $F_\delta(z)$ having the same limit regardless of the orientation of the edge z.

The Riemann BVP (9) is easily solved, and we arrive at the following conjecture.

CONJECTURE 2. *Let Ω be a simply connected domain (not equal to $\mathbb{C}$), let $z \in \Omega$, and let a, b be two distinct points on the boundary of Ω. We assume that the boundary of Ω is smooth near b. For $\delta > 0$, let F_δ be the holomorphic observable in the domain $(\Omega_\delta, a_\delta)$ approximating (Ω, a), and let z_δ be the closest point in Ω_δ to z. Then*

$$\lim_{\delta \to 0} \frac{F_\delta(z_\delta)}{F_\delta(b_\delta)} = \left(\frac{\phi'(z)}{\phi'(b)}\right)^{5/8}, \tag{10}$$

where Φ is a conformal map from Ω to the upper half-plane mapping a to ∞ and b to 0.

The right-hand side of (10) is well defined, since the conformal map ϕ is unique up to multiplication by a real factor. Proving this conjecture would be a major step toward Conjecture 1 and the derivation of critical exponents.

Other loop models. The self-avoiding walk model is a special case of the representation of loop $O(n)$-model (see, e.g., [6], [10]). A configuration ω of the loop model on a finite subdomain of $\mathbb{H}$ is a family of nonintersection loops together with one self-avoiding interface. The probability of a configuration is proportional to $x^{\#\text{ edges}} n^{\#\text{ loops}}$. The self-avoiding walk corresponds to the case $n = 0$. In [10], Nienhuis conjectured that the model undergoes a phase transition at the value $x_c = 1/\sqrt{2 + \sqrt{2 - n}}$. A parafermionic observable can also be defined in the $O(n)$-models for $n \in [-2, 2]$ (see [12]) and equations (1) and (2) have natural counterparts in this case. Unfortunately, the presence of loops in the model prevents us from deriving the critical value rigorously for general values of n. Let us mention that the $n = 1$ case corresponds to the Ising model, where much more detailed studies are possible [2].

Conjecture 2 generalizes to interfaces in $O(n)$ models (see, e.g., [6]). We refer to [12] for details on the question.

Square lattice. Our argument does not directly apply to the self-avoiding walk on the square lattice, for which the value of the connective constant is different and currently unknown. Instead, it applies to the integrable parameters of the $O(n)$ model, but on the square lattice with $n = 0$, those do not lead to self-avoiding walks (neither for vertex, nor for edge definition). Rather one obtains walks that visit every edge at most oncebut are allowed to visit vertices twice at a multiplicative penalty.

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