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THE CONSTRUCTION OF A-SOLVABLE ABELIAN GROUPS

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1. Introduction

One of the many ways to investigate universal properties of a torsion-free abelian group A is to consider A as a left module over its endomorphism ring $E(A)$. This approach was initiated by Arnold and Lady in [8] for torsion-free abelian groups of finite rank, and extended to larger classes of groups by Arnold and Murley in [9]. Several others, including the author of this note, continued the discussion initiated in [8] and [9] to obtain further insight in the way in which an abelian group A and its endomorphism ring $E(A)$ are related, see for instance [11], [12], [16], [7], and [17]. One of the main difficulties encountered in this approach to the structure problem of abelian groups is the generality of the arising classes of groups. As desirable as this generality may be, it severely limits the tools available in the discussion. The perhaps most useful of these is the adjoint pair of functors $(\operatorname{Hom}(A,-), -\otimes_{E(A)} A)$ between the category of abelian groups and the category of right $E(A)$-modules. The way these functors are used in the discussion of abelian groups is by considering full subcategories of the category of abelian groups on which the functors induce category equivalences. The largest of these classes is $\mathcal{C}_A$, the class of *A-solvable abelian groups.* It is equivalent under these functors to a class of right $E(A)$-modules, which is denoted by $\mathcal{M}_A$.

Arnold and Murley showed that $\mathcal{C}_A$ contains the class of A-projective abelian groups if A is self-small, i.e. the functor $\operatorname{Hom}(A,-)$ preserves direct sums of copies of A. In addition, they investigated under which conditions locally A-projective groups are A-solvable. While it is possible to obtain a satisfactory insight in the categorical properties of the class $\mathcal{C}_A$ [3], it proved difficult to construct examples of A-solvable groups which are not subgroups of A^I for some index-set I, unless A is completely decomposable. In the case that $A \subseteq \mathbb{Q}$, Warfield showed that every torsion-free group G of finite rank with $IT(G) \geqslant \mathit{type}\, A$ is A-solvable [19]. A first step in the direction of general existence theorem for A-solvable groups was taken in [5], where we constructed A-solvable groups which are not subgroups of products of A in the case that $E(A)$ is a hereditary ring whose quasi-endomorphism ring $\mathbb{Q}E(A)$ is semi-simple Artinian.

It is one of the goals of this paper to prove such an existence theorem for more general classes of A-solvable groups than those constructed in [5]. In particular, we do not impose any immediate restrictions on A as in [5] except for the standard requirement that A is self-small and faithfully flat as an $E(A)$-module.

Theorem 1.1. *(ZFC + $V = L$) Let A be a torsion-free abelian group which is self-small and faithfully flat as an $E(A)$-module. There exists a proper class of pairwise non-isomorphic A-solvable groups G with $\mathrm{Hom}(G, A) = 0$ whose endomorphism ring is the center of $E(A)$.*

The groups in this theorem are constructed as colimits of a directed system of A-projective groups in the category of A-solvable groups. The categorical results which are needed in this construction are consequences of a more general discussion in Sections 2 and 3 which investigates when limits and colimits exist in the category of A-solvable groups. By [18], this question is equivalent to whether or not $\mathcal{C}_A$ is a preabelian category with direct sums and products. The question whether $\mathcal{C}_A$ is preabelian has been addressed in [3] in the case that A is an indecomposable generalized rank 1 group. Although this paper uses several results of [3] in a more general setting, no new proofs are given unless the originally used arguments do not carry over. Theorem 2.3 shows that limits and colimits exist in the category of A-solvable groups provided $\mathcal{M}_A$ is the torsion-free class of a torsion-theory of right $E(A)$-modules.

While Proposition 2.2 shows that the colimit of a functor $\mathcal{F}$ between a small category and $\mathcal{C}_A$, if it exists in $\mathcal{C}_A$, is the largest A-solvable epimorphic image of the colimit of $\mathcal{F}$ in the category of abelian groups, Theorem 2.3 does not give a similar description for limits in $\mathcal{C}_A$. In Section 3, we investigate when the $\mathcal{C}_A$-limit of a functor $\mathcal{F}$ is isomorphic to the largest A-solvable subgroup of its limit in the category of abelian groups. Theorem 3.2 gives a complete answer to this question and relates our results to work by Gruson and Raynaud in [15] concerning tensor-products and Cartesian products.

Consider abelian groups A and G. Composition of maps induces a right $E(A)$-module-structure on $H_A(G) = \mathrm{Hom}(A,G)$. Since A is a left $E(A)$-module, $T_A(M) = M \otimes_{E(A)} A$ defines a functor from the category of right $E(A)$-modules, $\mathscr{M}_{E(A)}$, to the category of abelian groups, $\mathscr{A}b$, which is a right adjoint to H_A. The natural transformations associated with the adjoint pair (H_A, T_A) are denoted by θ and φ respectively. The functors H_A and T_A restrict to a category equivalence between $\mathscr{C}_A$ and the category $\mathscr{M}_A$ of right $E(A)$-modules M for which φ_M is an isomorphism.

Lemma 2.1. [2, Lemma 2.1 and Theorem 2.2] *Let A be a self-small abelian group which is flat as an $E(A)$-module.*

i) *An exact sequence $0 \to B \overset{\alpha}{\to} C \overset{\beta}{\to} G \to 0$ of abelian groups, in which C is A-solvable, induces an exact sequence $0 \to T_AH_A(B) \overset{\theta_B}{\to} B \overset{\delta}{\to} T_A(M) \overset{\theta}{\to} G \to 0$ in which $M = \mathrm{im}\, H_A(\beta)$ and $\theta\colon T_A(M) \to G$ is the evaluation map.*

ii) *$\mathscr{C}_A$ is A-closed; i.e. it is closed with respect to subgroups and finite direct sums, and kernels of homomorphisms between A-solvable groups are A-solvable.*

Consider a functor $\mathscr{F}$ from a small category I into $\mathscr{C}_A$. The colimit of $\mathscr{F}$ in the categories $\mathscr{C}_A$ and $\mathscr{A}b$ is defined as in [18], from which the following notation is taken:

Let $\mu_j\colon \mathscr{F}(j) \to \bigoplus_{i\in I} \mathscr{F}(i)$ be the embedding into the j^{th}-coordinate. For an I-morphism $\lambda\colon i \to j$ define $\delta_\lambda = \mu_j\mathscr{F}(\lambda) - \mu_i$. In the category of abelian groups, $\varinjlim_{\mathscr{A}b} \mathscr{F} \cong \big[\bigoplus_{i\in I} \mathscr{F}(i)\big]/B$ where $B = \langle \mathrm{im}\,\delta_\lambda \mid \lambda \in \mathrm{Mor}(I)\rangle$. The associated compatible family of maps $\varphi_i\colon \mathscr{F}(i) \to \varinjlim_{\mathscr{A}b} \mathscr{F}$ is given by $\varphi_i(x) = \delta_i(x) + B$ for all $x \in \mathscr{F}_i$.

If we try to compute the colimit of the same functor $\mathscr{F}$ in $\mathscr{C}_A$ as the cokernel of the embedding $B \to \bigoplus_I \mathscr{F}_i$, we encounter the problem that this map not always is in $\mathscr{C}_A$ since $\mathscr{C}_A$ need not be closed under arbitrary direct sums even if $A \subseteq \mathbb{Q}$ [3]. While this prohibits a direct application of the cokernel construction of [3], it can be modified to prove the following result. Because of the similarity of the proofs, we only present those parts where modifications of [3] are necessary and refer to [3] otherwise.

Proposition 2.2. *Let A be a self-small abelian group which is flat as an $E(A)$-module. A functor $\mathscr{F}$ from a small category I into $\mathscr{C}_A$ has a colimit in $\mathscr{C}_A$ if and only if there is a smallest subgroup V of $\bigoplus_{i\in I} \mathscr{F}(i)$ such that $B \subseteq V$ and $\big[\bigoplus_{i\in I} \mathscr{F}(i)\big]/V \in \mathscr{C}_A$.*

Proof. Assume that the A-solvable group G together with a compatible family of maps $\sigma_i \in \mathrm{Hom}(\mathscr{F}(i), G)$ is the colimit of $\mathscr{F}$ in $\mathscr{C}_A$. To simplify our notation, we write H for the colimit of $\mathcal{F}$ in $\mathcal{A}b$. There is a homomorphism $\sigma\colon H \to G$ with $\sigma_i = \sigma\varphi_i$ for all $i \in I$. Since H is A-generated, the group $\sigma(H)$ is A-solvable by Lemma 2.1. To show that $\sigma(H)$ together with the maps σ_i is the colimit of $\mathcal{F}$ in $\mathcal{C}_A$, we consider an A-solvable group K and a compatible family of maps $\lambda_i \in \mathrm{Hom}(\mathcal{F}_i, K)$. There exists a unique map $\lambda\colon G \to K$ with $\lambda_i = \lambda\sigma_i$ for all $i \in I$. We can, in addition, find a unique map $\varrho\colon H \to K$ with $\lambda_i = \varrho\varphi_i$ for all $i \in I$. We denote the restriction of λ to $\sigma(H)$ by ε, and observe $\varepsilon\sigma_i = \lambda\sigma_i = \lambda_i$. If $\delta \in \mathrm{Hom}(\sigma(H), K)$ also satisfies $\delta\sigma_i = \lambda_i$ for each i, then $(\delta\sigma)\varphi_i = \delta\sigma_i = \lambda_i = \lambda\sigma_i = (\lambda\sigma)\varphi_i$ yields $\delta\sigma = \varrho = \lambda\sigma$. For every $x \in \sigma(H)$, we choose $h \in H$ with $x = \sigma(h)$. Then, $\delta(x) = \delta\sigma(h) = \lambda\sigma(h) = \varepsilon(x)$. This shows that $\sigma(H)$ indeed is the colimit of $\mathcal{F}$ in $\mathcal{C}_A$.

Let V be the subgroup of $\bigoplus_{i\in I} \mathcal{F}(i)$ which contains B and satisfies $\ker\sigma = V/B$. Obviously, $[\bigoplus_{i\in I} \mathcal{F}(i)]/V$ is A-solvable. Suppose that U is another subgroup of $\bigoplus_{i\in I} \mathcal{F}(i)$ such that $K = \bigoplus_I [\mathcal{F}_i]/U$ is A-solvable. Define a map $\pi\colon H \to K$ by $\pi(x + B) = x + U$. For all $i, j \in I$ and all I-morphisms $\lambda\colon i \to j$, we have $\pi[\varphi_j \mathcal{F}(\lambda)] = \pi\varphi_i$. Hence, $\{\pi\varphi_i \mid i \in I\}$ is a compatible family of maps in $\mathcal{C}_A$. There exists a unique map $\psi\colon \sigma(H) \to K$ with $\pi\varphi_i = \psi\sigma_i$ for all $i \in I$. Since H is the colimit of $\mathcal{F}$ in $\mathcal{A}b$ and $\pi\varphi_i = (\psi\sigma)\varphi_i$ for each $i \in I$, we obtain $\pi = \psi\sigma$. Therefore, $V/B = \ker\sigma \subseteq \ker\pi = U/B$. Conversely, suppose that there exists a smallest subgroup V of $\bigoplus_{i\in I} \mathcal{F}(i)$ with the required properties. Denote the canonical projection of H onto $G = [\bigoplus_{i\in I} \mathcal{F}(i)]/V$ by π. If we set $\sigma_i = \pi\varphi_i$, then it is routine to check that G together with the maps $\{\sigma_i \mid i \in I\}$ is the colimit of $\mathcal{F}$ in $\mathcal{C}_A$. (see [3]) □

Theorem 2.3. *The following conditions are equivalent for a self-small abelian group A:*

a) *$\mathcal{M}_A$ is the torsion-free class of some torsion-theory of right $E(A)$-modules.*

b) i) *A is faithfully flat as an $E(A)$-module.*

   ii) *$\mathcal{C}_A$ is a cocomplete category.*

   iii) *$\mathcal{C}_A$ is a complete category with $\varprojlim_{\mathcal{C}_A} \mathcal{F} \cong T_A H_A(\varprojlim_{\mathcal{A}b} \mathcal{F})$ for all functors $\mathcal{F}$ from a small category into $\mathcal{C}_A$.*

P r o o f. a) $\Rightarrow$ b): Since $\mathcal{M}_A$ is closed with respect to submodules, A is faithfully flat by [6]. To show the last two conditions in b), we consider a family $\{G_i \mid i \in I\}$ of A-solvable groups, and first establish that $T_A H_A(\prod_{i\in I} G_i)$ is the $\mathcal{C}_A$-product of this family. Since $H_A(G_i) \in \mathcal{M}_A$, and $\mathcal{M}_A$ is closed with respect to products, we obtain that $T_A H_A(\prod_I G_i)$ is A-solvable. Define maps $\lambda_j\colon T_A H_A(\prod_I G_i) \to G_j$ by $\lambda_j = \theta_{G_i} T_A H_A(\pi_i)$ where $\pi_j\colon \Pi_I G_i \to G_j$ denotes the projection onto the j^{th}-coordinate.

If $B \in \mathscr{C}_A$ and $\{\alpha_i: B \to G_i \mid i \in I\}$ is a family of $\mathscr{C}_A$-morphisms, then the maps $H_A(\alpha_i)$ induce a unique map $\tilde{\alpha}: H_A(B) \to H_A(\prod_I G_i)$ with $H_A(\pi_i)\tilde{\alpha} = H_A(\alpha_i)$. We set $\alpha = T_A(\tilde{\alpha})\theta_B^{-1}$, and obtain $\lambda_i\alpha = \theta_{G_i}T_A(H_A(\pi_i)\tilde{\alpha})\theta_B^{-1} = \alpha_i$.

It remains to show the uniqueness of α. Suppose that the map $\beta: B \to T_AH_A(\Pi_I G_i)$ satisfies $\lambda_i\beta = \alpha_i$ for all $i \in I$. Then $H_A(\lambda_i)H_A(\beta) = H_A(\alpha_i)$. Observe that, for an abelian group H and a right $E(A)$-module M, we have $H_A(\theta_H)\varphi_{H_A(H)} = \mathrm{id}_{H_A(H)}$ and $\theta_{T_A(M)}T_A(\varphi_M) = \mathrm{id}_{T_A(M)}$. Therefore

$$\begin{aligned} H_A(\lambda_i)H_A(\beta) &= H_A(\theta_{G_i})H_AT_AH_A(\pi_i)H_A(\beta) \\ &= \varphi^{-1}_{H_A(G_i)}H_AT_AH_A(\pi_i)H_A(\beta) = H_A(\pi_i)\varphi^{-1}_{H_A(\Pi_I G_i)}H_A(\beta); \end{aligned}$$

and we obtain $\tilde{\alpha} = \varphi^{-1}_{H_A(\Pi_I G_i)}H_A(\beta)$ because of the uniqueness of $\tilde{\alpha}$. Thus, $\alpha = T_A(\varphi^{-1}_{H_A(\Pi_I G_i)})T_AH_A(\beta)\theta_B^{-1} = \theta_{T_AH_A(\Pi_I G_i)}T_AH_A(\beta)\theta_B^{-1} = \beta$ since the diagram

$$\begin{array}{ccc} T_AH_A(B) & \xrightarrow{T_AH_A(\beta)} & T_AH_AT_AH_A(\Pi_I G_i) \\ \wr\big\downarrow \theta_B & & \wr\big\downarrow \theta_{T_AH_A(\Pi_I G_i)} \\ B & \xrightarrow{\beta} & T_AH_A(\Pi_I G_i) \end{array}$$

commutes.

Since $\bigoplus_{i\in I} H_A(G_i) \subseteq \prod_{i\in I} H_A(G_i)$ and the last module is an element of $\mathscr{M}_A$ by what has been shown, we obtain that $\mathscr{M}_A$ contains $\bigoplus_{i\in I} H_A(G_i)$ because of a). Thus $\bigoplus_{i\in I} G_i \cong T_A\big(\bigoplus_{i\in I} H_A(G_i)\big)$ is A-solvable. Therefore, $\mathscr{C}_A$ is closed with respect to direct sums. To establish that $\mathscr{C}_A$ has cokernels, we consider a map $\varphi: G \to L$ where L is A-solvable, and show that there is a smallest subgroup V of L such that $\varphi(G) \subseteq V$ and L/V is A-solvable. Consider the family $\mathscr{U} = \{U \subseteq H \mid \varphi(G) \subseteq U \text{ and } L/U \in \mathscr{C}_A\}$; and observe that $K = T_AH_A\big(\prod_{\mathscr{U}} L/U\big)$ is A-solvable by what has been shown so far. The projection maps $L \to L/U$ induce a map $\lambda: L \to K$, whose kernel is a subgroup of L with the desired properties.

Finally, consider a functor $\mathscr{F}: I \to \mathscr{C}_A$, and set let $G_i = \mathscr{F}(i)$ and $P = T_AH_A\big(\prod_I G_i\big)$. If $\delta: i(\delta) \to j(\delta)$ is an I-morphism, then define $\sigma_\delta: P \to G_{j(\delta)}$ to be the map $\mathscr{F}(\delta)\lambda_{i(\delta)} - \lambda_{j(\delta)}$ where the λ's are defined as in the first paragraph of this proof. Since $\mathscr{C}_A$ has kernels and products, the limit of $\mathscr{F}$ in $\mathscr{C}_A$ exists by [18], and is the kernel of the map $\sigma = T_A(\varepsilon)\theta_P^{-1}: P \to T_AH_A\big(\prod_\delta G_{j(\delta)}\big)$ where $\varepsilon: H_A(P) \to H_A(\prod_\delta G_{j(\delta)})$ is induced by the maps $H_A(\sigma_\delta)$ by the universal property of a product of right $E(A)$-modules.

In the category of abelian groups, the limit of $\mathscr{F}$ is the kernel of the map $\tau : \prod_I G_i \to \prod_\delta G_{j(\delta)}$ which is induced by the mappings $\tau_\delta = \mathscr{F}(\delta)\pi_{i(\delta)} - \pi_{j(\delta)}$. We obtain $H_A(\sigma_\delta) = [H_A(\mathscr{F}(\delta))H_A(\pi_{i(\delta)}) - H_A(\pi_{j(\delta)})]\varphi^{-1}_{H_A(P)}$. This shows $H_A(\sigma_\delta)\varphi_{H_A(P)} = H_A(\tau_\delta) = H_A(\pi_\delta)H_A(\tau)$ which in turn yields $\varepsilon = H_A(\tau)\varphi^{-1}_{H_A(P)}$. Since θ_P is an isomorphism and A is flat,

$$\ker\sigma \cong T_A(\ker\varepsilon) \cong T_A(\ker\varepsilon\varphi_{H_A(P)}) \cong T_A(\ker H_A(\tau)) \cong T_A H_A(\ker\tau).$$

This shows that part iii) of condition b) holds.

b) $\Rightarrow$ a): The class $\mathscr{M}_A$ is closed with respect to submodules by [6]. If $\{M_i \mid i \in I\}$ is a family of modules in $\mathscr{M}_A$, then we can find A-solvable groups $\{G_i \mid i \in I\}$ with $M_i \cong H_A(G_i)$. We obtain $T_A\big(\prod_I M_i\big) \cong T_A H_A\big(\prod_I G_i\big)$ which is A-solvable by b). Another application of [6] yields $\prod_I M_i \in \mathscr{M}_A$. The fact that $\mathscr{M}_A$ is closed with respect to extensions is an immediate consequence of the 3-Lemma. □

3. A-solvability and the Mittag-Loefler-condition

The results of the last section raise the question which conditions have to be satisfied by a torsion-free abelian group A to ensure that $S_A\big(\prod_I G_i\big)$ is A-solvable for all families of A-solvable groups $\{G_i\}_{i\in I}$. Following [15], we say that a left R-module A satisfies the *Mittag-Loefler-condition (ML) with respect to a class $\mathscr{M}$ of right R-modules* if A is the direct limit of a filtration $\{F_i, \mu_i^j : F_i \to F_j \mid i, j \in I \text{ with } i \leqslant j\}$ of finitely presented modules satisfying

(∗) For every $i \in I$, there is $j \in I$ with $j \geqslant i$ such that $\ker(1_M \otimes \mu_i) \subseteq \ker(1_M \otimes \mu_i^j)$ for all $M \in \mathscr{M}$.

In [15], the following result was proved:

Lemma 3.1. *The following conditions are equivalent for a left R-module A and a family of right R-modules $\mathscr{M}$:*

a) *A satisfies ML with respect to $\mathscr{M}$.*

b) *Condition (∗) holds for any filtration of finitely presented left R-modules whose direct limit is A.*

c) *If $\{U_i \mid i \in I\}$ is a family of elements of $\mathscr{M}$, then the natural map $\sigma_A : \big[\prod_I U_i\big] \otimes_R A \to \prod_I [U_i \otimes_R A]$ is one-to-one.*

Using this result, we obtain:

Theorem 3.2. *The following conditions are equivalent for a self-small abelian group A which is faithfully flat as an $E(A)$-module:*

a) *A satisfies ML with respect to $\mathcal{M}_A$.*

b) i) *$\mathcal{M}_A$ is the torsion-free class of some torsion-theory on $\mathcal{M}_{E(A)}$.*

ii) *If $\{U_i \mid i \in I\}$ is a family of A-balanced, A-generated subgroups of an A-solvable group G, then $\bigcap_{i\in I} U_i$ is A-generated.*

c) *$\mathcal{C}_A$ is a cocomplete category; and $\varprojlim_{\mathcal{C}_A} \mathcal{F} = S_A(\varprojlim_{\mathcal{A}b} \mathcal{F})$ for all functors $\mathcal{F}$ from a small category into $\mathcal{C}_A$.*

d) *$S_A(\prod_I G_i)$ is A-solvable for all families $\{G_i \mid i \in I\}$ of A-solvable groups.*

Proof. a) $\Rightarrow$ d): By Lemma 3.1, the natural map $\sigma_A : T_A(\prod_I M_i) \to \prod_I T_A(M_i)$ is one-to-one for all families $\{M_i \mid i \in I\} \subseteq \mathcal{M}_A$. If $\lambda\colon \prod_I M_i \to H_A(\prod_I T_A(M_i))$ denotes the natural isomorphism, then $H_A(\sigma_A)\varphi_{\Pi_I M_i} = \lambda$ yields that $H_A(\sigma_A)$ is onto. Since it also is a monomorphism, the map $\varphi_{\Pi_I M_i}$ is an isomorphism too. The same holds for the first vertical map and the map forming the top-row of the following commutative diagram:

$$\begin{array}{ccccc} & & T_AH_AT_A\left(\prod_I M_i\right) & \xrightarrow{T_AH_A(\sigma_A)} & T_AH_A\left(\prod_I T_A(M_i)\right) \\ & & \Big\downarrow \theta_{T_A(\Pi_I M_i)} & & \Big\downarrow \theta_{\Pi_I T_A(M_i)} \\ 0 & \longrightarrow & T_A\left(\prod_I M_i\right) & \xrightarrow{\sigma} & \prod_I T_A(M_i) \end{array}$$

Thus, $\theta_{\Pi_I T_A(M_i)}$ is an monomorphism, and $S_A(\prod_I T_A(M_i))$ is A-solvable.

d) $\Rightarrow$ c): Since $S_A(\prod_I G_i)$ is A-solvable for all families $\{G_i \mid i \in I\}$ of A-solvable groups, we obtain $S_A(\varprojlim_{\mathcal{A}b} \mathcal{F})$ is A-solvable for all functors $\mathcal{F}$ from a small category into $\mathcal{C}_A$. The arguments in the proof of implication b) $\Rightarrow$ a) of Theorem 2.3 can be used to show that $\mathcal{M}_A$ is the torsion-free class of some torsion-theory. Theorem 2.3 yields that $\mathcal{C}_A$ is cocomplete, and $\varprojlim_{\mathcal{C}_A} \mathcal{F} \cong T_AH_A(\varprojlim_{\mathcal{A}b} \mathcal{F}) \cong S_A(\varprojlim_{\mathcal{A}b} \mathcal{F})$ by what has been shown.

c) $\Rightarrow$ b): In view of Theorem 2.3, it remains to verify condition ii): Since U_i is an A-balanced, A-generated subgroup of G, the group G/U_i is A-solvable, and $S_A(\prod_I G/U_i)$ is A-solvable by c) since products are inverse limits. The projection maps $G \to G/U_i$ induce an $\mathcal{C}_A$-homomorphism $G \to S_A(\prod_I G/U_i)$ whose kernel is $\bigcap_I U_i$. Since A is faithfully flat, the latter group is A-generated.

b) $\Rightarrow$ a): Let $\{G_i \mid i \in I\}$ be a family of A-solvable groups. We write $P = \prod_I G_i$ and observe that $T_A H_A(P)$ is an A-solvable abelian group by Theorem 2.3. In order to show that $S_A(P)$ is A-solvable, we consider the A-balanced exact sequence $0 \to \ker\theta_P \to T_A H_A(P) \xrightarrow{\theta_P} S_A(P) \to 0$. Since $T_A H_A(P)$ is A-solvable, the same holds for $S_A(P)$ once we have shown that $\ker\theta_P$ is A-generated in view of Lemma 2.1.i.

Let $\pi_i \colon P \to G_i$ be the projection onto the i^{th}-coordinate. Suppose $x \in \ker\theta_P$. Since $\theta_{G_i} T_A H_A(\pi_i) = \pi_i \theta_P$, we obtain $x \in \ker T_A H_A(\pi_i)$ for all $i \in I$ since G_i is A-solvable. On the other hand, if $x \in \ker T_A H_A(\pi_i)$ for all $i \in I$, then $\pi_i \theta_P(x) = 0$, which is only possible if $\theta_P(x) = 0$. Thus, $\ker\theta_P = \bigcap_I \ker T_A H_A(\pi_i)$. But $\ker T_A H_A(\pi_i)$ is a direct summand of the A-solvable group $T_A H_A(P)$. By b), $\ker\theta_P$ is A-generated.

Let $\lambda \colon P \to \prod_I T_A H_A(G_i)$ be the isomorphism which is coordinatewise induced by the maps θ_{G_i}. We identify the right $E(A)$-modules $H_A(\prod_I (G_i)$ and $\prod_I H_A(G_i)$ and observe that $\lambda\theta_P = \sigma_A$. Since θ_P is a monomorphism, the same holds for σ_A. By Lemma 3.1, A satisfies ML with respect to $\mathcal{M}_A$. $\square$

In the case that A has finite rank or is a generalized rank 1 group, the last result can be improved. In order to do this, the following technical result is needed:

Lemma 3.3. *Let A be a self-small torsion-free abelian group which is faithfully flat as an $E(A)$-module. Then, $\bigoplus_\omega \mathbb{Q} \in \mathcal{C}_A$ iff A is a homogeneous, completely decomposable group of finite rank.*

Proof. Suppose $\bigoplus_\omega \mathbb{Q} \in \mathcal{C}_A$. If A has infinite rank, then there is a subgroup B of A with $A/B \cong \mathbb{Q}$, and the sequence $0 \to \bigoplus_\omega B \xrightarrow{\alpha} \bigoplus_\omega A \xrightarrow{\beta} \bigoplus_\omega \mathbb{Q} \to 0$, which is induced coordinatewise, is A-balanced since A is faithfully flat. There is an epimorphism $\delta \colon A \to \bigoplus_\omega \mathbb{Q}$, which factors through β, say $\delta = \beta\varepsilon$. Since A is self-small, $\varepsilon(A) \subseteq \bigoplus_n A$ for some $n < \omega$, which results in a contradiction. Hence, A has finite rank. If U is a pure rank 1 subgroup of A, then A/U is an A-generated subgroup of the A-solvable group $\bigoplus_\omega \mathbb{Q}$. Since A is flat, we obtain that U is A-solvable. The inclusions $U \subseteq A$ induce an epimorphism $\varepsilon \colon G = \oplus\{U \mid U$ is a pure rank 1 subgroup of $A\} \to A$. Since $S_A(G) = G$ and A is faithfully flat, the map ε splits; and A is completely decomposable.

Write $A = A_1^{m_1} \oplus \ldots \oplus A_s^{m_s}$ where the A_i's are pairwise non-isomorphic rank 1 groups. Let U be a pure rank 1 subgroup of the A-projective group $A_1 \oplus \ldots \oplus A_s$ which is generated by an element $(a_1, \ldots, a_s)$ with $a_i \neq 0$ for all i. Then, *type* $U \leqslant$ *type* A_i for $i = 1, \ldots, s$. Since $[A_1 \oplus \ldots \oplus A_s]/U \subseteq \bigoplus_\omega \mathbb{Q}$ is A-generated, we obtain that U is A-generated too. Then, $\mathrm{Hom}(A_i, U) \neq 0$ for some i, and $U \cong A_i$. Without loss of generality, we may assume $i = 1$. This shows that A is an epimorphic image of $\bigoplus_I A_1$ for some index-set I. As before, this epimorphism splits; and A is homogeneous.

The converse is obvious. $\square$

Proposition 3.4. *Let A be a torsion-free abelian group such that $E(A)^n$ satisfies the DCC for $\mathbb{Z}$-pure submodules U with $E(A)^n/U \in \mathcal{M}_A$. Then, $S_A(\Pi_I G_i)$ is A-solvable for all families of torsion-free A-solvable groups $\{G_i \mid i \in I\}$.*

Proof. For a finite subset J of I, let $\pi_J \colon \prod_I G_i \to \bigoplus_J G_i$ be a canonical projection with kernel $\prod_{I \setminus J} G_i$. We consider a map $\varphi \colon A^m \to \prod_I G_i$ for some $m < \omega$, and assume $\ker \varphi \neq \ker \pi_J \varphi$ for all finite subsets J of I. Suppose that we have selected indices $\{i_1, \ldots, i_n\} \subseteq I$. If $U_n = \bigcap_{j=1}^{n} \ker \pi_{i_j} \varphi$, then $\ker \varphi \neq U_n$; and there is $a_{n+1} \in U_n \setminus \ker \varphi$. Choose an index $i_{n+1} \in I$ with $\pi_{i_{n+1}} \varphi(a_{n+1}) \neq 0$. We obtain that U_{n+1} is a proper subset of U_n.

Since $A^m/U_n \subseteq \bigoplus_{j=1}^{n} G_{i_j}$ and the G_i's are torsion-free, we have that U_n is a pure, A-generated, A-balanced subgroup of A^m. Therefore, $\{H_A(U_n) \mid n < \omega\}$ is an infinite strictly descending chain of $\mathbb{Z}$-pure submodules of $H_A(A^n)$ with $H_A(A^n)/H_A(U_n) \in \mathcal{M}_A$ for all $n < \omega$. However, such a chain cannot exist.

Therefore, we can find a finite subset J of I such that $\operatorname{im} \varphi$ is isomorphic to a subgroup of $\bigoplus_{j \in J} G_j$. This shows that $\operatorname{im} \varphi$ is A-solvable and the same holds for G. $\square$

The last result in particular shows that $\mathcal{C}_A$ is closed with respect to direct sums of torsion-free groups if A is as in Proposition 3.4.

Corollary 3.5. *Let A be a torsion-free, self-small abelian group which is faithfully flat as an $E(A)$-module, but not homogeneous completely decomposable of finite rank. The following conditions are equivalent if $E(A)/pE(A)$ is Artinian for all primes p of $\mathbb{Z}$, and $E(A)^n$ has the DCC for $\mathbb{Z}$-pure right submodules U with $E(A)^n/U \in \mathcal{M}_A$:*

a) *A satisfies ML with respect to $\mathcal{M}_A$; and $\mathcal{C}_A$ does not contain J_p for any prime p of $\mathbb{Z}$.*

b) *$\mathcal{C}_A$ is cocomplete, and does not contain J_p for any prime p of $\mathbb{Z}$.*

c) *$\mathcal{M}_A$ is the torsion-free class of some torsion-theory on $\mathcal{M}_{E(A)}$; and J_p is not A-solvable for any prime p of $\mathbb{Z}$.*

d) *If p is a prime of $\mathbb{Z}$ with $r_p(E(A)) < \infty$, then $r_p(E(A)) < [r_p(A)]^2$.*

Proof. a) $\Rightarrow$ c) is an immediate consequence of Theorem 3.2; while c) $\Rightarrow$ b) follows from Theorem 2.3.

b) $\Rightarrow$ d): Condition d) can be verified as in [3], once we have shown that the elements of $\mathscr{C}_A$ are torsion-free. If G is an A-solvable group such that $G[p] \neq 0$ for some prime p, then $\mathbb{Z}/p\mathbb{Z}$ is A-solvable by Lemma 2.1; and $A \neq pA$. To show that $\mathscr{C}_A$ contains all bounded p-groups, it is necessary to verify that A has finite p-rank.

Since cocomplete categories have cokernels, we obtain that multiplication by p on A has a $\mathscr{C}_A$-cokernel which is of the form A/V where V is the smallest subgroup of A containing pA such that A/V is A-solvable. We choose a $\mathbb{Z}/p\mathbb{Z}$-basis $\{e_i \mid i \in I\}$ of A/pA. For every finite subset J of I, we can find a subgroup U_J of A containing pA such that $A/pA = \langle e_j \mid j \in J\rangle \oplus U_J/pA$. Since $A/U_J \cong \bigoplus_J \mathbb{Z}/p\mathbb{Z}$ is A-solvable, we have $V \subseteq U_J$. For $x \in A \setminus pA$, there is a finite subset J_0 of I, with $x \in \langle e_j \mid j \in J_0\rangle$. Hence, $x \notin U_{J_0}$ and $\bigcap_{\{J \subseteq I \mid |J|<\infty\}} U_J = pA$. Therefore, $A/pA \cong \bigoplus_I \mathbb{Z}/p\mathbb{Z}$ is A-solvable. Consider the exact sequence $0 \to U \to A \to \mathbb{Z}/p\mathbb{Z} \to 0$ which induces $0 \to \bigoplus_I U \xrightarrow{\alpha} \bigoplus_I A \xrightarrow{\beta} \bigoplus_I \mathbb{Z}/p\mathbb{Z} \to 0$ coordinatewise. Let $\delta\colon A \to \bigoplus_I \mathbb{Z}/p\mathbb{Z}$ be an epimorphism. Since the last sequence is A-balanced, there is a map $\psi \in H_A(\bigoplus_I A)$ with $\beta\psi = \delta$. The fact that A is self-small yields $\psi(A) \subseteq \bigoplus_J A$ for some finite subset J of I. Consequently, $\delta(A) \subseteq \bigoplus_J \mathbb{Z}/p\mathbb{Z}$; and I has to be finite. Since A has finite p-rank, every family of cyclic p-groups is A-small. As in [3], $\mathscr{C}_A$ is closed with respect to direct sums of A-small families, and, therefore, contains all bounded p-groups. Consider the map $\hat{\varphi}\colon A^\omega \to A^\omega$ which is defined by $\hat{\varphi}((a_n)_{n<\omega}) = (p^n a_n)_{n<\omega}$. As in [3], $\hat{\varphi}$ induces an endomorphism φ of the group $G = S_A(A^\omega)$ which has $G/\varphi(G)$ as its $\mathscr{C}_A$-cokernel. Observe that G is A-solvable by Proposition 3.4. Moreover, $H_A(G/\varphi(G)) \cong \prod_{n<\omega} [E(A)/p^n E(A)]$ as a right $E(A)$-module.

Let U be the submodule of $H_A(G/\varphi(G))$ which corresponds to $\varprojlim E(A)/p^n E(A)$. As in [11, Proposition 39.4 and Example 12.2], the additive group of U is torsion-free, reduced, algebraically compact, and p-local. Since $U \subseteq H_A(G/\varphi(G)) \in \mathscr{M}_A$, the map φ_U is an isomorphism by [6]. Thus, $T_A(U)$ is a torsion-free, A-solvable group. If it were not cotorsion-free, then it would have a direct summand isomorphic to $\mathbb{Q}$ or J_p either of which is not possible by the hypotheses. Thus, $T_A(U)$ is cotorsion-free, and the same holds for $U \cong H_A T_A(U)$ which results in a contradiction. This shows that the elements of $\mathscr{C}_A$ are torsion-free.

d) $\Rightarrow$ a): Assume that J_p is A-solvable. By [13], the exact sequence $0 \to J_p \xrightarrow{p} J_p \to \mathbb{Z}/p\mathbb{Z} \to 0$ is A-balanced. This shows that $\mathbb{Z}/p\mathbb{Z}$ is A-solvable which is not possible since A solvable groups have to be torsion-free by d) as in [3]. By Proposition 3.4, $S_A(\Pi_I G_i)$ is A-solvable for all groups $G_i \in \mathscr{C}_A$. Now apply Theorem 3.2. □

If we assume in addition that $E(A)$ is right Noetherian and $\mathbb{Q}E(A)$ is right Artinian, then the restrictions with respect to J_p in the last result can be removed. We write

$$\Gamma(A) = \{p \mid r_p(A) < \infty \text{ and } r_p(E(A)) = [r_p(A)]^2\}.$$

Corollary 3.6. *Let A be a self-small torsion-free abelian group which is faithfully flat as an $E(A)$-module. The following conditions are equivalent if $E(A)/pE(A)$ is Artinian and $\mathbb{Q}E(A)$ is Artinian:*

a) *A satisfies ML with respect to $\mathcal{M}_A$.*

b) *$\mathcal{C}_A$ contains $S_A(G)$ for all reduced algebraic compact groups G with $G = pG$ for all primes $p \in \Gamma(A)$.*

Proof. a) $\Rightarrow$ b): Let $p \in \Gamma(A)$. As in [3], we obtain that $\mathcal{C}_A$ contains all bounded p-groups. The proof of the last corollary can be adopted to show that $\mathcal{C}_A$ also contains J_p since $\mathcal{C}_A$ is cocomplete by Theorem 2.3. Every reduced algebraically compact group G is a direct summand of a group which is a product of cocyclic groups and copies of p-adic integers. Write $G \oplus H = \Pi_I G_i = C$ where each G_i is either a cyclic q-group or a group of q-adic integers for some prime q of $\mathbb{Z}$. Consider a decomposition $C = D \oplus E$ where D consists of all those components of C associated with primes in $\Gamma(A)$, while E consists of the remaining components. Let $\delta\colon C \to E$ be a projection whose kernel is D. Since E is reduced, and $G = pG$ for all primes not in $\Gamma(A)$, we have $G \subseteq \ker \delta = D$. Thus, we may assume that the primes q in the definition of G_i are taken from $\Gamma(A)$. Since each of the G_i is A-solvable, we obtain that $S_A(C)$ is A-solvable by Theorem 3.2. This shows that $S_A(G)$ is A-solvable.

b) $\Rightarrow$ a): By Theorem 3.2, it is enough to show that $S_A(\Pi_I G_i)$ is A-solvable for all A-solvable groups G_i. To show this, we consider a map $\varphi\colon A^m \to \Pi_I G_i$ for some $m < \omega$. For a subset J of I, let $\pi_J\colon \prod_I G_i \to \prod_J G_i$ be a projection on $\Pi_J G_i$ whose kernel is $\Pi_{I\setminus J} G_i$.

Suppose that there is no finite subset J of I with the following property: $\pi_i(\ker \pi_J \varphi)$ is torsion for all $i \in I \setminus J$. Assume that we have chosen a finite subset J_n of I and write $U_n = \ker \pi_{J_n}\varphi$. There is $i_{n+1} \in I \setminus J_n$ such that $\pi_{i_{n+1}}\varphi(U_n)$ is not torsion. Choose an element $x \in U_n$ such that $\pi_{i_{n+1}}\varphi(x)$ has infinite order. We set $J_{n+1} = J_n \cup \{i_{n+1}\}$. Then, $x \in U_n \setminus U_{n+1}$ has infinite order.

Moreover, since $A^m/U_n \subseteq \bigoplus_{j=1}^{n} G_{i_j}$ and the latter group is A-solvable, we obtain that U_n is an A-generated subgroup of A^m. Let W_n be the $\mathbb{Z}$-purification of $H_A(U_n)$ in $H_A(A^m)$. Then, $T_A(W_n)$ is the $\mathbb{Z}$-purification of $T_A H_A(U_n)$ in $T_A H_A(A^m)$. In particular, the $\mathbb{Z}$-purification V_n of U_n is an A-generated subgroup of A^m. Since U_n/U_{n+1} is not torsion, we obtain that $V_n/V_{n+1} \neq 0$. Hence, $\{\mathbb{Q}H_A(V_n) \mid n < \omega\}$ is a strictly descending chain of submodules of $\mathbb{Q}E(A)^m$ of infinite length. Since $\mathbb{Q}E(A)$ is Artinian, this is not possible.

Hence, we can find a finite subset J of I with the required property. Write $H = \bigoplus_{i \in J} G_i$, which is A-solvable and $K = \Pi_{I \setminus J} \pi_i \varphi(A^m)$. Then, $\varphi(A^m) \subseteq H \oplus S_A(K)$. It is enough to show that $S_A(K)$ is A-solvable. Let $X_i = \pi_i \varphi(A^m)$ for $i \in I \setminus J$. Since $X_i \subseteq G_i$ is A-generated, we have that X_i is A-solvable. We choose an A-generated subgroup B_i of A^m with $A^m/B_i \cong X_i$. Since A is faithfully flat, B_i is an A-balanced subgroup of A^m. If $H_A(X_i)$ were not torsion, then the same would be true for $T_A H_A(X_i)$ by the faithfulness of A, which is not possible. Since $H_A(X_i)$ is an epimorphic image of $E(A)^m$, it is finitely generated, and hence bounded. Thus the same holds for X_i. Therefore, X_i is contained in an algebraically compact group Y_i which is bounded by the same integer s as X_i. Since X_i is A-solvable, s is a product of primes from $\Gamma(A)$. Thus, $Y = \Pi_{I \setminus J} Y_i$ is an algebraically compact group, which is divisible for all primes not in $\Gamma(A)$, containing K. By b), $S_A(Y)$ is A-solvable; and the same holds for $S_A(K)$. Consequently, $\varphi(A^m)$ is A-solvable. □

E x a m p l e 3.1. The conditions on A in the hypothesis of Corollary 3.5 are satisfied in each of the following cases:

a) A is faithfully flat as an $E(A)$-module; and $E(A)$ has finite rank.

b) A is a generalized rank 1 group which is not homogeneous completely decomposable.

P r o o f. a) is obvious.

b) Since $E(A)$ satisfies the restricted minimum condition, it is enough to show that $E(A)^n$ has the DCC for pure submodules U such that $E(A)^n/U \in \mathcal{M}_A$. If $\{U_n \mid n < \omega\}$ is an infinite descending chain of such submodules, then there is $m < \omega$ such that $V_m = U_{m+1}/U_m$ is singular. Otherwise, $E(A)^n$ would have infinite Goldie-dimension, which is not possible. Since U_m is pure in $E(A)^n$, and $E(A)$ is right Noetherian, we obtain that V_m is a finitely generated, singular torsion-free $E(A)$-module. The fact that $E(A)$ has the restricted minimum condition yields that V_m is Artinian, which is only possible if its additive group is divisible. Then, $T_A(V_m)$ is a divisible, torsion-free A-solvable group. In particular, $\mathbb{Q}$ is A-solvable. We may assume that A has infinite rank. This yields the inequalities $|H_A(\mathbb{Q})| = 2^{|A|} > |A| \geqslant |E(A)|$. On the other hand there exists an exact sequence $A \to \mathbb{Q} \to 0$ which is A-balanced. Thus, $H_A(\mathbb{Q})$ is an epimorphic image of $E(A)$, which is not possible by the previous inequalities. □

Even in the case that $\mathcal{C}_A$ is cocomplete, the limit of a functor $\mathcal{F}$ need not be isomorphic to its colimit in $\mathcal{A}b$.

Proposition 3.7. *Let A be a torsion-free abelian group which has a semi-prime, two-sided Noetherian endomorphism ring of Krull dimension at most 1. If $\mathcal{C}_A$ is* *cocomplete and does not contain J_p for any prime p, then there exists a functor $\mathscr{F}$ from a small category into $\mathscr{C}_A$ whose colimit in $\mathscr{C}_A$ is not isomorphic to its colimit in $\mathscr{A}b$.*

Proof. Observe that A satisfies the hypotheses of Corollary 3.5. Let I be the set of positive integers, and set $\mathscr{F}(i) = A$. If i divides j, then set $\mathrm{Mor}_I(i,j) = \{\lambda\}$, and define $\mathscr{F}(\lambda)$ to be multiplication by j/i. Obviously, the colimit of $\mathscr{F}$ in $\mathscr{A}b$ is the injective hull of A. Suppose that $G = [\oplus\mathscr{F}(n)]/U \in \mathscr{C}_A$. By Corollary 3.5, G is a torsion-free divisible group. If $G \neq 0$, then A is a homogeneous completely decomposable group. In this case the group J_p is A-solvable which is not possible. Thus, the colimit of $\mathscr{F}$ is 0 in $\mathscr{C}_A$. □

We conclude this section with an example of groups satisfying the Mittag-Loefler condition:

Example 3.2. Let A be a cotorsion-free abelian group which is constructed by [10, Theorem 3.3]. Then, A satisfies ML with respect to $\mathscr{M}_A$.

Proof. The group A in [DG, Theorem 3.3] is constructed in such a way that A is the direct limit of a family of finitely generated free submodules U such that A/U is flat. Let P be a finitely presented module, and $\sigma\colon P \to A$ be a map. Then, $\sigma(P) \subseteq U$ for some finitely generated, free submodule U of A such that A/U is flat. Let τ be σ viewed as a map from P to U, and $\iota\colon U \to A$ be the inclusion map. Then, $\iota\tau = \sigma$ yields $\ker 1_M \otimes \tau \subseteq \ker 1_M \otimes \sigma$ for all right $E(A)$-modules M. Since A/U is flat, the map $1_M \otimes \iota\colon M \otimes_{E(A)} U \to T_A(M)$ is a monomorphism. Thus, $\ker 1_M \otimes \sigma = \ker 1_M \otimes [\iota\tau] \subseteq \ker 1_M \otimes \tau$. By [15], A satisfies ML with respect to the class of all $E(A)$-modules. □

In contrast to the last result, torsion-free groups of finite rank which are constructed by Corner's Theorem need not satisfy ML with respect to $\mathscr{M}_A$:

Example 3.3. Let A be a torsion-free abelian group of rank 2 whose endomorphism ring is $\mathbb{Z}_p$. Then, A does not satisfy ML with respect to $\mathscr{M}_A$.

Proof. By [3], the category $\mathscr{C}_A$ is not preabelian, and hence not cocomplete. By what has been shown, A cannot satisfy ML with respect to $\mathscr{M}_A$. □

Consider a functor $\mathscr{F}\colon I \to \mathscr{C}_A$ where I is a small category. While the results of the last sections discuss when $\mathscr{F}$ has a colimit in $\mathscr{C}_A$, this section addresses the question under when the colimit of $\mathscr{F}$ in the category of abelian groups is its colimit in $\mathscr{C}_A$. We want to remind the reader of the notational conventions for colimits which we have introduced following Lemma 2.1.

Theorem 4.1. *Let A be a torsion-free abelian group which is faithfully flat as an $E(A)$-module, and $\mathscr{F}$ a functor from a small category I into $\mathscr{C}_A$ such that $\{\mathscr{F}(i) \mid i \in I\}$ is A-small. The following conditions are equivalent:*

a) $G = \varinjlim_{\mathscr{A}b} \mathscr{F}$ *is A-solvable.*

b) $H_A(G)$ *together with the family* $\{H_A(\varphi_i) \mid i \in I\}$ *induced by the compatible maps* $\varphi_i\colon \mathscr{F}(i) \to G_i$ *is the colimit of the functor* $H_A\mathscr{F}$ *in the category of right* $E(A)$*-modules.*

Proof. a) $\Rightarrow$ b): Let M be the colimit of the functor $H_A\mathscr{F}$ in the category of right $E(A)$-modules where $\{\psi_i \mid i \in I\}$ denotes the compatible family of maps which is obtained as in [18]. As in Section 2, M admits an exact sequence $0 \to B^* \xrightarrow{\iota} \bigoplus_{i\in I} H_A\mathscr{F}(i) \to M \to 0$. We may assume that ι is an inclusion map and the submodule B^* of $\bigoplus_{i\in I} H_A\mathscr{F}(i)$ is generated by the images of the maps $\varepsilon_\lambda = \mu_{t(\lambda)}H_A\mathscr{F}(\lambda) - \mu_{s(\lambda)}$ where $\lambda\colon s(\lambda) \to t(\lambda)$ is an I-map, and μ_j is the embedding into the j^{th}-coordinate.

On the other hand, since A is faithfully flat as an $E(A)$-module, and the groups G and $\bigoplus_{i\in I} \mathscr{F}(i)$ are A-solvable, the induced sequence $0 \to H_A(B) \xrightarrow{H_A(\alpha)} H_A\big(\bigoplus_{i\in I} \mathscr{F}(i)\big) \to H_A(G) \to 0$ is exact. Moreover, the natural map $\delta\colon \bigoplus_{i\in I} H_A\mathscr{F}(i) \to H_A\big(\bigoplus_{i\in I} \mathscr{F}(i)\big)$ is an isomorphism since $\{\mathscr{F}(i) \mid i \in I\}$ is A-small. If we have shown $\delta(B^*) = H_A(B)$, then δ induces an isomorphism $\tilde{\delta}\colon M \to H_A(G)$ with $\tilde{\delta}\psi_i = H_A(\varphi_i)$ for all $i \in I$. This proves b).

Observe $\delta\mu_i(\alpha) = H_A(\delta_i)(\alpha) = \delta_i\alpha$ for all $\alpha \in H_A(\mathscr{F}(i))$. For every $\varphi \in H_A\mathscr{F}(s(\lambda))$ and $a \in A$, we, hence, obtain

$$[\delta\varepsilon_\lambda(\varphi)](a) = [\delta_{t(\lambda)}H_A\mathscr{F}(\lambda)(\varphi)](a) - \delta_{s(\lambda)}\varphi(a) = [\delta_{t(\lambda)}\mathscr{F}(\lambda) - \delta_{s(\lambda)}](\varphi(a)) \in B.$$

which shows $\delta(B^*) \subseteq H_A(B)$. To establish the converse of this inclusion, we observe that the group B is A-solvable by Lemma 2.1. We define a map $\theta\colon T_A(\delta(B^*)) \to B$ by $\theta(\alpha \otimes a) = \alpha(a)$. The group B is generated by elements of the form $[\delta_{t(\lambda)}\mathscr{F}(\lambda) - \delta_{t(\lambda)}](x)$ where $x \in \mathscr{F}(s(\lambda))$. We choose $a_1, \dots, a_n \in A$ and $\varrho_1, \dots, \varrho_n \in H_A\mathscr{F}(s(\lambda))$

with $x = \sum_{i=1}^{n} \varrho_i(a_i)$ and observe that

$$[\delta_{t(\lambda)}\mathscr{F}(\lambda) - \delta_{s(\lambda)}](x) = \sum_{i=1}^{n}[\delta\varepsilon_\lambda(\varrho_i)(a_i)] = \theta\Big(\sum_{i=1}^{n}[\delta\varepsilon_\lambda(\varrho_i)] \otimes a_i\Big) \in \operatorname{im}\theta.$$

This shows that θ is onto. If $\varepsilon\colon \delta(B^*) \to H_A(B)$ is the inclusion-map, then $\theta_B T_A(\varepsilon) = \theta$ yields that $T_A(\varepsilon)$ is an epimorphism. Since the sequence $T_A(\delta(B^*)) \overset{T_A(\varepsilon)}{\to} T_A H_A(B) \to T_A(H_A(B)/\delta(B^*)) \to 0$ is exact, we obtain $T_A(H_A(B)/\delta(B^*)) = 0$ which yields $H_A(B) = \delta(B^*)$ since A is faithfully flat as an $E(A)$-module.

b) $\Rightarrow$ a): We consider the exact sequence $0 \to B \overset{\alpha}{\to} \bigoplus_{i\in I} \mathscr{F}(i) \overset{\beta}{\to} G \to 0$. Since $\{\mathscr{F}(i) \mid i \in I\}$ is A-small, the center-term in the sequence is A-solvable, and the same holds for B as an A-generated subgroup of an A-solvable group. By Lemma 2.1, it is enough to show that the sequence is A-balanced.

Let $\varphi \in H_A(G)$. Since $H_A(G) = \varinjlim H_A\mathscr{F}$, we have $H_A(G) = \langle \operatorname{im} H_A(\varphi_i) \mid i \in I\rangle$. We choose $i_1, \dots, i_n \in I$ and $\psi_j \in H_A\mathscr{F}(i_j)$ with $\varphi = \sum_{j=1}^{n} [H_A(\varphi_{i_j})](\psi_j) = \sum_{j=1}^{n} \varphi_{i_j}\psi_j$. For all $a \in A$, we obtain $\varphi(a) = \sum_{j=1}^{n} \varphi_{i_j}\psi_j(a) = \sum_{j=1}^{n} \psi_j(a) + B = \sum_{j=1}^{n} \beta\psi_j(a)$. Hence, $\varphi = H_A(\beta)(\sum_{j=1}^{n} \psi_j)$, and the sequence is A-balanced. □

In particular, the last result applies in the following situation:

Corollary 4.2. *Let A be an abelian group which is faithfully flat as an $E(A)$-module, and κ a cardinal with $|A| < cf(\kappa)$. An abelian group G of cardinality κ is A-solvable if it is the union of an strictly ascending chain $\{G_\nu \mid \nu < \kappa\}$, of A-solvable subgroups.*

Proof. Let $\iota_\nu\colon G_\nu \to G$ and $\iota_\nu^\mu\colon G_\nu \to G_\mu$ for $\nu \leqslant \mu$ be the inclusion maps. Since G is the colimit in $\mathscr{A}b$ of the G_ν's, it is A-solvable by Theorem 4.1 once we have shown that $H_A(G)$ is the colimit of the system $\{H_A(G_\nu), H_A(\iota_\nu^\mu) \mid \nu \leqslant \mu < \kappa\}$. To simplify our notation we set $U_\nu = \operatorname{im} H_A(\iota_\nu) \subseteq H_A(G)$. The corresponding inclusions are denoted by ε_ν and ε_ν^μ. For all $\nu < \mu < \kappa$, we have $H_A(\iota_\mu)H_A(\iota_\nu^\mu) = \varepsilon_\nu^\mu H_A(\iota_\nu)$ and $H_A(\iota_\nu) = \varepsilon_\nu H_A(\iota_\nu)$.

The family $\{U_\nu \mid v < \kappa\}$ is an ascending chain of submodules of $H_A(G)$. If $\varphi \in H_A(G)$, then $\varphi(A) \subseteq G_\nu$ for some ν since $|\varphi(A)| \leqslant |A| < cf(\kappa)$. Therefore, the U_ν's form an ascending chain whose union is $H_A(G)$. If M is a right $E(A)$-module for which we can find maps $\sigma_\nu\colon H_A(G_\nu) \to M$ with $\sigma_\mu H_A(\iota_\nu^\mu) = \sigma_\nu$, then we have $\sigma_\mu[H_A(\iota_\mu)]^{-1}\varepsilon_\nu^\mu = \sigma_\nu[H_A(\iota_\nu)]^{-1}$ since $H_A(\iota_\nu)$ is an isomorphism between $H_A(G_\nu)$

and U_ν. There is a unique map $\beta\colon H_A(G) \to M$ with $\beta\varepsilon_\nu = \sigma_\nu[H_A(\iota_\nu)]^{-1}$. Thus, $\sigma_\nu = \beta\varepsilon_\nu H_A(\iota_\nu) = \beta H_A(\iota_\nu)$. To show the uniqueness of β, assume $\gamma H_A(\iota_\nu) = \sigma_\nu$ for all $\nu < \kappa$. We have $\gamma\varepsilon_\nu = \sigma_\nu[H_A(\iota_\nu)]^{-1} = \beta\varepsilon_\nu$. Since β is unique with this property, we obtain that $H_A(G)$ is the colimit of the system under consideration. □

Corollary 4.3. *Let A be a self-small abelian group which is faithfully flat as an $E(A)$-module. The following conditions are equivalent for an abelian group G:*

a) *G is not A-solvable.*

b) *If G is the union of a strictly ascending chain $\{G_\nu \mid \nu < \kappa\}$ of A-solvable subgroups, then $\aleph_0 \leqslant cf(\kappa) \leqslant |A|$.*

We are now able to prove the existence theorem for A-solvable groups:

Theorem 4.4. *($ZFC + \nabla_\kappa$) Let A be a self-small cotorsion-free abelian group which is faithfully flat as an $E(A)$-module, S a cotorsion-free ring containing $E(A)$ such that $S^{\mathrm{op}} \in \mathcal{M}_A$ as an $E(A)$-module and κ a regular cardinal number with $\kappa > \sup\{|A|, |S|\}$. There exist 2^κ pairwise non-isomorphic cotorsion-free A-solvable groups G of cardinality κ such that $\mathrm{Hom}(G, A) = 0$ and $E(G) \cong C_S(E(A))$, the centralizer of $E(A)$ in S.*

Proof. Let R be the opposite ring of S. By [10, Theorem 3.2], there exist 2^κ pairwise non-isomorphic cotorsion-free left R-modules M of cardinality κ such that $R = E_{\mathbb{Z}}(A)$. Moreover, M can be chosen in such a way that it has a κ-filtration $\{M_\nu \mid \nu < \kappa\}$ of free submodules (which is the way that M has been constructed in [10].) To show that $G = T_A(M)$ is A-solvable, we set $G_\nu = T_A(M_\nu)$. Since $S^{\mathrm{op}} \in \mathcal{M}_A$, we obtain that G_ν is A-solvable. The family $\{G_\nu \mid \nu < \kappa\}$ is a smooth ascending chain whose union is G. We observe $|G_\nu| \leqslant \aleph_0|A|\,|M_\nu| < \kappa$ and $G_{\nu+1}/G_\nu \cong T_A(M_{\nu+1}/M_\nu)$ is non-zero since A is faithful. Thus, G has cardinality κ, and is A-solvable.

By the Adjoint-Functor-Theorem, the map

$$\psi_G\colon E_{\mathbb{Z}}(G) \to \mathrm{Hom}_{E(A)}(M, H_AT_A(M)),$$

which is defined by $[[\psi_G(\varphi)](m)](a) = \varphi(m \odot a) = [\varphi\varphi_M(m)](a)$, is an isomorphism. Since A is faithfully flat as an $E(A)$-module, $M \in \mathcal{M}(A)$ by [6]. Hence, there is an induced isomorphism $\sigma\colon \mathrm{Hom}_{E(A)}(M, H_AT_A(M)) \to E_{E(A)}(M)$ which is defined by $\sigma(\alpha) = \varphi_M^{-1}\alpha$. The composition of these two isomorphism satisfies $\sigma\psi_G(\alpha) = \varphi_M^{-1}\alpha\varphi_M$ for all $\varphi \in E_{\mathbb{Z}}(G)$. Hence, $E_{\mathbb{Z}}(G) \cong E_{E(A)}(M_{E(A)}) = E_R({}_RM) = C_S(R)$ as rings.

Assume $\mathrm{Hom}(G, A) \neq 0$. We write $G_\nu \cong \bigoplus_{\kappa_\nu} T_A(R)$ and observe $\kappa_\nu \leqslant |\bigoplus_{\kappa_\nu} \mathrm{Hom}(G, A)| \leqslant |\mathrm{Hom}(G, G_\nu)| \leqslant |E(G)| < \kappa$. On the other hand, $|T_A(R)| < \kappa$

yields $\sup\{\kappa_\nu \mid \nu < \kappa\} = \kappa$. The resulting contradiction shows $\mathrm{Hom}(G, A) = 0$. Finally, since A is faithfully flat and $G = T_A(M)$, [6] guarantees that non-isomorphic choices for M yield non-isomorphic groups G. □

The condition that $S^{\mathrm{op}} \in \mathcal{M}_A$ is, for instance, satisfied if S is contained in a free $E(A)$-module, e.g. $S \subseteq E(A)[x]$ or $S \subseteq \mathrm{Mat}_n(E(A))$ for some $n < \omega$, or S is an $E(A)$-order in the case that $\mathbb{Q}E(A)$ is semi-simple Artinian. In the latter case, we obtain additional insights in the structure of A-solvable abelian groups:

Corollary 4.5. *Let A be a torsion-free abelian group which is faithfully flat as an $E(A)$-module and has a semi-simple Artinian quasi-endomorphism ring. The following conditions are equivalent for a torsion-free abelian group G with $r_0(A)) < cf(|G|)$:*

a) *G is A-solvable.*

b) *G is the union of a smooth, strictly ascending chain of pure A-solvable subgroups.*

Proof. a) ⇒ b): Choose an essential submodule M of $H_A(G)$ which is the direct sum of cyclic submodules, say $M = \bigoplus_{\nu<\kappa} U_\nu$. We set $M_\alpha = \bigoplus_{\nu\leqslant\alpha} U_\nu$, and denote its $\mathbb{Z}$-purification in $H_A(G)$ by N_α. Since G is A-solvable, $H_A(G) \in \mathcal{M}_A$. Moreover, $\mathcal{M}_A$ is closed with respect to submodules by [6]. Thus, setting $G_\nu = T_A(N_\nu)$ yields a smooth ascending chain of subgroups of G such that $G/G_\nu \cong T_A(H_A(G)/N_\nu)$ is a torsion-free abelian group.

b) ⇒ a): Let $X \subseteq A$ be a subset with $|X| = r_0(A)$ and $A/\langle X\rangle$ torsion. If $\varphi \in H_A(G)$, then there is $\nu < \kappa$ with $\varphi(X) \subseteq G_\nu$. Since G_ν is pure in G, we obtain $\varphi(A) \subseteq G_\nu$, and $H_A(G)$ is the union of the modules $H_A(G_\nu)$. As in the proof of Corollary 4.2, G is A-solvable. □

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