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The Construction of Analytic Diffeomorphisms for Exact Robot Navigation on Star Worlds

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Comments

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The Construction of Analytic Diffeomorphisms for Exact Robot Navigation on Star Worlds

An invited paper for presentation in the
1989 IEEE International Conference on Robotics and Automation

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Abstract

A *navigation function* is a scalar valued function on a robot configuration space which encodes the task of moving to a desired destination without hitting any obstacles. Our program of research concerns the construction of navigation functions on a family of configuration spaces whose “geometric expressiveness” is rich enough for navigation amidst real world obstacles. A *sphere world* is a compact connected subset of E^n whose boundary is the finite union of disjoint $(n - 1)$ -spheres. In previous work we have constructed navigation functions for every sphere world. In this paper we embark upon the task of extending the construction of navigation functions to “star worlds”. A *star world* is a compact connected subset of E^n obtained by removing from a compact *star shaped set* a finite number of smaller disjoint open *star shaped sets*. This paper introduces a family of transformations from any star world into a suitable sphere world model, and demonstrates that these transformations are actually analytic diffeomorphisms. Since the defining properties of navigation functions are invariant under diffeomorphism, this construction, in composition with the previously developed navigation function on the corresponding model sphere world, immediately induces a navigation function on the star world.

1 Introduction

We seek a solution to the following problem in robotics. A kinematic chain, actuated by idealized bounded torque motors, is allowed to move in a cluttered workspace. Contained within the *joint space* — an analytic manifold with boundary which forms the configuration space of the kinematic chain — is the *free space*, \mathcal{F} — the set of all configurations which do not involve intersection with any of the “obstacles” cluttering the workspace. Given any “destination point” in the interior of \mathcal{F} to which it is desired to move the robot, find a control law which maps the robot state — position and velocity — into torque inputs at each joint, such that the resulting closed loop robotic system trajectories move to the destination from any initial configuration without hitting the obstacles.

The purely geometric problem of constructing a path between two points in a space obstructed by sets with arbitrary polynomial boundary (given perfect information) has already been completely solved by Schwartz and Sharir[19]. Moreover, a near optimally efficient solution has recently been offered by Canny[3] for this class of problems as well. The motivation for the present direction of inquiry (beyond its apparent academic interest) is the desire to incorporate explicitly aspects of the control problem — the construction of feedback compensators for a well characterized class of dynamical systems in the presence of well characterized constraints — in the planning phase of robot navigation problems. That is, the geometrical “find path” problem is generalized to the search for a family of paths in \mathcal{F} (the one-parameter group of a gradient flow) that provide a feedback law for the physical robot: the navigation task is reformulated as a dynamical “find control law” problem.

This formulation may seem to be at odds with the largely unspoken understanding that now prevails in the field of robotics to the effect that methods of task planning ought to be distinct from methods of control. The former belong to the realm of geometry and logic whereas the

latter inhabit the the earthier domain of engineering analysis; geometry is usually associated with off-line computation whereas everyone knows that control must be accomplished in real-time; the one is a “high level” activity whereas the other is at a “low level”. Nevertheless, the determination to combine planning and control techniques is not unknown within the robotics literature. The idea of using “potential functions” for the specification of robot tasks with a view of the control problems in mind was pioneered by Khatib[8] in the context of obstacle avoidance. Fundamental work of Hogan[7] in the context of force control further advanced the interest in this approach. A similar methodology has been developed independently by Arimoto in Japan[1], and by Soviet investigators as well[17]. We appeal to this small but growing body of existing literature — which includes both experimental as well as theoretical results — for the justification of our problem statement, and proceed with the discussion of its solution.

The negative gradient vector field of a scalar valued function which is transverse (exterior directed) on the boundary of the free space, and which has a single attractor at the destination point gives rise to a flow which moves almost all initial conditions toward that desired point. Thus, a suitably chosen scalar valued “cost” function solves the geometric problem of finding paths to the destination in free space from almost every initial condition (convergence from *every* initial condition is precluded in general, by the topology of \mathcal{F} [12]). Moreover, interpreting the cost function as an artificial potential energy, it can be shown that a gradient vector field on \mathcal{F} “lifts naturally” to a Lagrangian vector field on the phase space of \mathcal{F} , $T\mathcal{F}$, describing the robot’s Newtonian dynamics when subjected to a suitable feedback compensating control law[9]. Under certain additional regularity conditions, the Lagrangian system “inherits” the limit properties of the gradient system, and an explicitly specified portion of $T\mathcal{F}$, including $\mathcal{F} \times \{0\}$ — all zero velocity states, is positive invariant with respect to the lifted flow[10]. Thus, a further constrained cost function solves the robot navigation and the attendant control problems simultaneously.

In a recent paper[12], we propose a formal specification and provide a motivating discussion for a subclass of scalar valued functions on \mathcal{F} — the class of *navigation functions* — which achieves the stated two-fold goal (again, up to the limits that the topology of \mathcal{F} allows). Given a connected and compact n -dimensional analytic manifold with boundary, \mathcal{M} , considered as a “model space”, we show that if one constructs a navigation function on \mathcal{M} , then this construction induces a navigation function on any manifold in its analytic diffeomorphism class. This suggests the consideration of a distinguished space — the “*sphere world*” — a compact connected subset of E^n obtained by removing from a closed n -disc a finite number of smaller disjoint open n -discs, representing “obstacles”. The same paper concludes with the explicit construction of a family of navigation functions for any sphere world.

In this paper we embark upon the task of extending the construction on sphere worlds, considered as “model spaces”, to other members of their analytic diffeomorphism class. The problem of constructing a navigation function on such a space, \mathcal{F} , reduces to the construction of an analytic diffeomorphism from \mathcal{F} onto the corresponding sphere world, \mathcal{M} . We report in this paper on such a construction for a specific subclass — the *star worlds* — each of whose members is a compact connected subset of E^n obtained by removing from a compact *star shaped set* a finite number of smaller disjoint open *star shaped sets*. This transformation of a star world onto the corresponding sphere world, induces navigation functions on a much larger class than the original sphere worlds. For example, any convex set is star shaped (although a star shaped

set can be, in general, non-convex). This advances our program of research toward the goal of developing “geometric expressiveness” rich enough for navigation amidst real world obstacles.

A paper presently in preparation concerns the application of these results to obstacles in E^n comprised of finite unions of star shaped sets. We suspect that this class forms a dense subset of the entire homeomorphism equivalence class of the sphere worlds. However, we currently do not know how “far away” the latter class of topological sphere worlds lies from the most general realistic problem — the class of configuration spaces mentioned above which arise when a general kinematic chain operates in a cluttered environment.

The paper is organized as follows. This introductory section continues with a formal statement of the problem at hand, and a specification of the assumptions concerning the available information. In the next section, we define the class of star worlds, and present an explicit two-parameter family of analytic functions defined on an arbitrary star world. In Section 3 we prove Theorem 1 : for any star world, two lower bounds on the parameters are specified in terms of the boundary locations, guaranteeing that each member of this family whose parameter values comply with these bounds is an analytic diffeomorphism onto a suitably constructed sphere world. Finally, in the concluding section we discuss the computational complexity of this procedure. Appendix A contains a brief exposition of the technical terms used, details of some proofs are given in Appendix B , and in Appendix C , we present some preliminary ideas concerning the representation of “star shapes”.

1.1 Problem Statement

We will start by defining the workspace and the obstacles.

Definition 1 Let β_j $j \in \{0, \dots, M\}$, be real valued analytic functions on E^n , for which zero is a regular value.

The robot workspace, \mathcal{W} , is a connected and compact n -dimensional submanifold of E^n satisfying

$$\overset{\circ}{\mathcal{W}} \subset \{q \in E^n : \beta_0(q) > 0\} \quad \text{and} \quad \partial\mathcal{W} \subset \{q \in E^n : \beta_0(q) = 0\}.$$

An obstacle, \mathcal{O}_j , is the interior of a connected and compact n -dimensional submanifold of E^n such that $\overline{\mathcal{O}_j} \subset \overset{\circ}{\mathcal{W}}$, and

$$\mathcal{W} - \overline{\mathcal{O}_j} \subset \{q \in E^n : \beta_j(q) > 0\} \quad \text{and} \quad \partial\mathcal{O}_j \subset \{q \in E^n : \beta_j(q) = 0\} \quad j \in \{1, \dots, M\},$$

satisfying,

$$\overline{\mathcal{O}_i} \cap \overline{\mathcal{O}_j} = \emptyset \quad 1 \leq i < j \leq M. \quad (1)$$

The free space is

$$\mathcal{F} \triangleq \mathcal{W} - \bigcup_{j=1}^M \mathcal{O}_j.$$

It will prove convenient to refer to the complement of \mathcal{W} in E^n as the zero'th obstacle. Note that in the definition above, the boundary of the j^{th} obstacle, $\partial\mathcal{O}_j$, being a collection of connected components of a regular $(n - 1)$ -surface in E^n , is an $(n - 1)$ -dimensional analytic manifold, according to the Implicit Function Theorem.

Definition 2 *In the special case in which the robot workspace and each obstacle removed from it is an n -disc in E^n ,*

$$\mathcal{W} = \{q \in E^n : \underbrace{\rho_0^2 - \|q - q_0\|^2}_{\widehat{\beta}_0} \geq 0\},$$

and

$$\mathcal{O}_j = \{q \in E^n : \underbrace{\|q - q_j\|^2 - \rho_j^2}_{\widehat{\beta}_j} < 0\} \quad j = 1 \dots M,$$

the resulting free space,

$$\mathcal{M} \triangleq \mathcal{W} - \bigcup_{j=1}^M \mathcal{O}_j,$$

is an n -dimensional sphere world with M obstacles.

Given a pair $(\mathcal{F}, \mathcal{M})_{n,M}$ of connected and compact n -dimensional analytic manifolds with $M + 1$ boundary components, both considered as subsets of E^n , where \mathcal{M} is an n -dimensional sphere world with M obstacles and \mathcal{F} is a general n -dimensional free space with M obstacles, let $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{F}}$ denote some open neighborhoods about \mathcal{M} and \mathcal{F} , respectively, in E^n . We seek a transformation h from $\tilde{\mathcal{F}} \subset E^n$ into $\tilde{\mathcal{M}} \subset E^n$ satisfying,

1. $h|_{\mathcal{F}}$ is an analytic diffeomorphism from \mathcal{F} onto \mathcal{M} ;
2. in each space there is a distinguished interior point — the destination point — $q_d \in \overset{\circ}{\mathcal{F}}$ and $p_d \in \overset{\circ}{\mathcal{M}}$, such that $h(q_d) = p_d$.

The motivation for this problem is most simply provided by reference to the following definition and fact which obtains from application of the chain rule, for example, as discussed in [12]. Given a map $\varphi : \mathcal{M} \rightarrow [0, 1]$, using the terminology of M. Morse[16], we say that φ is *polar* if it has a unique minimum on \mathcal{M} . Using the terminology of M. Hirsch[6], we say that φ is *admissible* if it attains its maximal value (uniformly) exactly on all the boundary components — in our case, $\partial\mathcal{M} = \varphi^{-1}(1)$ and $\overset{\circ}{\mathcal{M}} = \varphi^{-1}(0, 1)$.

Definition 3 ([12], Definition 1) *Let $\mathcal{M} \subset E^n$ be a compact connected n -dimensional analytic manifold with boundary. A map $\varphi : \mathcal{M} \rightarrow [0, 1]$, is a navigation function if it is*

1. Analytic on \mathcal{M} ;
2. Polar on \mathcal{M} , with minimum at $p_d \in \overset{\circ}{\mathcal{M}}$;
3. Morse on \mathcal{M} ;
4. Admissible on \mathcal{M} .

It is shown in [10, 11] that control laws resulting from navigation functions define closed loop robotic systems whose trajectories approach the destination without intersecting obstacles, starting in an open dense set of initial states. In general, this is the “strongest” convergence behavior that the topology of the underlying free space allows, as we have shown in [12]. Moreover, we have shown as well that smooth navigation functions exist on any smooth manifold with boundary — hence it makes sense to attempt analytic constructions in specific cases. In particular, we have shown how to do so on any sphere.

Proposition 1.1 ([12], Proposition 2.6) *Let $\varphi: \mathcal{M} \rightarrow [0, 1]$ be a navigation function on \mathcal{M} , and let $h: \mathcal{F} \rightarrow \mathcal{M}$ be analytic. If h is an analytic diffeomorphism, then*

$$\tilde{\varphi} \triangleq \varphi \circ h,$$

is a navigation function on \mathcal{F} .

Thus, since we already know how to construct navigation functions on any sphere world, if a suitable sphere world model, \mathcal{M} , and an analytic diffeomorphism, h , can be found, we obtain a navigation function on \mathcal{F} as well.

1.2 The Available Information

We assume perfect information. That is, for a given star world (Definition 5 below), \mathcal{F} , all the “obstacle functions”,

$$\beta_j \quad j \in \{0, \dots, M\},$$

as well as the obstacle center points, $q_j \quad j \in \{0, \dots, M\}$, are known.

Moreover, for each obstacle, we assume the knowledge of an upper bound, E_j , on the image of the obstacle function, β_j , and an upper bound, E_d , on the distance from the destination point, γ_d , which guarantees that

$$\beta_i^{-1}[0, E_i] \cap \beta_j^{-1}[0, E_j] = \emptyset \quad \text{and} \quad \gamma_d^{-1}[0, E_d] \cap \beta_j^{-1}[0, E_j] = \emptyset \quad i, j \in \{0, \dots, M\} \quad i \neq j.$$

That is, the “ E_j -thickened” boundary components still do not intersect, nor do they overlap the destination. Further, we will unhesitatingly make use of upper and lower bounds attained by various continuous functions on various compact sets without ever computing them explicitly. Finally, we define the notion of a “strictly star shaped obstacle” (Definition 5), and require the knowledge of a lower bound on the defining inequality (for example, $\gamma_i(\epsilon)$ in Lemma 3.3 ,

eq. (30)) for each obstacle. In general, the extraction of these geometrical features from the knowledge of the obstacle functions, β_j $j \in \{0, \dots, M\}$, may prove to be computationally intensive. However, in Appendix C we present a family of star shaped obstacle functions — homogeneous polynomials — for which explicit formulas for the various bounds are given.

The model sphere world, \mathcal{M} , is explicitly constructed from this data. That is, we determine (p_j, ρ_j) , the center and radius of the j^{th} sphere, according to the center and minimum “radius” (the minimal distance from q_j to the j^{th} obstacle boundary) of the j^{th} star shaped obstacle. This in turn determines the model space “obstacle functions”,

$$\hat{\beta}_j \quad j \in \{0, \dots, M\},$$

as well as the navigation function on \mathcal{M} , $\hat{\varphi}$, as constructed in our previous paper[12]. The transformation is then constructed in terms of the given star world and the derived model sphere world geometrical parameters.

In both spaces we explicitly assume that each obstacle contributes a distinct boundary component — the obstacles do not intersect each other. This assumption implies in turn that the resulting spaces are connected. Also, we require that the destination point be specified as an interior point. Once the location of the boundary components is given, the verification of the latter assumption is straightforward.

In the robotics setting, the connectedness of the free space is not a realistic assumption. Certainly, the robot initial configuration in joint space determines a specific connected component of its free space, yet this might not include the destination point. At the present, we exclude this possibility: our only test of connectedness is the application of the construction to the initial configuration. If the resulting trajectory does not arrive at the destination point, we may conclude with probability one that the destination is not in the same component as the initial configuration.

2 Construction of the Transformation

In this section we define the star worlds, and present an explicit two-parameter family of analytic functions, each of whose members is a candidate diffeomorphism of a star world onto a particular model sphere world.

2.1 Star Worlds and Their Models

Definition 4 *A set $S \subset E^n$ with non-empty interior is star shaped (at x_0) if there exists a point $x_0 \in S$ such that for all $x \in S$, the line segment joining x_0 and x is contained in S .*

If, in addition, ∂S is a regularly embedded analytic manifold, then S is a regular star shaped set.

Any star shaped set is path-connected, and it can be shown[2] that any open star shaped set is homeomorphic to the open unit n -disc. According to the definition of an obstacle (Definition 1), if $q \in \partial \mathcal{O}_j$, the j^{th} obstacle boundary, then $\nabla \beta_j(q)$ $j \in \{1, \dots, M\}$ is directed outward with respect to \mathcal{O}_j , and if $q \in \partial \mathcal{O}_0$, then $-\nabla \beta_0(q)$ is directed outward with respect to the robot workspace $\mathcal{W} = \mathcal{O}_0^c$.

Definition 5 *An obstacle, \mathcal{O}_j (Definition 1), is strictly star shaped if there is a point $q_j \in \mathcal{O}_j$ such that for all $q \in \partial \mathcal{O}_j$ the outward directed gradient, $\nabla \beta_j(q)$, satisfies*

$$\nabla \beta_j \cdot (q - q_j) > 0. \quad (2)$$

If the robot workspace, \mathcal{W} , and all the obstacles removed from it, are such strictly star shaped obstacles, then the resulting free space, \mathcal{F} , is called a star world.

The connection between the classes of strictly star shaped obstacles and regular star shaped sets is drawn in the following Lemma.

Lemma 2.1 *If \mathcal{O}_j is a strictly star shaped obstacle, then \mathcal{O}_j is a regular star shaped set (at q_j). Moreover, for each $q \in \partial \mathcal{O}_j$, the line segment joining q_j and q intersects $\partial \mathcal{O}_j$ only at q .*

The proof is given in Appendix B . According to this Lemma, the collection of strictly star shaped obstacles constitutes “almost all” the bounded star shaped sets whose boundary is a regular surface.

In the class of star worlds, a distinguished member is the *sphere world*, each of whose boundary components is a scaled and translated version of the unit sphere, S^{n-1} .

2.2 The Transformation

In this section we present a two-parameter family, each of whose members is a map induced by a specified pair $(\mathcal{F}, \mathcal{M})_{M,n}$.

Denote the omitted product, $\prod_{i=0, i \neq j}^M \beta_i$, by $\bar{\beta}_j$.

Definition 6 The analytic switches, σ_j $j \in \{0, \dots, M\}$, are the real valued functions defined on \mathcal{F} by

$$\sigma_j(q, \lambda) \triangleq \frac{x}{x + \lambda} \circ \frac{\rho_d \bar{\beta}_j}{\beta_j} = \frac{\rho_d \bar{\beta}_j}{\rho_d \bar{\beta}_j + \lambda \beta_j}, \quad (3)$$

where λ is a fixed positive real number.

Assuming that the obstacles closures are disjoint (equation (1)), the j^{th} “switch”, σ_j , attains a uniform value of 1 on the j^{th} obstacle boundary, vanishes on any other obstacle boundary, and maps the interior of the free space to the open interval (0, 1). In the deformation scheme, sufficiently close to the j^{th} obstacle boundary, these “switches” provide a means by which the deformation problem is reduced to the simpler problem of mapping the boundary of *one* star shaped obstacle onto *one* sphere.

Definition 7 The star set deforming factors, ν_j $j \in \{0, \dots, M\}$, are the real valued functions defined on \mathcal{F} by

$$\nu_j(q, \kappa) \triangleq \rho_j \frac{(1 + \beta_j(q))^\kappa}{\|q - q_j\|} \quad j \in \{1, \dots, M\} \quad \text{and} \quad \nu_0(q, \kappa) \triangleq \rho_0 \frac{(1 - \beta_0(q))^\kappa}{\|q - q_0\|}, \quad (4)$$

where κ is a fixed positive real number.

Each ν_j scales the ray starting at the center point of the j^{th} obstacle, q_j , through its unique intersection with the boundary point $q \in \partial \mathcal{O}_j$, in such a way that q is mapped to the corresponding point on the j^{th} sphere, $\hat{\mathcal{O}}_j$. The overall effect is that $\partial \mathcal{O}_j$ is deformed “along the rays” originating at the center point of \mathcal{O}_j onto the corresponding sphere in model space.

Let $\tilde{\mathcal{F}}$ denote some open set in E^n containing \mathcal{F} , the free space.

Definition 8 The star world transformation, $h_{\lambda, \kappa}$, is a member of the 2-parameter family of $C^{(\omega)}$ maps from $\tilde{\mathcal{F}} \subset E^n$ into E^n , defined by

$$h_{\lambda, \kappa}(q) \triangleq \sum_{j=0}^M \sigma_j(q) [\nu_j(q) \cdot (q - q_j) + p_j] + \sigma_d(q) [(q - q_d) + p_d], \quad (5)$$

where σ_j is the j^{th} analytic switch, σ_d is defined by

$$\sigma_d \triangleq 1 - \sum_{j=0}^M \sigma_j, \quad (6)$$

and ν_j is the j^{th} star set deforming factor.

The “destination switch”, σ_d , assures that $h_{\lambda,\kappa}(q_d) = p_d$, that is, the star world destination point is contained in the inverse image of the sphere world destination point, a necessary condition for our method to work, since the cost function on the sphere world has a unique minimum at p_d .

Remark: This definition assumes no relation whatsoever between the location and diameter of the model space and workspace obstacles: it is only the number of boundary components in each space that counts. Nevertheless, in the proof we impose two additional constraints on the model space, \mathcal{M} . The first assumes that the center of an obstacle in model space is identical to the corresponding center in workspace, the second ensures that the deforming factors, ν_j $j \in \{0, \dots, M\}$, are bounded. We do not know whether these conditions are actually necessary for the desired result.

Definition 9 *Given any star world, \mathcal{F} , the corresponding sphere world, \mathcal{M} , satisfies the placement condition if*

$$p_j = q_j \quad j \in \{0, \dots, M\}, \quad \text{and} \quad p_d = q_d. \quad (7)$$

Intuitively, if $p_d = q_d$, away from the obstacles the transformation “looks like” the identity map,

$$h_{\lambda,\kappa}(q) \cong \sigma_d(q)\text{id}(q) \cong q,$$

provided that the parameter λ is sufficiently large, as will be made precise later.

Let $\mathcal{S}_j(\epsilon)$ denote an “ ϵ -neighborhood” — a thickened neighborhood in \mathcal{F} about the j^{th} boundary component, $\partial\mathcal{O}_j$ — defined by

$$\mathcal{S}_j(\epsilon) \triangleq \{q \in \mathcal{F} : 0 \leq \beta_j(q) \leq \epsilon\} \quad j \in \{0, \dots, M\}, \quad (8)$$

where ϵ is a positive constant.

Definition 10 *Given any star world, \mathcal{F} , the corresponding sphere world, \mathcal{M} , satisfies the containment condition if*

$$\nu_j(q) \leq 1 \quad \text{for all } q \in \mathcal{S}_i(\epsilon) \quad j \in \{1, \dots, M\}, \quad (9)$$

and

$$\nu_0(q) \geq 1 \quad \text{for all } q \in \mathcal{S}_0(\epsilon),$$

where $\epsilon > 0$ is small enough to guarantee that

$$\mathcal{S}_j(\epsilon) \subset \overset{\circ}{\mathcal{F}} \cup \partial\mathcal{O}_j \quad j \in \{0, \dots, M\}.$$

Remark: Evaluating $h_{\lambda,\kappa}$ at a boundary point, $q \in \partial\mathcal{O}_j$, yields,

$$h_{\lambda,\kappa}(q) = \nu_j(q)(q - q_j) + p_j.$$

If the placement condition is satisfied, then the containment condition implies that

$$\|h_{\lambda,\kappa}(q) - q_0\| \geq \|q - q_0\| \quad \text{and} \quad \|h_{\lambda,\kappa}(q) - q_j\| \leq \|q - q_j\| \quad j \in \{1, \dots, M\}.$$

Geometrically this means that if the j^{th} sphere, $\partial\hat{O}_j$, is considered as being isometrically embedded in the star world, then it is contained in the j^{th} obstacle. The zero'th sphere satisfies the opposite containment relation.

Finally, given a star world, \mathcal{F} , we derive a model sphere world, \mathcal{M} , which will serve as the image space of \mathcal{F} under $h_{\lambda,\kappa}$.

Definition 11 *Given an n -dimensional star world, \mathcal{F} , a suitable sphere world, \mathcal{M} , is an n -dimensional sphere world (Definition 2) such that \mathcal{M}*

1. *has the same number of boundary components as \mathcal{F} ;*
2. *satisfies the placement condition;*
3. *satisfies the containment condition.*

3 Proof of Correctness

In the sequel, unless otherwise stated, the spaces \mathcal{X} and \mathcal{Y} denote n -dimensional compact connected analytic manifolds with $M + 1$ boundary components, which are also subsets of E^n . Denote the j^{th} boundary component of \mathcal{X} and \mathcal{Y} by $\partial_j \mathcal{X}$ and $\partial_j \mathcal{Y}$ respectively: each boundary component is a compact $(n - 1)$ -dimensional connected manifold with no boundary. Also, $h \in C^{(q)}[\mathcal{X}, \mathcal{Y}]$ means that \mathcal{X} and \mathcal{Y} have open neighborhoods in E^n , $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ respectively, such that $h \in C^{(q)}[\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}]$.

3.1 $h_{\lambda, \kappa}$ is an Analytic Diffeomorphism if Its Jacobian is Non-Singular

We first characterize an analytic diffeomorphism, h , in terms of its Jacobian and its behavior on the boundary components, then we show that the construction of the previous section, $h_{\lambda, \kappa}$, satisfies these conditions provided it has a non-singular Jacobian on its domain.

Proposition 3.1 *A map, $h \in C^{(\omega)}[\mathcal{X}, E^n]$, is an analytic diffeomorphism onto \mathcal{Y} if and only if*

1. h has a non-singular Jacobian on \mathcal{X} ;
2. h “preserves” boundaries — that is, $h(\partial_j \mathcal{X}) \subset \partial_j \mathcal{Y}$ $j \in \{0, \dots, M\}$;
3. the boundary components of \mathcal{X} and \mathcal{Y} are pairwise homeomorphic — $\partial_j \mathcal{X} \approx \partial_j \mathcal{Y}$ $j \in \{0, \dots, M\}$.

Proof: Assume that conditions 1 – 3 hold. We first show that for each j , $h|_{\partial_j \mathcal{X}}$ is a bijection onto $\partial_j \mathcal{Y}$. The first assumption above implies that there is an open neighborhood in E^n about \mathcal{X} , $\tilde{\mathcal{X}}$, on which h is analytic and has a non-singular Jacobian. Consequently, according to the Inverse Function Theorem[21], h is locally an analytic diffeomorphism. In particular, using the second assumption, $h|_{\partial_j \mathcal{X}}$ is locally an analytic diffeomorphism into $\partial_j \mathcal{Y}$ with their respective subspace topologies. According to [13, exercise 5.2.3], a local homeomorphism from a compact manifold into a connected one is surjective. Since $\partial_j \mathcal{X}$ is compact, h maps $\partial_j \mathcal{X}$ onto $\partial_j \mathcal{Y}$. It can be shown[13, exercise 5.2.4] that the pair $(\partial_j \mathcal{X}, h)$ is a covering space of $\partial_j \mathcal{Y}$. In [20, Corollary 3.3] it is shown that if the covering space is homeomorphic to the space being covered, then the number of sheets in the covering is one. Thus it follows from the third assumption that h is injective on $\partial_j \mathcal{X}$, and we conclude that h maps $\partial_j \mathcal{X}$ homeomorphically onto $\partial_j \mathcal{Y}$.

We now show that $h(\overset{\circ}{\mathcal{X}}) \cap \partial \mathcal{Y} = \emptyset$. Suppose to the contrary, that there exists a point $q \in \overset{\circ}{\mathcal{X}}$ such that $p_1 \triangleq h(q) \in \partial \mathcal{Y}$. Since h maps $\partial \mathcal{X}$ onto $\partial \mathcal{Y}$, it must be that

$$\#h^{-1}(p_1) \geq 2,$$

where $\#h^{-1}(p)$ denotes the number of points in the inverse image $h^{-1}(p)$ (it can be shown that whenever \mathcal{X} is compact, $\#h^{-1}(p)$ is finite). It further follows from the Inverse Function

Theorem that h is an open map on $\tilde{\mathcal{X}}$ (open sets in $\tilde{\mathcal{X}}$ are mapped to open sets in $h(\tilde{\mathcal{X}})$), therefore $h(\overset{\circ}{\mathcal{X}})$ is open in $h(\tilde{\mathcal{X}})$. Moreover, $h(\mathcal{X})$ — the image of a compact space under a continuous map — is compact, therefore closed, and it follows that for any point $p_2 \in \partial h(\mathcal{X})$, $h^{-1}(p_2) \cap \overset{\circ}{\mathcal{X}} = \emptyset$. Since we have already shown that h is injective on $\partial\mathcal{X}$, we have

$$\#h^{-1}(p_2) = 1.$$

Now, $h(\mathcal{X})$ — the image of a connected set under a continuous map — is connected. According to the first assumption, $h(\mathcal{X})$ consists of regular points of h , and it can be shown[15, pp 8-9] that this implies that for all $p \in h(\mathcal{X})$, $\#h^{-1}(p)$ is constant. But this contradicts $h^{-1}(p_1) \neq h^{-1}(p_2)$ above. Thus $h(\overset{\circ}{\mathcal{X}}) \cap \partial\mathcal{Y} = \emptyset$.

This in turn implies that

$$h(\overset{\circ}{\mathcal{X}}) \subset \overset{\circ}{\mathcal{Y}} \quad \text{or} \quad h(\overset{\circ}{\mathcal{X}}) \subset E^n - \mathcal{Y},$$

otherwise, since $h(\overset{\circ}{\mathcal{X}})$ is connected, a path in $h(\overset{\circ}{\mathcal{X}})$ passing through $\partial\mathcal{Y}$ could be found. We will show that only the first alternative is possible: to do so, it is more convenient to first show that $h(\mathcal{X}) \subset \mathcal{Y}$. The boundary of \mathcal{X} , $\partial\mathcal{X}$ — a compact surface in E^n which has $M + 1$ components — decomposes E^n into $M + 2$ disjoint connected components[5].

Consider two cases. If \mathcal{X} and \mathcal{Y} have only one boundary component and $h(\mathcal{X}) \subset E^n - \overset{\circ}{\mathcal{Y}}$, then it must be that h maps \mathcal{X} onto $E^n - \overset{\circ}{\mathcal{Y}}$, otherwise $h(\overset{\circ}{\mathcal{X}})$ is not open. But $E^n - \overset{\circ}{\mathcal{Y}}$ is unbounded, therefore not compact — an impossibility, since h is continuous. If \mathcal{X} and \mathcal{Y} have more than one boundary component and $h(\mathcal{X}) \subset E^n - \overset{\circ}{\mathcal{Y}}$, then since h maps $\partial_j\mathcal{X}$ into $\partial_j\mathcal{Y}$, it must be that $h(\mathcal{X})$ is not connected, which is also impossible. Thus $h(\mathcal{X}) \subset \mathcal{Y}$. We can now conclude that $h(\overset{\circ}{\mathcal{X}}) \subset \overset{\circ}{\mathcal{Y}}$, for $h(\overset{\circ}{\mathcal{X}})$ is open in $h(\tilde{\mathcal{X}})$ and $h(\overset{\circ}{\mathcal{X}}) \cap \partial\mathcal{Y} = \emptyset$.

In Lemma A.1 we show that if h satisfies the first assumption above, and “preserves” interior and boundary,

$$h(\partial\mathcal{X}) \subset \partial\mathcal{Y} \quad \text{and} \quad h(\overset{\circ}{\mathcal{X}}) \subset \overset{\circ}{\mathcal{Y}},$$

then h is a local homeomorphism from \mathcal{X} into \mathcal{Y} with their respective subspace topologies (of \mathcal{X} in $\tilde{\mathcal{X}}$ and of \mathcal{Y} in $\tilde{\mathcal{Y}} \triangleq h(\tilde{\mathcal{X}})$). Using[13] again, in consequence of \mathcal{X} being compact, h maps \mathcal{X} onto \mathcal{Y} , and, using[15] again, in consequence of \mathcal{Y} being a connected set of regular values of h , h is injective on \mathcal{X}^1 . Thus h maps \mathcal{X} homeomorphically onto \mathcal{Y} . Since h is locally an analytic diffeomorphism, we have proved the ‘if direction’.

Conversely, if h is an analytic diffeomorphism then it has a non-singular Jacobian — so condition 1 obtains. In particular, it is a local homeomorphism. Using local homology groups, one can show[14] that

$$h(\overset{\circ}{\mathcal{X}}) \subset \overset{\circ}{\mathcal{Y}} \quad \text{and} \quad h(\partial\mathcal{X}) \subset \partial\mathcal{Y}.$$

Therefore h maps $\partial\mathcal{X}$ onto $\partial\mathcal{Y}$. If h maps two boundary components of \mathcal{X} onto the same boundary component of \mathcal{Y} , then h is not injective. Thus the second condition obtains. Last,

¹ (\mathcal{X}, h) is a covering space of \mathcal{Y} , and it can be shown that if \mathcal{Y} is simply connected, as in our case for $n > 2$, then the number of sheets is one.

$h|_{\partial_j \mathcal{X}}$ — the restriction of a continuous map to a subset of its domain — is continuous in the subspace topology, and it can be shown[20] that $h|_{\partial_j \mathcal{X}}$, being a continuous bijection on a compact space, is a homeomorphism, which yields the third condition above.

□

The following corollary, whose proof essentially relates the conditions of Proposition 3.1 to the structure of this section, constitutes the central contribution of the paper.

Theorem 1 *For any star world, \mathcal{F} , possessing a valid arrangement (equation (1)), there exist two positive real numbers K, Λ , and a suitable model sphere world (Definition 11), \mathcal{M} , such that if $\kappa \geq K$ and $\lambda \geq \Lambda$, then*

$$h_{\lambda,\kappa} : \mathcal{F} \rightarrow \mathcal{M},$$

is an analytic diffeomorphism.

Proof: We must show that $h_{\lambda,\kappa}$ is an analytic bijection with analytic inverse. Clearly, if \mathcal{F} has a valid arrangement then $h_{\lambda,\kappa}$ — constructed from quotients of analytic functions none of whose denominators vanishes on some open neighborhood about \mathcal{F} — is analytic.

In Section 3.2 below it is shown that for any star world, \mathcal{F} , with a valid arrangement, there exist two positive real numbers K and Λ and a suitable model sphere world, \mathcal{M} , such that for all $\kappa \geq K$ and $\lambda \geq \Lambda$ the Jacobian of $h_{\lambda,\kappa} : \mathcal{F} \rightarrow \mathcal{M}$ is non-singular on \mathcal{F} (as a consequence it is non-singular on some open neighborhood in E^n about $\mathcal{F}, \tilde{\mathcal{F}}$).

We now show that $h_{\lambda,\kappa}$ maps the j^{th} boundary component of $\mathcal{F}, \partial_j \mathcal{F}$, into the j^{th} boundary component of $\mathcal{M}, \partial_j \mathcal{M}$. If $q \in \partial_j \mathcal{F}$ then, for a valid arrangement of the star world,

$$\beta_j(q) = 0 \quad \text{and} \quad \beta_k(q) > 0 \quad k \in \{0, \dots, M\} \quad k \neq j.$$

Substituting in the definition of $h_{\lambda,\kappa}$ yields,

$$h_{\lambda,\kappa} |_{\partial_j \mathcal{F}} = \frac{\rho_j}{\|q - q_j\|} (q - q_j) + p_j,$$

which implies that $(\hat{\beta}_j \circ h_{\lambda,\kappa})(q)$, the j^{th} sphere function (Definition 2), vanishes as well. Thus,

$$h_{\lambda,\kappa}(\partial_j \mathcal{F}) \subset \partial_j \mathcal{M} \quad j \in \{0, \dots, M\}. \quad (10)$$

Finally, in our construction each boundary component of \mathcal{F} is the boundary of an open star shaped set. It can be shown that any open star shaped set in E^n is homeomorphic to the n -disc[2]. Since in the corresponding sphere world each boundary component is an $(n - 1)$ -sphere,

$$\partial_j \mathcal{F} \approx \partial_j \mathcal{M} \quad j \in \{0, \dots, M\}.$$

Therefore, according to Proposition 3.1, $h_{\lambda,\kappa}$ is an analytic diffeomorphism from \mathcal{F} onto \mathcal{M} .

□

3.2 The Jacobian of $h_{\lambda,\kappa}$ is Non-Singular

In the sequel, subscripts denoting the dependence of the analytic function $h_{\lambda,\kappa}$ on the parameters λ and κ will be omitted. It is further understood that any derivative of $\sigma_j(q, \lambda)$ or $\nu_j(q, \kappa)$ is with respect to the position vector q , the parameters λ and κ being held constant.

We distinguish in the star world the set “away from the obstacles”,

$$\mathcal{A}(\epsilon) \triangleq \{q \in \mathcal{F} : \beta_j(q) > \epsilon \quad j \in \{0, \dots, M\}\}.$$

and denote its complement in \mathcal{F} by

$$\mathcal{A}^c(\epsilon) \triangleq \mathcal{F} - \mathcal{A}(\epsilon) = \bigcup_{j=0}^M \mathcal{S}_j(\epsilon),$$

where $\mathcal{S}_j(\epsilon)$ is the “thickened” j^{th} boundary component, defined in equation (8) as

$$\mathcal{S}_j(\epsilon) = \{q \in \mathcal{F} : 0 \leq \beta_j \leq \epsilon\} \quad j \in \{0, \dots, M\},$$

where ϵ is a positive constant, small enough to guarantee that

$$\mathcal{S}_j(\epsilon) \subset \overset{\circ}{\mathcal{F}} \bigcup \partial \mathcal{O}_j \quad j \in \{0, \dots, M\}. \quad (11)$$

In consequence of the assumption that the obstacle closures are non-intersecting, such an ϵ exists.

We are now ready to prove the non-singularity of the Jacobian on the set “away” from the obstacles, $\mathcal{A}(\epsilon)$.

In the Appendix we prove in Lemma B.3 and Lemma B.4 that for any $\epsilon > 0$ and any $\delta > 0$, σ_j and $\|\sigma_j\|$ can be made to be smaller than δ on the set “away” from the j^{th} obstacle boundary, $\mathcal{F} - \mathcal{S}_j(\epsilon)$, by choosing

$$\lambda(\epsilon, \delta) \geq N_{0j}(\epsilon, \delta) \quad \text{and} \quad \lambda(\epsilon, \delta) \geq N_{1j}(\epsilon, \delta),$$

respectively, where $N_{0j}(\epsilon, \delta)$, $N_{1j}(\epsilon, \delta)$ are fixed positive real numbers. As a consequence, it is possible to guarantee that the Jacobian of $h_{\lambda,\kappa}$ is non-singular on $\mathcal{A}(\epsilon)$, a statement which is made precise as follows.

Lemma 3.2 *For any $\epsilon > 0$ and any $\kappa > 0$ there exists a positive real number $\Lambda_0(\epsilon, \kappa)$ such that if $\lambda \geq \Lambda_0$ then Dh is non-singular on $\mathcal{A}(\epsilon)$.*

Proof: Under the placement condition (Definition 9), the Jacobian of $h_{\lambda,\kappa}$ is shown in Lemma B.1 to be

$$Dh(q) = \sum_{j=0}^M \left\{ \sigma_j \nu_j I + \sigma_j (q - q_j) \nabla \nu_j^T + (\nu_j - 1) (q - q_j) \nabla \sigma_j^T \right\} + \sigma_d I.$$

Let \hat{x} be any unit vector based at $q \in \mathcal{A}(\epsilon)$, that is, $\hat{x} \in T_q \mathcal{A}(\epsilon)$. Evaluating the Jacobian along \hat{x} yields,

$$[Dh(q)]\hat{x} = \left(\sum_{j=0}^M \sigma_j \nu_j + \sigma_d \right) \hat{x} + \overbrace{\left[\sum_{j=0}^M \left\{ \sigma_j (q - q_j) \nabla \nu_j^T + (\nu_j - 1)(q - q_j) \nabla \sigma_j^T \right\} \right]}^{w(\sigma_j, \nabla \sigma_j; \kappa)} \hat{x},$$

where $w(\sigma_j, \nabla \sigma_j; \kappa)$ is a shorthand notation for $w(\sigma_j, \nabla \sigma_j; \kappa)$ $j \in \{0, \dots, M\}$. We will use the positive magnitude of σ_d to dominate w . First, note that the latter may be bounded above in magnitude by

$$\|w(\sigma_j, \nabla \sigma_j; \kappa)\| < \frac{1}{2}.$$

For, choosing

$$\lambda > \max_{j \in \{0, \dots, M\}} \{N_{0j}(\epsilon, \delta), N_{1j}(\epsilon, \delta)\},$$

it follows from Lemma B.3 and Lemma B.4 that σ_j and $\|\nabla \sigma_j\|$ are bounded by $\delta > 0$, hence,

$$\begin{aligned} \|w(\sigma_j, \nabla \sigma_j; \kappa)\| &\leq \sum_{j=0}^M \|q - q_j\| (\sigma_j \|\nabla \nu_j\| + |\nu_j - 1| \|\nabla \sigma_j\|) \\ &< \delta \sum_{j=0}^M \|q - q_j\| (\|\nabla \nu_j\| + |\nu_j - 1|). \end{aligned}$$

A sufficient condition on δ for the desired inequality is thus

$$\delta \leq \frac{1}{2 \max_{\mathcal{F}} \left\{ \sum_{j=0}^M \|q - q_j\| (\|\nabla \nu_j(q, \kappa)\| + |\nu_j(q, \kappa) - 1|) \right\}} \triangleq \delta'_0(\kappa). \quad (12)$$

Note that ν_j is analytic on the compact set \mathcal{F} , and in consequence both ν_j and $\|\nabla \nu_j\|$ are bounded, thus, there is no problem with the definition of δ'_0 .

On the other hand, note that σ_d may be bounded below in magnitude by $\frac{1}{2}$, for, again according to Lemma B.3 and Lemma B.4, by choosing λ sufficiently large, we may impose on the switches the condition

$$\sigma_j \leq \frac{1}{2} \frac{1}{1 + M} \quad \text{for all } q \in \mathcal{A}(\epsilon), \quad j \in \{0, \dots, M\}; \quad (13)$$

which implies that

$$\sigma_d = 1 - \sum_{j=0}^M \sigma_j \geq \frac{1}{2}.$$

Finally, choosing

$$\delta_0(\kappa) = \min \left\{ \delta'_0(\kappa), \frac{1}{2} \frac{1}{1 + M} \right\},$$

the desired Λ_0 is

$$\Lambda_0(\epsilon, \kappa) \triangleq \max_{j \in \{0, \dots, M\}} \{N_{0j}(\epsilon, \delta_0(\kappa)), N_{1j}(\epsilon, \delta_0(\kappa))\}, \quad (14)$$

and the result follows.

□

We now turn our attention to the set $\mathcal{A}^c(\epsilon)$.

$Dh(q)$ maps $T_q\mathcal{F}$ into $T_{h(q)}h(\mathcal{F})$, both of which are n -dimensional vector spaces, and therefore isomorphic to E^n . Since the domain and image of h — the star world, \mathcal{F} , and its model sphere world, \mathcal{M} — are both n -dimensional submanifolds of E^n , we are free to identify points in the two spaces, for example, as we have done in the placement condition (Definition 9). For the same reason, we will not distinguish points in the tangent space from points in the original base spaces.

In consequence of equation (2) , for sufficiently small ϵ , both tangent spaces admit the direct sum decomposition,

$$T_q\mathcal{F}, T_{h(q)}h(\mathcal{F}) = \langle q - q_i \rangle \oplus \langle \nabla\beta_i(q) \rangle^\perp \quad \text{for all } q \in \mathcal{S}_i(\epsilon), \quad (15)$$

where $\langle \cdot \rangle$ denotes the “span of”, and $(\cdot)^\perp$ denotes the “orthogonal complement of”.

Thus, any unit vector $\hat{x} \in T_q\mathcal{F}$ can be uniquely written as

$$\hat{x} = x_1 + x_2,$$

where $x_1 \in \langle q - q_i \rangle$ and $x_2 \in \langle \nabla\beta_i \rangle^\perp$.

Now, rearrange the terms in the equation for $Dh(q)$ (equation (23)) , so as to reflect the negligible effect of all the neighboring obstacles on the Jacobian near the i^{th} obstacle boundary:

$$\begin{aligned} Dh(q) = & \left\{ (\sigma_i\nu_i)I + \sigma_i(q - q_i)\nabla\nu_i^T + (\nu_i - 1)(q - q_i)\nabla\sigma_i^T \right\} + (1 - \sigma_i)I \\ & + \underbrace{\sum_{j=0, j \neq i}^M \left\{ \sigma_j(\nu_j - 1)I + \sigma_j[(q - q_j)\nabla\nu_j^T] + (\nu_j - 1)[(q - q_j)\nabla\sigma_j^T] \right\}}_{X_i(\sigma_j, \nabla\sigma_j; \kappa)}. \end{aligned} \quad (16)$$

where $X_i(\sigma_j, \nabla\sigma_j; \kappa)$ is a shorthand notation for $X_i(\sigma_j, \nabla\sigma_j; \kappa)$ $j \in \{0, \dots, M\}$, $j \neq i$.

The image of a unit vector, $\hat{x} \in T_q\mathcal{F}$, is

$$[Dh(q)]\hat{x} = (\sigma_i\nu_i)\hat{x} + \sigma_i(\nabla\nu_i \cdot \hat{x})(q - q_i) + (\nu_i - 1)(\nabla\sigma_i \cdot \hat{x})(q - q_i) + (1 - \sigma_i)\hat{x} + X_i\hat{x}. \quad (17)$$

The magnitude of the vector $X_i\hat{x} \in T_{h(q)}h(\mathcal{F})$ is bounded by $\|X_i(\sigma_j, \nabla\sigma_j; \kappa)\|$, which can be made arbitrarily small, as we show in Lemma B.5 .

For any choice of a test direction $\hat{x} \in T_q\mathcal{S}_i(\epsilon)$, one of the following two relations holds,

$$\text{either } \frac{\|x_1\|}{\|x_2\|} > 2, \quad \text{or } \frac{\|x_1\|}{\|x_2\|} \leq 2.$$

The following Lemma constructs formulas which designate, as a function of the geometrical data, a region for the parameters ϵ , λ and κ , which is later shown to guarantee a non-singular Jacobian of $h_{\lambda, \kappa}$ on \mathcal{S}_i .

Lemma 3.3 For any star world, \mathcal{F} , if \mathcal{M} is a suitable sphere world (Definition 11), then there exist $\epsilon_i, K_i(\epsilon)$ and $N_{2i}(\epsilon, \kappa)$, positive real numbers, such that for all $\epsilon \leq \epsilon_i$ and for all $q \in \mathcal{S}_i(\epsilon)$, whenever $\hat{x} \in T_q \mathcal{S}_i(\epsilon)$, $\|\hat{x}\| = 1$, satisfies

$$\frac{\|x_1\|}{\|x_2\|} > 2,$$

we have,

(i) if $\kappa \geq K_i(\epsilon)$ then

$$-1 + \kappa \frac{\nabla \beta_i \cdot (q - q_i)}{1 + \beta_i} \geq 0 \quad i \in \{1, \dots, M\} \quad \text{and} \quad -1 - \kappa \frac{\nabla \beta_0 \cdot (q - q_0)}{1 - \beta_0} \geq 0;$$

(ii) that

$$(\nu_i - 1)(\nabla \sigma_i \cdot \hat{x})((q - q_i) \cdot x_1) \geq 0;$$

(iii) if $\lambda \geq N_{2i}(\epsilon, \kappa)$, then

$$\|X_i\| \leq \frac{1}{2} \left((\widehat{q - q_i}) \cdot \widehat{\nabla \beta_i} \right)^2 \left(\frac{1}{2} \sigma_i \nu_i + (1 - \sigma_i) \right) \|x_1\|;$$

and whenever

$$\frac{\|x_1\|}{\|x_2\|} \leq 2,$$

(iv) if $\lambda \geq N_{2i}(\epsilon, \kappa)$, then

$$\|X_i\| \leq \frac{1}{2} \left((\widehat{q - q_i}) \cdot \widehat{\nabla \beta_i} \right)^2 (\sigma_i \nu_i + (1 - \sigma_i)) \|x_2\|,$$

where \hat{v} denotes the unit magnitude vector $v/\|v\|$.

The proof is given in Appendix B.2 .

Lemma 3.3 assumes the existence of a suitable model sphere world, \mathcal{M} , which, in particular, must satisfy the containment condition (Definition 10). The following Lemma specifies an upper bound for the radius of the i^{th} sphere (a lower bound for the 0^{th} sphere) in \mathcal{M} , guaranteeing that the containment condition is satisfied.

Lemma 3.4 For any $\epsilon > 0$ and $\kappa > 0$, there exist positive real numbers $R_i(\epsilon, \kappa)$ $i \in \{0, \dots, M\}$, such that for all $\rho_i \leq R_i$,

$$\nu_i(q, \kappa) \leq 1 \quad q \in \mathcal{S}_i(\epsilon), \quad i \in \{1, \dots, M\},$$

and for all $\rho_0 \geq R_0$,

$$\nu_0(q, \kappa) \geq 1 \quad q \in \mathcal{S}_0(\epsilon).$$

Proof: According to its definition in equation (4) ,

$$\nu_i(q, \kappa) = \rho_i \frac{(1 + \beta_i(q))^\kappa}{\|q - q_i\|} \leq 1,$$

if

$$\rho_i \leq \frac{\|q - q_i\|}{(1 + \beta_i(q))^\kappa},$$

which is implied by the condition

$$\rho_i \leq \frac{\min_{q \in \partial \mathcal{O}_i} \{\|q - q_i\|\}}{(1 + \epsilon)^\kappa} \triangleq R_i(\epsilon, \kappa) \quad i \in \{1, \dots, M\}, \quad (18)$$

since $0 \leq \beta_i \leq \epsilon$ and

$$\min_{p \in \partial \mathcal{O}_i} \{\|p - q_i\|\} \leq \|q - q_i\| \quad \text{for all } q \in \mathcal{F}.$$

In the case of ν_0 , using a similar argument, the condition

$$\rho_0 \geq \frac{\max_{q \in \partial \mathcal{O}_0} \{\|q - q_0\|\}}{(1 - \epsilon)^\kappa} \triangleq R_0(\epsilon, \kappa),$$

is sufficient to guarantee that $\nu_0 \geq 1$.

□

Remark: The last two Lemmas are interrelated. Lemma 3.3 assumes a suitable model sphere world, which, in particular, must satisfy equation (18) , specified in terms of the parameters ϵ and κ . Nevertheless, there is no problem, since in the proof of Lemma 3.3 , ϵ and κ are chosen according to equation (31) and equation (34) , both *independent* of the sphere world radii. Once these two parameter values are fixed, we can use Lemma 3.4 to pick a suitable sphere world, and then proceed to choose any other parameter values, for example, $N_{2i}(\epsilon, \kappa) \quad i \in \{0, \dots, M\}$, which depend on the geometrical parameters of both spaces.

Repeating equation (17) , the Jacobian of $h_{\lambda, \kappa}$ evaluated along the test direction \hat{x} ,

$$[Dh(q)]\hat{x} = (\sigma_i \nu_i)\hat{x} + \sigma_i(\nabla \nu_i \cdot \hat{x})(q - q_i) + (\nu_i - 1)(\nabla \sigma_i \cdot \hat{x})(q - q_i) + (1 - \sigma_i)\hat{x} + X_i \hat{x}.$$

The vector $X_i \hat{x} \in T_{h(q)}h(\mathcal{F})$ can be uniquely written as

$$X_i \hat{x} = y'_1 + y'_2,$$

where $y'_1 \in \langle q - q_i \rangle$ and $y'_2 \in \langle \nabla \beta_i \rangle^\perp$. Substituting $x_1 + x_2$ for \hat{x} , and $y'_1 + y'_2$ for $X_i \hat{x}$ yields,

$$[Dh(q)]\hat{x} = \overbrace{\{(\sigma_i \nu_i)x_1 + \sigma_i(\nabla \nu_i \cdot \hat{x})(q - q_i) + (\nu_i - 1)(\nabla \sigma_i \cdot \hat{x})(q - q_i) + (1 - \sigma_i)x_1 + y'_1\}}^{y_1} + \underbrace{\{(\sigma_i \nu_i + 1 - \sigma_i)x_2 + y'_2\}}_{y_2},$$

where $y_1 \in \langle q - q_i \rangle$ and $y_2 \in \langle \nabla \beta_i \rangle^\perp$.

We now show that for all $q \in \mathcal{S}_i(\epsilon)$ and for all $\hat{x} \in T_q \mathcal{S}_i(\epsilon)$ at least one of the vectors y_1, y_2 is non-zero. Since y_1 and y_2 are linearly independent by assumption, this implies that Dh is non-singular on $\mathcal{S}_i(\epsilon)$.

Corollary 3.5 *There exist $\epsilon_i, K_i(\epsilon)$ and $N_{2i}(\epsilon, \kappa)$, positive real numbers, such that for all $\epsilon \leq \epsilon_i$ and for all $q \in \mathcal{S}_i(\epsilon)$, if $\kappa \geq K_i(\epsilon)$ and $\lambda \geq N_{2i}(\epsilon, \kappa)$, then whenever $\hat{x} \in T_q \mathcal{S}_i(\epsilon)$ satisfies*

$$\frac{\|x_1\|}{\|x_2\|} > 2,$$

we have $y_1 \neq 0$, and whenever

$$\frac{\|x_1\|}{\|x_2\|} \leq 2,$$

we have $y_2 \neq 0$.

Proof: Using equation (24) for $\nabla \nu_i$,

$$\begin{aligned} \nabla \nu_i \cdot \hat{x} &= \nabla \nu_i \cdot (x_1 + x_2) \\ &= \frac{\nu_i}{\|q - q_i\|} \left(-(\widehat{q - q_i}) \cdot x_1 + \kappa \frac{\|q - q_i\|}{1 + \beta_i} \nabla \beta_i \cdot x_1 \right) - \frac{\nu_i}{\|q - q_i\|} (\widehat{q - q_i}) \cdot x_2, \end{aligned}$$

since $x_2 \perp \nabla \beta_i$. Expanding $\nabla \nu_i \cdot \hat{x}$ in the expression for y_1 ,

$$\begin{aligned} y_1 &= \underbrace{\left\{ \frac{1}{2}(\sigma_i \nu_i) x_1 - \sigma_i \nu_i ((\widehat{q - q_i}) \cdot x_2)(\widehat{q - q_i}) \right\}}_{(0)} + \underbrace{\left\{ \sigma_i \nu_i \left(-1 + \kappa \frac{\nabla \beta_i \cdot (q - q_i)}{1 + \beta_i} \right) x_1 \right.}_{(i)} \\ &\quad \left. + \underbrace{(\nu_i - 1)(\nabla \sigma_i \cdot \hat{x})(q - q_i)}_{(ii)} + \underbrace{\left\{ \frac{1}{2}(\sigma_i \nu_i) x_1 + (1 - \sigma_i) x_1 + y_1' \right\}}_{(iii)}, \end{aligned} \quad (19)$$

where we used the identities

$$((\widehat{q - q_i}) \cdot x_1)(\widehat{q - q_i}) = x_1 \quad \text{and} \quad (\nabla \beta_i \cdot x_1)(q - q_i) = (\nabla \beta_i \cdot (q - q_i)) x_1,$$

since by assumption $x_1 \in \langle q - q_i \rangle$.

We now use Lemma 3.3 to show that $y_1 \cdot x_1 > 0$, whenever $\frac{\|x_1\|}{\|x_2\|} > 2$.

The first term, labeled (0), is dominated by $\frac{1}{2}(\sigma_i \nu_i) x_1$ if

$$\frac{1}{2} \sigma_i \nu_i \|x_1\| > \sigma_i \nu_i \|x_2\|,$$

which is satisfied by the choice of the ratio $\|x_1\|/\|x_2\|$, and hence has a positive inner-product with x_1 .

The terms (i) and (ii) have a non-negative inner-product with x_1 according to assertions (i) and (ii) in Lemma 3.3.

A sufficient condition for x_1 to dominate in the term (iii), that is, to counteract the effect of the neighboring obstacles, is

$$\|y'_1\| \leq \left(\frac{1}{2} \sigma_i \nu_i + (1 - \sigma_i) \right) \|x_1\|,$$

but according to Lemma B.6 ,

$$\|y'_1\| \leq \frac{2}{\left((\widehat{q - q_i}) \cdot \widehat{\nabla \beta_i} \right)^2} \|X_i \hat{x}\|,$$

therefore, a sufficient condition is

$$\|X_i\| \leq \frac{1}{2} \left((\widehat{q - q_i}) \cdot \widehat{\nabla \beta_i} \right)^2 \left(\frac{1}{2} \sigma_i \nu_i + (1 - \sigma_i) \right) \|x_1\|,$$

which is assertion (iii) in Lemma 3.3 .

Turning our attention to the case in which $\|x_1\|/\|x_2\| \leq 2$, we would like to show that y_2 , which was defined to be,

$$y_2 = (\sigma_i \nu_i + 1 - \sigma_i) x_2 + y'_2,$$

is non-zero. By an argument similar to one given in the previous paragraph, a sufficient condition is

$$\|X_i\| \leq \frac{1}{2} \left((\widehat{q - q_i}) \cdot \widehat{\nabla \beta_i} \right)^2 (\sigma_i \nu_i - \sigma_i + 1) \|x_2\|,$$

which is the last assertion in Lemma 3.3 .

□

Remark: We show in Appendix C that if the i^{th} star shaped obstacle function, β_i , is *homogeneous*, then the parameter κ can be fixed such that the i^{th} star set deforming factor, ν_i , is constant along the rays originating at q_i . As a consequence, the term (i) in equation (19) vanishes identically.

Since we have considered the entirety of \mathcal{F} , the above results are summarized in the following Proposition.

Proposition 3.6 *Assuming a valid arrangement of \mathcal{F} , there exist three positive real numbers ϵ^* , $K(\epsilon)$ and $\Lambda(\epsilon, \kappa)$ such that if $\epsilon \leq \epsilon^*$, $\kappa \geq K(\epsilon)$, and $\lambda \geq \Lambda(\epsilon, \kappa)$, then there exists a suitable \mathcal{M} (Definition 11), such that the Jacobian of $h_{\lambda, \kappa} : \mathcal{F} \rightarrow \mathcal{M}$ is non-singular on \mathcal{F} .*

Proof: First consider the set “near” the obstacles, $\mathcal{A}^c(\epsilon)$. Let ϵ be chosen according to

$$\epsilon \leq \epsilon^* \triangleq \min_{j \in \{0, \dots, M\}} \{\epsilon_j\},$$

where each ϵ_j is defined in equation (34) as a function of the geometrical parameters of \mathcal{F} . Next, let the parameter κ be chosen according to

$$\kappa \geq K(\epsilon^*) \triangleq \max_{j \in \{0, \dots, M\}} \{K_j(\epsilon^*)\},$$

where $K_j(\epsilon)$ $j \in \{0, \dots, M\}$ are defined in equation (31). Now, choose a sphere world, \mathcal{M} , whose sphere center points satisfy the placement condition (Definition 10),

$$p_i = q_i \quad i \in \{0, \dots, M\} \quad \text{and} \quad p_d = q_d,$$

and whose radii satisfy the condition,

$$\rho_0 \geq R_0(\epsilon, \kappa) \quad \text{and} \quad \rho_i \leq R_i(\epsilon, \kappa) \quad i \in \{1, \dots, M\},$$

where $R_i(\epsilon, \kappa)$ $i \in \{0, \dots, M\}$ are defined in equation (18). In consequence of Lemma 3.4 \mathcal{M} satisfies the containment condition (Definition 10) and is a suitable model sphere world for \mathcal{F} .

Let λ be chosen according to

$$\lambda \geq \Lambda_1(\epsilon, \kappa(\epsilon)) \triangleq \max_{j \in \{0, \dots, M\}} \{N_{2j}(\epsilon, \kappa(\epsilon))\},$$

where N_{2j} $j \in \{0, \dots, M\}$ are defined in equation (38). Under this choice of the parameters λ and κ , Corollary 3.5 applies, and Dh is non-singular on $\mathcal{A}^c(\epsilon)$.

Now consider the set “away” from the obstacles, $\mathcal{A}(\epsilon)$. According to Lemma 3.2, for any $\epsilon > 0$ and any $\kappa > 0$, Dh is non-singular on this set, as long as $\lambda \geq \Lambda_0(\epsilon, \kappa)$, where $\Lambda_0(\epsilon, \kappa)$ is given in equation (14).

Finally, let

$$\Lambda(\epsilon, \kappa(\epsilon)) \triangleq \max\{\Lambda_0(\epsilon, \kappa(\epsilon)), \Lambda_1(\epsilon, \kappa(\epsilon))\},$$

and the result follows. □

4 Counting the Floating Point Operations

The computation involved has two parts. First, when presented with the data (specified in Section 1.2), describing a star world with a valid arrangement, \mathcal{F} , we construct a navigation function on \mathcal{F} , $\varphi = \hat{\varphi} \circ h_{\lambda, \kappa}$, by choosing a suitable sphere world, \mathcal{M} , constructing $\hat{\varphi}$ — the navigation function on \mathcal{M} , and choosing the parameters in $h_{\lambda, \kappa}$. The computational complexity of this part is analysed in Section 4.1. Second, the controller has to compute $\nabla\varphi$,

$$\nabla\varphi = \nabla(\hat{\varphi} \circ h_{\lambda, \kappa}) = [Dh_{\lambda, \kappa}]^T \nabla\hat{\varphi}(h_{\lambda, \kappa}), \quad (20)$$

and we analyze the computational complexity of this term in Section 4.2.

4.1 The Computation of the Parameters in $h_{\lambda, \kappa}$

The count of the floating point operations will be given in terms of M — the number of obstacles, and n — the dimension of the embedding Euclidean space. In this paper we defined the obstacle functions as the class of real valued analytic functions describing strictly star shaped obstacles (definition 5). In order to speak meaningfully about the number of floating point operations, we restrict the obstacle functions to the class of positive definite homogeneous polynomials of degree $k \in \mathbb{N}$. In Appendix C we show that each member of this class — which is essentially all the polynomials which satisfy the properties of a norm except, possibly, the triangular inequality — describes a strictly star shaped obstacle. It is important to note that this class serves only as an example. Although such functions generate a great variety of star shapes, we currently do not know whether this class is rich enough to represent “almost all” the star shaped obstacles.

Under this restriction, it turns out that in order to compute the parameters in $h_{\lambda, \kappa}$ for a star world, \mathcal{F} , the following data suffices,

1. for each obstacle, \mathcal{O}_j , its center point, q_j , the obstacle function, β_j , and its (homogeneous) degree k_j ;
2. for each obstacle, \mathcal{O}_j , a radius, loosely denoted by

$$\min_{q \in \partial\mathcal{O}_j} \{\|q - q_j\|\} \quad j \in \{0, \dots, M\},$$

such that the sphere with this radius, when centered at q_j , is contained in \mathcal{O}_j (contains \mathcal{O}_0 for the zeroth obstacle);

3. a destination point, $q_d \in \mathring{\mathcal{F}}$;
4. the upper bounds on the obstacle functions, $\{E_j\}_0^M$, and on the distance to q_d , E_d , which were specified in Section 1.2.

In Appendix C we provide practical formulas that render all the relevant terms in the computation of the parameters of $h_{\lambda, \kappa}$ in terms of this data. The steps in constructing $h_{\lambda, \kappa}$ from the data were summarized in Proposition 3.6, and are repeated here, together with the computational effort they require, a detailed analysis can be found in [18].

1. compute ϵ^* from the data — according to equation (35) , this step takes less than $10M$ operations;
2. find $K(\epsilon^*)$, a lower bound on the parameter κ in $h_{\lambda,\kappa}$ — using Lemma C.5 , this can be done in M operations;
3. choose a suitable sphere world for ϵ^* and $K(\epsilon^*)$ — according to equation (18) , this can be done in less than $5M$ operations; and then compute the sphere world parameter, k — a computation that we showed in [11] to take no more than $10M^2n$ operations;
4. find $\Lambda(\epsilon^*, K(\epsilon^*))$, a lower bound on the parameter λ in $h_{\lambda,\kappa}$ — according to equation (27) and (equation (38)) , this step involves less than $10(M + M^2)$ operations.

Summing up, the total number of floating point operations required is bounded by

$$5M^2n + 15M^2, \quad (21)$$

where $M + 1$ the number of obstacles.

Remark: The dimension of the space, n , appears only in the computation of the sphere world parameter, k . This is a consequence of the assumption we have made about the allowable obstacle functions — homogeneous polynomials, which enabled us to give explicit scalar bounds on all the required terms.

4.2 The Computation of $\nabla\varphi$

Using equation (20) , the computation of $\nabla\varphi(q)$ involves the following steps,

1. compute $p \triangleq h_{\lambda,\kappa}(q)$;
2. compute $\nabla\tilde{\varphi}(p)$;
3. compute $Dh_{\lambda,\kappa}(q)$;
4. multiply the matrix $Dh_{\lambda,\kappa}(q)^T$ by the vector $\nabla\tilde{\varphi}(p)$.

Denote the number of floating point operations required to compute the i^{th} obstacle function, β_i , and its gradient, $\nabla\beta_i$, by $\#(\beta_i)$ and $\#(\nabla\beta_i)$. The number of operations required will be given in terms of $\#(\beta_i)$, $\#(\nabla\beta_i)$, M and n .

From its definition (definition 8) , the computation of $h_{\lambda,\kappa}$ involves the summation of $M + 1$ terms, each of the form,

$$\sigma_i(q) [\nu_i(q) \cdot (q - q_i) + p_i].$$

According to their definition (equation (3)) , the computation of the analytic switches, $\{\sigma_j\}_0^M$, involves the product of $\{\beta_j\}_0^M$ and γ_d . Therefore it takes less than $5M + \sum_{j=0}^M \#(\beta_j)$ operations

to compute the analytic switches. Next, according to its definition (equation (4)), the computation of the i^{th} star deforming factor, ν_i , involves, roughly, the quotient of β_i with $\|q - q_i\|$. Therefore it takes less than $5Mn$ additional operations to compute $\{\nu_j\}_0^M$. We conclude that it takes no more than $10Mn + \sum_{j=0}^M \#(\beta_j)$ operations to compute $h_{\lambda,\kappa}(q)$.

In the second step — computing $\nabla\tilde{\varphi}(p)$ — we refer the reader to [11], in which we showed that it takes no more than $10(M^2 + Mn)$ operations to compute $\nabla\tilde{\varphi}(p)$.

Using equation (23) for $Dh_{\lambda,\kappa}$, the computation in the third step involves the summation of $M + 1$ terms, each of the form

$$\sigma_i \nu_i I + \sigma_i (q - q_i) \nabla \nu_i^T + (\nu_i - 1) (q - q_i) \nabla \sigma_i^T. \quad (22)$$

Rewriting $\nabla \nu_i$ from equation (24),

$$\nabla \nu_i(q, \kappa) = \frac{\nu_i}{\|q - q_i\|} \left(\kappa \frac{\|q - q_i\|}{1 + \beta_i} \nabla \beta_i - (q - q_i) \right).$$

Since ν_i and $\|q - q_i\|$ were already computed, it takes $\#(\nabla \beta_i) + 10n$ additional operations to compute $\nabla \nu_i$. Rewriting $\nabla \sigma_i$ (equation (25)),

$$\begin{aligned} \nabla \sigma_i &= \frac{\lambda}{(\gamma_d \beta_i + \lambda \beta_i)^2} (\beta_i \nabla(\gamma_d \bar{\beta}_i) - \gamma_d \bar{\beta}_i \nabla \beta_i) \\ &= \frac{\lambda}{(\gamma_d \beta_i + \lambda \beta_i)^2} \left(\sum_{j=0, j \neq i}^M \gamma_d \bar{\beta}_j \nabla \beta_j + \beta \nabla \gamma_d - \gamma_d \bar{\beta}_i \nabla \beta_i \right). \end{aligned}$$

Since β_j and $\nabla \beta_j$ were already computed, it takes $10Mn$ additional operations to compute $\nabla \sigma_i$. Thus, the computation of (22) takes no more than $5n^2$ additional operations, and we conclude that the computation of $Dh_{\lambda,\kappa}(q)$ is bounded by $10(Mn^2 + \sum_{j=0}^M \#(\nabla \beta_j))$ additional operations.

Summing up (the fourth step takes $5n^2$ operations), we conclude that it takes no more than

$$10 \left\{ Mn^2 + M^2 + \sum_{j=0}^M (\#(\beta_j) + \#(\nabla \beta_j)) \right\}$$

operations to compute $\nabla \varphi(q)$.

Remark: If we instantiate the obstacle functions within the class of polynomials of degree k or less, then, in general, each such polynomial consists of k homogeneous polynomials, each of whom can have no more than $\binom{n+k-1}{k}$ terms, therefore,

$$\#(\beta_i) = k \binom{n+k-1}{k} \quad \text{and} \quad \#(\nabla \beta_i) = n(k-2) \binom{n+k-2}{k-1}.$$

Thus, if we relate k to the “geometric complexity”, and n to the number of degrees of freedom of the underlying kinematic chain, then, assuming that $k \geq n$, the computation involved is proportional to k^n , i.e. exponential in the number of degrees of freedom and polynomial in the geometric complexity.

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A Some Basic Topological Facts

Basic to this discussion is the topological notion of a locally homeomorphic function, therefore it is best to state its definition.

Definition 12 ([13]) *A continuous map between two topological spaces, $h : \mathcal{X} \rightarrow \mathcal{Y}$, is a local homeomorphism if each point $x \in \mathcal{X}$ has an open neighborhood \mathcal{U} such that $h(\mathcal{U})$ is open in \mathcal{Y} and h maps \mathcal{U} homeomorphically onto $h(\mathcal{U})$.*

Let $\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}$ denote n -dimensional manifolds without boundary, and let $\mathcal{X} \subset \tilde{\mathcal{X}}, \mathcal{Y} \subset \tilde{\mathcal{Y}}$ be n -dimensional compact submanifolds with $M + 1$ boundary components. Suppose that $h \in C^{(\omega)}[\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}]$, if the Jacobian of h is non-singular on $\tilde{\mathcal{X}}$, then the Inverse Function Theorem guarantees that h is locally a smooth diffeomorphism: in particular, h is a local homeomorphism.

Lemma A.1 *If $h : \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}$ is a local homeomorphism, such that*

$$h(\partial\mathcal{X}) \subset \partial\mathcal{Y} \quad \text{and} \quad h(\overset{\circ}{\mathcal{X}}) \subset \overset{\circ}{\mathcal{Y}},$$

then $h|_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{Y}$ is a local homeomorphism in the subspace topology.

A proof can be found in [18].

Remark: Using local homology groups, one can show[14] that if $h : \mathcal{X} \rightarrow \mathcal{Y}$ is a local homeomorphism between n -dimensional manifolds with boundary, then

$$h(\overset{\circ}{\mathcal{X}}) \subset \overset{\circ}{\mathcal{Y}} \quad \text{and} \quad h(\partial\mathcal{X}) \subset \partial\mathcal{Y}.$$

B Computational Details

B.1 Some General Computations

First, we give a proof of Lemma 2.1 .

Lemma 2.1 *If \mathcal{O}_j is a strictly star shaped obstacle (Definition 5), then \mathcal{O}_j is a regular star shaped set (at q_j). Moreover, for each $q \in \partial\mathcal{O}_j$, the line segment joining q_j and q intersects $\partial\mathcal{O}_j$ only at q .*

Proof: We have to show that for each $q \in \mathcal{O}_j$, the line segment joining q_j and q is contained in \mathcal{O}_j . Let $r : [0, 1] \rightarrow E^n$ be a continuous parametrization of this line segment, defined by,

$$r(\lambda) \triangleq q_j + \lambda(q - q_j).$$

Suppose to the contrary, that there exists $\lambda_1 \in (0, 1)$ such that $q' \triangleq r(\lambda_1) \notin \mathcal{O}_j$. We loose no generality by assuming that

$$\lambda_1 = \inf\{\lambda > 0 : r(\lambda) \notin \mathcal{O}_j\},$$

which implies that $q' \in \partial\mathcal{O}_j$, and $(\beta_j \circ r)(\lambda_1) = 0$. According to equation (2) in the definition of a strictly star shaped obstacle,

$$\nabla\beta_j(q') \cdot (q' - q_j) > 0,$$

and it follows by continuity argument that there exists $\epsilon > 0$ such that

$$\left(\frac{d}{d\lambda}\beta_j \circ r\right)(\lambda_1 - \epsilon, \lambda_1 + \epsilon) > 0.$$

Now suppose that r does not cross $\partial\mathcal{O}_j$ into $(\overline{\mathcal{O}_j})^c$ at λ_1 , that is, there exists $\lambda^* \in (\lambda_1, \lambda_1 + \epsilon)$ such that

$$(\beta_j \circ r)(\lambda^*) \leq 0.$$

According to the Mean Value Theorem,

$$(\beta_j \circ r)(\lambda^*) - (\beta_j \circ r)(\lambda_1) = (\lambda^* - \lambda_1) \frac{d}{d\lambda}(\beta_j \circ r)(s) > 0,$$

for some $s \in (\lambda_1, \lambda^*)$, which implies that $(\beta_j \circ r)(\lambda^*) > 0$, a contradiction. Therefore, $(\beta_j \circ r)(\lambda_1, \lambda_1 + \epsilon) > 0$, and it follows that

$$r(\lambda_1, \lambda_1 + \epsilon) \subset (\overline{\mathcal{O}_j})^c.$$

We now show that r cannot cross $\partial\mathcal{O}_j$ again. Suppose to the contrary, that there exists $\lambda_2 > \lambda_1$ such that $q'' \triangleq r(\lambda_2) \in \partial\mathcal{O}_j$, that is, $(\beta_j \circ r)(\lambda_2) = 0$. We loose no generality by assuming that

$$\lambda_2 = \inf\{\lambda > \lambda_1 : r(\lambda) \in \partial\mathcal{O}_j\}.$$

It follows that

$$r(\lambda_1, \lambda_2) \subset (\overline{\mathcal{O}_j})^c,$$

which in turn implies that

$$(\beta_j \circ r)((\lambda_2 - \epsilon, \lambda_2)) > 0,$$

for some $\epsilon > 0$. By assumption,

$$\nabla\beta_j(q'') \cdot (q'' - q_j) > 0,$$

and it follows by an argument similar to the one given above that

$$(\beta_j \circ r)(\lambda_2) > (\beta_j \circ r)(\lambda_2 - \frac{1}{2}\epsilon) > 0,$$

a contradiction. Therefore, the line segment joining q_j and q cannot cross $\partial\mathcal{O}_j$ if $q \in \mathcal{O}_j$, and crosses $\partial\mathcal{O}_j$ only at q if $q \in \partial\mathcal{O}_j$.

□

The following technical Lemma computes the Jacobian of $h_{\lambda, \kappa}$,

Lemma B.1 *Assuming the placement condition (Definition 9), the Jacobian of the star world deformation, $h_{\lambda, \kappa}$, is*

$$Dh(q) = \sum_{j=0}^M \left\{ \sigma_j \nu_j I + \sigma_j (q - q_j) \nabla \nu_j^T + (\nu_j - 1)(q - q_j) \nabla \sigma_j^T \right\} + \sigma_d I. \quad (23)$$

Proof: Using equation (5), the Jacobian of $h_{\lambda, \kappa}$ is

$$Dh = \sum_{j=0}^M \left\{ \sigma_j \nu_j I + \sigma_j (q - q_j) \nabla \nu_j^T + (\nu_j (q - q_j) + p_j) \nabla \sigma_j^T \right\} + \sigma_d I + ((q - q_d) + p_d) \nabla \sigma_d^T.$$

As σ_d was defined to be $\sigma_d = 1 - \sum_{j=0}^M \sigma_j$, substituting $\nabla \sigma_d$ obtains,

$$Dh = \sum_{j=0}^M \left\{ \sigma_j \nu_j I + \sigma_j (q - q_j) \nabla \nu_j^T + [(\nu_j (q - q_j) + p_j) - ((q - q_d) + p_d)] \nabla \sigma_j^T \right\} + \sigma_d I,$$

assuming the placement condition, that is, $p_j = q_j$ $j \in \{0, \dots, M\}$ and $p_d = q_d$, obtains

$$Dh = \sum_{j=0}^M \left\{ \sigma_j \nu_j I + \sigma_j (q - q_j) \nabla \nu_j^T + (\nu_j - 1)(q - q_j) \nabla \sigma_j^T \right\} + \sigma_d I.$$

□

Denote a unit magnitude vector, v say, by \hat{v} . The following Lemma gives a formula for the gradient of the star set deforming factors.

Lemma B.2 *The gradient of the star set deforming factor (Definition 8), is*

$$\nabla \nu_j(q, \kappa) = \frac{\nu_j}{\|q - q_j\|} \left(\kappa \frac{\|q - q_j\|}{1 + \beta_j} \nabla \beta_j - (\widehat{q - q_j}) \right), \quad (24)$$

for $j \in \{1, \dots, M\}$, and

$$\nabla \nu_0(q, \kappa) = \frac{\nu_0}{\|q - q_0\|} \left(-\kappa \frac{\|q - q_0\|}{1 - \beta_0} \nabla \beta_0 - (\widehat{q - q_0}) \right),$$

when $j = 0$.

Proof: The j^{th} analytic switch was defined in Definition 7 to be

$$\nu_j(q, \kappa) = \rho_j \frac{(1 + \beta_j(q))^\kappa}{\|q - q_j\|} \quad i \in \{1, \dots, M\} \quad \text{and} \quad \nu_0(q, \kappa) = \rho_0 \frac{(1 - \beta_0(q))^\kappa}{\|q - q_0\|},$$

thus,

$$\begin{aligned}\nabla \nu_j &= \frac{\rho_j}{\|q - q_j\|^2} \left(\kappa \|q - q_j\| (1 + \beta_j)^{\kappa-1} \nabla \beta_j - \frac{(1 + \beta_j)^\kappa}{\|q - q_j\|} (q - q_j) \right) \\ &= \frac{\rho_j (1 + \beta_j)^\kappa}{\|q - q_j\|^2} \left(\frac{\kappa \|q - q_j\|}{1 + \beta_j} \nabla \beta_j - \frac{1}{\|q - q_j\|} (q - q_j) \right) \\ &= \frac{\nu_j}{\|q - q_j\|} \left(\kappa \frac{\|q - q_j\|}{1 + \beta_j} \nabla \beta_j - \widehat{(q - q_j)} \right).\end{aligned}$$

The case where $j = 0$ yields a similar result.

□

We now show that if the parameter λ is large enough, the j^{th} switch, σ_j , and its normed gradient, $\|\nabla \sigma_j\|$, can be made arbitrarily small on $\mathcal{F} - \mathcal{S}_i(\epsilon)$.

Lemma B.3 *For any $\epsilon > 0$ and $\delta > 0$ and $j \in \{0, \dots, M\}$, there exists a positive real number $N_{0j}(\epsilon, \delta)$ such that if $\lambda \geq N_{0j}$ then*

$$\sigma_j(q, \lambda) \leq \delta \quad \text{for all } q \in \mathcal{F} - \mathcal{S}_i(\epsilon).$$

Proof:

$$\sigma_j = \frac{\rho_d \bar{\beta}_j}{\rho_d \bar{\beta}_j + \lambda \beta_j} \leq \frac{\rho_d \bar{\beta}_j}{\lambda \beta_j},$$

for the term on the right to be less than δ , λ must satisfy,

$$\lambda \geq \frac{1}{\delta} \frac{\rho_d \bar{\beta}_j}{\beta_j}.$$

Since $\beta_j > \epsilon$ in $\mathcal{F} - \mathcal{S}(\epsilon)$, a sufficient condition on λ is

$$\lambda \geq \frac{1}{\delta \epsilon} \max_{\mathcal{F}} \{\rho_d \bar{\beta}_j\} \triangleq N_{0j}(\epsilon, \delta).$$

□

Lemma B.4 *For any $\epsilon > 0$ and $\delta > 0$ and $j \in \{0, \dots, M\}$ there exists a positive real number $N_{1j}(\epsilon, \delta)$ such that if $\lambda \geq N_{1j}$ then*

$$\|\nabla \sigma_j(q, \lambda)\| \leq \delta \quad \text{for all } q \in \mathcal{F} - \mathcal{S}_i(\epsilon).$$

Proof:

$$\nabla \sigma_j = \frac{\lambda}{(\rho_d \bar{\beta}_j + \lambda \beta_j)^2} (\beta_j \nabla(\rho_d \bar{\beta}_j) - \rho_d \bar{\beta}_j \nabla \beta_j), \quad (25)$$

which implies that

$$\|\nabla \sigma_j\| \leq \frac{\lambda}{(\rho_d \bar{\beta}_j + \lambda \beta_j)^2} (\beta_j \|\nabla(\rho_d \bar{\beta}_j)\| + \rho_d \bar{\beta}_j \|\nabla \beta_j\|),$$

and since $\frac{\lambda\beta_j}{\rho_d\beta_j+\lambda\beta_j} \leq 1$,

$$\begin{aligned} \|\nabla\sigma_j\| &\leq \frac{1}{\rho_d\beta_j+\lambda\beta_j} \|\nabla(\rho_d\bar{\beta}_j)\| + \frac{1}{\beta_j} \frac{\rho_d\bar{\beta}_j}{\rho_d\beta_j+\lambda\beta_j} \|\nabla\beta_j\| \\ &\leq \frac{1}{\lambda\beta_j} \|\nabla(\rho_d\bar{\beta}_j)\| + \frac{\rho_d\bar{\beta}_j}{\lambda\beta_j^2} \|\nabla\beta_j\|. \end{aligned}$$

The assumption that $\beta_j > \epsilon$ in $\mathcal{F} - \mathcal{S}(\epsilon)$ yields as a sufficient condition,

$$\lambda \geq \frac{1}{\delta} \left(\frac{1}{\epsilon} \max_{\mathcal{F}} \{\|\nabla(\rho_d\bar{\beta}_j)\|\} + \frac{1}{\epsilon^2} \max_{\mathcal{F}} \{\rho_d\bar{\beta}_j \|\nabla\beta_j\|\} \right) \triangleq N_{1j}(\epsilon, \delta).$$

□

B.2 The Set “Near” the i^{th} Obstacle

The results of this Section are concerned with $\mathcal{S}_i(\epsilon)$ — the set “near” the i^{th} obstacle boundary.

Lemma B.5 *For any $\epsilon > 0$, any $\kappa > 0$, and any $\delta > 0$ there exists a positive real number $N_2(\epsilon, \kappa, \delta)$ such that if $\lambda \geq N_2$ then*

$$\|X_i(\sigma_j, \nabla\sigma_j; \kappa)\| \leq \delta \quad \text{for all } q \in \mathcal{S}_i(\epsilon), \quad \text{where } j \in \{0, \dots, M\} \quad \text{and } j \neq i.$$

Proof: The norm of $X_i(\sigma_j, \nabla\sigma_j; \kappa)$ can be bounded as follows,

$$\|X_i(\sigma_j, \nabla\sigma_j; \kappa)\| \leq \sum_{j=0, j \neq i}^M \{ \sigma_j |\nu_j - 1| + \|q - q_j\| \|\nabla\nu_j\| + |\nu_j - 1| \|q - q_j\| \|\nabla\sigma_j\| \},$$

since $0 \leq \sigma_j \leq 1$. According to Lemma B.3 and Lemma B.4, for any $\epsilon > 0$ and any $\delta > 0$, σ_j and $\|\sigma_j\|$ can be made to be smaller than δ on $\mathcal{F} - \mathcal{S}_j(\epsilon)$, by choosing

$$\lambda(\epsilon, \delta) \geq N_{0j}(\epsilon, \delta) \quad \text{and} \quad \lambda(\epsilon, \delta) \geq N_{1j}(\epsilon, \delta),$$

respectively, where $N_{0j}(\epsilon, \delta), N_{1j}(\epsilon, \delta)$ are fixed positive real numbers. Now, assuming that

$$\lambda \geq \max_{j \in \{0, \dots, M\}, j \neq i} \{N_{0j}(\epsilon, \delta_1), N_{1j}(\epsilon, \delta_1)\},$$

where δ_1 is yet to be fixed as a function of the required δ and the parameter κ , it follows that

$$\|X_i(\sigma_j, \nabla\sigma_j; \kappa)\| \leq \delta_1 \sum_{j=0, j \neq i}^M \{ |\nu_j - 1| + \|q - q_j\| \|\nabla\nu_j\| + |\nu_j - 1| \|q - q_j\| \},$$

and a sufficient condition to guarantee that $\|X_i(\sigma_j, \nabla\sigma_j; \kappa)\| \leq \delta$ is

$$\delta_1(\kappa, \delta) = \frac{\delta}{\max_{\mathcal{F}} \{ \sum_{j=0, j \neq i}^M \{ |\nu_j - 1| + \|q - q_j\| \|\nabla\nu_j\| + |\nu_j - 1| \|q - q_j\| \} \}}. \quad (26)$$

Therefore, choose

$$N_{2i}(\epsilon, \kappa, \delta) \triangleq \max_{j \in \{0, \dots, M\}, j \neq i} \{N_{0j}(\epsilon, \delta_1(\kappa, \delta)), N_{1j}(\epsilon, \delta_1(\kappa, \delta))\}, \quad (27)$$

and the result follows. \square

Given two subspaces, \mathcal{S}_1 and \mathcal{S}_2 , such that $E^n = \mathcal{S}_1 \oplus \mathcal{S}_2$, any vector $x \in E^n$ can be uniquely written as

$$x = x_1 + x_2 \quad \text{such that } x_1 \in \mathcal{S}_1 \text{ and } x_2 \in \mathcal{S}_2.$$

In general, it is not true that $\|x\| \geq \|x_j\|$ $j = 1, 2$; nevertheless, a bound on $\|x_j\|$ $j = 1, 2$ in terms of $\|x\|$ and the "angle" between the x_1 and x_2 can be specified. Denote by \hat{x}_j the unit magnitude vector $x_j/\|x_j\|$.

Lemma B.6 *Let $E^n = \mathcal{S}_1 \oplus \mathcal{S}_2$. If $x = x_1 + x_2$, $x_i \in \mathcal{S}_i$, then*

$$\|x_j\|^2 \leq \frac{2}{1 - (\hat{x}_1 \cdot \hat{x}_2)^2} \|x\|^2 \quad j = 1, 2. \quad (28)$$

If, in addition,

$$\dim \mathcal{S}_1 = 1 \quad \text{and} \quad \dim \mathcal{S}_2 = n - 1 \quad n \geq 2,$$

then

$$\|x_j\|^2 \leq \frac{2}{(\hat{x}_1 \cdot \hat{v})^2} \|x\|^2 \quad j = 1, 2;$$

where $\hat{v} \in \mathcal{S}_2^\perp$.

Proof:

$$\begin{aligned} \|x\|^2 &= \|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + 2\|x_1\|\|x_2\|(\hat{x}_1 \cdot \hat{x}_2) \\ &= (\|x_1\| \|x_2\|) \underbrace{\begin{bmatrix} 1 & \hat{x}_1 \cdot \hat{x}_2 \\ \hat{x}_1 \cdot \hat{x}_2 & 1 \end{bmatrix}}_{(*)} \begin{pmatrix} \|x_1\| \\ \|x_2\| \end{pmatrix}. \end{aligned}$$

The smallest eigenvalue of the matrix $(*)$ is

$$\lambda_{\min} = 1 - |\hat{x}_1 \cdot \hat{x}_2| \geq \frac{1}{2}[1 - (\hat{x}_1 \cdot \hat{x}_2)^2],$$

thus,

$$\|x\|^2 \geq \lambda_{\min} (\|x_1\|^2 + \|x_2\|^2) \geq \frac{1}{2}[1 - (\hat{x}_1 \cdot \hat{x}_2)^2] \|x_j\|^2 \quad j = 1, 2;$$

and the result follows.

Turning our attention to the second assertion, we use the fact that in the special case $\dim \mathcal{S}_1 = 1$, \mathcal{S}_1 attains its smallest "angle" with any vector $\hat{x}_2 \in \mathcal{S}_2$ when \hat{x}_2 is in the

orthogonal projection of \mathcal{S}_1 onto \mathcal{S}_2 . Specifically, if $\hat{v} \in \mathcal{S}_2^\perp$, we first show that for any $\hat{x}_2 \in \mathcal{S}_2$,

$$(\hat{x}_1 \cdot \hat{x}_2)^2 + (\hat{x}_1 \cdot \hat{v})^2 \leq 1. \quad (29)$$

Let $\{\hat{x}_2, \hat{x}_3, \dots, \hat{x}_n\}$ be an orthonormal basis for \mathcal{S}_2 . \hat{x}_1 can be written as

$$\hat{x}_1 = \sum_{i=2}^n (\hat{x}_1 \cdot \hat{x}_i) \hat{x}_i + (\hat{x}_1 \cdot \hat{v}) \hat{v}.$$

Taking the inner product of both sides with \hat{x}_1 yields

$$1 = \sum_{i=2}^n (\hat{x}_1 \cdot \hat{x}_i)^2 + (\hat{x}_1 \cdot \hat{v})^2 \geq (\hat{x}_1 \cdot \hat{x}_2)^2 + (\hat{x}_1 \cdot \hat{v})^2.$$

Substituting for $1 - (\hat{x}_1 \cdot \hat{x}_2)^2$ in equation (28) according to equation (29) yields the second assertion. □

The following Lemma is used to derive a sufficient condition for the non-singularity of Dh on $\mathcal{S}_i(\epsilon)$. Recall (equation (15)) that for sufficiently small ϵ ,

$$T_q \mathcal{F} = \langle q - q_i \rangle \oplus \langle \nabla \beta_i(q) \rangle^\perp \quad \text{for all } q \in \mathcal{S}_i(\epsilon),$$

and as a consequence, any unit vector $\hat{x} \in T_q \mathcal{F}$ can be uniquely written as

$$\hat{x} = x_1 + x_2,$$

where $x_1 \in \langle q - q_i \rangle$ and $x_2 \in \langle \nabla \beta_i \rangle^\perp$.

Lemma 3.9 For any star world, \mathcal{F} , if \mathcal{M} is a suitable sphere world (Definition 11), then there exist $\epsilon_i, K_i(\epsilon)$ and $N_{2i}(\epsilon, \kappa)$, positive real numbers, such that for all $\epsilon \leq \epsilon_i$ and for all $q \in \mathcal{S}_i(\epsilon)$, whenever $\hat{x} \in T_q \mathcal{S}_i(\epsilon)$, $\|\hat{x}\| = 1$, satisfies

$$\frac{\|x_1\|}{\|x_2\|} > 2,$$

we have,

(i) if $\kappa \geq K_i(\epsilon)$ then

$$-1 + \kappa \frac{\nabla \beta_i \cdot (q - q_i)}{1 + \beta_i} \geq 0 \quad i \in \{1, \dots, M\} \quad \text{and} \quad -1 - \kappa \frac{\nabla \beta_0 \cdot (q - q_0)}{1 - \beta_0} \geq 0;$$

(ii) that

$$(\nu_i - 1)(\nabla \sigma_i \cdot \hat{x})((q - q_i) \cdot x_1) \geq 0;$$

(iii) if $\lambda \geq N_{2i}(\epsilon, \kappa)$, then

$$\|X_i\| \leq \frac{1}{2} \left((\widehat{q - q_i}) \cdot \widehat{\nabla \beta_i} \right)^2 \left(\frac{1}{2} \sigma_i \nu_i + (1 - \sigma_i) \right) \|x_1\|;$$

and whenever

$$\frac{\|x_1\|}{\|x_2\|} \leq 2,$$

(iv) if $\lambda \geq N_{2i}(\epsilon, \kappa)$, then

$$\|X_i\| \leq \frac{1}{2} \left((\widehat{q - q_i}) \cdot \widehat{\nabla \beta_i} \right)^2 (\sigma_i \nu_i + (1 - \sigma_i)) \|x_2\|,$$

where \hat{v} denotes the unit magnitude vector $v/\|v\|$.

Proof: In the proof we consider only $i \in \{1, \dots, M\}$, for which by hypothesis $\nu_i \leq 1$; the proof for $i = 0$, corresponding to the 0^{th} obstacle, for which $\nu_0 \geq 1$, follows similar lines and will be mentioned along the way.

Clearly,

$$-1 + \kappa \frac{\nabla \beta_i \cdot (q - q_i)}{1 + \beta_i} \geq 0,$$

holds true if for all $q \in \mathcal{S}_i(\epsilon)$,

$$\kappa \geq \frac{1 + \beta_i}{\nabla \beta_i \cdot (q - q_i)}.$$

By hypothesis $\beta_i \leq \epsilon$ on $\mathcal{S}_i(\epsilon)$, moreover,

$$\gamma_i(\epsilon) \triangleq \min_{\mathcal{S}_i(\epsilon)} \{ \nabla \beta_i \cdot (q - q_i) \}, \quad (30)$$

is a positive non-increasing function of ϵ on a suitably small interval $[0, \epsilon_i]$, according to the assumption of a star shaped obstacle (equation (2)). Thus, when $\epsilon < \epsilon_i$, a sufficient condition for κ will be,

$$\kappa \geq \frac{1 + \epsilon}{\gamma_i(\epsilon)} \triangleq K_i(\epsilon), \quad (31)$$

where ϵ is yet to be fixed.

In the special case $i = 0$, we define

$$\gamma_0(\epsilon) \triangleq \min_{\mathcal{S}_0(\epsilon)} \{ -\nabla \beta_0 \cdot (q - q_0) \},$$

which is also a positive non-increasing function of ϵ on a suitably small interval $[0, \epsilon_0]$, and a sufficient condition for κ will be,

$$\kappa \geq \frac{1 - \epsilon}{\gamma_0(\epsilon)} \triangleq K_0(\epsilon).$$

The second assertion can be satisfied by “shrinking” $\mathcal{S}_i(\epsilon)$ about the i^{th} obstacle boundary — i.e. by making ϵ small enough, in correspondence with our assumption that ϵ must be less than some ϵ_i . Repeating equation (25) ,

$$\nabla\sigma_i = \frac{\lambda}{(\rho_d\bar{\beta}_i + \lambda\beta_i)^2} (\beta_i\nabla(\rho_d\bar{\beta}_i) - \rho_d\bar{\beta}_i\nabla\beta_i) ,$$

thus,

$$(\nu_i - 1)(\nabla\sigma_i \cdot \hat{x})((q - q_i) \cdot x_1) = \frac{\lambda(\nu_i - 1)}{(\rho_d\bar{\beta}_i + \lambda\beta_i)^2} \{ \beta_i\nabla(\rho_d\bar{\beta}_i) \cdot \hat{x}((q - q_i) \cdot x_1) - \underbrace{\rho_d\bar{\beta}_i\nabla\beta_i \cdot x_1}_{\dagger}((q - q_i) \cdot x_1) \}. \quad (32)$$

In consequence of the assumption $\nabla\beta_i \cdot (q - q_i) > 0$, the term (†) is positive, since

$$\begin{aligned} (\nabla\beta_i \cdot x_1)((q - q_i) \cdot x_1) &= \nabla\beta_i^T [x_1 x_1^T] (q - q_i) \\ &= (\nabla\beta_i \cdot (q - q_i)) (q - q_i)^T [x_1 x_1^T] (q - q_i) \\ &= \nabla\beta_i \cdot (q - q_i) \|x_1\|^2 > 0. \end{aligned}$$

According to the containment condition (equation (9)) , $\nu_i \leq 1$, therefore it is sufficient that

$$\beta_i \|\nabla(\rho_d\bar{\beta}_i)\| \|q - q_i\| \|x_1\| \leq \rho_d\bar{\beta}_i \|x_1\|^2 \nabla\beta_i \cdot (q - q_i), \quad (33)$$

which, is implied by

$$\epsilon \|\nabla(\rho_d\bar{\beta}_i)\| \|q - q_i\| \leq \rho_d\bar{\beta}_i \|x_1\| \gamma_i(\epsilon),$$

which, in turn, follows from

$$\epsilon \|\nabla(\rho_d\bar{\beta}_i)\| \|q - q_i\| \leq \rho_d\bar{\beta}_i \gamma_i(\epsilon) \frac{2}{\sqrt{5}},$$

since $\|x_1\|^2 + \|x_2\|^2 = 1$ and the assumption $\|x_1\| > 2\|x_2\|$ implies that $\|x_1\| > \frac{2}{\sqrt{5}}$. We now require a number ϵ_i such that $\epsilon \in (0, \epsilon_i)$ implies

$$\frac{\epsilon}{\gamma_i(\epsilon)} \leq \frac{2}{\sqrt{5}} \min \left\{ \frac{\rho_d\bar{\beta}_i}{\|\nabla(\rho_d\bar{\beta}_i)\| \|q - q_i\|} \right\} = \frac{2}{\sqrt{5}} \min \left\{ \frac{1}{\|\sum_{j=0, j \neq i}^M \frac{\nabla\beta_j}{\beta_j} + \frac{\nabla\rho_d}{\rho_d}\| \|q - q_i\|} \right\},$$

which yields, as a sufficient condition,

$$\epsilon < E_i \quad \text{and} \quad \frac{\epsilon}{\gamma_i(\epsilon)} \leq \frac{2}{\sqrt{5}} \frac{1}{\sum_{j=0, j \neq i}^M \frac{\max_{\mathcal{F}} \{\|\nabla\beta_j\|\}}{E_j} + \frac{\max_{\mathcal{F}} \{\|\nabla\rho_d\|\}}{E_d}} \min \{\|q - q_i\|\}, \quad (34)$$

since, by assumption, for all $q \in \mathcal{S}_i(\epsilon)$,

$$\gamma_d > E_d \quad \text{and} \quad \beta_j > E_j \quad j \neq i.$$

In consequence of the assumption that $\nabla\beta_i \cdot (q - q_i) > 0$ for all $q \in \partial\mathcal{O}_i$ (equation (11)), as ϵ gets smaller,

$$\gamma_i(\epsilon) = \min_{\mathcal{S}_i(\epsilon)} \{\nabla\beta_i \cdot (q - q_i)\},$$

is non-decreasing ($\mathcal{S}_i(\epsilon_1) \subset \mathcal{S}_i(\epsilon_2)$ whenever $\epsilon_1 < \epsilon_2$), and is positive for ϵ sufficiently small. It follows that $\epsilon/\gamma_i(\epsilon)$ is decreasing as ϵ gets smaller. Therefore, while the practical computation of ϵ'_i may prove to be non-trivial ², there is no problem with the formula itself. A sufficient condition for (34) is

$$\epsilon < \left\{ E_i, \epsilon \leq \frac{2}{\sqrt{5}} \frac{\gamma_i(\epsilon)}{\sum_{j=0, j \neq i}^M \frac{\|\nabla\beta_j\|}{E_j} + \frac{\|\nabla\rho_d\|}{E_d}} \min_{\mathcal{S}_i(E_i)} \{\|q - q_i\|\} \right\} \triangleq \epsilon_i. \quad (35)$$

In the special case $i = 0$, in consequence of the assumption $\nabla\beta_0 \cdot (q - q_0) < 0$, the term (†) in equation (32) is negative. According to the containment condition (equation (9)), $\nu_0 \geq 1$, and equation (33) becomes

$$\beta_0 \|\nabla(\rho_d \bar{\beta}_0)\| \|q - q_0\| \|x_1\| \leq -\rho_d \bar{\beta}_0 \|x_1\|^2 \nabla\beta_0 \cdot (q - q_0),$$

which implies that equation (34) holds for $i = 0$ as well.

In the third assertion,

$$\|X_i\| \leq \frac{1}{2} \left((\widehat{q - q_i}) \cdot \widehat{\nabla\beta_i} \right)^2 \underbrace{\left(\frac{1}{2} \sigma_i \nu_i + (1 - \sigma_i) \right)}_{(*)} \|x_1\|,$$

according to the containment condition (equation (9)), the expression (*) can be bounded from below by,

$$\frac{1}{2} \sigma_i \nu_i + (1 - \sigma_i) = \sigma_i \left(\frac{1}{2} \nu_i - 1 \right) + 1 \geq \left(\frac{1}{2} \nu_i - 1 \right) + 1 = \frac{1}{2} \nu_i, \quad (36)$$

since, by definition, $0 \leq \sigma_i \leq 1$. Thus, a sufficient condition will be

$$\|X_i\| \leq \frac{1}{2\sqrt{5}} \left((\widehat{q - q_i}) \cdot \widehat{\nabla\beta_i} \right)^2 \nu_i(q) \quad \text{for all } q \in \mathcal{S}_i(\epsilon), \quad (37)$$

since $\|x_1\| > \frac{2}{\sqrt{5}}$. According to its definition (Definition 7), ν_i can be bounded from below on $\mathcal{S}_i(\epsilon)$ as follows,

$$\nu_i = \rho_j \frac{(1 + \beta_j(q))^\kappa}{\|q - q_j\|} \geq \frac{\rho_j}{\max_{\mathcal{S}_i(\epsilon)} \{\|q - q_j\|\}}.$$

Therefore, the bound in equation (37) is implied by

$$\|X_i\| \leq \frac{1}{2\sqrt{5}} \left(\min_{q \in \mathcal{S}_i(\epsilon)} \{(\widehat{q - q_i}) \cdot \widehat{\nabla\beta_i}\} \right)^2 \frac{\rho_j}{\max_{\mathcal{F}} \{\|q - q_j\|\}} \triangleq \delta_i. \quad (38)$$

²In the special case where each star shaped obstacle is represented by one of the functions suggested in Appendix C, the condition $\nabla\beta_i \cdot (q - q_i) > 0$ is automatically satisfied for any $\epsilon > 0$ and $q \in \mathcal{F}$.

According to Lemma B.5 , there exists a positive real number $N_{2i}(\epsilon, \kappa, \delta_i)$ such that if $\lambda \geq N_{2i}$ then the inequality above holds.

In the special case $i = 0$, equation (36) becomes

$$\frac{1}{2}\sigma_0\nu_0 + (1 - \sigma_0) \geq 1 - \frac{1}{2}\sigma_0 \geq \frac{1}{2},$$

since $\nu_0 \geq 1$ and $0 \leq \sigma_0 \leq 1$. Therefore, a sufficient condition will be

$$\|X_0\| \leq \frac{1}{2\sqrt{5}} \min_{q \in \mathcal{S}_0(\epsilon)} \{((q - q_0) \cdot \widehat{\nabla\beta_0})^2\} \triangleq \delta_0. \quad (39)$$

In the last assertion,

$$\|X_i\| \leq \frac{1}{2} \left((q - q_i) \cdot \widehat{\nabla\beta_i} \right)^2 \underbrace{(\sigma_i\nu_i + (1 - \sigma_i))}_{(**)} \|x_2\|,$$

according to the containment condition (equation (9)) and the fact that $0 \leq \sigma_i \leq 1$, the expression (**) can be bounded from below by

$$\sigma_i\nu_i - \sigma_i + 1 \geq \nu_i. \quad (40)$$

Since by assumption $\|x_2\| \geq \frac{1}{\sqrt{5}}$. It follows that a sufficient condition is

$$\|X_i\| \leq \frac{1}{2\sqrt{5}} \left((q - q_i) \cdot \widehat{\nabla\beta_i} \right)^2 \nu_i \quad \text{for all } q \in \mathcal{S}_i(\epsilon),$$

which is identical to the condition given in equation (37) above.

In the special case $i = 0$, in which $\nu_0 \geq 1$, equation (40) becomes

$$\sigma_0\nu_0 - \sigma_0 + 1 \geq 1 > \frac{1}{2},$$

which yields a condition identical to the one given in equation (39) above.

□

C Representing Star Shaped Obstacles

In this section we discuss the utility of representing star shaped obstacles using homogeneous functions. In this paper we are concerned with strictly star shaped obstacles (Definition 5), characterized by the requirement

$$\nabla\beta_i \cdot (q - q_i) > 0 \quad \text{for all } q \in \partial\mathcal{O}_i \subset \beta_i^{-1}(0). \quad (41)$$

Although, using continuity arguments, it is guaranteed that this condition holds on some open neighborhood of $\partial\mathcal{O}_i$, the boundary of the i^{th} obstacle, the computation of its extent may be

hard. However, for the suggested representation, we show that this condition is automatically satisfied, and as a consequence it does not add any computational burden. Furthermore, we will show that for a certain subfamily of homogeneous function, which include any analytic norm, the knowledge of the function degree and the radius of an n -disc contained in the corresponding obstacle (containing, for the zeroth obstacle), is sufficient to compute all the derived parameters, thus eliminating the need to resort to numerical computation.

All the omitted proofs in the sequel can be found in [18].

Definition 13 *Let $\gamma \in C^{(0)}[E^n, \mathbb{R}]$, and let σ be a fixed real number. The function γ is homogeneous of degree σ (at $q = 0$) if*

$$\gamma(\lambda q) = \lambda^\sigma \gamma(q) \quad \text{for every } q \neq 0 \text{ and } \lambda > 0.$$

Any homogeneous function of degree σ which belongs to the class $C^{(1)}$ on $E^n - \{0\}$ has the following property [4, exercise 3.3.8],

$$\nabla \gamma(q) \cdot q = \sigma \gamma(q) \quad \text{for all } q \neq 0, \quad (42)$$

and conversely.

Thus, if we let the i^{th} obstacle function to be

$$\beta_i(q) = \gamma(q) - \delta \quad \delta > 0,$$

such that γ is a homogeneous function of degree $\sigma > 0$, and is analytic on $E^n - \{0\}$, then

$$\nabla \beta_i \cdot (q - q_i) = \sigma \beta_i(q) > 0 \quad \text{for all } q \in \beta_i^{-1}(-\delta, \infty) \supset \mathcal{W} - \mathcal{O}_i,$$

that is, equation (41) is satisfied everywhere outside the i^{th} obstacle, \mathcal{O}_i .

The following Lemmas will be used in the Proposition below to designate a subfamily of homogeneous functions each of whose members represents a strictly star shaped obstacle. We assume in the sequel that $\gamma \in C^{(0)}[E^n, \mathbb{R}]$ and that γ is not identically zero.

Lemma C.1 *Let γ be homogeneous of degree $\sigma > 0$ (at $q = 0$). For all $\delta \neq 0$,*

$$\gamma^{-1}(\delta) \cap \{0\} = \emptyset.$$

Thus equation (42) applies at any point in the level set $\gamma^{-1}(\delta)$, as long as $\delta \neq 0$, a fact that we use to show that $\gamma^{-1}(\delta)$ is a compact regular surface.

Let \mathcal{D} be a neighborhood about the origin in E^n . A function $\gamma : \mathcal{D} \rightarrow \mathbb{R}$ is *definite* (at $q = 0$) if

$$\gamma(q) \neq 0 \quad \text{for all } q \neq 0.$$

Lemma C.2 *Let γ be a homogeneous function of degree $\sigma > 0$ (at $q = 0$). If $\gamma^{-1}(0) = \{0\}$ then γ is definite (at $q = 0$).*

The following Lemma is essentially a generalization of the equivalence of norms on E^n to homogeneous functions.

Lemma C.3 *Let γ be homogeneous of degree $\sigma > 0$ (at $q = 0$), and analytic on $E^n - \{0\}$. For all $\delta \neq 0$ such that $\gamma^{-1}(\delta) \neq \emptyset$, the level set $\gamma^{-1}(\delta)$ is a compact regular $(n - 1)$ -surface if and only if*

$$\gamma^{-1}(0) = \{0\}.$$

We are now ready to show that a definite homogeneous function describes a strictly star shaped obstacle. In the sequel, $\gamma \in C^{(0)}[E^n, \mathbb{R}]$ and is analytic on $E^n - \{0\}$.

Proposition C.4 *Let γ be homogeneous of degree $\sigma > 0$, and $\gamma^{-1}(0) = \{0\}$. If the i^{th} obstacle function, β_i , is defined to be*

$$\beta_i(q) \triangleq \begin{cases} \gamma(q) - \delta & \text{if } \gamma \text{ is positive definite} \\ \delta - \gamma(q) & \text{otherwise} \end{cases},$$

for some $\delta > 0$, then β_i describes a strictly star shaped obstacle (definition 5), centered at the origin.

Example: Let P be a positive definite symmetric matrix, the polynomial in two variables, $\gamma : E^2 \rightarrow \mathbb{R}$, defined by,

$$\gamma(x, y) \triangleq (q^T P q) \circ \begin{pmatrix} x^2 \\ y^2 \end{pmatrix},$$

is positive definite and homogeneous of positive degree, therefore for all $\delta > 0$, $\gamma^{-1}(-\infty, \delta)$ is strictly star shaped.

Assuming that the i^{th} obstacle is represented by a homogeneous function of positive degree, the following Lemmas provide formulas to compute all its corresponding parameters in $h_{\lambda, \kappa}$. We start by showing that in this case the i^{th} star set deforming factor, ν_i , has an especially simple structure.

Lemma C.5 *If the i^{th} obstacle function is*

$$\beta_i = \gamma - 1,$$

where γ is a positive definite (at $q = q_i$) homogeneous function of degree $\sigma > 0$, and the parameter κ in the i^{th} star set deforming factor (definition 7), $\nu_i(q, \kappa)$, is chosen as

$$\kappa = \frac{1}{\sigma},$$

then ν_i is constant along the rays originating at q_i .

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