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THE CONSTRUCTION OF UTILITY FUNCTIONS FROM EXPENDITURE DATA

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In considering the behavior of the consumer, a market is assumed which offers some n goods for purchase at certain prices in whatever quantities. A purchase requires an expenditure of money

$$e = \pi_1 \xi_n + \ldots + \pi_n \xi_n = p'x$$

which is determined as the scalar product of the vector

$$x = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

of quantities, which shows the <u>composition</u> of the purchase, and the vector

$$p = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_n \end{pmatrix},$$

of prices which prevail.

The classical assumption about the consumer is that any purchase is such as to give a maximum of utility for the money spent.

The consumer is supposed to attach a number $\varphi(x)$ to any purchase according

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to its composition x, which is the measure of the utility, to the effect that a purchase with composition x made at prices p, and therefore requiring an expenditure $e = p'x$, is such as to satisfy the maximum utility condition

$$\varphi(x) = \max \{\varphi(y) | p'y \leq e\} ,$$

An equivalent statement of this condition is

$$\varphi(x) = \max \{\varphi(y) | u'y \leq 1\}$$

where $u = p/e$ is the vector of prices divided by expenditure, that is with expenditure taken as the unit of money, and is to be called the <u>balance</u> vector, corresponding to those prices and that expenditure. The fundamental property required for a <u>utility function</u> $\varphi(x)$ is that, given a balance u, any composition x which is determined by condition of maximum utility satisfies $u'x = 1$, so that

$$u'y \leq 1 \Longrightarrow \varphi(y) \leq \varphi(x)$$

and

$$\varphi(y) \geq \varphi(x) \Longrightarrow u'y \geq 1$$

Such an assumption cannot represent deliberations on the part of the consumer. The real consumer is generally unaware of an attachment to any such function φ, indeed may deny, by intention and by manifest behavior, any such an attachment. Then if φ is to have a definite existence, it would have to be in the store of analytical constructions of those who entertain the assumption, and based on data of observation. In the earliest form used by Gossen,[1] Jevons,[2] Menger,[3] and Walras,[4] it was assumed that the utility of a composition of goods was a sum of utilities for the separate goods:

$$\varphi(x) = \varphi_1(x_1) + \dots + \varphi_n(x_n) .$$

Then Edgeworth [5] considered a general function

$$\varphi(x) = \varphi(x_1, \dots, x_n) .$$

He also considered the indifference surfaces, the level surfaces, φ = constant, of the utility function. But the now familiar approached, which has divorcement from numerical utility, and deals only with indifference surfaces, thus prefigured by Edgeworth, and also by Antonelli[6] and Fisher, [7] was established by Pareto.[8] Before Pareto, the utility analysis involved in demand theory dealt with utility and utility differences as measurable quantities.[9] By showing numerical utility to be inessential, Pareto brought relief to the discomfort of authors who had to assume a measurable utility, the measurability of which was held in universal doubt.[10] Nevertheless it can still be held that the classical utility is the analytically convenient concept; and also it is the more complete concept, in that it is essential to explain, even if apparently not all, then at least some phenomena of behavior. For there is an apparent absurdity in denying a measurable utility to goods, and not also to money, since, in a fundamental respect, the value of money arises from what can be got with it. While the concept of a utility measure for goods has been displaced from discussions, and has been replaced by the preference relation, it has at the same time become entirely acceptable and even customary to attach measurable utility to money, in the sense of Bernoulli,[11] and von Neumann and Morgenstern,[12] beyond the preference for more money rather than less, which is all that can otherwise be assumed. However, here the concern is with the utility function only as a measure of preference for deciding the better and the worse between collections of goods. But, never in the extended history of the hypothesis has even such a function been shown. The revealed preferences principle of Samuelson,[13] which has been elaborated by Houthakker,[14] gives basis for a condition for the rejection of the hypothesis. It is a condition which corresponds to choice giving an absolute maximum of utility, and is therefore, in regard to the general hypothesis, too strong. The principle has still been wanted by which the hypothesis may be both accepted or rejected, on the basis of any observed acts of choice of the consumer, which must necessarily be finite in number; and, in the case of acceptance, general method is wanted for the actual construction of a utility function which will realize the hypothesis for the data.

This problem is going to be considered here for price-quantity data which can in principle be directly observed, and must therefore be finite. The general problem, which arises when this finiteness restriction is removed, can be approached in one way on the basis of the results which are going to be obtained. But also there is another way of approach for it, which is independent of the present approach for the finite problem and which, incidentally, leads to another solution for the finite problem. These discussions will be taken up in a subsequent paper. The general problem appears more general then that which comes within the investigations of Samuelson, Houthakker, Uzawa and others,[15] which involves a demand system and therefore quantities for every price situation; that is, a complete system of data. For the data could be assumed infinite, but not necessarily complete. Also, even with completeness, the usual assumption of a single valued demand system can be dropped. Or, if a single valued function is assumed, the

Lipschitz-type condition assumed by Uzawa,[16] and therefore also the differentiability assumed by other writers, can be dropped. Also, the cyclical condition assumed by Houthakker has the same scope as the cyclical consistency condition which is to be considered here, if the system is single valued, but otherwise it is too restrictive. In the familiar investigations, the assumptions have been such as to yield just one functionally independent utility function. In the finite problem, and even in the infinite problem with completeness assumed, there is no such essential uniqueness.

While the results for finite data do not immediately give results for complete data, such as a demand system, also the investigations on demand systems, such as have been conducted by Samuelson, Houthakker, Uzawa and others, or conditions for the existence of a utility function, have no scope for the finite problem now to be considered. They depend on a continuous, if not a differentiable structure, which can have no meaning here, in the discrete, finite case; and they leave the problem of establishing criteria by which any finite expenditure data can be considered to be in accordance with some complete demand system which satisfies the appropriate conditions.

Let it be supposed that the consumer has been observed on some k occasions of purchase, and the expenditure data obtained for each occasion r $(r = 1, \ldots, k)$ provide the pair of vectors (x_r, p_r), which give the composition of purchase, and the prevailing prices. Hence the expenditure is $e_r = p_r' x_r$ and the balance vector is $u_r = p_r/e_r$; and, by definition, $u_r' x_r = 1$. Let $E_r = (x_r | u_r)$ define the <u>expenditure figure</u> for occasion r, and $E = \{E_r | r = 1, \ldots, n\}$ the <u>expenditure configuration</u> constructed from the data.

Only through this configuration does the utility hypothesis have bearing on the data.

The utility hypothesis applied to the configuration E asserts that there exists a utility function φ such that

$$\varphi(x_r) = \max \{\varphi(x) | u_r' x \leq 1\} \qquad (r = 1, \ldots, n)$$

in which case the function φ can be said to exhibit the utility hypothesis for E or to be a uitility function for E . The data E can be said to have the property of utility consistency if the utility hypothesis can be exhibited for it by some function, in other words if it has a utility function.

Now there is the problem of deciding, for any given expenditure configuration E , whether or not it has the property of utility consistency, and, if it has, of constructing a utility function for it. This is the problem which is going to be considered.

If utility consistency holds for E , some utility function φ exists for it, and then

$$u_r' x_s \leq 1 \Rightarrow \varphi(x_r) \geq \varphi(x_s)$$

and

$$u_r' x_s \leq 1 \wedge \varphi(x_r) = \varphi(x_s) \Rightarrow u_r' x_s = 1$$

for all r, s = 1 , ..., k . Hence, for all r, s, ..., q = 1, ..., k

$$u_r' x_s \leq 1 \wedge u_s' x_t \leq 1 \wedge \ldots \wedge u_q' x_r \leq 1$$

$$\Rightarrow \varphi(x_r) \geq \varphi(x_s) \geq \ldots \geq \varphi(x_q) \geq \varphi(x_r)$$

$$\Rightarrow \varphi(x_r) = \varphi(x_s) = \ldots = \varphi(x_q)$$

Hence

$$u_r' x_s \leq 1 \wedge u_s' x_t \leq 1 \wedge \dots \wedge u_q' x_r \leq 1$$

$$\Longrightarrow u_r' x_s = u_s' x_t = \dots = u_q' x_r = 1$$

This condition will define the property of *cyclical consistency* for E. It has been shown to be an obvious necessary condition for utility consistency, and it is going to be proved also sufficient. In order to do this, some other consistency conditions will be introduced for E, and finally they will all be proved equivalent. Define $D_{rs} = u_r' x_s - 1$, which may be called the *cross-coefficient*, from E_r to E_s. The cross-coefficients altogether define the *cross-structure* D for the expenditure configuration E.

The cyclical consistency condition now has the statement

$$D_{rs} \leq 0, D_{st} \leq 0 , \dots, D_{qr} \leq 0 \quad D_{rs} = D_{st} = \dots = D_{qr} = 0$$

for all $r, s, t, \dots, q = 1, \dots, k$.
Since a multiple cycle is just a conjunction of simple cycles, and since $D_{rr} = 0$, there is no restriction in assuming $r, s, t, \dots, q = 1, \dots, k$ all distinct.

Let a new consistency condition now be defined for E, again through its cross-structure D, by the existence of numbers λ_r $(r = 1, \dots, k)$, to be called multipliers for E, satisfying the system of inequalities

$$\lambda_r > 0, \lambda_r D_{rs} + \lambda_s D_{st} + \dots + \lambda_q D_{qr} \geq 0 ,$$

for all $r, s, t, \dots, q = 1, \dots, k$.
The consistency of this system of inequalities, in other words the existence of multipliers for E, will define the condition of *multiplier consistency* for E.

Again, the same condition is obtained if $r, s, t, \ldots, q = 1, \ldots, k$ are taken to be distinct.

It is obvious that multiplier consistency implies cyclical consistency. For

$$\lambda_r > 0 \wedge D_{rs} \leq 0 \Rightarrow \lambda_r D_{rs} \leq 0 ,$$

and

$$\lambda_r D_{rs} \leq 0 \wedge \lambda_s D_{st} \leq 0 \wedge \ldots \wedge \lambda_q D_{qr} \leq 0$$

with

$$\lambda_r D_{rs} + \lambda_s D_{st} + \ldots + \lambda_q D_{qr} \geq 0$$

implies

$$\lambda_r D_{rs} = \lambda_s D_{st} = \ldots = \lambda_q D_{qr} = 0 ,$$

which, with $\lambda_r > 0$, implies

$$D_{rs} = D_{st} = \ldots = D_{qr} = 0 .$$

Therefore multiplier consistency implies cyclical consistency. Also the converse is true, as will be shown.

Now let still another condition be defined for E through its cross-structure, by the existence of numbers λ_r, φ_r $(r = 1, \ldots, k)$, to be called <u>multipliers</u> and <u>levels</u>, satisfying the system of inequalities.

$\lambda_r > 0$, $\lambda_r D_{rs} \geq \varphi_s - \varphi_r$ $(r, s, = 1, \ldots, k)$. The consistency of this system of of inequalities will define the condition of <u>level consistency</u> for E .

It is obvious that level consistency implies multiplier consistency, and moreover that any multipliers which realize the level consistency condition also realizes the multiplier consistency condition. For, from

$$\lambda_r D_{rs} \geq \varphi_s - \varphi_r$$

follows

$$\lambda_r D_{rs} + \lambda_s D_{st} + \dots + \lambda_q D_{qr} \geq \varphi_s - \varphi_r + \varphi_t - \varphi_s + \dots + \varphi_q - \varphi_r = 0$$

It will be shown that, conversely, multiplier consistency implies level consistency, and moreover, that any set of multiplies which realized the multiplier consistency condition can be joined with a set of levels to realize the level consistency condition.

The following theorem is going to be proved.

Theorem: The three conditions of cyclical, multiplier and level consistency on the cross-structure of an expenditure configuration are all equivalent, and are implied by the condition of utility consistency for the configuration.

It has been seen that utility consistency for the configuration E implies cyclical consistency for its cross-structure D . Also it has been seen that level consistency implies multiplier consistency and that multiplies consistency implies cyclical consistency, for D . Hence it remains to be shown that cyclical consistency implies multiplier consistency, and that multiplier consistency implies level consistency, and then the theorem will have been proved.

Introduce the relation W defined by

$$rWs \equiv D_{rs} \leq 0 ,$$

it being reflexive, since $D_{rr} = 0$, and then $R = \overrightarrow{W}$, the transitive closure of W , this being transitive and such that $W \subset R$, from the form of its definition, and reflexive, since W is reflexive. Then $P = R \cap \bar{R}'$, the antisymmetric part of R, is antisymmetric, from the form of the definition, and transitive, since R is transitive. Hence it is an order. In case it is not an total order, there always exist an total order which is a refinement of it, that is $R \subset T$ where T is an ideal order, and $T \subset \bar{T}'$, since T is antisymmetric. Without loss in generality, it can be supposed that the occasions are so ordered that $r\,T\,s \equiv r < s$.

Now cyclical consistency is equivalent to the condition

$$D_{rs} \leq 0 \wedge C_{st} \leq 0 \wedge \dots \wedge D_{pq} \leq 0 \Rightarrow D_{qr} \geq 0$$

which can be stated

$$R \subset M'$$

where M is the relation defined by

$$r\,M\,s \equiv D_{rs} \geq 0$$

which is such that

$$r\,M\,s \Leftarrow D_{rs} > 0 \Leftrightarrow r\,\bar{W}\,s$$

so that

$$\bar{W} \subset M$$

Now cyclical consistency gives $R \subset M'$; and the definition of R gives $W \subset R$, so that $\bar{R}' \subset \bar{W}' \subset M'$. Hence $R \cup \bar{R}' \subset M'$. But $R \cap \bar{R}' \subset T \subset \bar{T}'$ so that $T \subset \bar{R}' \cup R$. Hence $T \subset M'$, or equivalently

$$r < s \Rightarrow D_{sr} \geq 0$$

Now assume, as an inductive hypothesis, that, at an $(m-1)^{th}$ stage, multipliers $\lambda_r > 0 (1 \leq r < m)$ have been found such that

$$\lambda_r D_{rs} + \lambda_s D_{st} + \dots + \lambda_q D_{qr} \geq 0 \ (1 \leq r, \dots, q < m)$$

Then, for the m^{th} stage to be attained, it is required to find a multiplier $\lambda_m > 0$ such that

$$\lambda_r D_{rs} + \lambda_s D_{st} + \dots + \lambda_q D_{qm} + \lambda_m D_{mr} \geq 0 \ (1 \leq r, \dots, q \leq m - 1)$$

But $D_{mr} \geq 0$ if $r < m$. Hence let

$$\mu_m = - \min \left\{ \frac{\lambda_r D_{rs} + \lambda_s D_{st} + \dots + \lambda_q D_{qm}}{D_{mr}} \,\middle|\, 1 \leq r, \dots, q < m ; D_{mr} > 0 \right\}$$

Then any $\lambda_m \geq \max \{0, \mu_m\}$ is as required. Hence the m^{th} stage is attainable from the $(m-1)^{th}$. The second stage can obviously be attained, since, with any $\lambda_1 > 0$, there only has to be taken a $\lambda_2 > 0$ such that $\lambda_1 D_{12} + \lambda_2 D_{21} \geq 0$, which is possible since $D_{12} < 0$ and $D_{21} < 0$ is impossible, by the hypothesis of cyclical consistency. It follows by induction that the k^{th} stage is attainable, that is, multipliers can be found which realize the multiplier consistency condition. The proof that cyclical consistency implies multiplier consistency is now complete.

To prove that multiplier consistency implies unit consistency, assume a set of multipliers λ_r and let $a_{rs} = \lambda_r D_{rs}$. Then

$$a_{rs} + a_{st} + \dots + a_{qr} \geq 0 ,$$

for all distinct $r, s, t, \ldots, q$. It is now going to be proved that there exist numbers φ_r such that

$$(a): \quad a_{rs} \geq \varphi_r - \varphi_s \ (r \neq s) ,$$

whence the level consistency condition will have been shown.

Let

$$a_{r\ell m \ldots ps} = a_{r\ell} + a_{\ell m} + \ldots + a_{ps} ,$$

and let

$$A_{rs} = \min_{\ell, m, \ldots, p} a_{r\ell m \ldots ps}$$

Then

$$a_{rs} \geq A_{rs} .$$

Also,

$$A_{rs} + A_{sr} \geq 0 \quad \text{and} \quad A_{rs} + A_{st} \geq A_{rt} .$$

Consider the system

$$(A): \quad A_{rs} \geq \varphi_r - \varphi_s \quad (r \neq s) .$$

Any solution φ_r of (A) is a solution of (a), since $a_{rs} \geq A_{rs}$. Also, any solution φ_r of (a) is a solution of (A). For

$$a_{r\ell} \geq \varphi_r - \varphi_\ell , \quad a_{\ell m} \geq \varphi_\ell , \ldots, a_{ps} \geq \varphi_p - \varphi_s ;$$

whence, by addition,

$$a_{r\ell m \ldots ps} \geq \varphi_r - \varphi_\ell - \varphi_m + \ldots + \varphi_p - \varphi_s$$
$$= \varphi_r - \varphi_s ,$$

and therefore

$$A_{rs} \geq \varphi_r - \varphi_s .$$

It follows that the consistency of (a), which has to be shown, is equivalent to that of (A), which will be shown now.

The proof depends on an extension property of solution of the subsystems of (A). Thus, assume a solution φ_r $(r < m)$ has been found for the subsystem

$$(A, m\text{-}1): \quad A_{rs} \geq \varphi_r - \varphi_s \; (r \neq s \; ; \; r, s < m) .$$

It will be shown that it can be extended by an element φ_m to a solution of (A, m) .

Thus, there is to be found a number φ_m such that

$$A_{rm} \geq \varphi_r - \varphi_m , \quad a_{ms} \geq \varphi_m - \varphi_s \; (r, s < m) ,$$

that is,

$$A_{ms} + \varphi_s \geq \varphi_m \geq \varphi_r - A_{rm} .$$

So the condition that such a φ_m can be found is

$$A_{mq} + \varphi_q \geq \varphi_p - A_{pm} ,$$

where

$$\varphi_p - A_{pm} = \max_r \{\varphi_r - A_{rm}\} , \quad A_{mq} + \varphi_q = \min_r \{A_{mq} + \varphi_q\} .$$

But if $p = q$, this is equivalent to

$$A_{mq} + A_{qm} \geq 0 ,$$

which is verified by hypothesis; and if $p \neq q$, it is equivalent to

$$A_{pm} + A_{mq} \geq \varphi_p - \varphi_q ,$$

which is verified, since, by hypothesis,

$$A_{pm} + A_{mq} \geq A_{pq} , \quad A_{pq} \geq \varphi_p - \varphi_q .$$

Since the system (A, 2) trivially has a solution, it follows by induction that the system (A) = (A, k) has a solution, and is thus consistent.

<u>THEOREM:</u> <u>If</u> $E = \{E_r | r = 1, \ldots, n\}$ <u>is any expenditure configuration, with figures</u> $E_r = (x_r | u_r)$ $(u_r' x_r = 1)$ <u>and cross-coefficients</u> $D_{rs} = u_r' x_s - 1$, <u>and if</u> λ_r , φ_r <u>are any multipliers and levels, being such that</u>

$$\lambda_r > 0 , \quad \lambda_r D_{rs} \geq \varphi_s - \varphi_r \qquad (r, s - 1, \ldots, n)$$

<u>and if</u> $g_r = u_r \lambda_r$, <u>and</u> $g_r(x) = \varphi_r + g_r' (x - x_r)$, <u>then</u>

$$\varphi(x) = \min \{\varphi_r(x) | r = 1, \ldots, n\}$$

<u>is a function which realizes the utility hypothesis for</u> E .

Since $g_r = u_r \lambda_r$ and $D_{rs} = u_r' x_{rs} - 1$,

from

$$\lambda_r > 0 , \quad \lambda_r D_{rs} \geq \varphi_s - \varphi_r$$

follows

$$g_r > 0 , \quad g_r' (x_s - x_r) \geq \varphi_s - \varphi_r$$

and thus

$$\varphi_r + g_r' (x_s - x_r) \geq \varphi_s = \varphi_s + g_s' (x_s - x_s)$$

Therefore

$$\varphi(x_s) = \min \{\varphi_r + g_r'(x_s - x_r) | r = 1, \ldots, n\}$$

$$= \varphi_s$$

Also if $\varphi(x) > \varphi_s$, then

$$\varphi_r + g_r(x - x_r) > \varphi_s \qquad (r = 1, \ldots, n)$$

so that

$$\varphi_s + \lambda_s(u_s' x - 1) > \varphi_s ,$$

and, since $\lambda_s > 0$, this implies $u_s' x > 1$. Hence $u_s' x \leq 1 \Rightarrow \varphi(x) \leq \varphi_s$. Therefore, since $\varphi(x_s) = \varphi_s$ and $u_s' x_s = 1$, it appears that

$$\varphi(x_s) = \max \{\varphi(x) | u_s' x \leq 1\}$$

Now

$$\varphi_r(x) = \varphi_r + g_r' (x - x_r)$$

$$= \varphi_r + \lambda_r (u_r' x - 1)$$

so that

$$\varphi_r(x_s) = \varphi_r + \lambda_r D_{rs} \geq \varphi_s , \quad \varphi_s(x_s) = \varphi_s$$

Hence

$$\varphi(x_s) = \min \{\varphi_r(x_s) | r = 1, \ldots, n\}$$
$$= \varphi_s$$

Also, $\varphi(x) \geq \varphi_s$ implies $\varphi_s(x) \geq \varphi_s$, which, since $\lambda_s > 0$, is equivalent to $u_s' x \geq 1$. Therefore $u_s' x < 1$ implies $\varphi(x) < \varphi_s$. Accordingly,

$$\max \{\varphi(x) | u_s' x \leq 1\} = \varphi_s$$

and

$$u_s' x \leq 1 \wedge \varphi(x) = \varphi_s \Rightarrow u_s' x = 1$$

The function φ therefore realizes the utility hypothesis for the configuration.

Since level consistency is the condition for the existence of the λ_r, φ_r, there follows:

COROLLARY: <u>For an expenditure configuration to have the property of utility consistency it is sufficient that its cross-structure name the property of level consistency.</u>

But, by the previous theorem, level consistency is necessary for utility consistency and is equivalent to cyclical consistency, whence

COROLLARY: <u>The cyclical consistency condition is necessary and sufficient for the utility consistency of a finite expenditure configuration.</u>

Some remarks may now be made on the form of the function $\varphi(x)$ which has been constructed. The functions $\varphi_r(x)$ are linear, and therefore concave and they have gradients $g_r > 0$, so they are increasing functions. Therefore, $\varphi(x)$, since it is the minimum of increasing concave functions, is an increasing concave function. Its level surfaces $\{x|\varphi(x) = \varphi\}$ are the convex polyhedral surfaces which are the boundaries of the convex polyhedral regions $\{x|\varphi(x) \geq \varphi\}$ defined by the inequalities $\varphi_r(x) \geq \varphi$, or equivalently

$$u_r' x \geq 1 + \frac{\varphi - \varphi_r}{\lambda_r} \qquad (r = 1, \ldots, n)$$

The region $\Omega_s = \{x|\varphi(x) = \varphi_s(x)\}$, in which $\varphi(x)$ coincides with $\varphi_r(x)$, is a polyhedral region, which is the projection, in x-space, of the face in which $\varphi_s(x) = \varphi$ cuts the boundaries of the region in (x, φ)-space defined by these inequalities. Since $\varphi(x_s) = \varphi_s = \varphi_s(x_s)$, as has been seen, it appears that $x_s \in \Omega_s$. Also

$$\Omega_s = \{x|\varphi_s(x) \geq \varphi_s(x) \; ; \; r = 1, \ldots, n\} ,$$

Hence Ω_r is defined by the inequalities

$$\varphi_r + \lambda_r \,(u_r' x - 1) \geq \varphi_s + \lambda_s \,(u_s' x - 1) \quad (r = 1, \ldots, n)$$

Thus for a point to belong to two of the cells, say $x \in \Omega_s \cap \Omega_t$, it is required that

$$\varphi_s + \lambda_s \,(u_s' x - 1) = \varphi_t + \lambda_t \,(u_t' x - 1)$$

Hence, in a regular case, these cells can only intersect on their boundaries. The regions Ω_r thus constitute a dissection of the x-space into polyhedral cells. In the relative interior of each cell Ω_r, the function $\varphi(x)$ is differentiable, and has constant gradient $g(x) = g_r \ (x \in \Omega_r)$.

Now an index-number formula will be shown, which is made intelligible by the construction of this utility function. Given any utility function $\varphi(x)$, the cost of living index with r and s as base and current periods has the determination

$$\rho_{sr} = \min \{u_s' x \,|\, \varphi(x) \geq \varphi_r\} .$$

where $u_s' x = p_s' x / p_s' x_s$, and $\varphi_r = \varphi(x_r)$. Hence, with determination relative to the function $\varphi(x)$ which has been constructed,

$$\rho_{sr} = \min \{u_s' x \,|\, u_t' x \geq 1 + \frac{\varphi_r - \varphi_t}{\lambda_t} ; \quad t = 1, \ldots, k\} .$$

It can be seen that the realization of the utility hypothesis by a utility function φ which is concave and has gradient g implies level consistency. For the concavity is equivalent to the condition

$$\varphi(y) - \varphi(x) \leq g(x)' (y - x) ,$$

and Gossen's Law, that preference and price directions coincide in equilibrium, gives $g = u\lambda$, where $\lambda = g'x$ since $u'x = 1$. Hence, with $\varphi(x_r) = \varphi_r$, $g(x_r) = u_r \lambda_r$, there follows

$$\varphi_s - \varphi_r \leq \lambda_r U_r(x_s - x_r)$$

and thus

$$\lambda_r D_{rs} \geq \varphi_s - \varphi_r$$

By an easy enlargement, the present results can be made to encompass the point of view of Pareto of preference as a relation divorced from a numerical measure.

An expenditure figure $E_r = (x_r | u_r)$ is considered as the choice $(x_r | W_{u_r})$, of x_r from among all compositions in the set $W_{u_r} = \{x | u_r' x \leq 1\}$; and the preferences <u>immediate</u> in this choice form the set

$$R_r = \{(x_r, x) | x \in W_{u_r}\} = (x_r, W_{u_r})$$

If these belong to a relation R, for all r, then

$$\bigcup_{r=1, \ldots, k} R_r \subset R$$

and if R is transitive, that is $\overrightarrow{R} \subset R$ where $\overrightarrow{R}$ is the transitive closure, this is equivalent to

$$R_E \subset R ,$$

where

$$R_E = \overrightarrow{\bigcup_{r=1, \ldots, n}} R_r$$

can define the preferences <u>implicit</u> in the configuration $E = \{E_r | r=1, \ldots, n\}$. Any preference relation which can be an hypothesis for E , in that it is reflexive and transitive and contains all the preferences in the choices shown by E , is <u>revealed</u> to the extent of containing R_E.

Now let $\textcircled{>}$ stand for the relation by which one composition is greater than another. That is $x \textcircled{>} y$ means every quantity in x is at least the corresponding quantity in y , and not all are the same. In any admissible preference hypothesis R , it is to be assumed that the greater is exclusively preferred to the lesser so that

$$x R y \Longrightarrow \sim . \; y \textcircled{<} x$$

That is $R \subset \overline{\textcircled{>}}'$, and, with $R_E \subset R$, this gives

$$R_E \subset \overline{\textcircled{>}}'$$

It will now be seen for this condition, which can be called the <u>preference</u> <u>consistency</u> condition and is obviously implied by utility consistency, that it implies cyclical consistency. For it implies that

$$x_r R x_q \Longrightarrow \sim x_q \textcircled{>} x_r$$

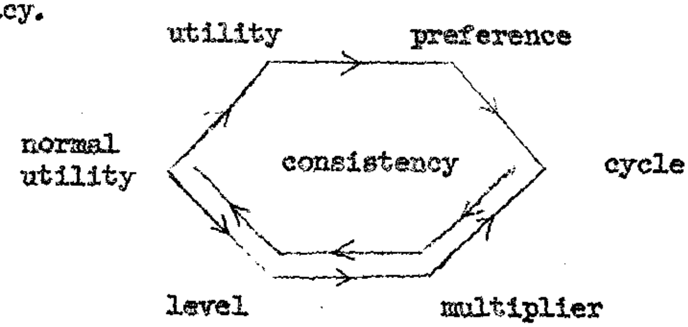
which implies, what is the same as cyclical consistency, that

$$D_{rs} \leq 1 \wedge D_{st} \leq 1 \wedge \ldots \wedge D_{qr} < 1 \quad \text{is impossible,}$$

since

$$x_q \circledR x_r \Longrightarrow D_{qr} < 1 .$$

Now if <u>normal utility consistency</u> is defined as utility consistency with realization by a concave utility function, and since, by virtue of the form of the function which has been shown constructible under level consistency, and by the implication of level consistency from the existence of such a function, the following implications are established, those on the outside in the diagram having been quite immediate, and those on the inside having been proved with less immediacy.



It follows therefore that all these six conditions are equivalent. The sparseness of the assumption in the utility consistency condition can be noted, as also the strictness of the normal utility consistency condition, which goes beyond addition of the familiar assumptions, that a utility function be continuous, increasing, and have concave levels, to the further assumption that it be concave. Also seen is the equivalence of the two approaches, involving preference as a relation and utility as a magnitude. The finiteness of the configuration E has been essential for methods used. Nevertheless it is possible to obtain analogous results without this restriction. Though they will be without the strict constructivity, which here has been followed throughout.

NOTES

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