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THE CONTACT STRUCTURES ON $\{SU(n+1) \times R/SU(n) \times R\}_\alpha$ AND $\{Sp(n) \times SU(2)/Sp(n-1) \times SU(2)\}_\alpha$ OF BERGER

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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M. Berger has classified all simply connected normal homogeneous Riemannian manifolds of positive curvature ([1]). In odd dimensional case there are seven classes of such structures:

(i) sphere of constant curvature,
(ii) $\{SU(n+1) \times R/SU(n) \times R\}_\alpha$ $(n \geqq 2, 0 < \alpha \leqq \pi/2)$,
(iii) $\{Sp(n) \times SU(2)/Sp(n-1) \times SU(2)\}_\alpha$ $(n \geqq 2, 0 < \alpha \leqq \pi/2)$,
(iv) $\{Sp(n) \times R/Sp(n-1) \times R\}_\alpha$ $(n \geqq 2, 0 < \alpha \leqq \pi/2)$,
(v) $SO(9)/SO(7)$,
(vi) $Sp(2)/SU(2)$,
(vii) $SU(5)/Sp(2) \times S^1$.

It is well known that (i) has a natural contact structure, that is, a Sasakian structure of constant curvature.

In the present note we shall show that (ii) and (iii) have also natural contact structures which relate closely to homogeneous Riemannian metrics, and that these contact structures define S^1 or S^3-fiberings of these manifolds over complex or quaternionic projective spaces. For example in case of (ii) we have the following.

THEOREM. *$\{SU(n+1) \times R/SU(n) \times R\}_\alpha$ has a structure of regualr compact simply connected Sasakian manifold of constant ϕ-holomorphic sectional curvature $4 - 3/4 \cdot \beta^2$, where we have put $\beta = \sqrt{2(n+1)/n} \sin \alpha$. Boothby-Wang's fibering is a principal circle bundle over complex projective space of constant holomorphic sectional curvature 4 (cf. Theorem 2.2).*

1. Preliminaries.

1.1 Let M^{2n+1} be a $(2n+1)$-dimensional Riemannian manifold. Let ∇ denote the covariant differentiation, and $R(X, Y)Z = \nabla_{[X,Y]_M}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z$ be the curvature tensor. In the following "[,]" denotes the bracket operation of Lie algebra of Lie group and "$[\ ,\ ]_M$" denotes the bracket of vector fields on the differentiable manifold M.

Now a 1-form η on M^{2n+1} is said to be a contact form if $\eta \wedge (d\eta)^n \neq 0$

holds everywhere. First we shall show

LEMMA 1.1. *Let ξ be a unit Killing vector field on M. Suppose that the linear map: $X \to R(\xi, X)\xi$ of M_m be of rank $2n$ at every point m of M, then ξ defines a contact structure on M. That is, if we put $\eta(X) = \langle \xi, X \rangle$, then η is a contact form.*

PROOF. We put $\phi(X) = -\nabla_X \xi$. Note that $\phi(\xi) = 0$ holds. Since ξ is a unit Killing vector field, by Ricci's identity we have

$$\begin{aligned} \phi^2(X) &= -\nabla_{\phi(X)}\xi = -\nabla_\xi \phi(X) + [\xi, \phi(X)]_M \\ &= \nabla_\xi \nabla_X \xi - [\xi, \nabla_X \xi]_M = R(X, \xi)\xi \,. \end{aligned} \tag{1.1}$$

By the assumption of the lemma, on $D = \xi^\perp$, ϕ^2 and consequently ϕ has maximal rank $2n$. On the other hand we get

$$d\eta(X, Y) = -2\langle \phi X, Y \rangle \,. \tag{1.2}$$

That is $(d\eta)^n \neq 0$ on D. Since $\eta(\xi) = 1$, $\eta \wedge (d\eta)^n \neq 0$ holds at every point of M. q.e.d.

1.2 If a unit Killing vector field ξ on M^{2n+1} satisfies

$$R(X, \xi)Y = k\{\langle X, Y \rangle \xi - \langle \xi, Y \rangle X\} \tag{1.3}$$

for some positive constant k, then $(M, \xi, \langle\,,\,\rangle)$ is called a Sasakian manifold ([4]). Note that in this case sectional curvature of the plane section containing ξ is a constant k. Next let ξ_1, ξ_2, ξ_3 be three contact structures on M. If $\langle \xi_i, \xi_j \rangle = \delta_{ij}$ and $[\xi_{\sigma(1)}, \xi_{\sigma(2)}]_M = c \operatorname{sgn} \sigma \cdot \xi_{\sigma(3)}$ hold, where c is a constant and σ is a permutation of $\{1, 2, 3\}$, we say that $\{\xi_1, \xi_2, \xi_3\}$ defines a contact 3-structure on M ([6]).

1.3 The curvature tensors of normal homogeneous Riemannian manifolds are well known. Let $M = G/H$ be a normal homogeneous Riemannian manifold, and $\mathfrak{G}$ (resp. $\mathfrak{H}$) be the Lie algebra of G (resp. H). Let $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ be an orthogonal decomposition. Then $\mathfrak{M}$ may be considered as the tangent space to M at $\pi(e)$, where π denotes the canonical projection $G \to G/H$ and e is the unit element of G. Now we have for $X, Y, Z \in \mathfrak{M}$

$$\begin{aligned} R(X, Y)Z = {} & [[X, Y]_{\mathfrak{H}}, Z] + 1/2 \cdot [[X, Y]_{\mathfrak{M}}, Z]_{\mathfrak{M}} \\ & + 1/4 \cdot [[Y, Z]_{\mathfrak{M}}, X]_{\mathfrak{M}} + 1/4 \cdot [[Z, X]_{\mathfrak{M}}, Y]_{\mathfrak{M}} \,, \end{aligned} \tag{1.4}$$

where $[X, Y]_{\mathfrak{H}}$ (resp. $[X, Y]_{\mathfrak{M}}$) denotes the $\mathfrak{H}$ (resp. $\mathfrak{M}$)-component of $[X, Y]$. (See K. Nomizu [3]).

2. $\{SU(n+1) \times R/SU(n) \times R\}_\alpha$. Throughout of this section we shall follow the notation of I. Chavel ([2]). Let $\mathfrak{A}_n$ be the Lie algebra of $SU(n+1)$. We choose on $\mathfrak{A}_n$ a bi-invariant metric $\langle X, Y \rangle = -1/2 \cdot \operatorname{trace} XY$.

Now if we put

$$A_{jk} = \sqrt{-1}(\varepsilon_{jj} - \varepsilon_{kk})$$
$$B_{jk} = (\varepsilon_{jk} - \varepsilon_{kj})$$
$$C_{jk} = \sqrt{-1}(\varepsilon_{jk} + \varepsilon_{kj})$$
$$S_j = \sqrt{2/\{j(j+1)\}}\sum_{l=1}^{j} lA_{l,l+1}\ ,$$

where ε_{jk} denotes the matrix $(\delta_{jr}\delta_{ks})_{1\leqq r,s\leqq n+1}$, then $\{S_1, \cdots, S_n; B_{ij}, C_{ij}; 1 \leqq i < j \leqq n+1\}$ forms an orthogonal frame for $\mathfrak{A}_n$. Let $\mathfrak{A}_{n-1}$ be the naturally imbedded subalgebra of $\mathfrak{A}_n$ with $[\mathfrak{A}_{n-1}, S_n] = 0$, and $\mathfrak{A}_n = \mathfrak{A}_{n-1} \oplus \mathfrak{M}$ be an orthogonal decomposition. Define

$$\mathfrak{G}_n = \mathfrak{A}_n \oplus R$$
$$\mathfrak{H}_\alpha = \{S_1, \cdots, S_{n-1}; \cos\alpha S_n + \sin\alpha D; B_{rj}, C_{rj}; 1 \leqq r < j \leqq n\}$$
$$\mathfrak{M}_\alpha = \{\sin\alpha S_n - \cos\alpha D; B_{jn+1}, C_{jn+1}; 1 \leqq j \leqq n\}\ ,$$

where D denotes the unit vector of R and $0 < \alpha \leqq \pi/2$. Then this decomposition $\mathfrak{H}_\alpha + \mathfrak{M}_\alpha$ of $\mathfrak{G}_n$ defines the simply connected normal homogeneous space $M_\alpha^{2n+1} = \{SU(n+1)\times R/SU(n)\times R\}_\alpha$ (See Berger [1] and Chavel [2]). Note that $M_{\pi/2}^{2n+1}$ may be considered as $SU(n+1)/SU(n)$. If we set $\xi = \sin\alpha S_n - \cos\alpha D$, $e_{2j-1} = B_{jn+1}$, $e_{2j} = C_{jn+1}$ then $\{\xi, e_{2j-1}, e_{2j}; 1 \leqq j \leqq n\}$ forms an orthonormal basis for $\mathfrak{M}_\alpha$. Note that $[\xi, \mathfrak{H}_\alpha] = 0$ holds.

Now we shall calculate the curvature tensor of M_α^{2n+1} at $\pi(e)$. Using the multiplication table of [2], we have by direct calculation

LEMMA 2.1. *If we put* $\beta = \sqrt{2(n+1)/n}\sin\alpha$, *we get*

(2.1) $R(\xi, e_{2j})\xi = \beta^2/4\cdot e_{2j}$

(2.2) $R(\xi, e_{2j})e_{2k} = -\beta^2/4\cdot\delta_{jk}\xi$

(2.3) $R(\xi, e_{2j})e_{2k-1} = 0$

(2.4) $R(\xi, e_{2j-1})\xi = \beta^2/4\cdot e_{2j-1}$

(2.5) $R(\xi, e_{2j-1})e_{2k} = 0$

(2.6) $R(\xi, e_{2j-1})e_{2k-1} = -\beta^2/4\cdot\delta_{jk}\xi$

(2.7) $R(e_{2j}, e_{2k})e_{2l} = \delta_{jl}e_{2k} - \delta_{kl}e_{2j}$

(2.8) $R(e_{2j}, e_{2k})e_{2l-1} = (1 - 1/4\cdot\beta^2)(\delta_{jl}e_{2k-1} - \delta_{kl}e_{2j-1})$

(2.9) $R(e_p, e_q)\xi = 0\ (p, q = 1, 2, \cdots, 2n)$

(2.10) $R(e_{2j}, e_{2k-1})e_{2l} = (2 - 1/2\cdot\beta^2)\delta_{jk}e_{2l-1} + \delta_{jl}e_{2k-1} + (1 - 1/4\cdot\beta^2)\delta_{kl}e_{2j-1}$

(2.11) $R(e_{2j}, e_{2k-1})e_{2l-1} = -(2 - 1/2\cdot\beta^2)\delta_{jk}e_{2l} - (1 - 1/4\cdot\beta^2)\delta_{jl}e_{2k} - \delta_{kl}e_{2j}$

(2.12) $R(e_{2j-1}, e_{2k-1})e_{2l} = (1 - 1/4\cdot\beta^2)(\delta_{jl}e_{2k} - \delta_{kl}e_{2j})$

(2.13) $R(e_{2j-1}, e_{2k-1})e_{2l-1} = \delta_{jl}e_{2k-1} - \delta_{kl}e_{2j-1}$.

Now ξ defines a left invariant vector field on $G_n = \exp\mathfrak{G}_n$. Since $[\xi, \mathfrak{H}_\alpha] = 0$ holds, $d\pi(\xi)$ defines a vector field on M_α^{2n+1} which will be also denoted by ξ. Then (2.1)$\sim$(2.6) show that the unit Killing vector field

ξ on M_α^{2n+1} defines a Sasakian structure on M_α^{2n+1}. In fact, if we put $\phi = -\nabla\xi$, we have (1.3) with $k = \beta^2/4$ and

$$\begin{aligned}
\phi e_{2j-1} &= \beta/2 \cdot e_{2j}, \quad \phi e_{2j} = -\beta/2 \cdot e_{2j-1}, \quad \phi(\xi) = 0 . \\
\phi^2 X &= \beta^2/4 \cdot (-X + \langle \xi, X \rangle \xi) \\
\langle \phi X, \phi Y \rangle &= \beta^2/4 \cdot (\langle X, Y \rangle - \langle \xi, X \rangle \langle \xi, Y \rangle) .
\end{aligned}$$

Next (2.7)$\sim$(2.13) show that M_α^{2n+1} is a Sasakian manifold of constant ϕ-holomorphic sectional curvature $4 - 3/4 \cdot \beta^2$. That is, every sectional curvature of the plane section defined by $\{X, \phi X\}$, $X \perp \xi$ is equal to a constant $4 - 3/4 \cdot \beta^2$. Note that $SU(n+1) \times R$ acts on M_α^{2n+1} as an automorphism group of this Sasakian structure. So ξ is a regular contact structure and we have the Boothby-Wang's fibering of M_α^{2n+1}. That is, M_α^{2n+1} is a principal circle bundle over complx projective space of constant holomorphic sectional curvature 4, as is easily seen. Thus we have the following theorem.

THEOREM 2.2. *$M_\alpha^{2n+1} = \{SU(n+1) \times R / SU(n) \times R\}_\alpha$ has a structure of regular compact simply connected Sasakian manifold of constant ϕ-holomorphic sectional curvature $4 - 3/4 \cdot \beta^2$. Boothby-Wang's fibering of M_α^{2n+1} is a principal circle bundle over complex projective space of constant holomorphic sectional curvature* 4.

REMARK 1. Note that the dimension of $SU(n+1) \times R$ is $(n+1)^2$. This is the maximal dimension for the automorphism group of a connected almost contact Riemannian manifold M^{2n+1}. A result of S. Tanno ([5]) shows that in this case M must be of constant ϕ-holomorphic sectional curvature.

REMARK 2. It is known that a Sasakian manifold of constant ϕ-holomorphic sectional curvature $4 - 3/4 \cdot \beta^2$ is of constant curvature if and only if $4 - 3/4 \cdot \beta^2$ is equal to $1/4 \cdot \beta^2$ (= the sectional curvature of the plane section containing ξ). But since $\beta = \sqrt{2(n+1)/n} \sin\alpha$, this explains why M_α^{2n+1} can not be of constant curvature.

REMARK 3. M_α^{2n+1} is diffeomorphic (D-homothetically deformable in the sense of S. Tanno ([5])) to a unit sphere.

3. $\{Sp(n) \times SU(2)/Sp(n-1) \times SU(2)\}_\alpha$.

3.1 Let $\mathfrak{S}_n$ be the Lie algebra of $Sp(n)$. We shall consider $Sp(n)$ as a subgroup of $U(2n)$ as usual and put

$$\begin{aligned}
A_i &= \sqrt{-1}(\varepsilon_{ii} - \varepsilon_{n+in+i}) \quad i = 1, \cdots, n \\
B_i &= \varepsilon_{in+i} - \varepsilon_{n+ii} \quad i = 1, \cdots, n \\
C_i &= \sqrt{-1}(\varepsilon_{in+i} + \varepsilon_{n+ii}) \quad i = 1, \cdots, n
\end{aligned}$$

$$
\begin{aligned}
D_{ik} &= \varepsilon_{ik} - \varepsilon_{ki} + \varepsilon_{n+in+k} - \varepsilon_{n+kn+i} \quad 1 \leqq i < k \leqq n \\
E_{ik} &= \sqrt{-1}(\varepsilon_{ik} + \varepsilon_{ki} - \varepsilon_{n+in+k} - \varepsilon_{n+kn+i}) \quad 1 \leqq i < k \leqq n \\
F_{ik} &= \varepsilon_{in+k} + \varepsilon_{kn+i} - \varepsilon_{n+ik} - \varepsilon_{n+ki} \quad 1 \leqq i < k \leqq n \\
G_{ik} &= \sqrt{-1}(\varepsilon_{in+k} + \varepsilon_{kn+i} + \varepsilon_{n+ik} + \varepsilon_{n+ki}) \quad 1 \leqq i < k \leqq n\,,
\end{aligned}
$$

where ε_{jk} denotes the matrix $(\delta_{jr}\delta_{ks})_{1\leqq r,s\leqq n}$. Then $\{A_i, B_i, C_i: 1 \leqq i \leqq n;$ $D_{ik}, E_{ik}, F_{ik}, G_{ik}: 1 \leqq i < k \leqq n\}$ forms a basis for $\mathfrak{E}_n$. First we list the multiplication table for completeness.

$$
\begin{aligned}
[A_i, A_j] &= 0 & [B_i, D_{jk}] &= \delta_{ij}F_{ik} - \delta_{ik}F_{ij} \\
[A_i, B_j] &= \delta_{ij}G_{ij} & [B_i, E_{jk}] &= -\delta_{ij}G_{ik} - \delta_{ik}G_{ij} \\
[A_i, C_j] &= -\delta_{ij}F_{ij} & [B_i, F_{jk}] &= -\delta_{ij}D_{ik} - \delta_{ik}D_{ij} \\
[A_i, D_{jk}] &= \delta_{ij}E_{ik} - \delta_{ik}E_{ij} & [B_i, G_{jk}] &= \delta_{ij}E_{jk} + \delta_{ik}E_{ij} \\
[A_i, E_{jk}] &= -\delta_{ij}D_{ik} - \delta_{ik}D_{ij} & [C_i, C_j] &= -\delta_{ij}D_{ij} \\
[A_i, F_{jk}] &= \delta_{ij}G_{ik} + \delta_{ik}G_{ij} & [C_i, D_{jk}] &= \delta_{ij}G_{ik} - \delta_{ik}G_{ij} \\
[A_j, G_{jk}] &= -\delta_{ij}F_{ik} - \delta_{ik}F_{ij} & [C_i, E_{jk}] &= \delta_{ij}F_{ik} + \delta_{ik}F_{ij} \\
[B_i, B_j] &= -\delta_{ij}D_{ij} & [C_i, F_{jk}] &= -\delta_{ij}E_{ik} - \delta_{jk}E_{ij} \\
[B_i, C_j] &= \delta_{ij}E_{ij} & [C_i, G_{jk}] &= -\delta_{ij}D_{ik} - \delta_{ik}D_{ij}
\end{aligned}
$$

$$
\begin{aligned}
[D_{ij}, D_{kl}] &= -\delta_{ik}D_{jl} + \delta_{il}D_{jk} + \delta_{jk}D_{il} - \delta_{jl}D_{ik} \\
[D_{ij}, E_{kl}] &= -\delta_{ik}E_{jl} - \delta_{il}E_{jk} + \delta_{jk}E_{il} + \delta_{jl}E_{ik} \\
[D_{ij}, F_{kl}] &= -\delta_{ik}F_{jl} - \delta_{il}F_{jk} + \delta_{jk}F_{il} + \delta_{jl}F_{ik} \\
[D_{ij}, G_{kl}] &= -\delta_{ik}G_{jl} - \delta_{il}G_{jk} + \delta_{jk}G_{il} + \delta_{jl}G_{ik} \\
[E_{ij}, E_{kl}] &= -\delta_{ik}D_{jl} - \delta_{il}D_{jk} - \delta_{jk}D_{il} - \delta_{jl}D_{ik} \\
[E_{ij}, F_{kl}] &= \delta_{ik}G_{jl} + \delta_{il}G_{jk} + \delta_{jk}G_{il} + \delta_{jl}G_{ik} \\
[E_{ij}, G_{kl}] &= -\delta_{ik}F_{jl} - \delta_{il}F_{jk} - \delta_{jk}F_{il} - \delta_{jl}F_{ik} \\
[F_{ij}, F_{kl}] &= -\delta_{ik}D_{jl} - \delta_{il}D_{jk} - \delta_{jk}D_{il} - \delta_{jl}D_{ik} \\
[F_{ij}, G_{kl}] &= \delta_{ik}E_{jl} + \delta_{il}E_{jk} + \delta_{jk}E_{il} + \delta_{jl}E_{ik} \\
[G_{ij}, G_{kl}] &= -\delta_{ik}D_{jl} - \delta_{il}D_{jk} - \delta_{jk}D_{il} - \delta_{jl}D_{ik}\,.
\end{aligned}
$$

Now we shall define $M_\alpha^{4n-1} = \{Sp(n)\times SU(2)/Sp(n-1)\times SU(2)\}_\alpha$ (See Berger [1] pp. 232). First choose the bi-invariant metrics $\langle X, Y\rangle = -1/2\cdot\text{trace } XY$ on $\mathfrak{E}_n$, and $\langle X, Y\rangle = -1/2\cdot\tan^2\alpha$ trace XY on $\tilde{\mathfrak{A}}_1$ (Lie algebra of $SU(2)$) with $0 < \alpha < \pi/2$. We set $\mathfrak{G}_n = \mathfrak{E}_n \oplus \tilde{\mathfrak{A}}_1$ where $\oplus$ denotes the direct orthogonal sum of ideals $\mathfrak{E}_n$ and $\tilde{\mathfrak{A}}_1$. Put $D_1 = A_n$, $D_2 = B_n$, $D_3 = C_n$ and $E_1 = \cot\alpha\ \tilde{A}$, $E_2 = \cot\alpha\ \tilde{B}$, $E_3 = \cot\alpha\ \tilde{C}$, where $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ is a basis of $\tilde{\mathfrak{A}}_1$ with $[\tilde{A}, \tilde{B}] = 2\tilde{C}$, $[\tilde{C}, \tilde{A}] = 2\tilde{B}$, $[\tilde{B}, \tilde{C}] = 2\tilde{A}$. Then we have

$$[D_{\sigma(1)}, D_{\sigma(2)}] = 2\ \text{sgn}\ \sigma D_{\sigma(3)}, \quad [E_{\sigma(1)}, E_{\sigma(2)}] = 2\cot\alpha\ \text{sgn}\ \sigma E_{\sigma(3)}\,.$$

Next set

$$
\begin{aligned}
\mathfrak{H}_\alpha = \{&H_i = \cos\alpha D_i + \sin\alpha E_i: 1 \leqq i \leqq 3;\ A_i, B_i, C_i: 1 \leqq i \leqq n-1\,; \\
&1/\sqrt{2}\cdot D_{ij}, 1/\sqrt{2}\cdot E_{ij}, 1/\sqrt{2}\cdot F_{ij}, 1/\sqrt{2}\cdot G_{ij}: 1 \leqq i < j \leqq n-1\}
\end{aligned}
$$

$$\mathfrak{M}_\alpha = \{\xi_i = \sin\alpha D_i - \cos\alpha E_i : 1 \leqq i \leqq 3 ;$$
$$1/\sqrt{2}\cdot D_{in}, 1/\sqrt{2}\cdot E_{in}, 1/\sqrt{2}\cdot F_{in}, 1/\sqrt{2}\cdot G_{in} : 1 \leqq i \leqq n-1\} .$$

Furthermore set $G_n = \exp \mathfrak{G}_n$, $H_\alpha = \exp \mathfrak{H}_\alpha$, and $M_\alpha^{4n-1} = G_n/H_\alpha$, where exp denotes the exponential mapping of the Lie algebra to the Lie group. Let $\pi: G_n \to M_\alpha^{4n-1}$ be the canonical projection. Then the tangent space of M_α^{4n-1} at $\pi(e)$ may be identified with $\mathfrak{M}_\alpha$ by $d\pi$. Now the normal homogeneous Riemannian metric on M_α^{4n-1} is obtained by restricting the metric on $\mathfrak{G}_n$ to $\mathfrak{M}_\alpha \times \mathfrak{M}_\alpha$ and next translating with G_n. This M_α^{4n-1} is the homogeneous space $\{Sp(n) \times SU(2)/Sp(n-1) \times SU(2)\}_\alpha$ of Berger (Berger's ξ is $\tan\alpha$ in our terminology.) Note that $M_{\pi/2}^{4n-1}$ may be identified with $Sp(n)/Sp(n-1)$.

3.2 $\{\xi_i : 1 \leqq i \leqq 3;\ e_{2i-1} = 1/\sqrt{2}\cdot D_{in},\ e_{2i} = 1/\sqrt{2}\cdot E_{in},\ f_{2i-1} = 1/\sqrt{2}\cdot F_{in},\ f_{2i} = 1/\sqrt{2}\cdot G_{in} : 1 \leqq i \leqq n-1\}$ forms an orthonormal basis for $\mathfrak{M}_\alpha$.

We shall calculate explicitly the curvature tensor of M_α^{4n-1} at $\pi(e)$. For this purpose we shall give some formulas which are all easily derived from the multiplication table.

(3.1)
$$\begin{array}{llllll}
[H_1, H_2] = 2\cos\alpha H_3 & [H_2, H_3] = 2\cos\alpha H_1 & [H_3, H_1] = 2\cos\alpha H_2 \\
[H_1, \xi_1] = 0 & [H_2, \xi_1] = -2\cos\alpha\xi_3 & [H_3, \xi_1] = 2\cos\alpha\xi_2 \\
[H_1, \xi_2] = 2\cos\alpha\xi_3 & [H_2, \xi_2] = 0 & [H_3, \xi_2] = -2\cos\alpha\xi_1 \\
[H_1, \xi_3] = -2\cos\alpha\xi_2 & [H_2, \xi_3] = 2\cos\alpha\xi_1 & [H_3, \xi_3] = 0 \\
[H_1, e_{2i-1}] = -\cos\alpha e_{2i} & [H_2, e_{2i-1}] = -\cos\alpha f_{2i-1} & [H_3, e_{2i-1}] = -\cos\alpha f_{2i} \\
[H_1, e_{2i}] = \cos\alpha e_{2i-1} & [H_2, e_{2i}] = -\cos\alpha f_{2i} & [H_3, e_{2i}] = \cos\alpha f_{2i-1} \\
[H_1, f_{2i-1}] = \cos\alpha f_{2i} & [H_2, f_{2i-1}] = \cos\alpha e_{2i-1} & [H_3, f_{2i-1}] = -\cos\alpha e_{2i} \\
[H_1, f_{2i}] = -\cos\alpha f_{2i-1} & [H_2, f_{2i}] = \cos\alpha e_{2i} & [H_3, f_{2i}] = \cos\alpha e_{2i-1}
\end{array}$$

(3.2)
$$[\xi_1, \xi_2] = -2\frac{\cos 2\alpha}{\sin\alpha}\xi_3 + 2\cos\alpha H_3$$
$$[\xi_2, \xi_3] = -2\frac{\cos 2\alpha}{\sin\alpha}\xi_1 + 2\cos\alpha H_1$$
$$[\xi_3, \xi_1] = -2\frac{\cos 2\alpha}{\sin\alpha}\xi_2 + 2\cos\alpha H_2$$
$$\begin{array}{llll}
[\xi_1, e_{2i-1}] = -\sin\alpha e_{2i} & [\xi_2, e_{2i-1}] = -\sin\alpha f_{2i-1} & [\xi_3, e_{2i-1}] = -\sin\alpha f_{2i} \\
[\xi_1, e_{2i}] = \sin\alpha e_{2i-1} & [\xi_2, e_{2i}] = -\sin\alpha f_{2i} & [\xi_3, e_{2i}] = \sin\alpha f_{2i-1} \\
[\xi_1, f_{2i-1}] = \sin\alpha f_{2i} & [\xi_2, f_{2i-1}] = \sin\alpha e_{2i-1} & [\xi_3, f_{2i-1}] = -\sin\alpha e_{2i} \\
[\xi_1, f_{2i}] = -\sin\alpha f_{2i-1} & [\xi_2, f_{2i}] = \sin\alpha e_{2i} & [\xi_3, f_{2i}] = \sin\alpha e_{2i-1}
\end{array}$$

(3.3)
$$\begin{array}{ll}
[e_{2j-1}, e_{2k-1}] = -1/2\cdot D_{jk} \in \mathfrak{H}_\alpha \\
[e_{2j-1}, e_{2k}] = -\delta_{jk}\sin\alpha\xi_1 - \delta_{jk}\cos\alpha H_1 + 1/2\cdot E_{jk} \\
[e_{2j-1}, f_{2k-1}] = -\delta_{jk}\sin\alpha\xi_2 - \delta_{jk}\cos\alpha H_2 + 1/2\cdot F_{jk} \\
[e_{2j-1}, f_{2k}] = -\delta_{jk}\sin\alpha\xi_3 - \delta_{jk}\cos\alpha H_3 + 1/2\cdot G_{jk}
\end{array}$$

$$
\begin{aligned}
[e_{2j}, e_{2k}] &= -1/2\cdot D_{jk} \in \mathfrak{H}_\alpha \\
[e_{2j}, f_{2k-1}] &= \delta_{jk}\sin\alpha\xi_3 + \delta_{jk}\cos\alpha H_3 + 1/2\cdot G_{jk} \\
[e_{2j}, f_{2k}] &= -\delta_{jk}\sin\alpha\xi_2 - \delta_{jk}\cos\alpha H_2 - 1/2\cdot F_{jk} \\
[f_{2j-1}, f_{2k-1}] &= -1/2\cdot D_{jk} \in \mathfrak{H}_\alpha \\
[f_{2j-1}, f_{2k}] &= \delta_{jk}\sin\alpha\xi_1 + \delta_{jk}\cos\alpha H_1 + 1/2\cdot E_{jk} \\
[f_{2j}, f_{2k}] &= -1/2\cdot D_{jk} \in \mathfrak{H}_\alpha .
\end{aligned}
$$

Now we shall calculate the curvature tensor of M_α^{4n-1} at $\pi(e)$. By (1.4) and (3.1)$\sim$(3.3) direct calculation gives the following.

LEMMA 3.1. *The curvature tensor of M_α^{4n-1} at $\pi(e)$ is given as follows:*

(3.4)
$$
\begin{aligned}
&R(\xi_i,\xi_j)\xi_i = 1/\sin^2\alpha\cdot\xi_j \quad (i\neq j) \\
&R(\xi_i,\xi_j)\xi_k = 0 \quad (i,j,k \text{ are distinct}) \\
&R(\xi_1,\xi_2)e_{2i-1} = -\beta/2\cdot f_{2i} \quad R(\xi_2,\xi_3)e_{2i-1} = -\beta/2\cdot e_{2i} \quad R(\xi_3,\xi_1)e_{2i-1} = -\beta/2\cdot f_{2i-1} \\
&\qquad e_{2i} = \beta/2\cdot f_{2i-1} \qquad\quad e_{2i} = \beta/2\cdot e_{2i-1} \qquad\quad e_{2i} = -\beta/2\cdot f_{2i} \\
&\qquad f_{2i-1} = -\beta/2\cdot e_{2i} \qquad f_{2i-1} = -\beta/2\cdot f_{2i} \qquad f_{2i-1} = \beta/2\cdot e_{2i-1} \\
&\qquad f_{2i} = \beta/2\cdot e_{2i-1} \qquad f_{2i} = \beta/2\cdot e_{2i-1} \qquad f_{2i} = \beta/2\cdot e_{2i}
\end{aligned}
$$

(3.5)
$$
\begin{aligned}
R(\xi_i, e_{2j-1})\xi_i &= \sin^2\alpha/4\cdot e_{2j-1} \\
R(\xi_i, e_{2j})\xi_i &= \sin^2\alpha/4\cdot e_{2j} \\
R(\xi_i, f_{2j-1})\xi_i &= \sin^2\alpha/4\cdot f_{2j-1} \\
R(\xi_i, f_{2j})\xi_i &= \sin^2\alpha/4\cdot f_{2j} \quad (i=1,2,3:\ 1\leqq j\leqq n)
\end{aligned}
$$

(3.6)
$$
\begin{aligned}
R(\xi_1, e_{2i-1})\xi_2 &= -\beta/4\cdot f_{2i} & R(\xi_2, f_{2i-1})\xi_1 &= \beta/4\cdot e_{2i} \\
\xi_3 &= \beta/4\cdot f_{2i-1} & \xi_3 &= \beta/4\cdot f_{2i} \\
R(\xi_1, e_{2i})\xi_2 &= \beta/4\cdot f_{2i-1} & R(\xi_2, f_{2i})\xi_1 &= -\beta/4\cdot e_{2i-1} \\
\xi_3 &= \beta/4\cdot f_{2i} & \xi_3 &= -\beta/4\cdot f_{2i-1} \\
R(\xi_1, f_{2i-1})\xi_2 &= -\beta/4\cdot e_{2i} & R(\xi_3, e_{2i-1})\xi_1 &= -\beta/4\cdot f_{2i-1} \\
\xi_3 &= -\beta/4\cdot e_{2i-1} & \xi_2 &= \beta/4\cdot e_{2i-1} \\
R(\xi_1, f_{2i})\xi_2 &= \beta/4\cdot e_{2i-1} & R(\xi_3, e_{2i})\xi_1 &= -\beta/4\cdot f_{2i} \\
\xi_3 &= -\beta/4\cdot e_{2i} & \xi_2 &= -\beta/4\cdot e_{2i-1} \\
R(\xi_2, e_{2i-1})\xi_1 &= \beta/4\cdot f_{2i} & R(\xi_3, f_{2i-1})\xi_1 &= \beta/4\cdot e_{2i-1} \\
\xi_3 &= -\beta/4\cdot e_{2i} & \xi_2 &= -\beta/4\cdot f_{2i} \\
R(\xi_2, e_{2i})\xi_1 &= -\beta/4\cdot f_{2i-1} & R(\xi_3, f_{2i})\xi_1 &= \beta/4\cdot e_{2i} \\
\xi_3 &= \beta/4\cdot e_{2i-1} & \xi_2 &= \beta/4\cdot f_{2i-1}
\end{aligned}
$$

(3.7)
$$
\begin{aligned}
R(\xi_1, e_{2i-1})e_{2j-1} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_1 & R(\xi_2, f_{2i-1})e_{2j-1} &= 0 \\
e_{2j} &= 0 & e_{2j} &= -\beta/4\cdot\delta_{ij}\xi_1 \\
f_{2j-1} &= -\beta/4\cdot\delta_{ij}\xi_3 & f_{2j-1} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_2 \\
f_{2j} &= \beta/4\cdot\delta_{ij}\xi_2 & f_{2j} &= -\beta/4\cdot\delta_{ij}\xi_3 \\
R(\xi_1, e_{2i})e_{2j-1} &= 0 & R(\xi_2, f_{2i})e_{2j-1} &= \beta/4\cdot\delta_{ij}\xi_1 \\
e_{2j} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_1 & e_{2j} &= 0 \\
f_{2j-1} &= -\beta/4\cdot\delta_{ij}\xi_2 & f_{2j-1} &= \beta/4\cdot\delta_{ij}\xi_3
\end{aligned}
$$

$$
\begin{aligned}
R(\xi_1, e_{2i})f_{2j} &= -\beta/4\cdot\delta_{ij}\xi_3 & R(\xi_2, f_{2i})f_{2j} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_2\\
R(\xi_1, f_{2i-1})e_{2j-1} &= \beta/4\cdot\delta_{ij}\xi_3 & R(\xi_3, e_{2i-1})e_{2j-1} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_3\\
e_{2j} &= \beta/4\cdot\delta_{ij}\xi_2 & e_{2j} &= -\beta/4\cdot\delta_{ij}\xi_2\\
f_{2j-1} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_1 & f_{2j-1} &= \beta/4\cdot\delta_{ij}\xi_1\\
f_{2j} &= 0 & f_{2j} &= 0\\
R(\xi_1, f_{2i})e_{2j-1} &= -\beta/4\cdot\delta_{ij}\xi_2 & R(\xi_3, e_{2i})e_{2j-1} &= \beta/4\cdot\delta_{ij}\xi_2\\
e_{2j} &= \beta/4\cdot\delta_{ij}\xi_3 & e_{2j} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_3\\
f_{2j-1} &= 0 & f_{2j-1} &= 0\\
f_{2j} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_1 & f_{2j} &= \beta/4\cdot\delta_{ij}\xi_1\\
R(\xi_2, e_{2i-1})e_{2j-1} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_2 & R(\xi_3, f_{2i-1})e_{2j-1} &= -\beta/4\cdot\delta_{ij}\xi_1\\
e_{2j} &= \beta/4\cdot\delta_{ij}\xi_3 & e_{2j} &= 0\\
f_{2j-1} &= 0 & f_{2j-1} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_3\\
f_{2j} &= -\beta/4\cdot\delta_{ij}\xi_1 & f_{2j} &= \beta/4\cdot\delta_{ij}\xi_2\\
R(\xi_2, e_{2i})e_{2j-1} &= -\beta/4\cdot\delta_{ij}\xi_3 & R(\xi_3, f_{2i})e_{2j-1} &= 0\\
e_{2j} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_2 & e_{2j} &= -\beta/4\cdot\delta_{ij}\xi_1\\
f_{2j-1} &= \beta/4\cdot\delta_{ij}\xi_1 & f_{2j-1} &= -\beta/4\cdot\delta_{ij}\xi_2\\
f_{2j} &= 0 & f_{2j} &= -\sin^2\alpha/4\cdot\delta_{ij}\xi_3
\end{aligned}
$$

(3.8)

$$
\begin{aligned}
R(e_{2i-1}, e_{2j-1})\xi_k &= 0\ (k=1,2,3) & R(e_{2i}, f_{2j-1})\xi_1 &= \beta/2\cdot\delta_{ij}\xi_2\\
R(e_{2i-1}, e_{2j})\xi_1 &= 0 & \xi_2 &= -\beta/2\cdot\delta_{ij}\xi_1\\
\xi_2 &= -\beta/2\cdot\delta_{ij}\xi_3 & \xi_3 &= 0\\
\xi_3 &= \beta/2\cdot\delta_{ij}\xi_2 & R(e_{2i}, f_{2j})\xi_1 &= \beta/2\cdot\delta_{ij}\xi_3\\
R(e_{2i-1}, f_{2j-1})\xi_1 &= \beta/2\cdot\delta_{ij}\xi_3 & \xi_2 &= 0\\
\xi_2 &= 0 & \xi_3 &= -\beta/2\cdot\delta_{ij}\xi_1\\
\xi_3 &= -\beta/2\cdot\delta_{ij}\xi_1 & R(f_{2i-1}, f_{2j-1})\xi_k &= 0\ (k=1,2,3)\\
R(e_{2i-1}, f_{2j})\xi_1 &= -\beta/2\cdot\delta_{ij}\xi_2 & R(f_{2i-1}, f_{2j})\xi_1 &= 0\\
\xi_2 &= \beta/2\cdot\delta_{ij}\xi_1 & \xi_2 &= \beta/2\cdot\delta_{ij}\xi_3\\
\xi_3 &= 0 & \xi_3 &= -\beta/2\cdot\delta_{ij}\xi_2\\
R(e_{2i}, e_{2j})\xi_k &= 0\ (k=1,2,3) & R(f_{2i}, f_{2j})\xi_k &= 0\ (k=1,2,3)
\end{aligned}
$$

(3.9)

$$
\begin{aligned}
R(e_{2i-1}, e_{2j-1})e_{2k-1} &= 1/2\cdot(\delta_{ik}e_{2j-1} - \delta_{jk}e_{2i-1})\\
e_{2k} &= \beta/4\cdot(\delta_{ik}e_{2j} - \delta_{jk}e_{2i})\\
f_{2k-1} &= \beta/4\cdot(\delta_{ik}f_{2j-1} - \delta_{jk}f_{2i-1})\\
f_{2k} &= \beta/4\cdot(\delta_{ik}f_{2j} - \delta_{jk}f_{2i})\\
R(e_{2i-1}, e_{2j})e_{2k-1} &= 1/4\cdot(2\delta_{ik}e_{2j} + \beta\delta_{jk}e_{2i} + 2\beta\delta_{ij}e_{2k})\\
e_{2k} &= -1/4\cdot(\beta\delta_{ik}e_{2j-1} + 2\delta_{jk}e_{2i-1} + 2\beta\delta_{ij}e_{2k-1})\\
f_{2k-1} &= \beta/4\cdot(\delta_{ik}f_{2j} + \delta_{jk}f_{2i} - 2\delta_{ij}f_{2k})\\
f_{2k} &= -\beta/4\cdot(\delta_{ik}f_{2j-1} + \delta_{jk}f_{2i-1} - 2\delta_{ij}f_{2k-1})\\
R(e_{2i-1}, f_{2j-1})e_{2k-1} &= 1/4\cdot(2\delta_{ik}f_{2j-1} + \beta\delta_{jk}f_{2i-1} + 2\beta\delta_{ij}f_{2k-1})\\
e_{2k} &= -\beta/4\cdot(\delta_{ik}f_{2j} + \delta_{jk}f_{2i} - 2\delta_{ij}f_{2k})\\
f_{2k-1} &= -1/4\cdot(\beta\delta_{ik}e_{2j-1} + 2\delta_{jk}e_{2i-1} + 2\beta\delta_{ij}e_{2k-1})
\end{aligned}
$$

$$
\begin{aligned}
R(e_{2i-1}, f_{2j-1})f_{2k} &= \beta/4\cdot(\delta_{ik}e_{2j} + \delta_{jk}e_{2i} - 2\delta_{ij}e_{2k})\\
R(e_{2i-1}, f_{2j})e_{2k-1} &= 1/4\cdot(2\delta_{ik}f_{2j} + \beta\delta_{jk}f_{2i} + 2\beta\delta_{ij}f_{2k})\\
e_{2k} &= \beta/4\cdot(\delta_{ik}f_{2j-1} + \delta_{jk}f_{2i-1} - 2\delta_{ij}f_{2k-1})\\
f_{2k-1} &= -\beta/4\cdot(\delta_{ik}e_{2j} + \delta_{jk}e_{2i} - 2\delta_{ij}e_{2k})\\
f_{2k} &= -1/4\cdot(\beta\delta_{ik}e_{2j-1} + 2\delta_{jk}e_{2i-1} + 2\beta\delta_{ij}e_{2k-1})\\
R(e_{2i}, e_{2j})e_{2k-1} &= \beta/4\cdot(\delta_{ik}e_{2j-1} - \delta_{jk}e_{2i-1})\\
e_{2k} &= 1/2\cdot(\delta_{ik}e_{2j} - \delta_{jk}e_{2i})\\
f_{2k-1} &= \beta/4\cdot(\delta_{ik}f_{2j-1} - \delta_{jk}f_{2i-1})\\
f_{2k} &= \beta/4\cdot(\delta_{ik}f_{2j} - \delta_{jk}f_{2i})\\
R(e_{2i}, f_{2j-1})e_{2k-1} &= \beta/4\cdot(\delta_{ik}f_{2j} + \delta_{jk}f_{2i} - 2\delta_{ij}f_{2k})\\
e_{2k} &= 1/4\cdot(2\delta_{ik}f_{2j-1} + \beta\delta_{jk}f_{2i-1} + 2\beta\delta_{ij}f_{2k-1})\\
f_{2k-1} &= -1/4\cdot(\beta\delta_{ik}e_{2j} + 2\delta_{jk}e_{2i} + 2\beta\delta_{ij}e_{2k})\\
f_{2k} &= -\beta/4\cdot(\delta_{ik}e_{2j-1} + \delta_{jk}e_{2i-1} - 2\delta_{ij}e_{2k-1})\\
R(e_{2i}, f_{2j})e_{2k-1} &= -\beta/4\cdot(\delta_{ik}f_{2j-1} + \delta_{jk}f_{2i-1} - 2\delta_{ij}f_{2k-1})\\
e_{2k} &= 1/4\cdot(2\delta_{ik}f_{2j} + \beta\delta_{jk}f_{2i} + 2\beta\delta_{ij}f_{2k})\\
f_{2k-1} &= \beta/4\cdot(\delta_{ik}e_{2j-1} + \delta_{jk}e_{2i-1} - 2\delta_{ij}e_{2k-1})\\
f_{2k} &= -1/4\cdot(\beta\delta_{ik}e_{2j} + 2\delta_{jk}e_{2i} + 2\beta\delta_{ij}e_{2k})\\
R(f_{2i-1}, f_{2j-1})e_{2k-1} &= \beta/4\cdot(\delta_{ik}e_{2j-1} - \delta_{jk}e_{2i-1})\\
e_{2k} &= \beta/4\cdot(\delta_{ik}e_{2j} - \delta_{jk}e_{2i})\\
f_{2k-1} &= 1/2\cdot(\delta_{ik}f_{2j-1} - \delta_{jk}f_{2i-1})\\
f_{2k} &= \beta/4\cdot(\delta_{ik}f_{2j} - \delta_{jk}f_{2i})\\
R(f_{2i-1}, f_{2j})e_{2k-1} &= \beta/4\cdot(\delta_{ik}e_{2j} + \delta_{jk}e_{2i} - 2\delta_{ij}e_{2k})\\
e_{2k} &= -\beta/4\cdot(\delta_{ik}e_{2j-1} + \delta_{jk}e_{2i-1} - 2\delta_{ij}e_{2k-1})\\
f_{2k-1} &= 1/4\cdot(2\delta_{ik}f_{2j} + \beta\delta_{jk}f_{2i} + 2\beta\delta_{ij}f_{2k})\\
f_{2k} &= 1/4\cdot(\beta\delta_{ik}f_{2j-1} + 2\delta_{jk}f_{2i-1} + 2\beta\delta_{ij}f_{2k-1})\\
R(f_{2i}, f_{2j})e_{2k-1} &= \beta/4\cdot(\delta_{ik}e_{2j-1} - \delta_{jk}e_{2i-1})\\
e_{2k} &= \beta/4\cdot(\delta_{ik}e_{2j} - \delta_{jk}e_{2i})\\
f_{2k-1} &= \beta/4\cdot(\delta_{ik}f_{2j-1} - \delta_{jk}f_{2i-1})\\
f_{2k} &= 1/2\cdot(\delta_{ik}f_{2j} - \delta_{jk}f_{2i})\ ,
\end{aligned}
$$

where we have put $\beta = 2-\sin^2\alpha$.

3.3 First we shall consider $M^{4n-1}_{\pi/2} = Sp(n)/Sp(n-1)$. In this case, since ad $\mathfrak{H}_{\pi/2}\xi_i = 0$ hold for $i = 1, 2, 3$ because of (3.2), ξ_1, ξ_2, ξ_3 define vector fields on $M^{4n-1}_{\pi/2}$ by $d\pi$, which will be denoted ξ_1, ξ_2, ξ_3 again. (3.4), (3.5), and Lemma 1.1 show that ξ_1, ξ_2, ξ_3 are contact structures on $M^{4n-1}_{\pi/2}$. Since ξ_i's are mutually orthogonal and $[\xi_{\sigma(1)}, \xi_{\sigma(2)}]_{M^{4n-1}_{\pi/2}} = 2\xi_{\sigma(3)}$ holds by (3.2), (ξ_1, ξ_2, ξ_3) defines a contact 3-structure on $M^{4n-1}_{\pi/2}$. Let $\mathfrak{D}$ denote the distribution defined by $\{\xi_1, \xi_2, \xi_3\}$. It is an involutive distribution by (3.2). Now we have

THEOREM 3.2. $M^{4n-1}_{\pi/2}$ *has a contact* 3*-structure* (ξ_1, ξ_2, ξ_3). *The maximal*

integral manifold of $\mathfrak{D}$ *is a totally geodesic submanifold which is a sphere* S^3 *of constant curvature* 1. $M^{4n-1}_{\pi/2}$ *has a structure of* S^3*-bundle over* $M^{4n-1}_{\pi/2}/S^3$ *which is a quaternionic projective space and whose metric is that of symmetric space of rank* 1 *with sectional curvature* $1/2 \leqq K(\sigma) \leqq 2$. *Furthermore curvatures of* $M^{4n-1}_{\pi/2}$ *satisfy the following*:

(i) *The sectional curvatures of the plane sections spanned by* ξ_i $(i = 1, 2, 3)$ *and* $X \perp \xi_1, \xi_2, \xi_3$ *are equal to a constant* 1/4.

(ii) $M^{4n-1}_{\pi/2}$ *is of constant* ϕ_i*-holomorphic sectional curvature* 5/4 *for* $i = 1, 2, 3$. *That is, every sectional curvature of plane section* $\{X, \phi_i(X)\}$, $X \perp \xi_1, \xi_2, \xi_3$ *is a constant* 5/4 *where we have put* $\phi_i = -\nabla\xi_i$.

PROOF. The integral manifold of $\mathfrak{D}$ may be identified with $M^3_{\pi/2} = SU(2)$ which is simply connected and of constant curvature 1. Next we shall show that $M^{4n-1}_{\pi/2}$ has a structure of S^3-bundle over quaternionic projective space. We put $G_n = Sp(n)$, $H = Sp(n-1)$, and $K = Sp(n-1) \times SU(2)$. Then we have the bundle structure (see Steenrod [7])

$$M^{4n-1}_{\pi/2} = G_n/H \to G_n/K = \text{the quaternionic projective space}.$$

The fibre of this bundle is $K/H = S^3$ (= the maximal integral manifold of $\mathfrak{D}$) and the group of this bundle is $SU(2)$.

The statements (i) and (ii) of the theorem may be proved by direct but very complicated calculation with full use of Lemma 3.1. We omitt the calculation. q.e.d.

3.4 Finally we shall consider $M^{4n-1}_\alpha = \{Sp(n) \times SU(2)/Sp(n-1) \times SU(2)\}_\alpha$. In this general case $\xi_1, \xi_2, \xi_3 \in \mathfrak{M}_\alpha$ do not define global vector fields on M^{4n-1}_α via $d\pi$. Take a neighborhood U of 0 in $\mathfrak{M}_\alpha$ which is mapped diffeomorphically onto a neighborhood V of $\pi(e)$ under $\pi \circ \exp | \mathfrak{M}_\alpha$. Then $\pi_* g_* \exp_* \xi_i$ defines a vector field $\xi_i^{(g)}$ over gV for $g \in G$. By the same arguments as 3.3, $\{\xi_1^{(g)}, \xi_2^{(g)}, \xi_3^{(g)}\}$ defines a contact 3-structure over gV. Since on $gV \cap g'V$, $L_{g'} \circ L_g^{-1}$ maps $\xi_i^{(g)}$ onto $\xi_i^{(g')}$, so $\{\xi_i^{(g)}, gV\}$ defines a contact structure in the wider sense for $i = 1, 2, 3$ (See S. Sasaki [4]). Now since ad $\mathfrak{H}_\alpha$ leaves the subspace spanned by ξ_1, ξ_2, ξ_3 invariant, $\{\xi_1, \xi_2, \xi_3\}$ defines a distribution $\mathfrak{D}_\alpha$ on M^{4n-1}_α by $d\pi$. Then we have the following theorem which may be proved by the same way as Theorem 3.2.

THEOREM 3.3. M^{4n-1}_α *has a contact* 3*-structure* $\{\xi_1^{(g)}, \xi_2^{(g)}, \xi_3^{(g)}, gV\}_{g \in G}$ *in the wider sense. The maximal integral manifold of* $\mathfrak{D}_\alpha$ *is a totally geodesic submanifold which is a sphere* S^3 *of constant curvature* $1/\sin^2 \alpha$. M^{4n-1}_α *has a structure of* S^3*-bundle over* M^{4n-1}_α/S^3 *which is a quaternionic projective space and whose metric is that of a symmetric space of rank* 1 *with sectional curvature* $1/2 \leqq K(\sigma) \leqq 2$. *Furthermore curvatures of* M^{4n-1}_α *satisfy the following*:

(i) *The sectional curvatures of the plane sections spanned by* $\xi_i^{(g)}$ *and* $X \perp \xi_1^{(g)}, \xi_2^{(g)}, \xi_3^{(g)}$ *are equal to a constant* $\sin^2\alpha/4$.

(ii) M_α^{4n-1} *is of constant* $\phi_i^{(g)}$*-holomorphic sectional curvature* $2-3/4\cdot\sin^2\alpha$ *for* $i=1,2,3$ *and* $g\in G$. *That is, every sectional curvature of the plane section* $\{X, \phi_i^{(g)}(X)\}$, $X\perp\xi_1^{(g)}, \xi_2^{(g)}, \xi_3^{(g)}$ *is a constant* $2-3/4\cdot\sin^2\alpha$, *where we have put* $\phi_i^{(g)} = -\nabla\xi_i^{(g)}$.

REMARK 1. The structure group of the bundle $M_\alpha^{4n-1}\to M_\alpha^{4n-1}/S^3$ is $SU(2)\times SU(2)$ unless $\alpha=\pi/2$.

REMARK 2. The contact structure $\xi_1^{(g)}, \xi_2^{(g)}, \xi_3^{(g)}$ are not contact metric structure in the sense of S. Sasaki ([4]). But they relate closely to the homogeneous Riemannian metrics as the above theorem shows.

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