THE CONTINUATION OF SECTIONS OF TORSION-FREE COHERENT ANALYTIC SHEAVES

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§1. **Introduction.** The purpose of this paper is to extend the continuation theorem of holomorphic functions, especially the generalized Hartogs-Osgood's theorem given in the previous papers [6] and [8], to the case of sections of torsion-free coherent analytic sheaves over a reduced complex space. In the following, we restrict ourselves only to reduced complex spaces.

For the local continuation, we have

THEOREM. *Let D be an open subset of a complex space X with structure sheaf $\mathcal{O}$ which is $*$-strongly s-concave at x. If a coherent $\mathcal{O}$-module $\mathcal{F}$ is torsion-free at x and a universal denominator u for $\mathcal{F}_x$ satisfies* 1. $\dim_x(\mathcal{F} / u\mathcal{F}) \geqq s$, *every section of $\mathcal{F}$ over D is uniquely continuable to a neighborhood of x* (see Def. 2. 8 in [6], p. 56, Def. 3.3 and Def. 4.1).

The proof is essentially due to W. Thimm [12], though he studied only the continuation of holomorphic functions on an open subset of an analytic subset in C^n to an analytically thin set. We shall state here the outline of the proof. We can regard $\mathcal{F}$ as a sub-$\mathcal{O}$-Module of the tensor product $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{M}$, where $\mathcal{M}$ denotes the sheaf of germs of meromorphic functions. While, $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{M}$ is isomorphic to the direct sum of several subsheaves of $\mathcal{M}$ with suitable conditions (§2). Therefore, any section of $\mathcal{F}$ over D can be identified with a system of meromorphic functions (f_i), which may be assumed to have a common denominator u depending on $\mathcal{F}_x$ only (§3). Roughly speaking, we call such u a universal denominator for $\mathcal{F}_x$. We continue the numerator of each f_i to a neighborhood of x as weakly holomorphic functions and obtain a system of meromorphic functions $(\tilde{f}_i)$. We shall show that such $(\tilde{f}_i)$ determines uniquely a continuation of f to a neighborhood of x under the assumption in the above theorem (§4).

For the global continuation, we can prove

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THEOREM. *Let X be a Stein space and $\mathscr{F}$ be a torsion-free coherent $\mathscr{O}$-Module over X with the property that for any $x \in X$ there exists a universal denominator u for $\mathscr{F}_x$ with* l. $\dim_x (\mathscr{F} / u\mathscr{F}) \geqq 1$. *If an open subset D of X and a compact subset K of D satisfy the condition that each irreducible component of D is irreducible in $D - K$, then every section of $\mathscr{F}$ over $D - K$ is uniquely continuable to D.*

We have also the analogous continuation theorems for a considerably general, not necessarily Stein complex space X if we assume the local irreducibility of X and for more general X's if we assume the suitable convexity of the domain D(§5).

Our assumption on $\mathscr{F}_x$ in the above theorems are closely related to the notion of homological codimension and also "starke homologische Codimension" in the sense of Scheja [11], p. 89. As to the continuation of sections of torsion-free coherent $\mathscr{O}$-Modules, our results are generalizations of the results in [3], [11] and [12] etc.

§2. The tensor product $\mathscr{F} \otimes_{\mathscr{O}} \mathscr{M}$. Let X be a complex space with structure sheaf $\mathscr{O}$ and $\mathscr{F}$ be a coherent $\mathscr{O}$-Module over X. We consider the tensor product $\mathscr{F} \otimes_{\mathscr{O}} \mathscr{M}$, where $\mathscr{M}$ denotes the sheaf of all germs of meromorphic functions on X.

Tensorizing the inclusion map $\mathscr{O} \to \mathscr{M}$ by $\mathscr{F}$, we get the canonical map $\tau: \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}} \mathscr{M}$.

(2. 1) *The kernel $T(\mathscr{F})$ of τ is a coherent sub-$\mathscr{O}$-Module of $\mathscr{F}$ and so the image $\tau(\mathscr{F})$ is a coherent sub-$\mathscr{O}$-Module of $\mathscr{F} \otimes_{\mathscr{O}} \mathscr{M}$* (c.f. Andreotti [2], Proposition 6, p. 14 and Houzel [5], Exposé 20, p. 2).

In fact, $T(\mathscr{F})$ is the sub-$\mathscr{O}$-Module of $\mathscr{F}$ such that each stalk $T(\mathscr{F})_x$ ($x \in X$) is the intersection of primary sub-$\mathscr{O}_x$-modules of $\mathscr{F}_x$ associated with minimal prime ideals in $\mathscr{O}_x$. Putting $\Pi = \{$all minimal prime Ideals in $\mathscr{O}\}$, we see easily it is nothing but the Π-component of a subsheaf (0) of $\mathscr{F}$ in the sense of Fujimoto [7]. The coherency of $T(\mathscr{F})$ is an immediate consequence of Theorem 6.3 in [7].

We say $\mathscr{F}$ is torsion-free at $x \in X$ if an $\mathscr{O}_x$-module $\mathscr{F}_x$ is torsion-free, namely, for any non-zero divisor u in $\mathscr{O}_x$ $uf = 0$ ($f \in \mathscr{F}_x$) implies $f = 0$ and $\mathscr{F}$ is torsion-free on X if $\mathscr{F}$ is torsion-free everywhere on X. Obviously, $\mathscr{F}$ is torsion-free at x if and only if $T(\mathscr{F})_x=(0)$. By (2. 1), the set $\{x\in X; \mathscr{F}$ is torsion-free at $x\}$ is open.

Suppose that X is irreducible at $x\in X$. Then $(\mathscr{F}\otimes_{\mathcal{O}}\mathscr{M})_x=\mathscr{F}_x\otimes_{\mathcal{O}_x}\mathscr{M}_x$ is considered as a module over the field $\mathscr{M}_x$ and any maximal $\mathcal{O}_x$-linearly independent family in $\mathscr{F}_x$ defines a base of $(\mathscr{F}\otimes_{\mathcal{O}}\mathscr{M})_x$ over $\mathscr{M}_x$. Therefore $\operatorname{rank}_x\mathscr{F}=\dim_{\mathscr{M}_x}(\mathscr{F}\otimes_{\mathcal{O}}\mathscr{M})_x$ is the number of elements of a maximal $\mathcal{O}_x$-linearly independent family in $\mathscr{F}_x$. Moreover, if X is locally irreducible and connected, $\operatorname{rank}_x\mathscr{F}$ is constant for any $x\in X$. In this case, the rank of $\mathscr{F}$ over X is defined as $\operatorname{rank}_x\mathscr{F}$ with an arbitrarily fixed $x\in X$. For an irreducible but not necessarily locally irreducible complex space we define $\operatorname{rank}_X\mathscr{F}=\operatorname{rank}_x\mathscr{F}$ with an arbitrarily fixed $x\in X-S(X)$, where $S(X)$ denotes the totality of all singularities of X.

LEMMA 2. 2. *If X is an irreducible Stein space and a coherent $\mathcal{O}$-Module $\mathscr{F}$ over X is of rank $k(\geqq 1)$, we can find k sections $f_1,\dots,f_k\in\Gamma(X,\mathscr{F})$ such that for each $x\in X$*

1^0, *$\{f_1,\dots,f_k\}$ defines an $\mathcal{O}_x$-linearly independent family in $\mathscr{F}_x$,*

2^0, *there exists a non-zero divisor a in $\mathcal{O}_x$ with $a\cdot\mathscr{F}_x\subseteqq\mathcal{O}_x(f_1,\dots,f_k)$.*

Proof. Firstly, we choose a normal point x_0 of X arbitrarily. Since X is Stein, $\mathscr{F}_{x_0}$ is generated by several elements in $\Gamma(X,\mathscr{F})$ as an $\mathcal{O}_{x_0}$-module. By a suitable choice of elements of these global sections, we can find $f_1,\dots,f_k$ in $\Gamma(X,\mathscr{F})$ which define a maximal $\mathcal{O}_{x_0}$-linearly independent family in $\mathscr{F}_{x_0}$. The family $\{f_1,\dots,f_k\}$ gives a base of the $\mathcal{O}_{x_0}$-module $(\mathscr{F}\otimes_{\mathcal{O}}\mathscr{M})_{x_0}$ and therefore satisfies the conditions 1^0 and 2^0 in Lemma 2. 2 for a special point $x_0\in X$. We want to prove this family $\{f_1,\dots,f_k\}$ satisfies the same conditions for every $x\in X$.

To examine the condition 1^0, we define the $\mathcal{O}$-Homomorphism $\tilde{f}:=(f_1,\dots,f_k)$: $\mathcal{O}^k\to\mathscr{F}$ as usual and consider the sheaf $\mathscr{N}:=\operatorname{Ker}(\tilde{f})$. Since $\mathscr{N}$ is a sub-$\mathcal{O}$-module of $\mathcal{O}^k$, the analytic set $|\mathscr{N}|:=\{x\in X; \mathscr{N}_x\neq(0)\}$ has the non-empty open kernel if $|\mathscr{N}|\neq\phi$. By the assumption of the irreducibility of X, $|\mathscr{N}|$ must be equal to X or empty. While, it cannot happen to be $|\mathscr{N}|=X$ because $x_0\notin|\mathscr{N}|$. Thus, $|\mathscr{N}|=\phi$ or $\mathscr{N}=(0)$. This shows that for each $x\in X$ $\tilde{f}$ gives the injective $\mathcal{O}_x$-homomorphism $\tilde{f}_x$: $\mathcal{O}_x^k\to\mathscr{F}_x$ and so $\{f_1,\dots,f_k\}$ defines an $\mathcal{O}_x$-linearly independent family in $\mathscr{F}_x$.

Next, we shall confirm the condition 2^0. Let X^* be the normalization of X with projection $\mu: X^* \to X$. We denote the sheaf of germs of holomorphic functions on X^* by $\mathcal{O}^*$ and of meromorphic functions on X^* by $\mathcal{M}^*$. The inverse image $\mathcal{F}^*: = \mu^*(\mathcal{F})$ of $\mathcal{F}$ by μ is a coherent $\mathcal{O}^*$-Module over X^*. Since X is normal at x_0, the section $f_1^*: = \mu^*(f_1), \dots, f_k^*: = \mu^*(f_k)$ in $\Gamma(X^*, \mathcal{F}^*)$ give a maximal $\mathcal{O}_{y_0}^*$-linearly independent family in $\mathcal{F}_{y_0}^*$ for $y_0 = \mu^{-1}(x_0)$. Therefore, by the same argument as above, $\{f_1^*, \dots, f_k^*\}$ defines an $\mathcal{O}_y^*$-linearly independent family in $\mathcal{F}_y^*$ for each $y \in X^*$. Moreover, since $\mathrm{rank}_y \mathcal{F}^*$ is constantly equal to $\mathrm{rank}_{y_0} \mathcal{F}^* = \mathrm{rank}_{x_0} \mathcal{F} = k, \{f_1^*, \dots, f_k^*\}$ gives a base of $(\mathcal{F}^* \otimes_{\mathcal{O}^*} \mathcal{M}^*)_y$ over the field $\mathcal{M}_y^*$. Consequently, for each $y \in X^*$ there exists a non-zero divisor $a^{(y)} \in \mathcal{O}_y^*$ with $a^{(y)} \cdot \mathcal{F}_y^* \subseteqq \mathcal{O}_y^*(f_1^*, \dots, f_k^*)$.

Now, let x be a point in X. There exists an arbitrarily small neighborhood U of x with the irreducible decomposition

$$U = U_1 \cup \dots \cup U_t$$

where U_i is irreducible at x. By y_i denoting the point in X^* which corresponds to U_i, we may assume $\mu^{-1}(U)$ is the disjoint union of connected neighborhoods V_i of y_i with $a^{(y_i)} \in \Gamma(V_i, \mathcal{O}^*)$ $(1 \leqq i \leqq t)$ and moreover there exists a universal denominator on U, namely, a holomorphic function u on U not vanishing identically on each U_i such that $u \cdot \mu_*(\mathcal{O}^*) \subseteqq \mathcal{O}$ on U for the direct image $\mu_*(\mathcal{O}^*)$ of $\mathcal{O}^*$ by μ. Putting $\tilde{a} = a^{(y_i)}$ on V_i, we define a section $\tilde{a} \in \Gamma(\mu^{-1}(U), \mathcal{O}^*) = \Gamma(U, \mu_*(\mathcal{O}^*))$. Thus we get a section $a': = u\tilde{a} \in \Gamma(U, \mathcal{O})$. On the other hand, there exists a non-zero divisor v in $\mathcal{O}_x$ such that $v \cdot T(\mathcal{F})_x = 0$. To complete the proof, it suffices to take the non-zero divisor $a = va'$, which will be shown to satisfy the condition $a \cdot \mathcal{F}_x \subseteqq \mathcal{O}_x(f_1, \dots, f_k)$. Take an arbitrary $f \in \mathcal{F}_x$. Since $a^{(y_i)} \cdot \mathcal{F}_{y_i}^* \subseteqq \mathcal{O}_{y_i}^*(f_1^*, \dots, f_k^*)$, we can write

$$a^{(y_i)} \mu^*(f) = b_1^{(i)} f_1^* + \cdots + b_k^{(i)} f_k^*$$

in $\mathcal{F}_{y_i}^*$ for suitable $b_l^{(i)} \in \mathcal{O}_{y_i}^*$ $(1 \leqq i \leqq t,\ 1 \leqq l \leqq k)$. Again, taking a sufficiently small neighborhood U' of x with $\mu^{-1}(U') = V_1' \cup \dots \cup V_t'$ and $b_l^{(i)} \in \Gamma(V_i', \mathcal{O}^*)$, we define $\tilde{b}_l \in \Gamma(\mu^{-1}(U'), \mathcal{O}^*) = \Gamma(U', \mu_*(\mathcal{O}^*))$ so as to be $\tilde{b}_l = b_l^{(i)}$ on V_i. Then, for $b_l: = u\tilde{b}_l \in \Gamma(U', \mathcal{O})$ we have

$$a' \mu^*(f) = b_1 f_1^* + \cdots + b_k f_k^*$$

in $\Gamma(\mu^{-1}(U'), \mathscr{F}^*)$. This shows the canonical image of $g := a'f - (b_1 f_1 + \cdots + b_k f_k) \in \Gamma(U', \mathscr{F})$ into $\Gamma(\mu^{-1}(U'), \mathscr{F}^*)$ is equal to zero. While, by the following lemma, $\Gamma(\mu^{-1}(U'), \mathscr{F}^*) = \Gamma(U', \mu_*(\mu^*(\mathscr{F})))$ is isomorphic to $\Gamma(U', \mathscr{F} \otimes_{\mathscr{O}} \mu_*(\mathscr{O}^*))$ and $\mu_*(\mathscr{O}^*)$ is considered as a subsheaf of $\mathscr{M}$ canonically. Thus $\tau(g)$ is also equal to 0 in $(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{M})_x$ and so $vg = 0$ because v is chosen as $v \cdot T(\mathscr{F})_x = (0)$. This asserts

$$af = (vb_1)f_1 + \cdots + (vb_k)f_k,$$

which is contained in $\mathscr{O}_x(f, \ldots, f_k)$. Consequently $a \cdot \mathscr{F}_x \subseteqq \mathscr{O}_x(f_1, \ldots, f_k)$.

LEMMA 2. 3. *Let X, Y be complex spaces with structure sheaves $\mathscr{O}_X, \mathscr{O}_Y$ respectively and $\nu: Y \to X$ be a proper nowhere degenerate holomorphic mapping. For an $\mathscr{O}_X$-Module $\mathscr{F}$ and an $\mathscr{O}_Y$-Module $\mathscr{G}$ the canonical $\mathscr{O}_X$-Homomorphism $\mathscr{F} \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}) \to \nu_*(\nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G})$ is an isomorphism.*

Proof. The canonical $\mathscr{O}$-Homomorphism is given as the composition of canonical $\mathscr{O}$-Homomorphisms $\rho: \mathscr{F} \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}) \to \nu_*(\nu^*(\mathscr{F})) \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G})$ and $\theta: \nu_*(\nu^*(\mathscr{F})) \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}) \to \nu_*(\nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G})$ (c.f. [9], Chap. 0, §4). It suffices to prove that for each $x \in X$ the stalk $(\mathscr{F} \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}))_x$ is isomorphic to $(\nu_*(\nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G}))_x$ canonically. We put $\nu^{-1}(x) = \{y_1, \ldots, y_t\}$. There exists an arbitrarily small neighborhood U of x such that $\nu^{-1}(U)$ is the disjoint union of neighborhoods V_i of y_i. For such a neighborhood U of x we have isomorphisms

$$\begin{aligned}
&\Gamma(U, \mathscr{F}) \otimes_{\Gamma(U, \mathscr{O}_X)} \Gamma(U, \nu_*(\mathscr{G})) \\
&\qquad \cong \Gamma(U, \mathscr{F}) \otimes_{\Gamma(U, \mathscr{O}_X)} \Gamma(\nu^{-1}(U), \mathscr{G}) \\
&\qquad \cong \Gamma(U, \mathscr{F}) \otimes_{\Gamma(U, \mathscr{O}_X)} (\oplus_{i=1}^t \Gamma(V_i, \mathscr{G})) \\
&\qquad \cong \oplus_{i=1}^t \Gamma(U, \mathscr{F}) \otimes_{\Gamma(U, \mathscr{O}_X)} \Gamma(V_i, \mathscr{G}).
\end{aligned}$$

As the inductive limit of these isomorphisms, we see $(\mathscr{F} \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}))_x$ is isomorphic to $\oplus_{i=1}^t (\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_{y_i})$. On the other hand,

$$\begin{aligned}
\Gamma(U, \nu_*(\nu^*(\mathscr{F})) \otimes_{\mathscr{O}_Y} \mathscr{G})) &= \Gamma(\nu^{-1}(U), \nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G}) \\
&\cong \oplus_{i=1}^t \Gamma(V_i, \nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G}).
\end{aligned}$$

Therefore we obtain

$$(\nu_*(\nu^*(\mathscr{F})\otimes_{\mathscr{O}_Y}\mathscr{G}))_x\cong\oplus_{i=1}^t(\nu^*(\mathscr{F})\otimes_{\mathscr{O}_Y}\mathscr{G})_{y_i}\ ,$$

which is also isomorphic to $\oplus_{i=1}^t(\mathscr{F}_x\otimes_{\mathscr{O}_{X,x}}\mathscr{O}_{Y,y_i})\otimes_{\mathscr{O}_{Y,y_i}}\mathscr{G}_{y_i}$ by definition and so to $\oplus_{i=1}^t(\mathscr{F}_x\otimes_{\mathscr{O}_{X,x}}\mathscr{G}_{y_i})$. It is easy to see these isomorphisms give the above canonical mapping. This completes the proof.

REMARK 1. In Lemma 2. 2, we can find a not identically vanishing $a\in\Gamma(X,\mathscr{O})$ with $a\cdot\mathscr{F}\subseteqq\mathscr{O}(f_1,\dots,f_k)$. In fact, the condition 2^0 shows the coherent Ideal $\mathscr{O}(f_1,\dots,f_k):\mathscr{F}$ with the stalks $(\mathscr{O}(f_1,\dots,f_k):\mathscr{F})_x=\{a\in\mathscr{O}_x;\ a\cdot\mathscr{F}_x\subseteqq\mathscr{O}_x(f_1,\dots,f_k)\}\ (x\in X)$ is not equal to zero. By the assumption that X is Stein, there exists a not identically vanishing $a\in\Gamma(X,\mathscr{O}(f_1,\dots,f_k):\mathscr{F})$, which is a desired section.

REMARK 2. In case of $\text{rank}_X\mathscr{F}=0$, Lemma 2. 2 is also valid in the sense that there exists a not identically vanishing $a\in\Gamma(X,\mathscr{O})$ with $a\cdot\mathscr{F}=(0)$.

Now, we study $\mathscr{F}\otimes_{\mathscr{O}}\mathscr{M}$ for a coherent $\mathscr{O}$-Module $\mathscr{F}$ over a not necessarily irreducible complex space. Let $X=\cup_\nu X_\nu$ be the irreducible decomposition and $i_\nu: X_\nu\to X$ be the canonical inclusion maps. We denote the sheaf of germs of holomorphic functions over X_ν by $\mathscr{O}_\nu$ and of meromorphic functions over X_ν by $\mathscr{M}_\nu$. The direct image $\tilde{\mathscr{M}}_\nu:=(i_\nu)_*(\mathscr{M}_\nu)$ is considered as an $\mathscr{M}$-Module over X. The inclusion maps $\tilde{\mathscr{M}}_\nu\to\mathscr{M}$ induce the canonical map $\iota:\oplus_\nu\tilde{\mathscr{M}}_\nu\to\mathscr{M}$.

(2. 4) *The map ι is an $\mathscr{M}$-Isomorphism.*

The problem is of local nature. For the proof, see Abhyankar [1], p. 157.

For each X_ν we can consider the rank of $(i_\nu)^*(\mathscr{F})$ over X_ν. We call it the rank of $\mathscr{F}$ over X_ν and denote it by $\text{rank}_{X_\nu}\mathscr{F}$.

PROPOSITION 2. 5. *Let X be a Stein space and $\mathscr{F}$ be a coherent $\mathscr{O}$-Module over X. Then there exists an $\mathscr{M}$-Isomorphism $\varphi:\mathscr{F}\otimes_{\mathscr{O}}\mathscr{M}\to\oplus_\nu\tilde{\mathscr{M}}_\nu^{k_\nu}$, where $k_\nu=\text{rank}_{X_\nu}\mathscr{F}$ and $\tilde{\mathscr{M}}_\nu^{k_\nu}$ denotes the k_ν-fold direct sum of $\tilde{\mathscr{M}}_\nu$.*

Proof. Firstly, we prove Proposition 2. 5 in case that X is irreducible. To this end, we take sections $f_1,\dots,f_k\in\Gamma(X,\mathscr{F})$ with the conditions in

Lemma 2. 2 and define an $\mathcal{O}$-Homomorphism $f=(f_1,\dots,f_k)\colon \mathcal{O}^k\to\mathcal{F}$ as usual. It induces an $\mathcal{M}$-Homomorphism $\tilde{f}\colon \mathcal{M}^k\to\mathcal{F}\otimes_{\mathcal{O}}\mathcal{M}$, which is injective by the condition 1⁰ and surjective by the condition 2⁰. The $\mathcal{M}$-Isomorphism $\varphi:=\tilde{f}^{-1}\colon \mathcal{F}\otimes_{\mathcal{O}}\mathcal{M}\to\mathcal{M}^k$ is a desired one.

Now, take a not necessarily irreducible space X with the irreducible decomposition $X=\cup_\nu X_\nu$. In the above notations $\mathcal{F}_\nu:=(i_\nu)^*(\mathcal{F})$ is a coherent $\mathcal{O}_\nu$-Module of rank k_ν over X_ν for each ν. By the above argument, there exists an $\mathcal{M}_\nu$-Isomorphism $\varphi_\nu\colon \mathcal{F}_\nu\otimes_{\mathcal{O}_\nu}\mathcal{M}_\nu\xrightarrow{\sim}\mathcal{M}_\nu{}^{k_\nu}$ and so $\tilde{\varphi}_\nu\colon (i_\nu)_*(\mathcal{F}_\nu\otimes_{\mathcal{O}_\nu}\mathcal{M}_\nu)\xrightarrow{\sim}\tilde{\mathcal{M}}_\nu{}^{k_\nu}$. On the other hand, by Lemma 2. 3, $(i_\nu)_*(\mathcal{F}_\nu\otimes_{\mathcal{O}_\nu}\mathcal{M}_\nu)$ is canonically isomorphic to $\mathcal{F}\otimes_{\mathcal{O}}\tilde{\mathcal{M}}_\nu$ and by (2. 4) $\mathcal{F}\otimes_{\mathcal{O}}\mathcal{M}$ is isomorphic to $\mathcal{F}\otimes_{\mathcal{O}}(\oplus_\nu\tilde{\mathcal{M}}_\nu)=\oplus_\nu\mathcal{F}\otimes_{\mathcal{O}}\tilde{\mathcal{M}}_\nu$. Consequently we obtain the $\mathcal{M}$-Isomorphism $\varphi\colon \mathcal{F}\otimes_{\mathcal{O}}\mathcal{M}\xrightarrow{\sim}\oplus_\nu\tilde{\mathcal{M}}_\nu{}^{k_\nu}$ by the composition of these isomorphisms.

§3. A property of a torsion-free coherent $\mathcal{O}$-Module. For a given complex space X, let $X=\cup_\nu X_\nu$ be the irreducible decomposition of X. As in the previous section, we put $\tilde{\mathcal{M}}_\nu:=(i_\nu)_*(\mathcal{M}_\nu)$ for the sheaf $\mathcal{M}_\nu$ of all germs of meromorphic functions of X_ν and the inclusion map $i_\nu\colon X_\nu\to X$.

LEMMA 3. 1. *Let $\mathcal{G}$ and $\mathcal{G}'$ be coherent sub-$\mathcal{O}$-Modules of $\oplus_\nu\tilde{\mathcal{M}}_\nu{}^{l_\nu}$ over a Stein space X, where l_ν are non-negative integers. If for each $x\in X$ there exists a non-zero divisor w in $\mathcal{O}_x$ with $w\mathcal{G}_x\subseteqq\mathcal{G}_x'$, then there exists a holomorphic function u with $u\cdot\mathcal{G}\subseteqq\mathcal{G}'$ on X which does not vanish identically on any X_ν.*

Proof. Firstly, we shall prove the sheaf $\mathcal{G}'\colon\mathcal{G}$ with the stalks $(\mathcal{G}'\colon\mathcal{G})_x:=\{u\in\mathcal{O}_x\,;\,u\cdot\mathcal{G}_x\subseteqq\mathcal{G}_x'\}$ $(x\in X)$ is a coherent Ideal over X. The problem is of local nature. We take an arbitrary point $x\in X$. By the assumption, we can find a non-zero divisor w in $\mathcal{O}_x$ with $w\cdot\mathcal{G}_x\subseteqq\mathcal{G}_x'$, which satisfies $w\cdot\mathcal{G}\subseteqq\mathcal{G}'$ on U and does not vanish identically on any irreducible component of U for a sufficiently small neighborhood U of x. For each $y\in U$, an element $a\in\mathcal{O}_x$ satisfies $a\cdot\mathcal{G}_y\subseteqq\mathcal{G}_y'$ if and only if $(aw)\mathcal{G}_y\subseteqq w\cdot\mathcal{G}_y'$ because w gives a non-zero in $\oplus_\nu(\mathcal{M}_\nu)_y^{l_\nu}$. This shows $\mathcal{G}'\colon\mathcal{G}=w\cdot\mathcal{G}'\colon w\cdot\mathcal{G}$, which is coherent over X since both $w\cdot\mathcal{G}$ and $w\cdot\mathcal{G}'$ are coherent sub-$\mathcal{O}$-Modules of $\mathcal{G}'$.

By the assumption, $(\mathscr{G}' : \mathscr{G})_x$ is not equal to (0) for any $x \in X$. While, since X is Stein, $\mathscr{G}' : \mathscr{G}$ is generated by global sections of $\mathscr{G}' : \mathscr{G}$ over X as an $\mathscr{O}$-Module. For each X_ν we choose a point $x_\nu \in X_\nu$ arbitrarily. We can find a suitable $u_\nu \in \Gamma(X, \mathscr{G}' : \mathscr{G})$ which does not vanish identically on a neighborhood of x_ν. Then u_ν does not vanish identically on any open subset of X_ν by the theorem of identity. Again, using the assumption that X is Stein, we can take a holomorphic function v_ν over X which vanishes identically on each $X_\mu (\mu \neq \nu)$ but not on X_ν. Since $\{X_\nu\}$ is a locally finite family, we can define a holomorphic function $u = \sum_\nu v_\nu u_\nu$, which belongs to $\Gamma(X, \mathscr{G}' : \mathscr{G})$. Ovbiously, u satisfies the desired conditions.

Now, let X_ν^* be the normalization of each irreducible component X_ν of X with projection $\mu_\nu : X_\nu^* \to X_\nu$ and structure sheaf $\mathscr{O}_\nu^*$. The direct image $(\mu_\nu)_*(\mathscr{O}_\nu^*)$ is nothing but the sheaf of all germs of weakly holomorphic functions on X_ν. We put $\tilde{\mathscr{O}}_\nu := (i_\nu)_*(\mu_\nu)_*(\mathscr{O}_\nu^*)$.

PROPOSITION 3. 2. *If X is Stein and a coherent $\mathscr{O}$-Module $\mathscr{F}$ over X is torsion-free, then there exists an injective $\mathscr{O}$-Homomorphism $\chi : \mathscr{F} \to \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$ ($k_\nu :=$* $\operatorname{rank}_{X_\nu} \mathscr{F}$*) such that $u \cdot (\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}) \subseteq \chi(\mathscr{F})$ on X for a suitable $u \in \Gamma(X, \mathscr{O})$ not identically vanishing on any X_ν.*

Proof. Since $\mathscr{F}$ is torsion-free on X, the canonical map $\tau : \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}} \mathscr{M}$ is injective and so $\psi := \varphi \cdot \tau$ is an injective $\mathscr{O}$-Homomorphism $\mathscr{F} \to \oplus_\nu \tilde{\mathscr{M}}_\nu^{k_\nu}$, where φ is an $\mathscr{M}$-Isomorphism given in Proposition 2. 5. For each ν, we take sections $g_1^{(\nu)}, \dots, g_{k_\nu}^{(\nu)}$ in $\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}} \mathscr{M})$ whose φ-images give a base of an $\mathscr{M}_y$-module $(\tilde{\mathscr{M}}_\nu)_y^{k_\nu}$ $(y \in X)$. We consider the coherent $\mathscr{O}$-Modules $\mathscr{G}_i^{(\nu)} := \mathscr{O}(g_i^{(\nu)})$ $(1 \leqq i \leqq k_\nu)$. Obviously, for each $x \in X$ there exists a non-zero divisor w in $\mathscr{O}_x$ with $w \cdot (\mathscr{G}_i^{(\nu)})_x \subseteq \tau(\mathscr{F})_x$. According to Lemma 3. 1, there exists a holomorphic function u on X with $u \cdot \mathscr{G}_i^{(\nu)} \subseteq \tau(\mathscr{F})$ not identically vanishing on any X_ν. This shows that $u g_i^{(\nu)} \subseteq \Gamma(X, \tau(\mathscr{F}))$ and so there exists $f_i^{(\nu)} \in \Gamma(X, \mathscr{F})$ with $\tau(f_i^{(\nu)}) = u\, g_i^{(\nu)}$ $(1 \leqq i \leqq k_\nu)$. Thus we get a system of sections $\{f_1^{(\nu)}, \dots, f_{k_\nu}^{(\nu)}\}$ in $\Gamma(X, \mathscr{F})$ whose ψ-image gives a base of an $\mathscr{M}_y$-module $(\tilde{\mathscr{M}}_\nu)_y^{k_\nu}$ for each $y \in X$. Renewing φ suitably if necessary, we may assume $\psi(f_i^{(\nu)})$ is the identity element in each $\tilde{\mathscr{M}}_\nu$.

Now, we consider two coherent sub-$\mathcal{O}$-Modules $\mathcal{G} := \psi(\mathcal{F})$ and $\mathcal{G}' := \oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}$ of $\oplus_\nu(\tilde{\mathcal{M}}_\nu)^{k_\nu}$. We shall prove $\mathcal{G}$ and $\mathcal{G}'$ satisfy the assumption in Lemma 3. 1. Take an arbitrary point $x \in X$. There exists a system of generators $\{g_1, \ldots, g_s\}$ of an $\mathcal{O}_x$-module $\mathcal{F}_x$. Since there exists only finitely many ν's with $x \in X_\nu$, say $X_1, \ldots, X_t$, we can write

$$\psi(g_j) = \sum_{1 \leqq \nu \leqq t} \sum_{1 \leqq i \leqq k_\nu} c_{i,j}^{(\nu)} \psi(f_i^{(\nu)})$$

where each $c_{i,j}^{(\nu)}$ is a section of $\tilde{\mathcal{M}}_\nu$ over a neighborhood U of x and considered as a meromorphic function on U vanishing identically on each $X_\mu \cap U (\mu \neq \nu)$. Moreover, taking sufficiently small U, we may assume that all $c_{i,j}^{(\nu)}$ have the common denominator w in $\mathcal{O}_x$ which does not vanish identically on any irreducible component of U. Obviously, $w \cdot \psi(g_j) \in \Gamma(U, \oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu})$ and hence for each $f \in \mathcal{F}_x$, if we write $f = a_1 g_1 + \cdots + a_s g_s$ $(a_j \in \mathcal{O}_x)$,

$$w\psi(f) = a_1 w\psi(g_1) + \cdots + a_s w\psi(g_s) \in (\oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu})_x.$$

This shows $w\psi(\mathcal{F}) \subseteqq \oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}$. Therefore we can apply Lemma 3. 1 to the coherent sheaves $\mathcal{G}$ and $\mathcal{G}'$. We obtain a holomorphic function u' with $u' \cdot \psi(\mathcal{F}) \subseteqq \oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}$ on X which does not vanish identically on any X_ν.

Multiplying each component of elements in $(\oplus_\nu \tilde{\mathcal{M}}_\nu)^{k_\nu}$ by the above u', we define an $\mathcal{M}$-Isomorphism $\tilde{u}' : \oplus_\nu(\tilde{\mathcal{M}}_\nu)^{k_\nu} \to \oplus (\tilde{\mathcal{M}}_\nu)^{k_\nu}$ on X. Then, the composite $\chi = \tilde{u}'\psi$ is an injective $\mathcal{O}$-Homomorphism of $\mathcal{F}$ into $\oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}$, because $\chi(\mathcal{F}) = u'\psi(\mathcal{F}) \subseteqq \oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}$. We shall prove such a map χ satisfies the condition in Proposition 3. 2. For our purpose, it suffices to show that the coherent sub-$\mathcal{O}$-Modules $\chi(\mathcal{F})$ and $\oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}$ of $\oplus_\nu \tilde{\mathcal{M}}_\nu^{k_\nu}$ satisfy the assumption in Lemma 3. 1, namely, for each $x \in X$ there exists a non-zero divisor w in $\mathcal{O}_x$ with $w \cdot (\oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu})_x \subseteqq \chi(\mathcal{F})_x$. To see this, we take a universal denominator w' in $\mathcal{O}_x$ and define a non-zero divisor $w = w'u'$. Since $\psi(f_i^{(\nu)})$ $(1 \leqq i \leqq k_\nu)$ give the canonical base of the $\mathcal{M}$-module $\oplus_\nu \tilde{\mathcal{M}}_\nu^{k_\nu}$, each $h \in (\oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu})_x \subseteqq \oplus_\nu \tilde{\mathcal{M}}_\nu^{k_\nu}$ can be written as

$$h = \sum_\nu \sum_{1 \leqq i \leqq k_\nu} h_i^{(\nu)} \psi(f_i^{(\nu)})$$

where $h_i^{(\nu)} \in (\tilde{\mathcal{O}}_\nu^{k_\nu})_x$. By the definition of universal denominator, each $\tilde{h}_i^{(\nu)} := w' h_i^{(\nu)}$ belongs to $\mathcal{O}_x$. Thus we have

$$\begin{aligned} w'u'h &= \sum_\nu \sum_{1 \leqq i \leqq k} w' h_i^{(\nu)} u' \psi(f_i^{(\nu)}) \\ &= \sum_\nu \sum_i \tilde{h}_i^{(\nu)} \chi(f_i^{(\nu)}) \\ &\in \mathcal{O}(\ldots, \chi(f_1^{(\nu)}), \ldots, \chi(f_{k_\nu}^{(\nu)}), \ldots) \subseteqq \chi(\mathcal{F}). \end{aligned}$$

This shows $w \cdot (\oplus_\nu \mathcal{O}_\nu^{k_\nu})_x \subseteqq \chi(\mathcal{F})_x$. The proof is completely accomplished.

For later uses, we give

DEFINITION 3. 3. For a coherent $\mathcal{O}$-Module $\mathcal{F}$ we shall say a non-zero divisor u in $\mathcal{O}_x$ to be a *universal denominator* for $\mathcal{F}_x$ $(x \in X)$ if for a suitable neighborhood U of x it satisfies the conditions as in Proposition 3. 2, that is, there exists an injective $\mathcal{O}$-Homomorphism $\chi: \mathcal{F}|U \to \oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}$ such that $u \cdot (\oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}) \subseteqq \chi(\mathcal{F})$ on U, where $\tilde{\mathcal{O}}_\nu$ is the sheaf defined as above for each irreducible component U_ν of U and k_ν is some non-negative integer.

As is easily proved, any k_ν satisfying the above conditions is necessarily equal to $\text{rank}_{U_\nu} \mathcal{F}$ and we can discuss the existence of universal denominators for $\mathcal{F}_x$ only if $\mathcal{F}$ is torsion-free at x. While, if $\mathcal{F}$ is torsion-free at $x \in X$, there exists at least one universal denominator for $\mathcal{F}_x$. For, in this case, we can take a Stein neighborhood U of x such that $\mathcal{F}$ is torsion-free on U and apply Proposition 3. 2 to the space U.

§4. **Local Continuation Theorem.** Now, we can generalize the theorems on the continuation of holomorphic functions given in [6], [8] and [10] etc. to the case of sections of torsion-free $\mathcal{O}$-Modules.

To see this, we need

DEFINITION 4. 1. By $\text{Ass}(\mathcal{F}_x)$, we denote the totality of all prime ideals associated with a reduced primary decomposition of sub-$\mathcal{O}_x$-module (0) of $\mathcal{F}_x$. If $k = \min\{\dim \mathcal{O}_x/\mathfrak{p}; \mathfrak{p} \in \text{Ass}(\mathcal{F}_x)\}$, we shall say that $\mathcal{F}$ is *of lower dimension* k at x and indicate this by $\text{l.}\dim_x \mathcal{F} = k$.

The first main theorem is given as follows.

THEOREM 4. 2 *Let D be an open subset of X which is *-strongly s-concave at $x \in X$ (see Definition* 2. 5 *in* [6]), *and $\mathcal{F}$ be a coherent $\mathcal{O}$-Module over X. Suppose that $\mathcal{F}$ is torsion-free at x and a universal denominator u for $\mathcal{F}_x$ satisfies*

l. $\dim_x(\mathscr{F}/u\mathscr{F}) \geqq s$. *Then, there exists a neighborhood U of x such that every section of $\mathscr{F}$ over D is uniquely continuable to a section of $\mathscr{F}$ over $D \cup U$.*

Proof. In case that $\mathscr{F}_x = (0)$, Theorem 4. 2 is trivial. We assume $\mathscr{F}_x \neq (0)$. By Definition 3. 3, we can find a neighborhood V_1 of x such that $\mathscr{F}$ is torsion-free on V_1 and there exists an injective $\mathscr{O}$-Homomorphism $\chi: \mathscr{F}|V_1 \to \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$ with $u\cdot(\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}) \subseteqq \chi(\mathscr{F})$ on V_1. Moreover, we may assume l. $\dim_y(\mathscr{F}/u\mathscr{F}) \geqq s$ for any $y \in V_1$. On the other hand, l. $\dim_x \mathscr{O} \geqq s+1$, or any irreducible component of X at x is of dimension $\geqq s+1$. Indeed, since u is a non-zero divisor in $\mathscr{O}_x$, for each $\mathfrak{p} \in \mathrm{Ass}(\mathscr{O}_x)$ $u \notin \mathfrak{p}$ and there exists some $\mathfrak{p}' \in \mathrm{Ass}(\mathscr{F}_x/u\mathscr{F}_x)$ with $\mathfrak{p} \cup (u) \subseteqq \mathfrak{p}'$. By definition, $\mathrm{depth}\,\mathfrak{p} \geqq s$ and hence $\mathrm{depth}\,\mathfrak{p}' \geqq s+1$ because $\mathfrak{p} \subsetneqq \mathfrak{p}'$. This shows l. $\dim_x \mathscr{O} \geqq s+1$. According to Proposition 6. 1 in [6], there exists a neighborhood V_2 of $x(V_2 \subseteqq V_1)$ such that every weakly holomorphic function on $D \cap V_2$ is uniquely continuable to V_2. Moreover, as in the proof of Proposition 6. 2 in [6], we can find another neighborhood V_3 of $x(V_3 \subseteqq V_2)$ with the property that any analytic set N of V_2 satisfies $N \cap D \neq \phi$ if $\dim N \cap V_3 \geqq s$. We shall prove that the neighborhood $U := V_3$ of x satisfies the desired condition.

Now, take an arbitrary $f \in \Gamma(D, \mathscr{F})$. The sheaf $\tilde{\mathscr{O}}_\nu$ can be regarded as the sheaf of all germs of weakly holomorphic functions vanishing identically on any irreducible component except the corresponding component X_ν. By the assumption of V_2, every section of $\tilde{\mathscr{O}}_\nu$ over $V_2 \cap D$ is uniquely continuable to V_2. Therefore, each component $g_i^{(\nu)}$ of $\chi(f) \in \Gamma(D \cap V_2, \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})$ has the unique continuation $h_i^{(\nu)} \in \Gamma(V_2, \tilde{\mathscr{O}}_\nu)$. Thus we get a section $h = \sum_\nu \sum_i h_i^{(\nu)} \in \oplus_\nu \Gamma(V_2, \tilde{\mathscr{O}}_\nu)^{k_\nu} = \Gamma(V_2, \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})$ with $h|V_2 \cap D = \chi(f)$. By definition, uh is contained in $\Gamma(V_2, \chi(\mathscr{F}))$. Since χ is injective, there exists a section $\tilde{h} \in \Gamma(V_2, \mathscr{F})$ with $\chi(\tilde{h}) = uh$ on V_2 and so $= u\chi(f) = \chi(uf)$ on $V_2 \cap D$. Again, using the injectivity of χ, we conclude $\tilde{h} = uf$ on $V_2 \cap D$. This shows the analytic set $N := |\mathscr{O}/u\mathscr{F} : (\tilde{h})|$ does not intersect D. While, in view of Lemma 2.4 in [7], N is of dimension $\geqq s$ everywhere on $V_2 \cap N$. By the property of U, $N \cap D = \phi$ implies $N \cap U = \phi$. Therefore, $u\mathscr{F} : (\tilde{h}) = \mathscr{O}$ on U or $\tilde{h} \in \Gamma(U, u\mathscr{F})$. Since u defines a non-zero divisor in $\mathscr{O}_y$ for any $y \in U$, there exists a section $\tilde{f} \in \Gamma(U, \mathscr{F})$ with $u\tilde{f} = \tilde{h} = uf$ on $U \cap D$, whence $\tilde{f} = f$ on $U \cap D$. This shows that $\tilde{f}$ is a continuation of f to U.

The uniqueness of the continuation is owing to the assumption that $\mathscr{F}$ is torsion-free on U. In fact, for any $x \in U$ every element in $\mathrm{Ass}(\mathscr{F}_x)$ is a minimal prime ideal of $\mathcal{O}_x$. Therefore, any section of $\mathscr{F}$ over U satisfies the theorem of identity on each irreducible component of U(c.f. [7], Theorem 5.1). The proof is carried out similarly to the case of holomorphic functions.

REMARK. If a torsion-free coherent $\mathcal{O}$-Module $\mathscr{F}$ satisfies $\mathrm{dih}_x\mathscr{F} \geqq s+1$ for a point $x \in X$ (see [3], p. 197), any non-zero divisor in $\mathcal{O}_x$, particularly any universal denominator for $\mathscr{F}_x$, satisfies $\mathrm{l.\,dim}_x(\mathscr{F}/u\mathscr{F}) \geqq s$. In fact, if $\mathrm{dih}_x\mathscr{F} \geqq s+1$, there exists an $\mathscr{F}_x$-sequence $u_1 := u, u_2, \dots, u_{s+1}$ in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_x$. For each $\mathfrak{p} \in \mathrm{Ass}(\mathscr{F}_x/u\mathscr{F}_x)$ we can find easily prime ideals $\mathfrak{p}_i \in \mathrm{Ass}(\mathscr{F}_x/(u_1, \dots, u_i)\mathscr{F}_x)$ with $\mathfrak{p}_1 = \mathfrak{p} \subsetneqq \mathfrak{p}_2 \subsetneqq \cdots \subsetneqq \mathfrak{p}_{s+1}$. Moreover, if the local ring $\mathcal{O}_x$ is normal and of dimension $\geqq s+1$, any non-zero divisor in $\mathcal{O}_x$ satisfies $\mathrm{l.\,dim}_x(\mathcal{O}/u\mathcal{O}) \geqq s$. Both of Proposition 6.1 and 6.2 in [6] are special cases of Theorem 4.2 in view of Proposition 3.2.

We can prove also the following local continuation theorem.

THEOREM 4.3. *If $\mathscr{F}$ is torsion-free at x and there exists a universal denominator u for $\mathscr{F}_x$ such that* $\mathrm{l.\,dim}_x(\mathscr{F}/u\mathscr{F}) \geqq s$, *then there exists an arbitrarily small neighborhood U of x such that for any open subset V of U and any analytic set M in V of dimension $\leqq s-1$ every section of $\mathscr{F}$ over $V-M$ is uniquely continuable to V.*

Proof. We may assume $\mathscr{F}_x \neq (0)$. By the assumption any irreducible component of X at x is of dimension $\geqq s+1$ and so $\dim_y X \geqq s+1$ for any y in a neighborhood U of x. For any locally analytic set M in U of dimension $\leqq s-1$ M is of codimension $\geqq (s+1)-(s-1)=2$. Therefore, every weakly holomorphic function on $V-M$ is uniquely continuable to V. While, any analytic set N with $\dim_x N \geqq s$ intersects $X-M$. From these facts, we can easily conclude Theorem 4.3 by the same arguments as in the proof of Theorem 4.2. We omit the details.

REMARK. As to the continuation of sections of torsion-free $\mathcal{O}$-Modules over a purely dimensional reduced complex space, Theorem 4.3 is an improvement of the implication (1) $\rightarrow$ (3) of Satz 10 in [11], p. 90. In fact, in this case, if we put $k=1$ in [11], Satz 10, the assertion (1) means that $\mathscr{G}_x=(0)$, or $\mathscr{G}_x$ is torsion-free and "any" non-zero divisor u in $\mathscr{O}_x$ satisfies $\text{l.}\dim_x(\mathscr{G}/u\mathscr{G})\geqq\dim_x X-1$. Then, there exists a neighborhood U of x such that for any open set $V\subseteqq U$ and analytic set M in V of codimension $\geqq 2$ the restriction mapping $\Gamma(V,\mathscr{G})\to\Gamma(V-M,\mathscr{G})$ is bijective by Theorem 4.3. This is the first half of the assertion (3) in [11] Satz 10.

§5. **Global Continuation Theorems.** As in the previous paper [6], using the local continuation theorems, we can show global continuation theorems of sections of a torsion-free coherent $\mathscr{O}$-module to sets with the non-empty open kernel.

We first generalize Theorem 7.6 in [6].

THEOREM 5.1. *Let X be a complex space where there exists a $*$-strongly s-convex function v and $\mathscr{F}$ be a torsion-free coherent $\mathscr{O}$-Module over X. Suppose that for any $x\in X$ there exists a universal denominator u for $\mathscr{F}_x$ with* $\text{l.}\dim_x(\mathscr{F}/u\mathscr{F})\geqq s$ *and an open subset B of X satisfies the conditions;*

1°. $\bar{B}\cap\{v>\lambda\}\Subset X$ *for any* λ,

2°. *for any $x\in\partial B$ $\{v>v(x)\}-B$ intersects any irreducible component of X at x.*

Then, every section of $\mathscr{F}$ over a connected neighborhood U of ∂B is uniquely continuable to $U\cup B$.

Proof. Using Theorem 4.2 and the theorem of identity for sections of $\mathscr{F}$, we can prove Theorem 5.1 by the analogous arguments as in the proof of Theorem 7.6 in [6]. The details are left to the reader.

Easily, we have

COROLLARY 5.2. *Suppose that there exists a $*$-strongly s-convex function v on X with $\{\lambda<v<\mu\}\Subset X$ for any $\lambda,\mu(\lambda<\mu)$. If a torsion-free coherent $\mathscr{O}$-Module $\mathscr{F}$ over X has a universal denominator u in $\mathscr{O}_x$ with* $\text{l.}\dim_x(\mathscr{F}/u\mathscr{F})\geqq s$ *for each $x\in X$, every section of $\mathscr{F}$ over $X_\lambda:=X\cap\{v>\lambda\}$ is uniquely continuable to the total X.*

REMARK. A strongly s-convex function is obviously $*$-strongly s-convex. As to the continuation of sections of torsion-free coherent $\mathscr{O}$-Modules, Corollary 5.2 is a generalization of Theorem 15 in [3], p. 254.

Let X be a complex space where there exists a $*$-strongly s-convex function v and $\mathscr{F}$ be a torsion-free coherent $\mathscr{O}$-Module over X. If X is locally irreducible, $\mathscr{F}$ is a hard sheaf over X (see Definition 8.1 in [6]), because any section of $\mathscr{F}$ satisfies the theorem of identity. Moreover, if for any $x \in X$ $\mathscr{F}_x$ has a universal denominator $u \in \mathscr{O}_x$ with $\mathrm{l.\,dim}_x(\mathscr{F}/u\mathscr{F}) \geqq s$, then each $x \in X$ has a fundamental system of neighborhoods $\mathfrak{U} = \{U\}$ such that the restriction map $\Gamma(U, \mathscr{F}) \to \Gamma(U \cap \{v > v(x)\}, \mathscr{F})$ is surjective for each U. This shows that v is admissible for $\mathscr{F}$ if $\{\lambda \leqq v \leqq \mu\}$ is compact for any $\lambda, \mu (\lambda < \mu)$ (see [6], p. 81). Now, we can apply the method in [6], §8 to give global continuation theorems of sections of $\mathscr{F}$.

As a generalization of Theorem 8.4 in [6], we have

THEOREM 5.3. *Let X be a connected, locally irreducible complex space of dimension n and v be a $*$-strongly s-convex function on X with the property that $\{\lambda \leqq v \leqq \mu\}$ is compact for any $\lambda, \mu (\lambda < \mu)$ and v is represented as $v = v'\varphi$ by a suitable nowhere degenerate holomorphic mapping φ of X into a purely n-dimensional complex manifold Y and a real analytic function v' on Y. Suppose that a coherent $\mathscr{O}$-Module $\mathscr{F}$ over X is torsion-free on X and has a universal denominator $u \in \mathscr{O}_x$ with* $\mathrm{l.\,dim}_x(\mathscr{F}/u\mathscr{F}) \geqq s$ *for any $x \in X$. Then for any open set $B \Subset X$ with the connected boundary ∂B every section of $\mathscr{F}$ over a connected neighborhood U of ∂B is uniquely continuable to $B \cup U$.*

Proof. As is seen above, v is admissible for $\mathscr{F}$. Take an open set D and a compact set K with $K \subsetneqq B \subsetneqq D$ such that $U := D - K$ is connected. In view of Theorem 1 in [10], p. 299, it suffices to show that there exists an open set B' with $K \subsetneqq B' \subsetneqq D$ such that $\partial B'$ is good for v in the sense of Kasahara [10]. By the assumption of local irreducibility of X, the normalization X^* of X is homeomorphic to X. For the proof, see [8], §7, [10], §5, Lemma 4 and [6], §8, Lemma 8.3.

For a Stein space X, we can prove the following continuation theorem without the assumption of the local irreducibility.

THEOREM 5.4. *If X is Stein and a torsion-free coherent $\mathscr{O}$-Module $\mathscr{F}$ over X has a universal denominator u in $\mathscr{O}_x$ with* $\mathrm{l.\,dim}_x(\mathscr{F}/u\mathscr{F}) \geqq 1$ *for each $x \in X$, then the restriction mapping $\rho^D_{D-K}: \Gamma(D, \mathscr{F}) \to \Gamma(D-K, \mathscr{F})$ is bijective for an open set D and a compact subset K of D satisfying the condition that each irreducible component of D is irreducible in $D-K$.*

Proof. In the notations as in §3, there exists an injective $\mathscr{O}$-Homomorphism $\chi: \mathscr{F} \rightarrow \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$ $(k_\nu = \mathrm{rank}_{X_\nu} \mathscr{F})$ and a holomorphic function u on X with $u \cdot (\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}) \subseteqq \chi(\mathscr{F})$ which does not vanish identically on any irreducible component by virtue of Proposition 3.2. Take a section $f \in \Gamma(D-K, \mathscr{F})$. We put

$$\chi(f) = (\ldots, h_1^{(\nu)}, \ldots, h_{k_\nu}^{(\nu)}, \ldots) \in \Gamma(D-K, \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})$$

where $h_i^{(\nu)} \in \Gamma(D-K, \tilde{\mathscr{O}}_\nu)$. By the assumption, each irreducible component of X is of dimension $\geqq 2$. According to Kasahara [10], every weakly holomorphic function is uniquely continuable to D. Especially, each $h_i^{(\nu)}$ has a continuation $\tilde{h}_i^{(\nu)} \in \Gamma(D, \tilde{\mathscr{O}}_\nu)$. We put

$$\tilde{h} = (\ldots, \tilde{h}_1^{(\nu)}, \ldots, \tilde{h}_{k_\nu}^{(\nu)}, \ldots) \in \Gamma(D, \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}).$$

By the property of u, $u\tilde{h} \in \Gamma(D, \chi(\mathscr{F}))$ and so there exists a section $g \in \Gamma(D, \mathscr{F})$ with $\chi(g) = u\tilde{h}$ on D and so $= u\chi(f) = \chi(uf)$ on $D-K$. Thus we have a section $g \in \Gamma(D, \mathscr{F})$ with $g = uf$ on $D-K$ by the injectivity of χ.

To complete the proof of the existence of the continuation, it suffices to show the existence of a section $\tilde{f} \in \Gamma(D, \mathscr{F})$ with $g = u\tilde{f}$ on D. For, such $\tilde{f}$ satisfies $\tilde{f} = f$ on $D-K$ because u defines a non-zero divisor in $\mathscr{O}_x$ for any $x \in X$. Now, since X is Stein, we know the existence of a strongly 1-convex function v on X. As usual, we consider the set $\Lambda = \{\lambda;$ there exists a section $\tilde{f}_\lambda$ of $\mathscr{F}$ over $D_\lambda := D \cap \{v > \lambda\}$ with $u\tilde{f}_\lambda = g$ on $D_\lambda\}$. Putting $\lambda_0 = \sup\{v(x); x \in K\}$, we see easily $\lambda_0 \in \Lambda$ and so $\Lambda \neq \phi$. Moreover, for $\lambda_1 = \inf \Lambda$ we have $\lambda_1 \in \Lambda$. We assume $\lambda_1 > -\infty$. In view of Theorem 4.2, each point $x \in \{v = \lambda_1\} \cap D$ has a neighborhood $U^{(x)}$ such that there exists a continuation $\tilde{f}^{(x)}$ of $\tilde{f}_{\lambda_1}$ to $U^{(x)}$. On $D-K$, $u\tilde{f}_{\lambda_1} = g = uf$ implies $\tilde{f}_{\lambda_1} = f$. For a point $x \in \{v = \lambda_1\} \cap (D-K)$ we can take $\tilde{f}^{(x)} = f$ on $U^{(x)} = D-K$. For a point $x \in \{v = \lambda_1\} \cap K$, taking $U^{(x)}$ satisfying that each connected component of $U^{(x)}$ intersects D_{λ_1}, we may assume $u\tilde{f}^{(x)} = g$ on $U^{(x)}$. Obviously, $\tilde{f}^{(x)} = \tilde{f}^{(x')}$ on $U^{(x)} \cap U^{(x')}$ if $U^{(x)} \cap U^{(x')} \neq \phi$. If we cover K by finitely many above $U^{(x)}$, it is easy to find $\lambda_2 \in \Lambda$ with $\lambda_2 < \lambda_1$. This is a contradiction. We conclude $\lambda_1 = -\infty$. This shows there exists a section $f \in \Gamma(D, \mathscr{F})$ with $g = u\tilde{f}$.

The uniqueness of the continuation is obvious by the theorem of identity for sections of torsion-free coherent $\mathscr{O}$-Modules. Theorem 5.4 is completely proved.

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