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The continuity of the rearrangement in $W^{1,p}(\mathbb{R})$

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The Continuity of the Rearrangement in $W^{1,p}(\mathbb{R})$.

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1. – Introduction.

Let, in the following, p be a real number such that $1 < p < +\infty$. Let u be a nonnegative function of $W^{1,p}(\mathbb{R})$. Let u^* be the rearrangement of u, that is the unique function u^* which is even, nonincreasing on $[0, +\infty]$ and such that:

for all $y \in \mathbb{R}$ meas $\{x|u^*(x) \geqslant y\} =$ meas $\{x|u(x) \geqslant y\}$ (meas A stands for the Lebesgue measure of A).

We know (see, for example [1] appendix 1, [2], [3], [4] p. 154, [5], [6], [7] and [8]) that u^* is in $W^{1,p}(\mathbb{R})$ and:

$$\int_{\mathbb{R}} \left|\frac{du^*}{dx}\right|^p dx \leqslant \int_{\mathbb{R}} \left|\frac{du}{dx}\right|^p dx . \tag{1}$$

Let $W^{1,p}_+(\mathbb{R})$ be the set of nonnegative functions of $W^{1,p}(\mathbb{R})$; the weak and the strong topologies of $W^{1,p}(\mathbb{R})$ induce two topologies on $W^{1,p}_+(\mathbb{R})$; we shall also call them weak and strong topologies respectively.

Let c be a positive real number and let:

$$\Phi_c(u) = \int_{\mathbb{R}} \left|\frac{du}{dx}\right|^p dx - c\int_{\mathbb{R}} \left|\frac{du^*}{dx}\right|^p dx , \qquad u \in W^{1,p}_+(\mathbb{R}) .$$

The purpose of this article is to prove the following theorem:

THEOREM. *Φ_c is weakly l.s.c. if and only if $c \leqslant 1/2^p$.*

COROLLARY. *The rearrangement is a continuous mapping from $W^{1,p}_+(\mathbb{R})$ into $W^{1,p}_+(\mathbb{R})$ for the strong topologies.*

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PROOF OF COROLLARY. Let $u_n \in W^{1,p}_+(\mathbf{R})$, $u_n \to u$ in $W^{1,p}(\mathbf{R})$.

Since the rearrangement is a continuous mapping from the set of non-negative functions of $L^p(\mathbf{R})$ into $L^p(\mathbf{R})$ (see appendix 0) we have:

$$u_n^* \to u^* \quad \text{in } L^p(\mathbf{R}) .$$

Therefore, using (1), we have $u_n^* \rightharpoonup u^*$ in $W^{1,p}(\mathbf{R})$ weakly. Let $c \in (0, 1/2^p]$.

$$\Phi_c(u) \leqslant \underline{\lim}\, \Phi_c(u_n) .$$

But

$$\int_{\mathbf{R}} \left|\frac{du_n}{dx}\right|^p dx \to \int_{\mathbf{R}} \left|\frac{du}{dx}\right|^p dx$$

hence

$$\overline{\lim} \int_{\mathbf{R}} \left|\frac{du_n^*}{dx}\right|^p dx \leqslant \int_{\mathbf{R}} \left|\frac{du^*}{dx}\right|^p dx$$

and therefore (since $1 < p < +\infty$ and $u_n^* \rightharpoonup u^*$ in $W^{1,p}(\mathbf{R})$)

$$u_n^* \to u^* \quad \text{in } W^{1,p}(\mathbf{R}) .$$

The proof of the theorem will be divided in two parts.

In part A we assume that $c \leqslant 1/2^p$ and we prove that Φ_c is weakly l.s.c.. In part B we assume that $c > 1/2^p$ and we construct a sequence u_n such that $u_n \rightharpoonup u$ in $W^{1,p}_+(\mathbf{R})$ and $\Phi_c(u) > \lim \Phi_c(u_n)$.

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2. – Proof of the theorem.

Part A. Here we assume that $c \leqslant 1/2^p$ and we prove that Φ_c is weakly l.s.c. Let $f \in W^{1,p}(\mathbf{R})$, we shall use the following notation

$$|f| = \left(\int_{\mathbf{R}} \left|\frac{df}{dx}\right|^p dx\right)^{1/p}.$$

Let u_n be a sequence of functions in $W^{1,p}_+(\mathbf{R})$ such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\mathbf{R}) \text{ when } n \to +\infty .$$

If $u = 0$, we have:

$$\Phi_c(u) \leqslant \varliminf \Phi_c(u_n) \quad \text{since } \Phi_c \geqslant 0 .$$

Therefore we may assume that $u \neq 0$.

Let v be in $W^{1,p}(\mathbf{R})$ and let:

$V(v) = \{y \in \mathbf{R} |$ there exists x in $u^{-1}(y)$ such that either v is not differentiable in x or v is derivable in x and $v'(x) = 0\}$.

One can prove (see appendix 1) that $V(v)$ is negligible for the Lebesgue measure (this is a little modification of Sard's theorem). Let $\eta > 0$; since $V(u)$ is negligible, there exist m and M, real numbers, such that

$$m \notin V(u), \quad M \notin V(u), \qquad 0 < m < M \tag{2}$$

$$M < \operatorname*{Max}_{x\in\mathbf{R}} u(x) \tag{3}$$

and if

$$g(x) = \operatorname{Min}\big(u(x), m\big)$$

$$f(x) = \operatorname{Max}\big(u(x), M\big) - M$$

we have:

$$|g|^p \leqslant \eta, \qquad |f|^p \leqslant \eta . \tag{4}$$

Let:

$$\begin{aligned} g_n(x) &= \operatorname{Min}\big(u_n(x), m\big) \\ f_n(x) &= \operatorname{Max}\big(u_n(x), M\big) - M \\ \bar{u}(x) &= \operatorname{Max}\Big(\operatorname{Min}\big(u(x), M\big), m\Big) - m \\ \bar{u}_n(x) &= \operatorname{Max}\Big(\operatorname{Min}\big(u_n(x), M\big), m\Big) - m . \end{aligned}$$

$\bar{u}$ and $\bar{u}_n$ are in $W^{1,p}_+(\mathbf{R})$ and:

$$\bar{u}_n \rightharpoonup \bar{u} \quad \text{in } W^{1,p}(\mathbf{R}) \text{ when } n \to +\infty .$$

For the moment being let us assume that:

$$\Phi_c(\bar{u}) \leqslant \varliminf_{n\to+\infty} \Phi_c(\bar{u}_n); \tag{5}$$

we have:

$$\begin{aligned} \Phi_c(u) &= \Phi_c(\bar{u}) + \Phi_c(g) + \Phi_c(f) \\ \Phi_c(u_n) &= \Phi_c(\bar{u}_n) + \Phi_c(g_n) + \Phi_c(f_n) . \end{aligned}$$

Using (4), (1) and (5), this yields

$$\Phi_c(u) \leqslant \varliminf_{n\to+\infty} \Phi_c(u_n) + 2\eta$$

and the theorem is proved.

It remains to prove (5); without any restriction we may assume that

$$\operatorname*{Max}_{x\in\mathbf{R}} u_n(x) > M\,. \quad \text{Let } \bar{M} = M - m\,.$$

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ be a sequence of r strictly positive numbers (r depends on ε) such that:

$$\sum_{i=1}^{r} \varepsilon_i = \bar{M}$$

Let

$$A(\varepsilon) = \left\{\sum_{i=1}^{k} \varepsilon_i \middle| 1 \leqslant k \leqslant r-1\right\}$$

$$\tilde{A}(\varepsilon) = A(\varepsilon) \cup \{0, \bar{M}\}\,.$$

We are going to define by induction a finite sequence of real numbers. Let

$$a_1 = \operatorname{Inf}\left\{x \middle| \bar{u}(x) \neq 0\right\}$$

(it is easy to see that a_1 exists). Assume that a_{i-1} is defined. Either:

$$\left\{x \middle| \bar{u}(x) \in \tilde{A}(\varepsilon) - \{\bar{u}(a_{i-1})\}\right\} \cap [a_{i-1}, +\infty) = \emptyset$$

then we stop here the sequence a_j; we have $\bar{u}(a_{i-1}) = 0$ and:

$$\bar{u}(x) < \varepsilon_1 \qquad \forall x \in [a_{i-1}, +\infty)$$

or:

$$\left\{x \middle| \bar{u}(x) \in \tilde{A}(\varepsilon) - \{\bar{u}(a_{i-1})\}\right\} \cap [a_{i-1}, +\infty) \neq \emptyset\,,$$

then we let:

$$a_i = \operatorname{Min}\left\{x \middle| \bar{u}(x) \in \tilde{A}(\varepsilon) - \{\bar{u}(a_{i-1})\} \text{ and } x \geqslant a_{i-1}\right\}.$$

We are going to prove that the sequence a_i has only a finite number of terms.

Let

$$\mu = \operatorname*{Min}_{1\leqslant j\leqslant r} \varepsilon_j; \qquad \mu > 0\,.$$

We have

$$\mu \leqslant |\overline{u}(a_{i+1}) - \overline{u}(a_i)|$$

but

$$|\overline{u}(a_{i+1}) - \overline{u}(a_i)| \leqslant \int_{a_i}^{a_i+1} |\overline{u}'(\tau)|\, d\tau \leqslant |\overline{u}|(a_{i+1} - a_i)^{1/q}$$

with

$$\frac{1}{p} + \frac{1}{q} = 1$$

therefore:

$$\mu \leqslant (a_{i+1} - a_i)^{1/q} |\overline{u}| \,. \tag{6}$$

Let $b = \operatorname{Sup} \{x | \overline{u}(x) \neq 0\}$; $b < +\infty$ and

$$\forall i \qquad a_i \leqslant b \tag{7}$$

then using (6) and (7) we see that the sequence (a_i) has only a finite number of terms. Let l be the number of terms of the sequence a_i. With $\overline{u}$ and the sequence a_i we are going to define a new function in $W^{1,p}_+(\mathbb{R})$ $P_\varepsilon \overline{u}$ as follows:

when $x \geqslant a_l$ let $(P_\varepsilon \overline{u})(x) = 0$

when $x \leqslant a_1$ let $(P_\varepsilon \overline{u})(x) = 0$

when $a_i < x \leqslant a_{i+1}$:

— either $\overline{u}(a_i) < \overline{u}(a_{i+1})$ then we let:

$$(P_\varepsilon \overline{u})(x) = \operatorname*{Max}_{y \in [a_i, x]} \overline{u}(y)$$

— or $\overline{u}(a_i) > \overline{u}(a_{i+1})$ then we let:

$$(P_\varepsilon \overline{u})(x) = \operatorname{Min} \big(\overline{u}(a_i), \operatorname*{Max}_{y \in [x, a_{i+1}]} \overline{u}(y)\big) \,.$$

It is easy to see that $P_\varepsilon \overline{u}$ is a continuous function; using appendix 2 we see that $P_\varepsilon \overline{u}|_{]a_i, a_{i+1}[} \in W^{1,p}\big((a_i, a_{i+1})\big)$ and

$$\int_{a_i}^{a_{i+1}} |(P_\varepsilon \overline{u})'|^p \, dx = \int_{a_i}^{a_{i+1}} |(P_\varepsilon \overline{u})'| |\overline{u}'|^{p-1} \, dx \,.$$

Thus $P_\varepsilon \overline{u} \in W^{1,p}_+(\mathbb{R})$ and

$$|P_\varepsilon \overline{u}|^p = \int_{\mathbf{R}} |(P_\varepsilon \overline{u})'||\overline{u}'|^{p-1}\,dx\,. \tag{8}$$

We are now going to define a_i^n and $P_\varepsilon \overline{u}_n$;

Let δ_0 be such that

$$u(a_1 - \delta_0) < m$$

$$u(a_l + \delta_0) < m$$

let

$$a_1^n = \operatorname{Inf}\left\{x|\overline{u}_n(x) \neq 0 \text{ and } a_1 - \delta_0 \leqslant x \leqslant a_l + \delta_0\right\}$$

a_1^n exists for n large enough and, always for n large enough,

$$\overline{u}_n(a_1^n) = 0\,.$$

Let us assume that a_{i-1}^n is defined.

Either:

$$\left\{x|\overline{u}_n(x) \in \tilde{A}(\varepsilon) - \{\overline{u}(a_{i-1})\}\right\} \cap [a_{i-1}^n, a_i + \delta_0) \neq \emptyset$$

then we stop here the sequence a_i^n we have $a_{i+1}^n \leqslant a_l + \delta_0$ and for n large enough (i.e. if $u_n(a_1 + \delta_0) < m$):

$$\overline{u}_n(a_{i-1}^n) = 0\,,$$

or:

$$\left\{x|\overline{u}_n(x) \in \tilde{A}(\varepsilon) - \{u_n(a_{i=1}^n)\}\right\} \cap [a_{i-1}^n, a_l + \delta_0[\neq \emptyset$$

and then we set

$$a_i^n = \operatorname{Min}\left\{x|\overline{u}_u(x) \in \tilde{A}(\varepsilon) - \{\overline{u}_n(a_{i-1})\} \text{ and } x \in [a_n^{i-1}, a_l + \delta_0[\right\}.$$

In the same way as for the sequence a_i, one can prove that the sequence a_i^n has only a finite number of terms and we define $P_\varepsilon \overline{u}$ from $(a_i^n)_i$ and $\overline{u}_n$ in the same way we have defined $P_\varepsilon \overline{u}$ from $(a_i)_i$ and $\overline{u}$. Let us remark that:

$$P_\varepsilon \overline{u}_n \in W^{1,p}(\mathbb{R})$$

and

$$\operatorname{Supp} P_\varepsilon \overline{u}_n \subset [a_1 - \delta_0, a_l + \delta_0]\,.$$

We are going to prove:

(9) $$P_\varepsilon \bar{u} \to \bar{u} \quad \text{in } W^{1,p}(\mathbf{R}) \text{ when } |\varepsilon| \to 0$$

(10) $$(P_\varepsilon \bar{u})^* \to (\bar{u})^* \quad \text{in } W^{1,p}(\mathbf{R}) \text{ when } |\varepsilon| \to 0$$

(11) $$\Phi_c(P_\varepsilon \bar{u}_n) \leqslant \Phi_c(\bar{u}_n)$$

(12) If $A(\varepsilon) \cap V(\bar{u}) = \emptyset$ then:

$$\Phi_c(P_\varepsilon \bar{u}) \leqslant \varliminf_{n \to +\infty} \Phi_c(P_\varepsilon \bar{u}_n) .$$

Before proving (9), (10), (11) and (12) we are going to explain how from (9), (10), (11) and (12) we can deduce (5). Let $\gamma > 0$; since $V(u)$ is negligible, from (9) and (10) we deduce that there exists a sequence $\varepsilon = (\varepsilon_i)_{1 \leqslant i \leqslant r}$ of strictly positive numbers with $\sum_{i=1}^{r} \varepsilon_i = \bar{M}$ such that

$$A(\varepsilon) \cap V(\bar{u}) = \emptyset$$

and:

(13) $$\Phi_c(P_\varepsilon \bar{u}) \geqslant \Phi_c(\bar{u}) - \gamma .$$

Using (11) and (12) we have:

(14) $$\Phi_c(P_\varepsilon \bar{u}) \leqslant \lim_{n \to +\infty} \Phi_c(\bar{u}_n) .$$

We use (13) and (14); we obtain

$$\Phi_c(\bar{u}) - \gamma \leqslant \lim_{n \to +\infty} \Phi_c(\bar{u}_n) \quad \forall \gamma > 0$$

which establishes (5).

It remains to prove (9), (10), (11), (12).

PROOF OF (9). (8) yields:

(15) $$|P_\varepsilon \bar{u}| \leqslant |\bar{u}| .$$

But there exists α in $\mathbf{R}$ such that

$$\operatorname{Supp} \bar{u} \subset [-\alpha, \alpha] .$$

Then we have:

(16) $$\operatorname{Supp} P_\varepsilon \bar{u} \subset [-\alpha, \alpha] .$$

From (15) and (16) it follows that $P_\varepsilon \bar{u}$ is bounded in $W^{1,p}(\mathbb{R})$. But it is easy to see that:

$$\|P_\varepsilon \bar{u} - \bar{u}\|_\infty \leqslant 2\varepsilon .$$

Then using (15) we have (9).

PROOF of (10). Since the rearrangement is a continuous mapping from the set of nonnegative functions of $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ it follows from (9) and (1) that (since $\exists c|\operatorname{Supp} P_\varepsilon \bar{u} \subset [-c, c]$):

$$(P_\varepsilon \bar{u})^* \rightharpoonup \bar{u}^* \quad \text{in } W^{1,p}(\mathbb{R}) \text{ when } |\varepsilon| \to 0 . \tag{17}$$

We are going to prove that:

$$\lim_{|\varepsilon| \to 0} |(P_\varepsilon \bar{u})^*| = |\bar{u}^*| . \tag{18}$$

Clearly (10) follows from (17) and (18).

Let ε^k with $|\varepsilon^k| \to 0$ when $k \to +\infty$.

Let

$$\bar{u}^k = P_{\varepsilon^k} \bar{u}$$

$$v^k(y) = -\operatorname{meas}\{x|\bar{u}^k(x) \geqslant y\}$$

$$v(y) \ = -\operatorname{meas}\{x|\bar{u}(x) \geqslant y\} .$$

We have (see appendix 3):

$$|(\bar{u}^k)^*|^p = 2^p \int_0^{\bar{M}} \frac{1}{[(v^k)'(y)]^{p-1}}\, dy \tag{19}$$

$$|\bar{u}^*|^p = 2^p \int_0^{\bar{M}} \frac{1}{(v'(y))^{p-1}}\, dy . \tag{20}$$

We are going to prove:

(21) there exists a function h of $L^1((0, \bar{M}))$ such that

$$\frac{1}{[(v^k)'(y)]^{p-1}} \leqslant h(y) \qquad \text{a.e. } y \in (0, \bar{M})$$

$$(v^k)'(y) \xrightarrow[(k \to +\infty)]{} v'(y) \qquad \text{a.e. } y \in (0, \bar{M}) . \tag{22}$$

Clearly (18) follows from (19), (20), (21) and (22).

PROOF OF (21) AND (22). Let

$$C =]0, \overline{M}[- \Big(\bigcup_{k\in\mathbf{N}} V(\overline{u}^k) \cup V(\overline{u}) \bigcup_{k\in\mathbf{N}} A(\varepsilon^k)\Big).$$

$[0, M] - C$ is negligible. Let $y \in C$. Using appendix 4 we see that v^k is differentiable in y and:

$$(v^k)'(y) = \sum_{x\in(\overline{u}^k)^{-1}(y)} \frac{1}{|(\overline{u}^k)'(x)|}$$

(remark: since $y \in C$, $(\overline{u}^k)^{-1}(y)$ is a finite set)

Then, using the convexity of t^{1-p} we have

$$\frac{1}{[(v^k)'(y)]^{p-1}} \leqslant \sum_{x\in(\overline{u}^k)^{-1}(y)} |(\overline{u}^k)'(x)|^{p-1}. \tag{23}$$

Let

$$h^k(y) = \sum_{x\in\overline{u}^{k-1}(y)} |(\overline{u}^k)'(x)|^{p-1}$$

On $[a_i, a_{i+1}]$ $\overline{u}^k$ is monotone; let θ_i^k be the unique function from $\overline{u}^k([a_i, a_{i+1}]) \cap C$ into $[a_i, a_{i+1}]$ such that:

$$\overline{u}^k \circ \theta_i^k = Id_{C\cap\overline{u}^k([a_i, a_{i+1}])}.$$

We have:

$$\int_a^{a_{i+1}} |(\overline{u}^k)'(x)|^p\, dx = \int_{\overline{u}^k([a_i, a_{i+1}])\cap C} |(\overline{u}^k)'(\theta_i^k(y))|^{p-1}\, dy.$$

Then it is easy to see that h^k is a measurable function and that

$$\int_0^{\overline{M}} h^k(y)\, dy = |\overline{u}^k|^p$$

but $(\overline{u}^k)' \to \overline{u}'$ in $L^p(\mathbb{R})$ when $k \to +\infty$, and thus

$$\int_0^{\overline{M}} h^k(y)\, dy \to |\overline{u}|^p \qquad (k \to +\infty).$$

Using Fatou's lemma we obtain

$$\int_0^{\overline{M}} \underline{\lim}_{k}\, h^k(y)\, dy \leqslant |\overline{u}|^p. \tag{24}$$

Let

$$h(y) = \sum_{x \in \overline{u}^{-1}(y)} |\overline{u}'(x)|^{p-1} .$$

We are going to prove that

(25) if $y \in C$, $(\overline{u}^k)^{-1}(y) \subset \overline{u}^{-1}(y)$

and if $x \in (\overline{u}^k)^{-1}(y)$ then $\overline{u}'(x) = (\overline{u}^k)'(x)$

(26) if $y \in C$, for k sufficiently large we have

$$(\overline{u}^k)^{-1}(y) = \overline{u}^{-1}(y) .$$

Before proving (25) and (26) we are going to deduce (21) and (22) from (25) and (26).

Using (25) we have:

$$h^k(y) \leqslant h(y)$$

Using (25) and (26) $h^k(y) \to h(y)$ $(k \to +\infty)$ $\forall y \in C$.

Using (24)

$$\int_0^M h(y)\,dy \leqslant |\overline{u}|^p$$

which gives (20).

(22) follows from (25), (26) and appendix 4.

PROOF OF (25). Let x be in $(\overline{u}^k)^{-1}(y)$, $a_i < x < a_{i+1}$; let us assume that, for example, $\overline{u}(a_i) < \overline{u}(a_{i+1})$ (the proof in the case $\overline{u}(a_i) > \overline{u}(a_{i+1})$ would be nearly the same).

Let z be in $[a_i, a_{i+1}]$

$$\overline{u}^k(z) = \operatorname*{Max}_{y \in [a_i, z]} \overline{u}(y) .$$

We have $\overline{u}^k(x) \geqslant \overline{u}(x)$; but if $\overline{u}^k(x) > \overline{u}(x)$ it is easy to see that $(\overline{u}^k)'(x) = 0$ in contradiction with $y \in C$ therefore $\overline{u}^k(x) = \overline{u}(x)$. We recall that $\overline{u}$ and $\overline{u}^k$ are differentiable in x (since $y \in C$). Let $\tau > 0$ with $x + \tau < a_{i+1}$

$$\frac{\overline{u}(x+\tau) - \overline{u}(x)}{\tau} \leqslant \frac{\overline{u}^k(x+\tau) - \overline{u}^k(x)}{\tau} \to (\overline{u}^k)'(x)$$

therefore

(27) $$\overline{u}'(x) \leqslant (\overline{u}^k)'(x) .$$

Let

$$\tau_n \to 0 \qquad \tau_n > 0 \qquad \text{with } x + \tau_n < a_{i+1}$$

$$\overline{u}^k(x + \tau_n) = \overline{u}(x + \bar{\tau}_n) \qquad \text{with } 0 \leqslant \bar{\tau}_n \leqslant \tau_n$$

$$0 \leqslant \frac{\overline{u}^k(x_n + \tau) - \overline{u}^k(x)}{\tau_n} = \frac{\overline{u}(x + \bar{\tau}_n) - \overline{u}(x)}{\tau \bar{\tau}_n} \cdot \frac{\bar{\tau}_n}{\tau_n}$$

$$\frac{\overline{u}^k(x + \tau_n) - \overline{u}^k(x)}{\tau_n} \to (\overline{u}^k)'(x) > 0 \, .$$

Hence:

$$(\overline{u}^k)'(x) \leqslant \overline{u}'(x) \, . \tag{28}$$

From (27) and (28) we deduce

$$(\overline{u}^k)'(x) = \overline{u}'(x) \, .$$

Thus (25) is proved.

PROOF OF (26). Let $y \in C$ and $x \in \overline{u}^{-1}(y)$; we are going to prove that if k is sufficiently large then $x \in (\overline{u}^k)^{-1}(y)$. Since $\overline{u}^{-1}(y)$ is a finite set this will prove (26). u is derivable in x and $\overline{u}'(x) \neq 0$ (since $y \in C$). Let us assume that for example $\overline{u}'(x) > 0$ (the proof in the case $\overline{u}'(x) < 0$ would be nearly the same). Let $\eta > 0$ such that:

$$z \in [x - \eta, x) \Rightarrow \overline{u}(z) < \overline{u}(x)$$

$$z \in (x, x + \eta] \Rightarrow \overline{u}(z) > \overline{u}(x) \, .$$

Let

$$\delta = \operatorname{Min}\big(\overline{u}(x + \eta) - \overline{u}(x), \overline{u}(x) - \overline{u}(x - \eta)\big) \, .$$

Let us assume that

$$|\varepsilon^k| < \frac{\delta}{2} \, . \tag{29}$$

Let a_i^k be the sequence used for definition of $\overline{u}^k$ (see above definition of a_i). It is easy to see, using (29), that if

$$a_i^k < x < a_{i+1}^k$$

then

$$x - \eta < a_i^k < a_{i+1}^k < x + \eta \, .$$

Then

$$\overline{u}^k(a_i^k) < \overline{u}(x) < \overline{u}^k(a_{i+1}^k)$$

and

$$\overline{u}^k(x) = \underset{v \in [a_i^k, x]}{\mathrm{Max}}\, \overline{u}(x)\,.$$

(26) is proved, and so (10) is proved.

PROOF OF (11). Now ε is fixed.
Using (15) with $\overline{u}_n$ instead of $\overline{u}$ we have

$$|P_\varepsilon \overline{u}_n| \leqslant |\overline{u}_n|\,.$$

Let

$$v_n(y) = -\ \mathrm{meas}\,\{x | \overline{u}_n(x) \geqslant y\}$$

$$w_n(y) = -\ \mathrm{meas}\,\{x | P_\varepsilon \overline{u}_n(x) \geqslant y\}\,.$$

Let $D =]0, m[- \big(V(\overline{u}_n) \cup V(P_\varepsilon \overline{u}_n) \cup A(\varepsilon)\big)$; $[0, m] \setminus D$ is negligible. Using Appendix 3, we know that, if $y \in D$, then v_n and w_n are differentiable in y and:

$$v_n'(y) = \sum_{x \in \overline{u}_n^{-1}(y)} \frac{1}{|(\overline{u}_n)'(x)|}$$

$$w_n'(y) = \sum_{x \in (P_\varepsilon \overline{u}_n)^{-1}(y)} \frac{1}{|(P_\varepsilon \overline{u}_n)'(x)|}\,.$$

But (see the proof of (25))

$$(P_\varepsilon \overline{u}_n)^{-1}(y) \subset \overline{u}_n^{-1}(y)$$

and if $x \in (P_\varepsilon \overline{u}_n)^{-1}(y)$, we have $(\overline{u}_n)'(x) = (P_\varepsilon \overline{u}_n)'(x)$ therefore

(30) $$w_n'(y) \leqslant v_n'(y)\,.$$

But (see appendix 3):

$$|\overline{u}_n^*| = \int_0^M \frac{2^p}{\big(v_n'(y)\big)^{p-1}}\, dy$$

and

$$|(P_\varepsilon \overline{u}_n)^*| = \int_0^M \frac{2^p}{\big(w_n'(y)\big)^{p-1}}\, dy\,.$$

Then (11) follows from (30).

PROOF OF (12). First we show that:

$$\lim_{n\to+\infty} a_1^n = a_1 . \tag{31}$$

PROOF OF (31). We have $y_n(a_1^n) = m$ and

$$a_1 - \delta_0 \leqslant a_1^n \leqslant a_l + \delta_0 .$$

We extract from the sequence a_1^n a convergent subsequence, (we shall also note a_1^n) such that:

$$a_1^n \to b \quad \text{ when } n \to +\infty .$$

We have $u(b) = m$.

Since $m \notin V(u)$, $\forall\delta > 0$ there exists x such that

$$u(x) > m \quad \text{ and } |b - x| < \delta .$$

Hence

$$a_1 \leqslant b . \tag{32}$$

But $u(a_1) = m$ and $m \notin V(u)$ then, $\forall\delta > 0$, there exists x' such that:

$$u(x') > m \quad \text{ and } |a_1 - x'| < \delta .$$

We have:

$$\lim_{n\to+\infty} u_n(x') = u(x') .$$

Thus for n sufficiently large

$$u_n(x') > m$$

and therefore (if $\delta < \delta_0$):

$$a_1^n \leqslant x' \leqslant a_1 + \delta .$$

Then:

$$b \leqslant a_1 . \tag{33}$$

Clearly (31) follows from (32) and (33).

Let l_n be the number of terms of sequence a_i^n.

We assume that:

$$A(\varepsilon) \cap V(\bar{u}) = \emptyset .$$

Using the arguments of the Proof of (32) it is easy to prove that there exists n_0

such that

$$n \geqslant n_0 \Rightarrow l_n = l$$

and

$$\lim_{n \to +\infty} a_i^n = a_i$$

and, then, there exists n_1 such that:

$$n \geqslant n_1 \Rightarrow l_n = 1 \quad \text{ and } \bar{u}_n(a_i^n) = \bar{u}(a_i) \;\; \forall i \in \varepsilon[1, l] .$$

Let x be a real number with $a_i < x < a_{i+1}$; for n sufficiently large, $a_i^n < x < a_{i+1}^n$,

$$\bar{u}_n(a_i^n) = \bar{u}(a_i) \quad \text{ and } \quad \bar{u}_n(a_{i+1}^n) = \bar{u}(a_{i+1}) .$$

Now using the definitions of $P_\varepsilon \bar{u}_n$ and $P_\varepsilon \bar{u}$ it is easy to see that:

$$P_\varepsilon \bar{u}_n(x) \to P_\varepsilon \bar{u}(x)$$

and the same method yields: if $x > a_1$ or $x < a_1$ then:

$$P_\varepsilon \bar{u}_n(x) = 0 = P_\varepsilon \bar{u}(x)$$

for n sufficiently large but (see (15) with $\bar{u}_n$ instead of $\bar{u}$) $P_\varepsilon \bar{u}_n'$ is bounded in $W^{1,p}(\mathbb{R})$. (Let us recall that $\|P_\varepsilon \bar{u}_n\|_\infty \leqslant \bar{M}$ and $\operatorname{Supp} P_\varepsilon u_n \subset [a_1 - \delta_0, a_l + \delta_0]$).
Then:

$$P_\varepsilon u \xrightarrow[n \to +\infty]{} P_\varepsilon \bar{u} \quad \text{ in } W^{1,p}(\mathbb{R}) .$$

For $i \in [1, 1]$ and γ in $W^{1,p}(\mathbb{R})$, let $F_i(\gamma)$ be the function of $W^{1,p}(\mathbb{R})$ defined by:

$$F_i(\gamma)(x) = \operatorname{Max}\Big(\operatorname{Min}\Big(\gamma(x), \sum_{j=0}^{i} \varepsilon_j\Big), \sum_{j=0}^{i-1} \varepsilon_i\Big) - \sum_{j=0}^{i-1} \varepsilon_j ,$$

with the convention $\varepsilon_0 = 0$. We have:

$$\Phi_c(P_\varepsilon \bar{u}_n) = \sum_{i=1}^{l} \Phi_c\big(F_i(P_\varepsilon \bar{u}_n)\big) .$$

and

$$F_i(P_\varepsilon \bar{u}_n) \rightharpoonup F_i(P_\varepsilon \bar{u}) \quad \text{ in } W^{1,p}(\mathbb{R}) .$$

Then using appendix 3 we see that (19) follows from the following lemma:

Lemma. *Let T and L be two positive real numbers; let k be a positive integer and $(\alpha_n^1, \alpha_n^2, \dots, \alpha_n^k)$ be a sequence of elements in $\big(W^{1,p}((0,T))\big)^k$ such that for each i in $[1,k]$:*

$$\alpha_n^i \text{ is nondecreasing}$$

$$\alpha_n^i(0) = 0 \qquad \alpha_n^i(T) = L$$

$$\alpha_n^i \xrightarrow[n\to+\infty]{} \alpha^i \quad \text{in } W^{1,p}((0,T)) .$$

Let

$$\beta_n^i(y) = -\operatorname{meas}\{x \in [0,T] | \alpha_n^i(x) \geq y\} ,$$

$$\beta^i(y) = -\operatorname{meas}\{x \in [0,T] | \alpha^i(x) \geq y\} .$$

Then

$$\sum_{i=1}^k \int_0^L \frac{dy}{(\beta_i'(y))^{p-1}} - c\int_0^L \frac{2^p}{\left(\sum_{i=1}^k \beta_i'(y)\right)^{p-1}}\, dy \leq \varliminf_{n\to+\infty}\left(\sum_{1=i}^k \int_0^L \frac{dy}{((\beta_i^n)'(y))^{p-1}} - c\int_0^L \frac{2^p\, dy}{\left(\sum_{i=1}^k (\beta_i^n)'(y)\right)^{p-1}}\right).$$

Proof of the lemma. Let m_n^i be the unique positive Radon measure on $[0, L]$ such that:

$$0 \leq y < y' \leq L \Rightarrow m_n^i([y, y'[) = \beta_n^i(y') - \beta_n^i(y) , \qquad m_n^i([0,L]) = T .$$

Let m^i be the unique positive Radon measure on $[0, L]$ such that:

$$0 \leq y < y' \leq L \Rightarrow m^i([y, y'[) = \beta^i(y') - \beta^i(y) , \qquad m^i([0,L]) = T .$$

Let φ be a continuous function from $[0, L]$ into R; we have:

$$\int_{[0,L]} \varphi(y)\, dm_n^i(y) = \int_0^T \varphi(\alpha_n^i(x))\, dx$$

$$\int_{[0,L]} \varphi(y)\, dm^i(y) = \int_0^T \varphi(\alpha^i(x))\, dx .$$

Since $\alpha_n^i \to \alpha$ in $W^{1,p}((0,T))$, $\alpha_n^i(x) \to \alpha^i(x)$, $\forall x \in [0, T]$.

Hence

$$\int_{[0,L]} \varphi(y)\, dm_n^i(y) \to \int_0^T \varphi(\alpha^i(x))\, dx$$

and:

$$\lim_{n\to+\infty} \int_{[0,L]} \varphi(y)\, dm_n^i(y) = \int_{[0,L]} \varphi(y)\, dm^i(y).$$

But

$$m_n^i = (\beta_n^i)'(y)\, dy + \nu_n^i \qquad m^i = (\beta^i)'(y)\, dy + \nu^i$$

where ν_n^i and dy are mutually singular, and, ν^i and dy are mutually singular. Therefore the lemma follows from appendix 6.

Part B. Here we assume that $c > 1/2^p$ and we construct a sequence u_n such that $u_n \rightharpoonup u$ in $W_+^{1,p}(\mathbb{R})$ and $\Phi_c(u) > \lim \Phi_c(u_n)$.

It follows from appendix 5 that there exist four real numbers t_1, t_2, s_1, s_2 such that:

$$0 < t_1, \quad 0 < t_2, \quad 0 < s_1, \quad 0 < s_2$$

and:

$$(34) \qquad \frac{1}{[(t_1+t_2)/2]^{p-1}} + \frac{1}{[(s_1+s_2)/2]^{p-1}} - \frac{2^p c}{[(s_1+t_1+s_2+t_2)/2]^{p-1}}$$
$$> \frac{1}{2}\left(\frac{1}{t_1^{p-1}} + \frac{1}{s_1^{p-1}} - \frac{2^p c}{(t_1+s_1)^{p-1}} + \frac{1}{t_2^{p-1}} + \frac{1}{s_2^{p-1}} - \frac{2^p c}{(t_2+s_2)^{p-1}}\right)$$

Let d_n and e_n be the functions from $]0, 1]$ into $\mathbb{R}$ defined by:

for x in $]0, 1]$ with $k/2^n < x \leqslant (k+1)/2^n$ where k is an integer we set:

— when k is odd: $d_n(y) = s_1,\ e_n(y) = -t_1$

— when k is even: $d_n(y) = s_2,\ e_n(y) = -t_2$.

Let

$$D_n(y) = \int_y^1 d_n(\tau)\, d\tau \qquad \text{for } y \in [0, 1]$$

$$E_n(y) = \int_y^1 e_n(\tau)\, d\tau \qquad \text{for } y \in [0, 1].$$

We have

$$D_n(0) = \frac{s_1 + s_2}{2} \qquad E_n(0) = -\frac{t_1 + t_2}{2}$$

and

$$(35) \qquad \begin{cases} \lim\limits_{n\to+\infty} D_n(y) = \dfrac{s_1 + s_2}{2}(1-y) & \forall y \in [0,1] \\ \lim\limits_{n\to+\infty} E_n(y) = -\dfrac{t_1 + t_2}{2}(1-y) & \forall y \in [0,1]. \end{cases}$$

We are going to define u_n:

when $x \geqslant (s_1 + s_2)/2$ let $u_n(x) = 0$

when $0 \leqslant x < (s_1 + s_2)/2$ let $u_n(x)$ be the only real number such that

$$D_n(u_n(x)) = x$$

when $-(t_1 + t_2)/2 < x < 0$ let $u_n(x)$ be the only real number such that

$$E_n(u_n(x)) = x$$

when $x < -(t_1 + t_2)/2$ let $u_n(x) = 0$.

It is easy, using (35), to prove that:

$$(36) \qquad \lim_{n\to+\infty} u_n(x) = u(x)$$

with

$$u(x) = 1 - \frac{2}{s_1 + s_2}x \quad \text{when } 0 \leqslant x \leqslant \frac{s_1 + s_2}{2}$$

$$u(x) = 1 + \frac{2}{t_1 + t_2}x \quad \text{when } -\frac{t_1 + t_2}{2} \leqslant x \leqslant 0$$

$$u(x) = 0 \quad \text{when } x > \frac{s_1 + s_2}{2} \quad \text{or } x < -\frac{t_1 + t_2}{2}.$$

We have

$$(37) \qquad |u_n|^p = \frac{1}{2}\left\{\left(\frac{1}{s_1^{p-1}} + \frac{1}{s_2^{p-1}}\right) + \left(\frac{1}{t_1^{p-1}} + \frac{1}{t_2^{p-1}}\right)\right\}.$$

Then u_n is bounded in $W^{1,p}(\mathbb{R})$ and using (36)

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\mathbb{R}) \text{ when } n \to +\infty.$$

An easy computation gives:

$$|u_n^*|^p = \frac{1}{2}\left\{\frac{2^p}{(s_1+t_1)^{p-1}} + \frac{2^p}{(s_2+t_2)^{p-1}}\right\} \tag{38}$$

$$|u|^p = \frac{1}{[(s_1+s_2)/2]^{p-1}} + \frac{1}{[(t_1+t_2)/2]^{p-1}} \tag{39}$$

$$|u^*|^p = \frac{1}{\left((s_1+s_2)/2 + (t_1+t_2)/2\right)^{p-1}}. \tag{40}$$

Using (34), (37), (38), (39) and (40) we have

$$\Phi_c(u) > \lim_{n\to+\infty} \Phi_c(u_n)\,.$$

Appendix 0.

Let $L^p_+(\mathbf{R})$ be the set of nonnegative functions of $L^p(\mathbf{R})$. Then we have the following (for $1 < p < +\infty$).

PROPOSITION. *The rearrangement is a continuous mapping from $L^p_+(\mathbf{R})$ into $L^p_+(\mathbf{R})$ (for the strong topologies).*

PROOF. First we recall that, if $u \in L^p_+(\mathbf{R})$, $u^* \in L^p_+(\mathbf{R})$ and:

$$\int (u^*)^p\, dx = \int u^p e\, dx$$

(see [5]).

Let $(u_n)_{i\in\mathbf{N}}$ be a sequence of functions of $L^p_+(\mathbf{R})$ such that

$$u_n \to u \quad \text{in } L^p(\mathbf{R})$$

We are going to prove that

$$u_n^* \to u^* \quad \text{in } L^p(\mathbf{R})\,.$$

Obviously we may assume that

$$u_n(x) \to u(x) \quad \text{a.e. } x \in \mathbf{R}$$

and

$$\exists h \in L^p_+(\mathbf{R}) \text{ such that } u_n(x) \leqslant h(x) \text{ a.e. } x \in \mathbf{R}\,.$$

Let f_n, f and g be the following functions

$$\begin{aligned} f_n(x) &= 1 \quad \text{if } u_n(x) > t \\ f_n(x) &= 0 \quad \text{if } u_n(x) \leqslant t \\ f(x) &= 1 \quad \text{if } u(x) > t \\ f(x) &= 0 \quad \text{if } u(x) \leqslant t \\ g(x) &= 1 \quad \text{if } h(x) > t \\ g(x) &= 0 \quad \text{if } h(x) \leqslant t\,. \end{aligned}$$

Then $f_n \to f$ a.e., $g \in L^1(\mathbf{R})$, $f_n \leqslant g$ a.e.

Therefore

$$\int f_n \to \int f\,.$$

Thus

$$\text{meas}\,\{x | u_n(x) > t\} \to \text{meas}\,\{x | u(x) > t\}\,.$$

Then the proposition follows easily from the definition of u_n^* and u^*, from:

$$\int (u_n^*)^p\,dx = \int u_n^p\,dx \to \int u^p\,dx - \int (u^*)^p\,dx$$

and

$$u_n^* \leqslant h^*\,.$$

Appendix 1.

Let u be an absolutely continuous function from $\mathbf{R}$ into $\mathbf{R}$. Let

$V'(0) = \{y|$ there exists x in $\mathbf{R}$ such that $u(x) = y$ and either u is not derivable in x or u is derivable in x and $u'(x) = 0\}$.

Then;

(41) $V(u)$ is negligible (for the Lebesgue measure).

PROOF. Let A be a measurable set; we are going to prove that:

$$\lambda^*\big(u(A)\big) \leqslant \int_A |u'(t)|\,dt \tag{42}$$

where

$$\lambda^*(B) = \text{Inf}\,\{\lambda(\Omega) | \Omega \text{ is an open set of } \mathbf{R} \text{ such that } B \subset \Omega\}$$

(λ is the Lebesgue measure).

Property (41) follows easily from (42) by taking

$A = \{x|u$ is not derivable in $x\} \cup \{x|u$ is derivable in x and $u'(x) = 0\}$.

Let $\varepsilon > 0$. There exists $\eta > 0$ such that:

(43) for any measurable set E such that $\lambda(E) < \eta$ then $\int_E |u'(\tau)|\,d\tau < \varepsilon$.

There exist two sequences of real numbers $(\alpha_i)_{i\in\mathbf{N}}$, $(\beta_i)_{i\in\mathbf{N}}$ such that

$$\alpha_i < \beta_i \qquad \forall i \in \mathbf{N}$$

$$]\alpha_i, \beta_i[\cap]\alpha_j, \beta_j[= \emptyset \qquad \text{if } i \neq j$$

and:

(44) $A \subset \Omega$ and $\lambda(\Omega - A) < \eta$ where $\Omega = \bigcup_{i\in\mathbf{N}}]\alpha_i, \beta_i[$.

Clearly

$$u(A) \subset \bigcup_{i\in\mathbf{N}} u(]\alpha_i, \beta_i[)$$

$$\lambda^*(u(A)) \leqslant \sum_{i\in\mathbf{N}} \lambda^*(u(]\alpha_i, \beta_i[))$$

but

$$\lambda^*(u(]\alpha_i, \beta_i[)) = \lambda(u(]\alpha_i, \beta_i[)) \leqslant \int_{\alpha_i}^{\beta_i} |u'(\tau)|\,d\tau$$

$$\lambda^*[u(A)] \leqslant \int_\Omega |u'(\tau)|\,d\tau = \int_A |u'(\tau)|\,d\tau + \int_{\Omega - A} |u'(\tau)|\,d\tau$$

we use (43) and (44):

$$\lambda^*(u(A)) \leqslant \int_A |u'(\tau)|\,d\tau + \varepsilon .$$

Hence (42) follows.

Appendix 2.

Let u be in $W^{1,p}((0, T))$; let

$$v(x) = \operatorname*{Max}_{y\in[0,x]} u(y)$$

then:

(45) v is in $W^{1,p}((0, T))$ and $|v|^p = \int_0^T v'(t)|u'(t)|^{p-1}\,dt$.

PROOF OF (45).

(45) is of course true when u is a polynomial function; let u_n be a sequence of polynomial functions such that:

$$u_n \to u \quad \text{in } W^{1,p}\big((0, T)\big) .$$

Let

$$v_n(x) = \operatorname*{Max}_{y\in[0,x]} u_n(y) .$$

We have

$$\lim_{n\to+\infty} v_n(x) = v(x) \qquad \forall x \in [0, T] . \tag{46}$$

Using (45) for v_n we have

$$|v_n| \leqslant |u_n| .$$

Then v_n is bounded in $W^{1,p}\big((0, T)\big)$; using (46) we have:

$$v \in W^{1,p}\big((0, T)\big) \text{ and } v_n \rightharpoonup v \text{ in } W^{1,p}\big((0, T)\big) \text{ when } n \to +\infty .$$

Let x be a point of $(0, T)$ such that v and u are differentiable in x. We are going to prove that:

$$v'(x)^p = v'(x)|u'(x)|^{p-1} . \tag{47}$$

This will prove (45).
Note that since v is nondecreasing, $v'(x) \geqslant 0$; if $v'(x) = 0$ (47) is of course true. Now let us assume that $v'(x) > 0$. We shall prove that $v(x) = u(x)$. Clearly $v(x) \geqslant u(x)$. Assume by contradiction that $v(x) > u(x)$; then there exists $\varepsilon > 0$ such that

$$[x, x + \varepsilon] \subset [0, T]$$

and

$$z \in [x, x + \varepsilon] \Rightarrow u(z) < v(x) .$$

Therefore

$$z \in [x, x + \varepsilon] \Rightarrow v(z) = v(x)$$

and so $v'(x) = 0$.

A contradiction with $v'(x) > 0$.

We have proved that $v(x) = u(x)$. Since $v \geqslant u$ and $v(x) = u(x)$, we have (47).

Appendix 3.

This appendix is due to T. Gallouët.

Let u be a nondecreasing function in $W^{1,p}((0,T))$ such that $u(0)=0$ and $u(T)=L$.

Let v the function from $[0,L]$ into $[-T,0]$ defined by

$$v(y) = -\operatorname{meas}\{x\in[0,T]|u(x)\geqslant y\};$$

v is a nondecreasing function and then derivable a.e. with $v'\geqslant 0$. Let $1/v'$ be the function from $[0,L]$ into $\mathbf{R}$ defined by:

$$\frac{1}{v'}(y) = \frac{1}{v'(y)} \qquad \text{if } v \text{ is differentiable in } y \text{ with } v'(y)\neq 0$$

$$\frac{1}{v'}(y) = \alpha \qquad \text{elsewhere } (\alpha\in\mathbf{R}^+\ \alpha \text{ is fixed}).$$

Then we have:

$$\int_0^L \left(\frac{1}{v'}\right)^{p-1} dy = |u|^p. \tag{48}$$

PROOF OF (48). We have

$$\{x\in[0,T]|u(x)\geqslant y\} = [\operatorname{Min} u^{-1}(y), T] \qquad \text{for } y\in[0,L].$$

Then

$$v(y) = -\left(T-\operatorname{Min} u^{-1}(y)\right) \tag{49}$$

and therefore:

$$u(v(y)+T) = y. \tag{50}$$

Since u is absolutely continuous and nondecreasing, we have:

$$\int_0^L \left(\frac{1}{v'}(y)\right)^{p-1} dy = \int_0^T \left(\frac{1}{v'}\right)^{p-1}\left(u(x)\right)\cdot u'(x)\,dx. \tag{51}$$

Let x be in $]0,T[$ such that u is derivable in x with $u'(x)\neq 0$.

We have:

$$x'<x \Rightarrow u(x')<u(x)$$

$$x'>x \Rightarrow u(x')>u(x).$$

Let $y=u(x)$ and h be such that $y+h$ and $y-h$ are in $(0,T)$. Using (50) we have:

$$\frac{v(y+h)-v(y)}{y+h-y}=\frac{u(v(y+h))-u(v(y))}{v(h+h)-v(y)};$$

but using (49) and (52) it is easy to see that

$$\lim_{h\to 0} v(y+h)=v(y)\,.$$

Then v is differentiable in y and $v'(y)=1/u'(x)\neq 0$. Then using (51) we have (48).

Appendix 4.

Let $u\in W^{1,p}(\mathbb{R})$, $u\geqslant 0$; let:

$$v(y)=-\operatorname{meas}\{x|u(x)\geqslant y\}.$$

If $y\notin V(u)$ and $y\in u(\mathbb{R})$ then v is derivable in y and:

$$v'(y)=\sum_{x\in u^{-1}(y)}\frac{1}{|u'(x)|}\,. \tag{53}$$

Proof of (53). First we remark that, since $y\notin V(u)$, $u^{-1}(y)$ has only a finite number of elements. On the other hand the number of elements of $u^{-1}(y)$ is even since $u\to 0$ at infinity. For simplicity we shall assume that $u^{-1}(y)$ has only two elements x_1, x_2 with $x_1<x_2$ and we shall prove only the right-differentiability. We have $u'(x_1)>0$, $u'(x_2)<0$.
Let $k>0$ be such that $u^{-1}(y+k)\neq\emptyset$ (if k is sufficiently small $u^{-1}(y+k)\neq\emptyset$).
Let

$$x_1(k)=\operatorname{Min}\{x|u(x)=y+k\}$$
$$x_2(k)=\operatorname{Max}\{x|u(x)=y+k\}\,.$$

We have

$$\lim_{k\to 0^+} x_i(k)=x_i \qquad \forall i\in\{1,2\}$$

and

$$u(z)\geqslant y+k \Rightarrow z\in[x_1(k),x_2(k)]\,.$$

Therefore $\operatorname{meas}\{x|u(x)\geqslant y+k\}\leqslant x_2(k)-x_1(k)$.
We have

$$u(x_i(k))=y+k=u(x_i)+u'(x_i)(x_i(k)-x_i)+(x_i(k)-x_i)\,\varepsilon_i(k)$$

with

$$\lim_{k\to 0^+} \varepsilon_i(k) = 0 \qquad \text{and } u(x_i) = y\,.$$

Thus:

$$\lim_{k\to 0^+} \frac{x_i(k) - x_i}{k} = \frac{1}{u'(x_i)}\,.$$

Therefore

$$\lim_{k\to 0^+} \frac{k}{v(y+k) - v(y)} \geqslant \frac{1}{u'(x_1)} - \frac{1}{u'(x_2)}\,. \tag{54}$$

Let

$$\bar{x}_1(k) = \mathrm{Max}\left\{x | u(x) = y + k \text{ et } x \leqslant \frac{x_1 + x_2}{2}\right\}$$

$$\bar{x}_2(k) = \mathrm{Min}\left\{x | u(x) = y + k \text{ et } x \geqslant \frac{x_1 + x_2}{2}\right\}$$

$\big(\bar{x}_i(k)$ is well defined if k is sufficiently small$\big)$.

We have

$$\lim_{k\to 0^+} \bar{x}_i(k) = x_i\,.$$

It is easy to see that if k is sufficiently small,

$$x \in [\bar{x}_1(k), \bar{x}_2(k)] \Rightarrow u(x) \geqslant y + k\,.$$

We have

$$\lim_{k\to 0^+} \bar{x}_i(k) = x_i\,.$$

as before we prove that

$$\lim_{k\to 0^+} \frac{\bar{x}_i(k) - x_i}{k} = \frac{1}{u'(x_i)}$$

and we have:

$$\mathrm{meas}\,\{x | u(x) \geqslant y + k\} \geqslant \bar{x}_2(k) - \bar{x}_1(k)\,.$$

Thus we have

$$\lim_{k\to 0^+} \frac{v(y+k) - v(y)}{k} \leqslant \frac{1}{u'(x_1)} - \frac{1}{u'(x_2)}\,. \tag{55}$$

Using (54) and (55) we have

$$\lim_{k\to 0^+} \frac{v(y+k) - v(y)}{k} = \frac{1}{|u'(x_1)|} + \frac{1}{|u'(x_2)|}\,.$$

Appendix 5.

Let d be a real number and let

$$\varphi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$$

$$(x_1, x_2, \ldots, x_n) \to \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{1}{x_i^{p-1}} - \frac{d}{\left(\displaystyle\sum_{i=1}^{n} x_i\right)^{p-1}} & \text{if } \forall i \ x_i > 0 \\ +\infty & \textit{elsewhere} \end{cases}$$

Then if $d \leqslant 1$ φ is convex and l.s.c. If $d > 1$ and $n = 2$ φ is not convex on $(\mathbb{R}^{+*})^n$.

PROOF. 1) $n = 2$.

φ is C^∞ on $(\mathbb{R}^{+*})^2$. Let $x_1 > 0$, $x_2 > 0$ we have:

$$\frac{\partial^2 \varphi}{\partial x_1^2} = p(p-1)\left\{\frac{1}{x_1^{p+1}} - \frac{d}{(x_1 + x_2)^{p+1}}\right\}$$

$$\frac{\partial^2 \varphi}{\partial x_2^2} = p(p-1)\left\{\frac{1}{x_2^{p+1}} - \frac{d}{(x_1 + x_2)^{p+1}}\right\}$$

$$\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = -p(p-1)\frac{d}{(x_1 + x_2)^{p+1}}$$

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} = p(p-1)\left\{\frac{1}{p_1^{p+1}} + \frac{1}{x_2^{p+1}} - \frac{2d}{(x_1 + x_2)^{p+1}}\right\} \geqslant 0 \quad \text{if } d \leqslant 1$$

$$\frac{\partial^2 \varphi}{\partial x_1^2} \cdot \frac{\partial^2 \varphi}{\partial x_2^2} - \left(\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}\right)^2 = p^2(p-1)^2\left\{\frac{(x_1 + x_2)^{p+1} - d(x_1^{p+1} + x_2^{p+1})}{x_1^{p+1} x_2^{p+1}(x_1 + x_2)^{p+1}}\right\} \geqslant 0 \quad \text{if } d \leqslant 1 .$$

Thus, if $d \leqslant 1$, φ is convex (and continuous) on $(\mathbb{R}^{+*})^2$; if $d > 1$ there exists $(x_1, x_2) \in (\mathbb{R}^{+*})^2$ such that

$$\left(\frac{\partial^2 \varphi}{\partial x_1^2} \frac{\partial^2 \varphi}{\partial x_2^2} - \left(\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}\right)^2\right)(x_1, x_2) < 0$$

and therefore φ is not convex on $(\mathbb{R}^{+*})^2$. We assume now $d \leqslant 1$. φ is convex on $(\mathbb{R}^{+*})^2$ and then φ is convex on $\mathbb{R}^2$. Iy is easy to see that φ is l.s.c. in (x_1, x_2) if $(x_1, x_2) \neq (0, 0)$. It remains to prove that φ is l.s.c. in $(0, 0)$.

We have

$$\varphi(x_1, x_2) \geqslant \frac{1}{x_1^{p-1}} \qquad \text{if } x_1 > 0$$

$$\varphi(x_1, x_2) = +\infty \quad \text{if } x_1 \leqslant 0 .$$

Thus if $(x_1^n, x_2^n) \to (0, 0)$ as $n \to +\infty$ we have

$$\lim_{n\to+\infty} \varphi(x_1^n, x_2^n) = +\infty = \varphi(0, 0) .$$

2) $n \geq 3$; we assume $d \leq 1$.

Since the mapping from $\mathbb{R}^n$ into $R \cup \{+\infty\}$ defined by:

$$(x_1 \dots x_n) \to \begin{cases} \left\{\left(\sum_{i=1}^{n} x_i\right)^{p-1}\right\}^{-1} & \text{if } x_i \geq 0 \ \sum_{i=1}^{n} x_i \neq 0 \\ +\infty & \text{elsewhere} \end{cases}$$

is convex l.s.c. We may assume that $d = 1$.

As for $n = 2$ it is easy to prove that φ is l.s.c. We are going to prove that φ is convex on $(w^{+*})^n$ by induction on n. We shall write φ_n instead of φ; we assume that φ_{n-1} is convex on $(\mathbb{R}^{+*})^{n-1}$.

Let

$$x = (x_1, x_2, \dots, x_n) \in (\mathbb{R}^{+*})^n$$

$$y = (y_1, y_2, \dots, y_n) \in (\mathbb{R}^{+*})^n$$

Let $t \in [0, 1]$, $\tilde{x} = (x_2, \dots, x_n)$, $\tilde{y} = (y_2, \dots, y_n)$

$$\varphi_n(tx + (1-t)y) = \varphi_2\left(t\left(x_1, \sum_{i=2}^{n} x_i\right) + (1-t)\left(y_1, \sum_{i=2}^{n} y_i\right)\right) + \varphi_{n-1}(t\tilde{x} + (1-t)\tilde{y})$$

φ_2 and φ_{n-1} are convex on $(\mathbb{R}^{+*})^2$ and $(\mathbb{R}^{+*})^{n-1}$; therefore

$$\begin{aligned} \varphi_n(tx + (1-t)y) &\leq t\varphi_2\left(x_1, \sum_{i=2}^{n} x_i\right) + (1-t)\varphi_2\left(y_1, \sum_{i=2}^{n} y_i\right) + t\varphi_{n-1}(\tilde{x}) + (1-t)\,\varphi_{n-1}(\tilde{y}) \\ &\leq t\varphi_n(x) + (1-t)\varphi_n(y) . \end{aligned}$$

Appendix 6.

Let K be a compact set of $\mathbb{R}$ and $C(K)$ be the set of the continuous functions from K into $\mathbb{R}$; for f in $C(K)$. Let

$$\|f\| = \operatorname*{Max}_{x\in K} |f(x)| .$$

$\| \ \|$ is a norm on $C(K)$; let M be the dual space of $C(K)$.

For m in M we have the decomposition:

$$m = f\,dx + \mu\,, \qquad f \in L^1(K)\,, \qquad \mu \in M$$

where $f\,dx$ and μ are mutually singular. We shall write:

$$f = R(m)\,.$$

Let F be the maping from M^n into $w \cup \{+\infty\}$ defined by:

$$F(m_1, m_2, \dots, m_n) = \int_K \varphi(Pm_1, \dots, Pm_n)\,dx$$

where φ is defined in the appendix 5. We assume (see the definition of φ) that $d \leqslant 1$.

Let $(m_{i,p})_{1\leqslant i\leqslant n,\, 0\leqslant p}$ be a sequence of elements in M^n such that:

$$\lim_{p\to+\infty} \int \theta\,dm_{i,p} = \int \theta\,dm_i \qquad \forall \theta \in C(K)\,,\ \forall i \in [1, n] \tag{56}$$

$$\int \theta\,dm_{i,p} \geqslant 0 \qquad \forall i \in [1, n]\ \forall p\ \ \forall \theta \in C(K)\ \text{ with } \theta \geqslant 0\,.$$

We are going to prove that:

$$F(m_1, \dots, m_n) \leqslant \varliminf_{p\to\infty} F(m_{1,p}, \dots, m_{n,p})\,. \tag{57}$$

Let

$$f_{i,p} = R(m_{i,p})\,, \qquad f_i = R(m_i)\,.$$

Let $r > 0$ and $f^r_{i,p}(x) = \operatorname{Min}\big(r, f_{i,p}(x)\big)$.

$$\|f^r_{i,p}\|_\infty \leqslant r\,.$$

Thus we can extract a subsequence which converges for the topology $\sigma(L^1, L^\infty)$ we shall denote also $f_{i,p}$ such a subsequence:

$$f^r_{i,p} \xrightarrow[(p\to+\infty)]{} g^r_i \qquad \sigma(L^1, L^\infty)\,.$$

Using appendix 5 we have:

$$\int_K \varphi(g^r_1, \dots, g^r_n)\,dx \leqslant \varliminf_{p\to+\infty} \int_K \varphi(f^r_{1,p}, \dots, f^r_{n,p})\,dx\,. \tag{58}$$

But it is easy to see that:

$$0 \leqslant \varphi(f^r_{1,p}, \dots, f^r_{n,p}) - \varphi(f_{1,p}, \dots, f_{n,p}) \leqslant \frac{n}{r^{p-1}}.$$

Thus

$$\int_K \varphi(f^r_{1,p}, \dots, f^r_{n,p})\, dx \leqslant F(m_{1,p}, \dots, m_{n,p}) + \frac{nL}{r^{p-1}} \tag{59}$$

where L is the Lebesgue measure of K.

Let $\theta \in C(K)$ with $\theta \geqslant 0$ and $i \in [1, n]$.

$$\int_K \theta g^r_i\, dx = \lim_{p\to+\infty} \int_K f^r_{i,p}\theta\, dx \leqslant \lim_{p\to+\infty} \int_K \theta\, dm_{i,p} = \int_K \theta\, dm_i .$$

Therefore

$$g^r_i \leqslant f_i .$$

But

$$x_i \leqslant x'_i \quad \forall i \in [1, n] \Rightarrow 0 \leqslant \varphi(x'_1, \dots, x'_p) \leqslant \varphi(x_1, \dots, x_p).$$

Hence

$$\int_K \varphi(f_1, \dots, f_n)\, dx \leqslant \int_K \varphi(g^r_1, \dots, g^r_n)\, dx . \tag{60}$$

Using (58), (59) and (60) we have, for every r in $\mathbb{R}^{+*}$.

$$F(m_1, \dots, m_n) \leqslant \varliminf_p F(m_{1,p}, \dots, m_{n,p}) + \frac{nL}{r^{p+1}}.$$

It gives (57).

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