The continuous classical Heisenberg ferromagnet equation with in-plane asymptotic conditions. II. IST and closed-form soliton solutions.

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Abstract

A new, general, closed-form soliton solution formula for the classical Heisenberg ferromagnet equation with in-plane asymptotic conditions is obtained by means of the Inverse Scattering Transform (IST) technique and the matrix triplet method. This formula encompasses the soliton solutions already known in the literature as well as a new class of soliton solutions (the so-called multipole solutions), allowing their classification and description. Examples from all classes are provided and discussed.

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This paper is dedicated to Prof Tommaso Ruggeri, on the occasion of his 70^{th} *birthday.*

1 Introduction

In this paper we show how to construct a formula containing all the reflectionless solutions of the classical, continuous Heisenberg ferromagnet chain equation [1–4],

$$\boldsymbol{m}_t = \boldsymbol{m} \wedge \boldsymbol{m}_{xx}, \tag{1.1a}$$

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to which we impose the in-plane asymptotic condition

$$\boldsymbol{m}(x) \to \cos(\gamma)\boldsymbol{e}_1 - \sin(\gamma)\boldsymbol{e}_2 \text{ as } x \to \pm\infty\,, \tag{1.1b}$$

where $\gamma \in [0, 2\pi)$ is a constant angle, as discussed in [5]. Here

$$\boldsymbol{m} : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^2\,, \quad \boldsymbol{m}(x,t) = \sum_{j=1}^{3} m_j(x,t)\,\boldsymbol{e}_j \tag{1.2}$$

is the magnetization vector at position x and time t, where the vectors $\boldsymbol{e}_j$, $j = 1, 2, 3$, are the standard Cartesian basis vectors for $\mathbb{R}^3$, $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ and then $\|\boldsymbol{m}(x,t)\| = 1$.

After the recent, first enucleation and experimental observation in a nano-contact spin-torque oscillator device of magnetic-droplet solitons [6–14], following their theoretical prediction [15–23], it has been theoretically shown in [24] how, as an extended magnetic thin film is reduced to a nano-wire with a nano-contact of fixed size at its center, the observed excited modes undergo transitions from a fully localized two-dimensional droplet into a pulsating one-dimensional droplet. This result has contributed to renew the interest in the study of low-dimensional magnetic solitons as a tool for better understanding the physics of ferromagnetic systems at the nano-meter length scale.

In this spirit, the present work aims at extending the analysis carried out in [25] for the classical, continuous Heisenberg ferromagnet equation with perpendicular ("easy-axis") asymptotic conditions, $\boldsymbol{m}(x) \to \boldsymbol{e}_3$ as $x \to \pm\infty$, by constructing a new, general formula which generates all reflectionless solutions of (1.1a) under condition (1.1b), allowing their classification.

Special soliton solutions of (1.1a) with (1.1b) have been also recently constructed by means of the method of the Darboux transformation [26, 27].

In the present work, to reach our goal, that is, to find a general formula for the soliton solutions of (1.1a) satisfying condition (1.1b), we apply the Inverse Scattering Transform (IST) [28–30] and the matrix triplet method [31–35] to (1.1a). For the sake of clarity let us briefly recall how the IST and the matrix triplet method work.

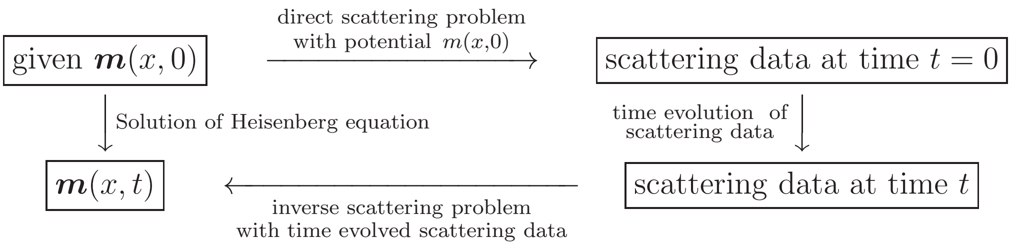
In the first part of this work [5], we have remarked that (1.1a) admits the following Lax pair representation

$$\begin{cases} V_x = \boldsymbol{A}\,V = [i\lambda(\boldsymbol{m}\cdot\boldsymbol{\sigma})]\,V \\[2ex] V_t = \boldsymbol{B}\,V = [-2i\lambda^2(\boldsymbol{m}\cdot\boldsymbol{\sigma}) - i\lambda(\boldsymbol{m}\wedge\boldsymbol{m}_x\cdot\boldsymbol{\sigma})]\,V\,, \end{cases} \tag{1.3}$$

It is well-known [28–30] that the knowledge of the Lax pair for (1.1a) assures that the Inverse Scattering Transform (IST) can be applied to solve the initial-value problem [2, 3],

$$\begin{cases} \boldsymbol{m}_t = \boldsymbol{m}\wedge\boldsymbol{m}_{xx} \\[2ex] \boldsymbol{m}(x,0) \quad \text{known}\,. \end{cases} \tag{1.4}$$

After the association of (1.1a) to (1.3), the following classical diagram shows how the IST works:



Let us recall that the initial datum $\boldsymbol{m}(x,0)$ which appears in (1.4) (and in the first box in the diagram) has to be considered as a coefficient in the first equation of system (1.3). However, in the first part of this work [5], we have developed the direct scattering problem – which consists of the construction of the scattering data when $\boldsymbol{m}(x,0)$ is assigned – for the first of (1.3) (horizontal top arrow in the above diagram), we have discussed the evolution of the scattering data (vertical right arrow in the above diagram) and, finally, we have formulated the inverse scattering problem – which consists of the reconstruction of the potential $\boldsymbol{m}(x)$ corresponding to a set of a given scattering data – in terms of certain Marchenko integral equations (horizontal down arrow in the above diagram). So, in the first part of this work [5] we have treated the arrows of the IST scheme.

In the present second part, we are interested in solving explicitly the inverse scattering problem when the reflection coefficient is identically zero, aiming at an explicit soliton solution formula for (1.1a) under condition (1.1b). We will present this soliton solution formula in the next section. In order to derive the formula, we employ the matrix triplet method. Indeed, if the reflection coefficient vanishes identically, there exists a triplet of matrices $\big(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\big)$, of sizes $2\bar{n}\times 2\bar{n}$, $2\bar{n}\times 2$, and $2\times 2\bar{n}$, respectively, such that the Marchenko kernel is given by

$$\boldsymbol{\Omega}_l(x+y,t)=\boldsymbol{C}e^{t\boldsymbol{H}}e^{-(x+y)\boldsymbol{A}}\boldsymbol{B}\,,$$

where the $2\bar{n}\times 2\bar{n}$ matrices $\boldsymbol{A}$ and $\boldsymbol{H}$ commute and $\boldsymbol{A}$ has only eigenvalues with positive real parts. Typically, $\boldsymbol{H}$ is a function of $\boldsymbol{A}$. After solving the Marchenko equation by separation of variables, in the next section we arrive at the solution of the initial-value problem in terms of the matrix triplet $\big(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\big)$ and the matrix $\boldsymbol{H}$, providing the time dependence via (2.4). The expression obtained can then be written in terms of elementary functions, and is particularly amenable to computer algebra.

2 Soliton solutions formula

In this section we construct an explicit soliton solution formula for equation (1.1a) under the asymptotic condition (1.1b). To this aim, we apply the IST method (see, for instance, [28–30] for more details on this method) combined with the matrix triplet technique, successfully used in [31–35] and more recently in [25] in the context of the classical Heisenberg ferromagnet equation.

2.1 Inverse scattering transform

Having presented in the first part of this work [5] the *direct scattering problem* (consisting in the construction of the scattering data when $\boldsymbol{m}(x,0)$ is known), the *inverse scattering problem* (amounting to the construction of $\boldsymbol{m}(x)$ when the scattering data are given), and the *time evolution of the scattering data* associated to the first equation in system (1.3), we are now ready to discuss how the IST allows us to obtain the solution to the initial value problem for (1.1a).

Using the initial condition $\boldsymbol{m}(x,0)$ given in (1.4) as a potential in the system (1.3), we develop the direct scattering theory as shown in the first part [5] and build the scattering data. For the sake of clarity, let us recall that the scattering data (see Section 3 of [5] for more details) evolve in time as follows

$$R(\lambda,t) = e^{-4i\lambda^2 t}\,R(\lambda,0)\,, \qquad T(\lambda,t) = T(\lambda,0)\,, \qquad N_j(t) = e^{-4ia_j^2 t}\,N_j(0)\,, \tag{2.1}$$

where $R(\lambda,t)$ is the reflection coefficient, $T(\lambda,t)$ the transmission coefficients, and N_j are the norming constants associated to the discrete eigenvalues ia_j (see Section 2 in [5] for a detailed description of the scattering data).

As we have seen in the first part of this work [5], the inverse scattering problem requires to solve the following Marchenko equation

$$\boldsymbol{L}(x,y) + \boldsymbol{\Omega}(x+y) + \int_x^\infty \mathrm{d}\xi\,\boldsymbol{L}(x,\xi)\,\boldsymbol{\Omega}(\xi+y) = 0_{2\times 2}\,, \tag{2.2}$$

where the kernel $\boldsymbol{\Omega}(x)$ of (2.2) is given by

$$\boldsymbol{\Omega}(x) = \begin{pmatrix} 0 & \Omega(x) \\ -\Omega(x)^* & 0 \end{pmatrix}, \quad \text{with} \quad \Omega(x) = \hat{R}(x) + \sum_{j=1}^{n} N_j\,e^{-a_j x}\,. \tag{2.3}$$

The solution of the Heisenberg equation (1.1a) under condition (1.1b) is then obtained by replacing $\boldsymbol{\Omega}(x)$ with $\boldsymbol{\Omega}(x;t)$ in the Marchenko equation (*i.e.* taking into account (2.1)) and using the relation

$$\boldsymbol{m}(x)\cdot\boldsymbol{\sigma} = U\left(I_2 + \tilde{\boldsymbol{L}}(x)^\dagger\right)\sigma_3\left(I_2 + \tilde{\boldsymbol{L}}(x)\right)U^{-1}\,, \tag{2.4}$$

where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -e^{i\gamma} \\ e^{-i\gamma} & 1 \end{pmatrix}, \tag{2.5}$$

$$\tilde{\boldsymbol{L}}(x) = \int_x^\infty \mathrm{d}\xi\,\boldsymbol{L}(x,\xi)\,, \tag{2.6}$$

and $\boldsymbol{\sigma}$ is the column vector whose entries are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that in the expression of $\Omega(x)$ with $\rho(x)$ we have denoted the Fourier transform of the reflection coefficient.

2.2 Matrix triplet method

In the remaining part of this work we will focus onto the reflectionless case, *i.e.* the case $R(\lambda) = 0$. In this case the expression for $\boldsymbol{\Omega}(x; 0)$ is obtained from the expression of the kernel given above and setting $\rho(x) = 0$. In particular, we can treat the situation where the discrete eigenvalues are not necessarily simple [36] by generalizing formula (2.3) as follows:

$$\Omega(x;t) = \sum_{j=1}^{n} \sum_{k=0}^{n_j-1} N_{jk}(t) \frac{x^k}{k!} e^{-a_j x} . \tag{2.7}$$

In (2.7), n is the number of discrete eigenvalues $\{ia_j\}_{j=1}^{n}$, namely the poles of the transmission coefficient $T(\lambda)$ in $\mathbb{C}^+$ (thus, satisfying $\mathrm{Re}(a_j) > 0$); the quantities a_j are obtained by multiplying the discrete eigenvalues by $-i$; n_j is the algebraic multiplicity of ia_j; and $\{N_{jk}(t)\}_{k=0}^{n_j-1}$, for all $j = 1, 2, ..., n$, are the (time-dependent) norming constants corresponding to ia_j, evolving in time according to (2.1).

To recover the solution of (1.4) we follow the three steps indicated here below.

a. Suppose that the scattering data, namely the discrete eigenvalues and the corresponding norming constants,

$$\{ia_j\}_{j=1}^{n} \quad \text{and} \quad \left\{ \{N_{jk}(t)\}_{k=0}^{n_j-1} \right\}_{j=1}^{n} ,$$

are given. Then, we construct $\boldsymbol{\Omega}(x)$ as in (2.3) and we let it evolve in time using (2.7):

$$\boldsymbol{\Omega}(x;t) = \begin{pmatrix} 0 & \Omega(x;t) \\ -\Omega(x;t)^* & 0 \end{pmatrix} . \tag{2.8}$$

b. We solve the Marchenko integral equation (2.2):

$$\boldsymbol{L}(x,y;t) + \boldsymbol{\Omega}(x+y;t) + \int_x^{\infty} \mathrm{d}\xi \, \boldsymbol{L}(x,\xi;t)\, \boldsymbol{\Omega}(\xi+y;t) = 0_{2\times2} .$$

where $\xi > x$ and the kernel $\boldsymbol{\Omega}(x,y)$ is given in (2.8).

c. We construct the potential $\boldsymbol{m}(x;t)$ by using formula(2.4):

$$\boldsymbol{m}(x) \cdot \boldsymbol{\sigma} = U_+ \left(I_2 + \tilde{\boldsymbol{L}}(x)^\dagger \right) \sigma_3 \left(I_2 + \tilde{\boldsymbol{L}}(x) \right) U_+^{-1}.,$$

where $\tilde{\boldsymbol{L}}(x) = \int_x^\infty \mathrm{d}\xi \, \boldsymbol{L}(x,\xi)$.

Let us follow the above procedure (an analogous procedure can be developed with the kernel $\overline{\boldsymbol{\Omega}}$, as per in formula (3.10b) of [5], and solving the corresponding Marchenko equation for $\overline{\boldsymbol{L}}$, that is, equation (3.11) of [5]). We start by disregarding the time dependence (*e.g.*, we construct $\boldsymbol{\Omega}(x)$ assuming no dependence on the time). We will subsequently show how to take the time dependence into account.

It is well known [37, 38] that it is possible to factorize a matrix function which is in the form (2.8) with (2.7) by using a suitable triplet of matrices. More precisely, let

$\bar{n} = \sum_{j=1}^{n} n_j$, and suppose that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a matrix triplet such that all the eigenvalues of the $2\bar{n} \times 2\bar{n}$ matrix $\mathcal{A}$ have positive real parts, $\mathcal{B}$ is a $2\bar{n} \times 2$ matrix, and $\mathcal{C}$ is a $2 \times 2\bar{n}$ matrix. We then set

$$\mathbf{\Omega}(x) = \begin{pmatrix} 0 & \Omega(x) \\ -\Omega(x)^* & 0 \end{pmatrix} \stackrel{\text{def}}{=} \mathcal{C}\, e^{-x\mathcal{A}}\, \mathcal{B} \,. \tag{2.9a}$$

Alternatively, equation (2.9a) can be written by setting

$$\Omega(x) = \sum_{j=1}^{n} \sum_{k=0}^{n_j - 1} c_{jk} \frac{x^k}{k!}\, e^{-a_j x} = C\, e^{-xA}\, B \,, \tag{2.9b}$$

with

$$\mathcal{A} = \begin{pmatrix} A & 0_{\bar{n}\times\bar{n}} \\ 0_{\bar{n}\times\bar{n}} & A^\dagger \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0_{\bar{n}\times 1} & B \\ -C^\dagger & 0_{\bar{n}\times 1} \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C & 0_{1\times\bar{n}} \\ 0_{1\times\bar{n}} & B^\dagger \end{pmatrix}. \tag{2.9c}$$

Here A is a $\bar{n} \times \bar{n}$ matrix whose n eigenvalues $\{a_j\}_{j=1}^{n}$ are obtained from the poles $\{ia_j\}_{j=1}^{n}$ of the transmission coefficient $T(\lambda)$ (namely the discrete eigenvalues) by multiplication by a factor $-i$ (a proof of this fact can be found in [25]); B is a $\bar{n} \times 1$ matrix; and C is a $1 \times \bar{n}$ matrix. Furthermore, we assume that the triplet (A, B, C) is a *minimal* triplet in the sense that the matrix order of A is minimal among all triplets representing the same Marchenko kernel by means of (2.9) [37, 38]. As the discrete eigenvalues $\{ia_j\}_{j=1}^{n}$ belong to the upper half-plane $\mathbb{C}^+$, we have $\mathrm{Re}(a_j) > 0$ for all j, namely all the eigenvalues of the matrix A have positive real parts: this fact is necessary in order to assure the convergence of the integrals in (2.11f). Moreover, we recall that the minimality of the triplet (A, B, C) entails that the geometric multiplicity of the eigenvalues of A be one (see [31]).

We observe that it is not restrictive (in fact, it is the typical choice) to set the triplet (A, B, C) as follows [38]:

$$A_{\bar{n}\times\bar{n}} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}, \quad B_{\bar{n}\times 1} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix}, \quad C_{1\times\bar{n}} = \begin{pmatrix} C_1 & C_2 & \cdots & C_n \end{pmatrix}, \tag{2.10a}$$

where A is in Jordan canonical form, with A_j being the Jordan block of dimension $n_j \times n_j$ corresponding to the discrete eigenvalue ia_j, B_j is a column vector of dimension n_j, typically chosen to be a vector of ones; and C_j is a row vector of dimension n_j, typically chosen to be the vector of the norming constants corresponding to the discrete eigenvalue ia_j,

$$C_j = \begin{pmatrix} c_{j,0} & c_{j,1} & \cdots & c_{j,n_j-1} \end{pmatrix}, \tag{2.10b}$$

so that the elements of C are chosen to be the $\bar{n}$ norming constants $\left\{ \{c_{jk}\}_{k=0}^{n_j-1} \right\}_{j=1}^{n}$.

For later convenience, we also introduce the matrix $\mathcal{P}$, which is – under the conditions satisfied by our triplet – the unique solution of the Sylvester equation

$$\mathcal{A}\,\mathcal{P} + \mathcal{P}\,\mathcal{A} = \mathcal{B}\,\mathcal{C} \,, \tag{2.11a}$$

namely

$$\mathcal{P} = \int_0^\infty \mathrm{d}\xi \, e^{-\xi\mathcal{A}} \, \mathcal{B}\, \mathcal{C} \, e^{-\xi\mathcal{A}}. \tag{2.11b}$$

Note that it is also possible to write $\mathcal{P}$ as

$$\mathcal{P} = \begin{pmatrix} 0_{\bar{n}\times\bar{n}} & N \\ -Q & 0_{\bar{n}\times\bar{n}} \end{pmatrix}, \tag{2.11c}$$

where N and Q solve the Lyapunov matrix equations

$$A^\dagger Q + Q\, A = C^\dagger C\,, \tag{2.11d}$$

$$A\, N + N\, A^\dagger = B\, B^\dagger\,, \tag{2.11e}$$

that is

$$N = \int_0^\infty \mathrm{d}\xi \, e^{-\xi\, A}\, B\, B^\dagger \, e^{-\xi\, A^\dagger}, \qquad Q = \int_0^\infty \mathrm{d}\xi \, e^{-\xi\, A^\dagger}\, C^\dagger\, C\, e^{-\xi\, A}. \tag{2.11f}$$

By the minimality of the triplet (A, B, C) [38, Sec.4.1], we see that N and Q are positive Hermitian matrices and then $\mathcal{P}$ is invertible and

$$\mathcal{P}^{-1} = \begin{pmatrix} 0_{\bar{n}\times\bar{n}} & -Q^{-1} \\ N^{-1} & 0_{\bar{n}\times\bar{n}} \end{pmatrix}. \tag{2.11g}$$

Now we are ready to express the solution $\boldsymbol{L}(x, y)$ of the Marchenko integral equation (2.2) in terms of the triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and of the matrix $\mathcal{P}$. Indeed, by substituting the expression of the kernel (2.9) into (2.2), we arrive at the following Marchenko equation

$$\boldsymbol{L}(x, y) + \mathcal{C}\, e^{-(x+y)\,\mathcal{A}}\, \mathcal{B} + \int_x^\infty \mathrm{d}\xi\, \boldsymbol{L}(x, \xi)\, \mathcal{C}\, e^{-(\xi+y)\,\mathcal{A}}\, \mathcal{B} = 0_{2\times 2}. \tag{2.12}$$

Equation (2.12) can be solved explicitly via separation of variables and we obtain (see [31, 35] for more details on the resolution of (2.12))

$$\boldsymbol{L}(x, y) = -\mathcal{C}e^{-x\mathcal{A}}[I_{2\bar{n}} + e^{-x\mathcal{A}}\mathcal{P}e^{-x\mathcal{A}}]^{-1}e^{-y\mathcal{A}}\mathcal{B}, \tag{2.13}$$

provided the inverse matrix exists for all $x \in \mathbb{R}$.

Finally, in order to reconstruct the solution of (1.4) we have to integrate (2.13) with respect to y, obtaining the explicit formula

$$\tilde{\boldsymbol{L}}(x) = -\mathcal{C}e^{-x\mathcal{A}}[I_{2\bar{n}} + e^{-x\mathcal{A}}\mathcal{P}e^{-x\mathcal{A}}]^{-1}e^{-x\mathcal{A}}\,\mathcal{A}^{-1}\,\mathcal{B}. \tag{2.14}$$

The right-hand side of (2.14) is now explicit and we can use such formula to recover the components $m_j(x)$, $j = 1, 2, 3$, of the vector $\boldsymbol{m}(x)$.

Let us now introduce the dependence on the time t. In order to recover it, we have to take into account the time evolution of the scattering data expressed by (2.1). Then the (reflectionless) Marchenko kernels become:

$$\Omega(x; t) = \sum_{j=1}^{n} \sum_{k=0}^{n_j-1} c_{jk}(t) \frac{x^k}{k!}\, e^{-a_j x} = C\, e^{-4itA^2}\, e^{-xA}\, B\,, \tag{2.15a}$$

$$\Omega(x;t)^* = \sum_{j=1}^{n}\sum_{k=0}^{n_j-1} c_{jk}^*(t)\frac{x^k}{k!}\,e^{-a_j^* x} = B^\dagger\, e^{-xA^\dagger}\, e^{4itA^{\dagger 2}t}\, C^\dagger\,. \tag{2.15b}$$

In other words, we may replace the matrix triplet (A,B,C) for the triplet (A,B,Ce^{-4itA^2}) in a such way that (2.1) are satisfied (A contains the discrete eigenvalues which are time independent and C the norming constants). Consequently, the explicit right-hand side of (2.14) can be written as follows:

$$\begin{aligned}\tilde{\boldsymbol{L}}(x;t) &= -\mathcal{C}(t)\,e^{-x\mathcal{A}}\,\left[I_{2\bar{n}} + e^{-x\mathcal{A}}\mathcal{P}(t)e^{-x\mathcal{A}}\right]^{-1}\,e^{-x\mathcal{A}}\,\mathcal{A}^{-1}\,\mathcal{B}(t) = \\ &= -\mathcal{C}(t)\,\left[e^{2\,x\mathcal{A}} + \mathcal{P}(t)\right]^{-1}\,\mathcal{A}^{-1}\,\mathcal{B}(t)\,,\end{aligned} \tag{2.16a}$$

where

$$\mathcal{B}(t) = \begin{pmatrix} 0_{\bar{n}\times 1} & B \\ -\left(Ce^{-4itA^2}\right)^\dagger & 0_{\bar{n}\times 1}\end{pmatrix}, \quad \mathcal{C}(t) = \begin{pmatrix} Ce^{-4itA^2} & 0_{1\times\bar{n}} \\ 0_{1\times\bar{n}} & B^\dagger\end{pmatrix}\,, \tag{2.16b}$$

and

$$\mathcal{P}(t) = \begin{pmatrix} 0_{\bar{n}\times\bar{n}} & N \\ -Q(t) & 0_{\bar{n}\times\bar{n}}\end{pmatrix}\,, \tag{2.16c}$$

with

$$Q(t) = \int_0^\infty dx\,e^{-xA^\dagger}\left(Ce^{-4itA^2}\right)^\dagger Ce^{-4itA^2}e^{-xA}, \tag{2.16d}$$

satisfying

$$A^\dagger Q(t) + Q(t)A = (Ce^{-4itA^2})^\dagger(Ce^{-4itA^2}). \tag{2.16e}$$

Finally, after some algebra, using (2.4) with (2.16) and taking into account that $\tilde{\boldsymbol{L}}(x,y)$ belongs to SU2, so that

$$\tilde{\boldsymbol{L}} = \begin{pmatrix}\tilde{L}_1 & -\tilde{L}_2^{\,*} \\ \tilde{L}_2 & \tilde{L}_1^*\end{pmatrix}\,, \quad \text{and} \quad \left[I_2 + \tilde{\boldsymbol{L}}\right]^{-1} = \left[I_2 + \tilde{\boldsymbol{L}}\right]^\dagger = \left[I_2 + \tilde{\boldsymbol{L}}^\dagger\right]\,,$$

we arrive at the following soliton solution formula of (1.1a) with asymptotic boundary conditions (1.1b)

$$m_1(x,t) = \sin^2(\gamma)\,\tilde{m}_1(x,t) + \cos(\gamma)\sin(\gamma)\,\tilde{m}_2(x,t) + \cos(\gamma)\,\tilde{m}_3(x,t)\,, \tag{2.17a}$$
$$m_2(x,t) = \cos(\gamma)\sin(\gamma)\,\tilde{m}_1(x,t) + \cos^2(\gamma)\,\tilde{m}_2(x,t) - \sin(\gamma)\,\tilde{m}_3(x,t)\,, \tag{2.17b}$$
$$m_3(x,t) = -\cos(\gamma)\,\tilde{m}_1(x,t) + \sin(\gamma)\,\tilde{m}_2(x,t)\,, \tag{2.17c}$$

where $(\tilde{m}_1(x,t), \tilde{m}_2(x,t), \tilde{m}_3(x,t))$ have the following explicit expression in terms of the elements of the matrix $\tilde{\boldsymbol{L}}(x;t)$

$$\tilde{m}_1(x,t) = -2\,\mathrm{Re}\Big((1+\tilde{L}_1(x,t))\,\tilde{L}_2(x,t)\Big)\,. \tag{2.18a}$$

$$\tilde{m}_2(x,t) = -2\,\mathrm{Im}\Big((1+\tilde{L}_1(x,t))\,\tilde{L}_2(x,t)\Big)\,, \tag{2.18b}$$

$$\tilde{m}_3(x,t) = 2\left|1+\tilde{L}_1(x,t)\right|^2 - 1\,. \tag{2.18c}$$

We note that formulae (2.18) give the soliton solutions of (1.1a) with the so-called easy-axis conditions (*i.e.*, $\boldsymbol{m}(x) \to \boldsymbol{e}_3$ as $x \to \pm\infty$, see [25]). Thus, equations (2.17) allow one to generate the soliton solutions of (1.1a) with boundary conditions (1.1b) when a soliton solutions of (1.1a) with easy-axis conditions is known.

We observe that the matrix

$$V = U\,\mathrm{diag}\left(e^{i\delta}, e^{-i\delta}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\delta} & -e^{i(\gamma-\delta)} \\ e^{i(\delta-\gamma)} & e^{-i\delta}\end{pmatrix}, \tag{2.19}$$

where δ is a constant and the expression of U is given by (2.5), is such that its columns form an orthonormal basis of eigenvectors of $\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2$, corresponding to the eigenvalues 1 and -1, respectively. Consequently, we can replace the matrix U for the matrix V in formula (2.4), obtaining a more general formula (indeed, a formula featuring the additional phase δ). Indeed, the more general reconstruction formula for the soliton solutions is

$$\boldsymbol{m}(x)\cdot\boldsymbol{\sigma} = V\left(I_2 + \tilde{\boldsymbol{L}}(x)^\dagger\right)\sigma_3\left(I_2 + \tilde{\boldsymbol{L}}(x)\right)V^{-1}, \tag{2.20}$$

where V is given by (2.19). By using (2.20) and after some straightforward computations, we arrive at the the more general soliton solution formula

$$m_1(x,t) = \sin(\gamma)\sin(\hat{\delta})\,\tilde{m}_1(x,t) + \sin(\gamma)\cos(\hat{\delta})\,\tilde{m}_2(x,t) + \cos(\gamma)\,\tilde{m}_3(x,t)\,, \tag{2.21a}$$

$$m_2(x,t) = \cos(\gamma)\sin(\hat{\delta})\,\tilde{m}_1(x,t) + \cos(\gamma)\cos(\hat{\delta})\,\tilde{m}_2(x,t) - \sin(\gamma)\,\tilde{m}_3(x,t)\,, \tag{2.21b}$$

$$m_3(x,t) = -\cos(\hat{\delta})\,\tilde{m}_1(x,t) + \sin(\hat{\delta})\,\tilde{m}_2(x,t)\,, \tag{2.21c}$$

where $\hat{\delta} = \gamma - 2\delta$.

We conclude this section with two remarks.

Remark 1. By following a procedure analogous to the one that has led to formulae (2.21), a similar generalization of the soliton solution formula can be obtained also in the easy-axis case studied in [25]. In that case what one gets is a rotation of the angle 2δ around the z-axis for the components m_1 and m_2.

Remark 2. It is worth observing that formulae (2.17) and (2.21), which have been obtained in this paper by means of the IST technique, can be obtained also (and straightforwardly) by employing the symmetries of the classical Heisenberg ferromagnet equation (1.1) (see, for instance, [39, 40]).

3 Classes of soliton solutions

In the present section we discuss classes of soliton solutions of (1.1), as resulting from the explicit formula (2.17) with (2.18) and (2.16). Moreover, similarly to [25], we provide several numerical examples, obtained by computing (on *MATLAB R2017a*) the terms $\tilde{L}_1$

and $\tilde{L}_2$ appearing in (2.18) using formulae (C.2a) and (C.2d) in [25] when x is large and negative, and formulae (C.2b) and (C.2e) in [25] when x is large and positive.

An immediate classification of the soliton solutions of (1.1) can be had by considering the algebraic multiplicity of the eigenvalues of the matrix A in the matrix triplet (A, B, C) in (2.9b). Propagating and stationary soliton solutions (the so-called *magnetic-droplet solitons*, see [16]) are associated to algebraically simple eigenvalues of A. Multiple-pole (or, more simply, *multipole*) soliton solutions are instead associated to eigenvalues of A having algebraic multiplicity larger than one (*i.e.*, *degenerate* eigenvalues). In the following, we choose A to be in Jordan canonical form as in (2.10): single eigenvalues on the main diagonal are associated to individual (stationary or propagating) solitons, whereas Jordan blocks of algebraic multiplicity $n_j > 1$ are associated to multipole solutions. No blocks are repeated, as the geometric multiplicity of each eigenvalue is one due to the minimality of the triplet [31, 38].

3.1 One-soliton solution

The one-soliton solution corresponds to the choice $n = 1$, $n_1 = 1$ in (2.9), so that $\bar{n} = 1$. If we set the matrix triplet (A, B, C) as

$$A = (a) \ , \quad B = (1) \ , \ \text{and} \ C = (c) \ ,$$

we get

$$N = \left(\frac{1}{2\,\mathrm{Re}(a)}\right) , \quad Q = \left(\frac{|c|^2}{2\,\mathrm{Re}(a)}\right) , \quad Q(t) = \left(\frac{|c|^2\, e^{-4\,i\,\mathrm{Re}(a^2)\,t}}{2\,\mathrm{Re}(a)}\right) ,$$

and from (2.16) we have

$$\tilde{L}_1 = -\frac{2\,|c|^2\,\mathrm{Re}(a)}{a^*\left(|c|^2 + 4\,\mathrm{Re}(a)^2\, e^{4\,\mathrm{Re}(a)\,(x-4\,\mathrm{Im}(a)\,t)}\right)} , \ \tilde{L}_2 = \frac{2\,c^*\,\mathrm{Re}(a)\, e^{2\,a\,(x+2\,i\,a\,t)}}{a^*\left(|c|^2 + 4\,\mathrm{Re}(a)^2\, e^{4\,\mathrm{Re}(a)\,(x-4\,\mathrm{Im}(a)\,t)}\right)} .$$

Then we set, without any loss of generality (see [25]),

$$a = p + i\,q\,, \quad p > 0\,, \tag{3.1a}$$

and

$$c \equiv c(p, q, x_0, \varphi_0) = \begin{cases} 2\,i\,p\,\mathrm{sign}(q)\,\left(\frac{p+i\,q}{p-i\,q}\right)\, e^{2\,(p+i\,q)\,x_0 - i\,\varphi_0} & \text{if } q \neq 0 \\ \\ 2\,p\,e^{2\,p\,x_0 - i\,\varphi_0} & \text{if } q = 0\,, \end{cases} \tag{3.1b}$$

for some $x_0, \varphi_0 \in \mathbb{R}$. From (2.18), after some simple algebra, we obtain the in-plane one soliton solution $(m_1, m_2, m(3))$, via (2.17), or alternatively via (2.21), with $(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$ given by:

$$\tilde{m}_+ = 2\,\frac{(p+i\,q) - (p-i\,q)\,e^{4\,p\,(x-4\,q\,t-x_0)}}{(p-i\,q)^2\,\left[1 + e^{4\,p\,(x-4\,q\,t-x_0)}\right]^2}\, e^{2\,(p+i\,q)\,(x-4\,q\,t) - 2\,p\,x_0 - i\arg(c)}\, e^{4\,i\,(p^2+q^2)\,t} , \tag{3.2a}$$

$$\tilde{m}_1 = \mathrm{Re}\,(\tilde{m}_+)\ ,\ \ \tilde{m}_2 = \mathrm{Im}\,(\tilde{m}_+)\ ,\ \ \tilde{m}_3 = 1 - \frac{2\,p^2\,\mathrm{sech}^2\Big(2\,p\,(x - 4\,q\,t - x_0)\Big)}{p^2 + q^2}\,. \qquad (3.2b)$$

This solution describes a localized, coherent magnetic configuration travelling at the constant speed

$$v = 4\,\mathrm{Im}(a) = 4\,q\,. \qquad (3.3a)$$

Furthermore, the exponent of the last exponential term in the right-hand side of (3.2a) is a phase factor depending only on the time t. Consequently, the space and time evolution of the magnetic configuration is entirely described in terms of the constant speed v and the constant frequency

$$\omega = 4\,|a|^2 = 4\,(p^2 + q^2)\,, \qquad (3.3b)$$

which, in turn, depend only on the real and imaginary parts of the eigenvalue a. By inverting (3.3a) and (3.3b),

$$p = \frac{1}{2}\sqrt{\omega - \frac{v^2}{4}}\,, \quad q = \frac{v}{4}\,, \qquad (3.3c)$$

we immediately obtain the well-known condition for localization (see [16]),

$$\omega \geq 0\,, \quad |v| \leq 2\sqrt{\omega}\,. \qquad (3.3d)$$

On the other hand, via (3.1b), the norming constant c can be used to give the initial ($t = 0$) position x_0 of the minimum of $\tilde{m}_3$ and the initial phase φ_0, see [25].

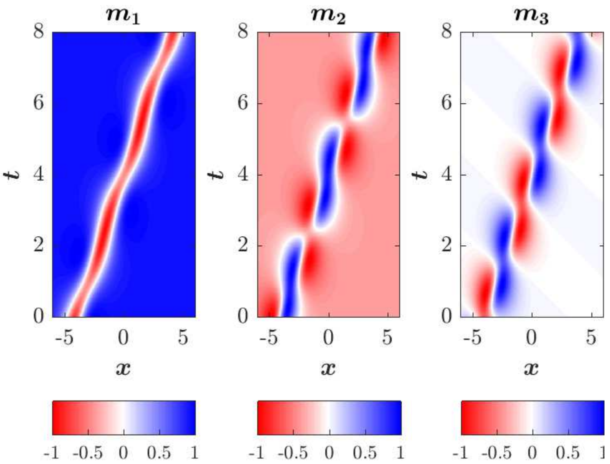
In Figure 1 we illustrate an in-plane, propagating, one-soliton solution for the choice $v = 1$, $\omega = 2$, $x_0 = -4$, and $\varphi_0 = 0$, entailing $p = \frac{\sqrt{7}}{4}$, $q = \frac{1}{4}$, and $c = \frac{-7+i\,3\sqrt{7}}{8}\,e^{-2\,(i+\sqrt{7})}$, for several choices of the asymptotic angle γ.

3.2 Multi-soliton and breather-like solutions

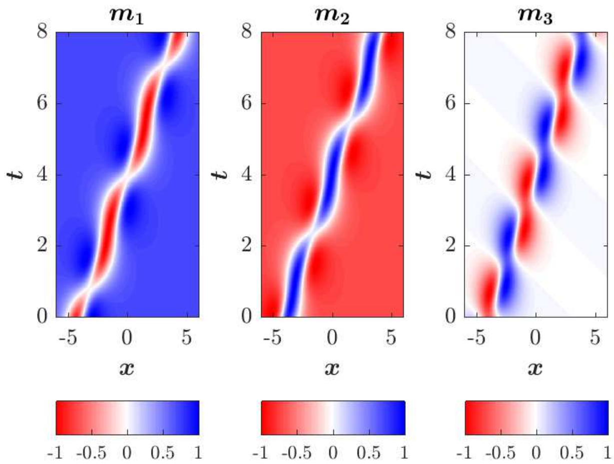
By combining two or more one-soliton solutions, namely choosing $n > 1$, and $n_j = 1$ for all j, $\bar{n} = n$ in (2.9), one can easily construct multi-soliton solutions. In this respect, we point out once more that formulae (2.17) as well as (2.21) are notably amenable to computer algebra, and allow to obtain explicit expressions (see [25]).

In particular, similarly to the easy-axis case [25], breather-like soliton solutions can be constructed out of two-soliton solutions, by creating two stationary, or two same-speed, propagating solitons close to each other. In the case of two stationary solitons ($v^{(1)} = v^{(2)} = 0$), namely, in the case of two real eigenvalues $a^1 = p_1$ and $a_2 = p_2$, $p_1 \neq p_2$, it is possible to show that, if the norming constants are chosen as follows

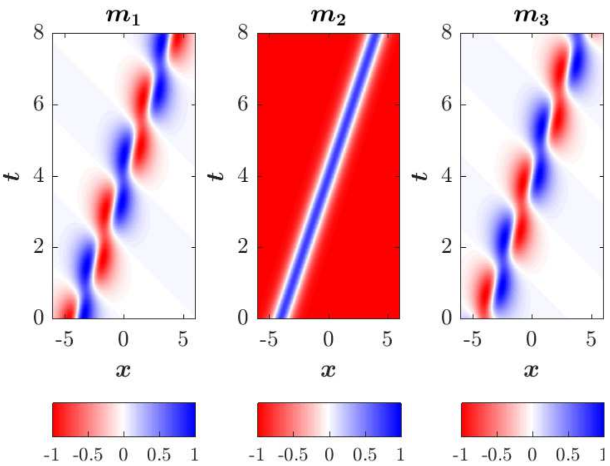
$$C = (c_1,\, c_2) = 2\sqrt{\frac{(p_1 + p_2)^2 + (q_1 - q_2)^2}{(p_1 - p_2)^2 + (q_1 - q_2)^2}}\,(p_1,\, p_2) \qquad (3.4)$$



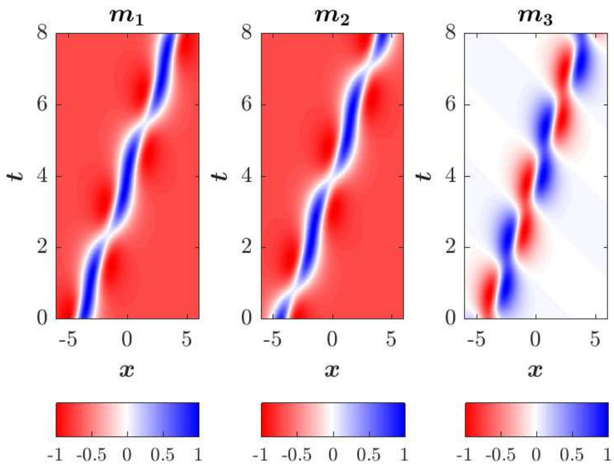
(a) $\gamma = \frac{\pi}{8}$



(b) $\gamma = \frac{\pi}{4}$



(c) $\gamma = \frac{\pi}{2}$



(d) $\gamma = \frac{3\pi}{4}$

Figure 1: Propagating, one-soliton solution, for different values of the asymptotic angle γ.

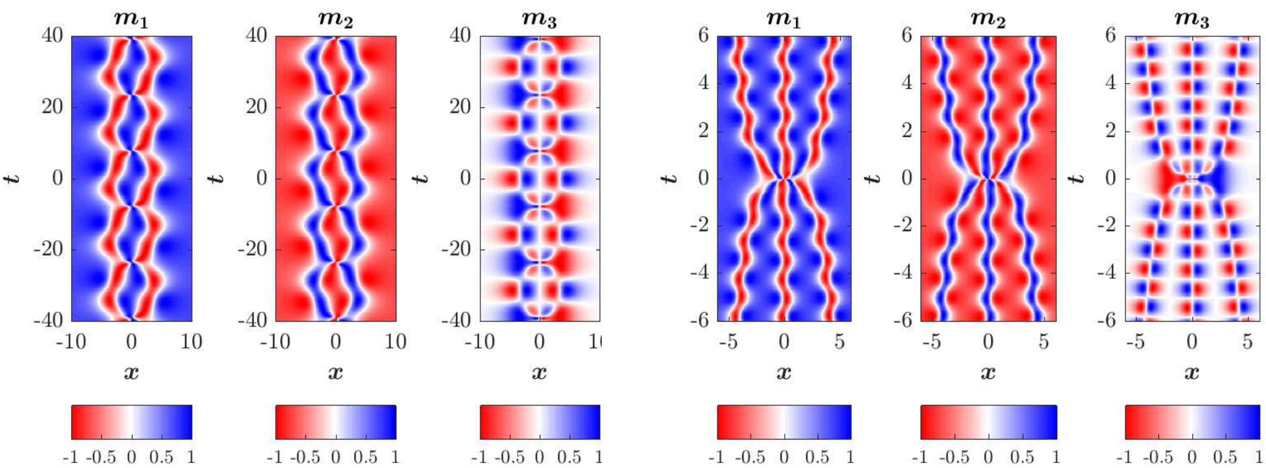
with $q_1 = 0$ and $q_2 = 0$, then a single, symmetrical, breather-like soliton solution is created, with m_1 and m_2 characterized by two identical, localized extrema oscillating in time around the origin with period

$$\nu = \frac{2\,\pi}{4\,(p_1 + p_2)\,(p_1 - p_2)}\,. \tag{3.5}$$

Figure 2(a) shows an example of such a breather-like soliton for $\gamma = \frac{\pi}{4}$, obtained via (3.3c) and (3.4) with

$$v^{(1)} = 0\,, \quad \omega^{(1)} = 0.8\,, \qquad \text{and} \qquad v^{(2)} = 0\,, \quad \omega^{(2)} = 0.4\,,$$

thus entailing an oscillation in time with period $\nu \simeq 15.71$. Similarly to the easy-axis case [25], propagating, breather-like solitons can be constructed in the same way as above, but assigning the same non-zero imaginary part to both the discrete eigenvalues. More generally, propagating or stationary, breather-like solitons can be had by creating two stationary, or two same-speed, propagating solitons close to one another: in other words, a single, breather-like soliton should always be regarded as a stable, periodic tangle of two interacting, but individual solitons, associated to two different eigenvalues in the matrix A (see [25]).



(a) Stationary, breather-like soliton

(b) A three-pole soliton solution ($n_1 = 3$)

Figure 2: Examples of breather-like and multipole solutions for $\gamma = \frac{\pi}{4}$.

3.3 Multipole solutions

If $n_j > 1$ for some j, then A features a Jordan block of order n_j, and one has multipole soliton solutions. Multipole solutions of (1.1) with (1.1a) are presented here for the first time. Their analysis can be achieved in analogy to the study of the multipole solutions of the nonlinear Schrödinger equation [41, 42], and is postponed to future investigation.

If $n = 1$, $n_1 = 2$, $\bar{n} = 2$ in (2.9), then we have a single two-pole soliton solution. In this case, it is possible to show [25] that, if the associated eigenvalue of A is real ($a = p$), so that $A = \left(\begin{smallmatrix} p & 1 \\ 0 & p \end{smallmatrix}\right)$, if B is chosen as a vector of ones, and if the norming constants in C are chosen as follows

$$C = (c_1,\, c_2) = \left(4\,p^2,\; 4\,p\,[1 + p\,(2\,x_0 - 1)]\right)\, e^{2\,p\,x_0 - i\,\varphi_0}\,, \tag{3.6}$$

then a single, symmetrical, two-pole soliton solution is created, with $m_1$and m_2 characterized by two minima, constituting two separated branches, that are expected to propagate in space at a velocity that varies logarithmically in time.

The same technique can be generalized to any value of the algebraic multiplicity n_j. For instance, if $n = 1$, $n_1 = 3$, $\bar{n} = 3$ in (2.9), then we have a single three-pole soliton solution: if the associated eigenvalue of A is real ($a = p$), so that $A = \left(\begin{smallmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{smallmatrix}\right)$, if B is chosen as a vector of ones, and if the norming constants in C are chosen as follows

$$C^T = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 8\,p^3 \\ 4\,p^2\,[3 + p\,(4\,x_0 - 2)] \\ 8\,p^2\,x_0\,(x_0 - 1) + 6\,p\,(2\,x_0 - 1) + 3 \end{pmatrix} e^{2\,p\,x_0 - i\,\varphi_0}\,, \tag{3.7}$$

then a single, symmetrical, three-pole soliton solution is created, with m_1 and m_2 characterized by three minima, constituting three separated branches, propagating in space at a velocity that varies logarithmically in time, and interacting in $x = x_0$ at $t = 0$. Figure 2(b) shows an example of such a solution with $\gamma = \frac{\pi}{4}$, obtained via (3.7) with $p = 1$, $x_0 = 0$, and $\varphi_0 = 0$.

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