The Continuous Part of a Markov Operator

SHU-TEH C. MOY*

Communicated by GIAN-CARLO ROTA

Summary. Two theorems are proved for a Markov operator. Theorem 1 states that, for a Markov operator P_1 which is strictly dominated by a conservative, ergodic Markov operator, $P_1^n 1 \downarrow 0$ a.e.. Theorem 2 is concerned with a Markov operator P which possesses a probability transition function. It is shown that if P is conservative and ergodic and if P_n is the continuous part of P^n then either $P_n = 0$ for all n or $P_n 1 \uparrow 1$ a.e..

Let X be a non-empty set, $\mathcal{B}$, a σ-algebra of subsets of X and λ, a σ-finite measure on $\mathcal{B}$. Let $L_\infty(\lambda)$ be the collection of all real valued, λ-essentially bounded, $\mathcal{B}$-measurable functions. For two functions f, g in $L_\infty(\lambda)$, $f = g$, $f < g$ are to mean that the equality and the inequality, respectively, are satisfied except on a λ-null set. Sometimes we still indicate = a.e. (λ) or < a.e. (λ) for emphasis. Let P be a linear operator on $L_\infty(\lambda)$ to $L_\infty(\lambda)$ satisfying the following conditions:

p1. if $f \geqq 0$ a.e.(λ) then $Pf \geqq 0$ a.e.(λ),
p2. if $f_n \downarrow 0$ a.e.(λ) then $Pf_n \downarrow 0$ a.e.(λ),
p3. $P1 \leqq 1$ a.e.(λ).

Such an operator is a *λ-measurable Markov operator* of E. Hopf or simply, a *Markov operator*. For any set A in $\mathcal{B}$, let 1_A represent the function which takes the value 1 on A and 0 on the complement A' of A, and I_A represent the Markov operator defined by

$$I_A f(x) = 1_A(x) f(x).$$

For an arbitrary Markov operator P we let

$$P_A = \sum_{n=0}^{\infty} P(I_{A'} P)^n.$$

P_A operating on nonnegative elements of $L_\infty(\lambda)$ has a well defined meaning. In particular $P_A 1_A$ is a nonnegative function which is $\leqq 1$ (*cf* Section VI of [6]). A set A in $\mathcal{B}$ is said to be *conservative* if for every λ-non-null subset B of A, $P_B 1_B = 1$ on B. P is said to be *conservative* if X is conservative. A set C in $\mathcal{B}$ is said to be *P-closed* if $P1_C = 1$ on C. A P-closed set C is *indecomposable* if

* This work was supported by National Science Foundation GP5270.

if it does not contain two disjoint, λ-non-null P-closed sets. A conservative Markov operator P is *ergodic* if X is indecomposable. A conservative, ergodic Markov operator P is characterized by the fact: for every λ-non-null set $A \in \mathcal{B}$, $P_A 1_A = 1$ a.e.(λ).

Theorem 1. *If P is a conservative, ergodic, Markov operator and P_1 is another Markov operator such that $P_1 \leqq P$ and $P - P_1 \neq 0$ then $P_1^n 1 \downarrow 0$ a.e.(λ).*

Proof. First we shall prove the theorem for the special case that P_1 is of the form $I_{A'}P$ where A is a λ-non-null set and A' is its complement.

The following equality can be easily proved by mathematical induction.

$$(1) \qquad P^n = I_A P^n + (I_{A'}P)I_A P^{n-1} + (I_{A'}P)^2 I_A P^{n-2} + \cdots + (I_{A'}P)^n I_A + (I_{A'}P)^n I_{A'} .$$

It follows from (1) that

$$P^n 1 = I_A P^n 1 + (I_{A'}P)I_A P^{n-1}1 + \cdots + (I_{A'}P)^n 1_A + (I_{A'}P)^n 1_{A'} .$$

Hence

$$1 = \{1_A + (I_{A'}P)1_A + \cdots + (I_{A'}P)^n 1_A\} + (I_{A'}P)^n 1_{A'}$$

and

$$(2) \qquad 1 - \{1_A + (I_{A'}P)1_A + \cdots + (I_{A'}P)^n 1_A\} = (I_{A'}P)^n 1_{A'} .$$

Since P is conservative and ergodic, $P_A 1_A = 1$ a.e.(λ), the left hand side of (2) approaches 0 a.e.(λ) as $n \rightarrow \infty$ so that $(I_{A'}P)^n 1_{A'} \downarrow 0$ a.e.(λ). Hence $(I_{A'}P)^n 1 = (I_{A'}P)^{n-1} I_{A'}P1 = (I_{A'}P)^{n-1} 1_{A'}$ and the conclusion of the theorem follows.

For the general case, there is a positive number ϵ and a λ-non-null set A such that $P_1 1 < 1 - \epsilon$ on A. Define another Markov operator $\bar{P}$ as follows.

$$\bar{P} = I_A P_1 + I_{A'}P.$$

Then $P_1 \leqq \bar{P} \leqq P$. It is sufficient to prove $\bar{P}^n 1 \downarrow 0$ a.e.(λ).

We shall now establish the following assertion: (I) For every positive integer k, there exist a positive number c_k and a sequence $\{g_n^{(k)}\}$ of functions such that $0 \leqq g_n^{(k)} \leqq c_k$, $\lim_{n\to\infty} g_n^{(k)} = 0$ a.e.(λ) and

$$(3) \qquad \bar{P}^n 1 \leqq (1 - \epsilon)^k + g_n^{(k)}.$$

Applying formula (1) to $\bar{P}$ and noting that $I_{A'}\bar{P} = I_{A'}P$, we obtain:

$$\bar{P}^n 1 = \{I_A \bar{P}^n + (I_{A'}P)I_A \bar{P}^{n-1} + \cdots + (I_{A'}P)^{n-1} I_A \bar{P}\}1 + (I_{A'}P)^n 1$$
$$\leqq \{I_A + (I_{A'}P)I_A + \cdots + (I_{A'}P)^{n-1} I_A\}\bar{P}1 + (I_{A'}P)^n 1.$$

Since $\bar{P}1 = P_1 1 < 1 - \epsilon$ on A, it follows

$$\bar{P}^n 1 \leqq \{1_A + (I_{A'}P)1_A + \cdots + (I_{A'}P)^{n-1} 1_A\}(1 - \epsilon) + (I_{A'}P)^n 1$$
$$\leqq (1 - \epsilon) + (I_{A'}P)^n 1.$$

Hence assertion (I) holds true for $k = 1$ with $c_1 = 1$, $g_n^{(1)} = (I_{A'}P)^n 1$. Now we assume that ($I$) holds true for k and proceed to show that (I) also holds for $k + 1$:

$$\begin{aligned}\bar{P}^n 1 &= \{I_A \bar{P}^n 1 + (I_{A'}P) I_A \bar{P}^{n-1} 1 + \cdots + (I_{A'}P)^{n-2} I_A \bar{P}^2 1\} + (I_{A'}P)^{n-1}\bar{P}1 \\ &\leqq I_A\bar{P}\{(1-\epsilon)^k + g_{n-1}^{(k)}\} + (I_{A'}P)I_A\bar{P}\{(1-\epsilon)^k + g_{n-2}^{(k)}\} \\ &\quad + \cdots + (I_{A'}P)^{n-2}I_A\bar{P}\{(1-\epsilon)^k + g_1^{(k)}\} + (I_{A'}P)^{n-1}\bar{P}1. \\ &= \{I_A\bar{P} + (I_{A'}P)I_A\bar{P} + \cdots + (I_{A'}P)^{n-2}I_A\bar{P}\}(1-\epsilon)^k \\ &\quad + \{I_A\bar{P}g_{n-1}^{(k)} + (I_{A'}P)I_A\bar{P}g_{n-2}^{(k)} + \cdots + (I_{A'}P)^{n-2}I_A\bar{P}g_1^{(k)}\} + (I_{A'}P)^{n-1}\bar{P}1.\end{aligned}$$

Let

$$f_n = I_A\bar{P}g_{n-1}^{(k)} + (I_{A'}P)I_A\bar{P}g_{n-2}^{(k)} + \cdots + (I_{A'}P)^{n-2}I_A\bar{P}g_1^{(k)},$$

and

$$g_n^{(k+1)} = f_n + (I_{A'}P)^{n-1}\bar{P}1,$$

then $\bar{P}^n 1 \leqq (1-\epsilon)^{(k+1)} + g_n^{(k+1)}$. Since $\lim_{n\to\infty} g_n^{(k)} =$ a.e.(λ) and $g_n^{(k)} \leqq c_k$ for $n = 1, 2, \cdots$, we have $\bar{P}g_n^{(k)} \leqq c_k$ for $n = 1, 2, \cdots$ and $\lim_{n\to\infty} \bar{P}g_n^{(k)} = 0$ a.e.(λ). Since $1 = 1_A + (I_{A'}P)1_A + (I_{A'}P)^2 1_A + \cdots$, we have $f_n \leqq c_k$ for $n = 1, 2, \cdots$ and $\lim_{n\to\infty} f_n = 0$ a.e.(λ). We have already proved that $\lim_{n\to\infty} (I_{A'}P)^n 1 = 0$ a.e.(λ). Hence $\lim_{n\to\infty} g_n^{(k+1)} = 0$ a.e.(λ) and $g_n^{(k+1)} \leqq c_{k+1}$ where $c_{k+1} = c_k + 1$. Thus (I) is shown to hold true for $k + 1$. Clearly $\lim_{n\to\infty} \bar{P}^n 1 \leqq (1-\epsilon)^k$ a.e.(λ) may be obtained by letting $n \to \infty$ in (3). Since k is an arbitrary positive integer, $\lim_{n\to\infty} \bar{P}^n 1 = 0$ a.e.(λ) and the proof of Theorem 1 is complete.

A real valued function $P(x, A)$ of two variables, $x \in X$, $A \in \mathcal{B}$, is called a *probability transition function* if the following conditions are satisfied.

T1. For every fixed $x \in X$, $P(x, \cdot)$ is a measure.
T2. For every fixed $A \in \mathcal{B}$, $P(\cdot, A)$ is an $\mathcal{B}$-measurable function.
T3. For all x, $P(x, X) \leqq 1$.

Define $P^{(n)}(x, A)$, $n = 1, 2, \cdots$, inductively by

$$P^{(1)}(x, A) = P(x, A),$$

$$P^{(n+1)}(x, A) = \int P^{(n)}(x, dy)P(y, A).$$

Then $P^{(n)}(x, A)$, $n = 1, 2, \cdots$, are also probability transition functions. A Markov operator is said to have a probability transition function $P(x, A)$ if, for all $f \in L_\infty(\lambda)$,

$$Pf(x) = \int P(x, dy)f(y)$$

for (λ) almost all x. If P has a probability transition function $P(x, A)$, then P^n has probability transition function $P^{(n)}(x, A)$. A λ-*continuous Markov* *operator* P is a Markov operator for which there is an $\mathcal{B} \times \mathcal{B}$ measurable function $p(x, y)$ such that $Pf(x) = \int p(x, y)f(y)\lambda(dy)$, [6], [7].

Let P be a Markov operator with a probability transition function $P(x, A)$. Define a measure η on $\mathcal{B} \times \mathcal{B}$ as follows. If E is an $\mathcal{B} \times \mathcal{B}$-measurable subset of $X \times X$,

$$\eta(E) = \int \lambda(dx) \int P(x, dy) 1_E(x, y) \tag{4}$$

where $1_E(x, y) = 1$ if $(x, y) \in E$, $= 0$ if $(x, y) \notin E$. η may be decomposed into two measures η_c and η_s,

$$\eta = \eta_c + \eta_s,$$

where η_c is absolutely continuous to $\lambda \times \lambda$ while η_s is singular to $\lambda \times \lambda$. Let $p_1(x, y)$ be a derivative of η_c with respect to $\lambda \times \lambda$ and let us define Markov operator P_1 by

$$P_1 f(x) = \int p_1(x, y) f(y) \lambda(dy). \tag{5}$$

Then P_1 is the largest λ-continuous Markov operator such that $P_1 \leqq P$, and is called the *λ-continuous part* of P ([8], Section III). Decomposition of probability transition functions with respect to a fixed measure has been used in the past by W. Doeblin [1] and J. L. Doob [3]. Here it is put in a slightly different setting.

Lemma 1. *Let P be a Markov operator with a probability transition function $P(x, A)$. If for (λ) almost all x, $P(x, \cdot)$ is absolutely continuous to λ, then P is λ-continuous. The converse is also true if $\mathcal{B}$ is generated by a countable collection.*

Proof. Let E_s be a set in $\mathcal{B} \times \mathcal{B}$ such that

$$\lambda \times \lambda(E_s) = 0 \quad \text{and} \quad \eta_s(E) = \eta(E \cap E_s)$$

for all $E \in \mathcal{B} \times \mathcal{B}$. Let

$$[E_s]^x = \{y: (x, y) \in E_s\}.$$

Then for (λ) almost all x, $\lambda([E_s]^x) = 0$. If for (λ) almost all x $P(x, \cdot)$ is absolutely continuous to λ. Then $P(x, [E_s]^x) = 0$ for (λ) almost all x so that $\eta_s(X \times X) = \eta(E_s) = \int \lambda(dx) P(x, [E_s]^x) = 0$. Hence $\eta_c = \eta$ and λ-continuity of P follows immediately.

The converse follows from Theorem 5 of [8].

Lemma 2. *Let P and P_1 be Markov operators with probability transition functions. If P_1 is λ-continuous then PP_1 and P_1P are λ-continuous.*

Proof. Let $P(x, A)$ be a probability transition function of P and P_1 be given by (5). Let $R = PP_1$. Then, for all $f \in L_\infty(\lambda)$,

$$Rf(x) = \int P(x, dz) \int p_1(z, y) f(y) \lambda(dy)$$

$$= \int \left[\int P(x, dz) p_1(z, y) \right] f(y) \lambda(dy)$$

so that R is λ-continuous. The operator $P_1 P$ has probability transition function

$$Q(x, A) = \int p_1(x, y) P 1_A(y) \lambda(dy).$$

If $p_1(x, \cdot)$ is λ-integrable (which is true for (λ) almost all x) then $Q(x, \cdot)$ is absolutely continuous to λ. The λ-continuity of $P_1 P$ follows from Lemma 1.

Theorem 2. *If P is a conservative, ergodic Markov operator with a probability transition function and P_n is the λ-continuous part of P^n, then either $P_n = 0$ for all n or $P_n 1 \uparrow 1$ a.e.(λ).*

Proof. By Lemma 2, $P_n P$ is λ-continuous. Since $P_n P \leqq P^{n-1}$, we have $P_n P \leqq P_{n+1}$. It follows that $P_n 1 = P_n P 1 \leqq P_{n+1} 1$. Hence the sequence $\{P_n 1\}$ is nondecreasing. Let the limit of the sequence be l. We also have $PP_n \leqq P_{n+1}$ for PP_n is λ-continuous by Lemma 2. Hence

$$PP_n 1 \leqq P_{n+1} 1. \tag{6}$$

Letting $n \to \infty$ in (6), we obtain

$$Pl \leqq l.$$

Since P is conservative and ergodic, l is a constant function. If $l = 0$, then $P_n 1 = 0$ for all n so that $P_n = 0$ for all n. If $l \neq 0$, then there is a positive integer r such that $P_r \neq 0$. Consider the Markov operator P^r. It is conservative and there is a positive integer δ such that the whole space X is partitioned into δ indecomposably P^r-closed sets (*cf.* Section II of [7]). Among these δ indecomposably P^r-closed sets, there is one set C such that $P_r I_C \neq 0$. Define measure λ_C on $\mathcal{B}$ by $\lambda_C(A) = \lambda(C \cap A)$. Since the values of the function $P^r f$ on C are not affected by the values of f outside C, P^r may be considered as a Markov operator on $L_\infty(\lambda_C)$ to $L_\infty(\lambda_C)$ which is then conservative and ergodic. For the same reason, P_r and $P^r - P_r$ may also be considered as Markov operators on $L_\infty(\lambda_C)$ to $L_\infty(\lambda_C)$. Applying Theorem 1 to $P^r - P_r$, we conclude that $(P^r - P_r)^n 1 \to 0$ a.e.(λ) on C. Write

$$P^{nr} = [P_r + (P^r - P_r)]^n. \tag{7}$$

If we expand the right side of (7), $(P^r - P_r)^n$ would be one term among $(n + 1)$ terms in the expansion, and is the only term which does not contain P_r as a factor. Hence $P^{nr} - (P^r - P_r)^n$ is λ-continuous by Lemma 2 so that

$$P_{nr} \geqq P^{nr} - (P^r - P_r)^n. \tag{8}$$

Since $P^{nr} 1 - (P^r - P_r)^n 1 \to 1$ a.e.(λ) on C, $l = 1$ a.e.(λ) on C by (8). Since l is a constant function, we have $l = 1$.

References

[1] W. Doeblin, Sur les propriétés asymtotiques de mouvement régis par certains types de chaines simples, *Bull. Math. Soc. Roum. Sci.*, **39** (1) (1937) 57–115; **39** (2) (1937) 3–61.
[2] J. L. Doob, *Stochastic Processes*, Wiley, New York, 1963.
[3] ———, Asymptotic properties of Markov transition probabilities, *Trans. Amer. Math. Soc.*, **63** (1948).
[4] Jacob Feldman, Subinvariant measures for Markoff operators, *Duke Math. J.*, **29** (1962) 71–98.
[5] Eberhard Hopf, The general temporally discrete Markoff process, *J. Rational Mech. Annal.*, **3** (1954) 13–45.
[6] Shu-Teh C. Moy, λ-continuous Markov chains, *Trans. Amer. Math. Soc.*, **117** (1965) 68–91
[7] ———, λ-continuous Markov chains II, *Trans. Amer. Math. Soc.*, **120** (1965) 83–107.
[8] ———, Period of an irreducible positive operator, *Illinois J. Math.*, **11** (1967) 24–39.

Syracuse University and
University of California, Santa Barbara.

Date communicated: December 18, 1967