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A CONVERGENCE RESULT FOR CRITICAL REVERSIBLE NEAREST PARTICLE SYSTEMS

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We consider critical reversible nearest particle systems. We assume that the associated renewal measure has large moments as well as some regularity conditions. It is shown that such processes, started from a nontrivial ergodic translation invariant distribution, converge in distribution to the upper invariant measure.

0. Introduction. In this paper we consider critical nearest particle systems and their distributional properties. A nearest particle system (NPS) is a spin system on $\{0,1\}^Z$ with flip rates given by

$$c(x,\eta)=\begin{cases}1, & \text{if } \eta(x)=1,\\ f(l_x,r_x), & \text{if } \eta(x)=0,\end{cases}$$

where $l_x = x-\sup\{y<x:\eta(y)=1\}$ and $r_x=\inf\{y>x:\eta(y)=1\}-x$ (either or both possibly ∞).

Of particular interest are the so-called *reversible* NPSs. These are systems where $f(l,r)$ is of the form $\beta(l)\beta(r)/\beta(l+r)$, $f(l,\infty)=f(\infty,l)=\beta(l)$, $f(\infty,\infty)=0$. If a NPS is reversible in the classical sense, then [see Liggett (1985)] $f(\cdot,\cdot)$ must be of this form. These processes are of mathematical interest because there is an array of reversible Markov chain techniques with which to analyze them. This paper considers reversible processes; we will also require the condition

$$(*)\qquad \frac{\beta(n)}{\beta(n+1)}\downarrow 1 \qquad \text{as } n\to\infty.$$

The convergence of the quotient to 1 is equivalent to the process being Feller, a naturally desirable property. The assumption of monotonic convergence down to 1 ensures that the process is *attractive*. This makes the process much more mathematically tractable. We also assume $\sum\beta(n)<\infty$.

We will assume that the process η_t is generated by a given Harris system (which will also generate auxiliary comparison processes as well): We suppose that we are given for each $x\in Z$ two Poisson processes D_x and B_x, independent of each other and independent over Z, with D_x of rate 1 and B_x of rate $M=(\beta(1))^2/\beta(2)$, the largest possible flip rate. The value (or spin) at site x can only change for t in either D_x or in B_x. The process D_x corresponds to flips of 1's or deaths of

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particles at site x and is simple. If $t \in D_x$ then $\eta_t(x) = 0$, irrespective of its value immediately preceding t. The Poisson process B_x corresponds to flips of a 0 to a 1 at site x and associated with each $t \in B_x$ is a random variable U_t uniform on [0, 1]. At time $t \in B_x$, we have $\eta_t(x) = 1$ either if $\eta_s(x)$ was equal to 1 immediately prior to t or if $U_t \leq c(x, \eta_{t-})/M$, for η_{t-} the state immediately before time t. If $\sum_x \eta_0(x) < \infty$, then the summability condition, $\sum_x \beta(x) < \infty$, ensures that, for all time t, $\sum_x \eta_t(x) < \infty$ and that the update times for the process form a.s. a discrete set. If $\sum_{x>0} \eta_0(x) = \sum_{x \leq 0} \eta_0(x) = \infty$, then a.s. for any $T > 0$ there exist $x_i^T \to \infty, x_{-i}^T \to -\infty$ as $i \to \infty$ so that, for all i, $\eta_0(x_i^T) = \eta_0(x_{-i}^T) = 1$ and $D_{x_i^T}, D_{x_{-i}^T}$ have no points on time interval $[0, T]$. It is clear that if we fix all sites outside spatial interval $[-n, n]$, while updating sites in $(-n, n)$ according to the above rules, then we obtain a process η_s^n. The existence of the x_i^T, x_{-i}^T's, above, shows that almost surely for every $T > 0, N > 0$, there exists $n_0 = n_0(\omega)$ so that $\forall n \geq n_0, \eta_s^n(x) = \eta_s^{n_0}(x)$ for $|x| \leq N, 0 \leq s \leq T$. We can thus define the infinite process η_t as the limit of the η_t^n's. The remaining case, where one of $\sum_{x>0} \eta_0(x), \sum_{x \leq 0} \eta_0(x)$ is finite and the other infinite, can be treated in a similar manner.

The attractiveness condition ensures that if the same Harris system generates two NPSs η_t and ξ_t and $\xi_0 \leq \eta_0$ with respect to the natural partial order, then $\xi_t \leq \eta_t$ for all times t. It also implies monotonicity relations between η and various finite comparison Markov chains, as detailed in Section 2.

We say that $\eta \in \{0, 1\}^Z$ is *finite* if $\sum_x \eta(x) < \infty$, that it is *semiinfinite* if one of $\sum_{x<0} \eta(x), \sum_{x>0} \eta(x)$ is infinite but not both and that it is *infinite* if both these terms are infinite. If η_0 is infinite, then for all t, η_t must be infinite, while the finiteness of $\sum \beta(n)$ ensures (as already noted) that if η_0 is finite (semiinfinite), then it will remain so for all subsequent times. We can thus speak of finite, semiinfinite and infinite NPSs.

We say that a NPS is *supercritical* if $\sum \beta(n) > 1$. In this case the finite and infinite systems survive. In the finite case this means that if $\tau = \inf\{t : \eta_t \equiv \mathbf{0}\}$, then $P^{\{0\}}[\tau = \infty] > 0$, where P^η denotes the probability for a process starting from initial configuration η. [See Griffeath and Liggett (1982), Liggett (1985).] For infinite systems survival means that there exists a nontrivial invariant distribution for the process, $\bar{\nu}$. In fact we have $\bar{\nu}$ [see Mountford (1997)], so that, for each η_0,

$$\eta_t \to P^{\eta_0}(\tau = \infty)\bar{\nu} + P^{\eta_0}(\tau < \infty)\delta_{\mathbf{0}}$$

in distribution. In particular all infinite systems must converge in distribution to the upper invariant distribution.

We say that a NPS is *subcritical* if $\sum \beta(n) < 1$. In this case finite systems must die out: $P^{\eta_0}(\tau = \infty) = 0$ for all finite η_0. Under the additional assumption of attractiveness, infinite subcritical systems die out in the sense that $\eta_t \to \delta_{\mathbf{0}}$ as t tends to infinity. In fact, under minimal assumptions, Mountford (1995) showed that for attractive subcritical systems $P(\eta_t(0) = 1)$ tended to zero exponentially fast.

The remaining case, $\sum \beta(n) = 1$, is called the critical case and is the subject of this paper. Griffeath and Liggett (1982) prove that in all cases the finite critical case NPS dies out. The infinite process, however, will survive if and only if $\sum n\beta(n) < \infty$. If this condition is satisfied, then the upper invariant distribution is the renewal measure on $\{0, 1\}^Z$ corresponding to probability measure on the integers, $\beta(\cdot)$. Subsequently we will denote this renewal measure by $\mathrm{Ren}(\beta)$.

We show:

THEOREM 1. *Let η_t be a critical, attractive, Feller, reversible NPS satisfying* $(**)$ *below with η_0 ergodic, translation invariant and of positive density. Then*

$$\eta_t \to \mathrm{Ren}(\beta),$$

the upper invariant distribution (*the renewal measure corresponding to β*).

REMARK 1. We note that, to show the above result, it will suffice to show that, for any fixed cylinder, increasing f and any positive ε arbitrarily small, that $E[f(\eta_t)]$ will exceed $\langle \mathrm{Ren}(\beta), f\rangle - \varepsilon$ for t large, where for any measure ν and continuous function f, $\langle \nu, f\rangle = \int f(\omega)\nu(d\omega)$.

REMARK 2. Liggett (1985) notes that, since Bernoulli of sufficiently high density stochastically dominates the associated renewal measure, an attractive, reversible NPS with such an initial distribution must converge to the upper invariant distribution.

REMARK 3. By the ergodic theorem it immediately follows that η_t converges in distribution to $\mathrm{Ren}(\beta)$ as t becomes large whenever η_0 has a translation-invariant distribution such that $P(\eta_0 \equiv 0) = 0$.

The condition $(**)$ below is a more extreme form of the assumptions used in Mountford and Sweet (1998). The high order moment condition is by no means optimal, but we feel that it is better to give a clear exposition showing the ideas than to become involved in technical issues which may obscure the central idea.

We assume

$$(**) \qquad n\left(\frac{\beta(n)}{\beta(n+1)} - 1\right) \to k \geq 120.$$

If $k = \infty$ we also assume that, for some $\alpha \in (0, 1)$,

$$0 < \liminf_{n\to\infty} n^\alpha \left(\frac{\beta(n)}{\beta(n+1)} - 1\right) \leq \limsup_{n\to\infty} n^\alpha \left(\frac{\beta(n)}{\beta(n+1)} - 1\right) < \infty.$$

These conditions (and in particular those for $k = \infty$) are far from necessary for the arguments in this paper to work. We use these simple cases to avoid undue technicalities in a paper already overburdened with technicalities.

It is easy to check the following lemma.

LEMMA 0.1. *The assumptions* $(**)$ (*including the extra assumption for the case* $k = \infty$) *imply*

$$(***) \qquad \sup_n \sum_{l+r=n} \frac{\beta(l)\beta(r)}{\beta(n)} \le V < \infty.$$

Observe that if k is finite, then for any $\delta > 0$ we have for large enough x that

$$\beta(x) \ll x^{-(k-\delta)} \quad \text{and} \quad \bar{\beta}(x) = \sum_{y \ge x} \beta(y) \ll x^{-(k-1-\delta)}.$$

The proofs in this paper assume tacitly that the constant k above is finite. The statements are all still valid for infinite k, and the proofs are typically easier. We leave the minimal changes to the reader.

As detailed above, our process (and auxiliary processes) is constructed via a system of Poisson processes. Repeatedly throughout the paper we will be using large deviation bounds for Poisson random variables; given this and the elementary nature of the following lemma we feel it is appropriate to introduce it here.

LEMMA 0.2. *For fixed* $c > 0$ *and* $\alpha \in (1/2, 1)$, *there exists a constant* K *so that, for* $y \ge 1$, *a Poisson random variable* X^y *of mean* y *satisfies*

$$P(|X^y - y| \ge cy^\alpha) < Ke^{-c^2 y^{2\alpha-1}/3}.$$

The proof follows from Stirling's formula and, for reasons of space as well as its elementary nature, is left to the reader.

We now give a brief sketch of the plan of attack.

The basic picture comes from Schinazi (1992), who considered semiinfinite critical reversible NPSs. A right-sided NPS is one for which, for all time t, $\sum_{x<0} \eta_t(x) = \infty$ and $\sum_{x>0} \eta_t(x) < \infty$. For such processes we denote the position of the rightmost particle at time $t[\sup\{x : \eta_t(x) = 1\}]$ by r_t (and similarly for left-sided semiinfinite processes the leftmost particle is denoted by l_t). We say a semiinfinite process is in equilibrium if, seen from its rightmost particle (respectively leftmost particle), the configuration is distributed as a renewal measure to the left (resp. right). It is easy to check that, for the one-sided process continually shifted so that its rightmost particle occupies the origin, this measure is invariant. We say a right semiinfinite process η^R is supported by $(-\infty, x]$ at time 0 if its rightmost particle at time 0 is at site x; η_0^R is in equilibrium supported on $(-\infty, x]$ if it is vacant on interval (x, ∞) and if on $(-\infty, x]$ its distribution is renewal measure conditioned on there being a particle at x.

Schinazi (1992) showed the following theorem.

THEOREM 0.1. *Let η^R be a semiinfinite process in equilibrium supported by $(-\infty, 0]$ at time 0. Then r_{N^2t}/N tends in distribution to a nontrivial Brownian motion as N tends to infinity.*

Purely to avoid too much notation we will make the following assumption which will be in force throughout the sequel:

(∗∗∗∗) the diffusion constant for the limiting Brownian motion is 1.

Theorem 0.1 is the starting point for this paper. It suggests the following picture of the evolution of an infinite critical NPS: After a short time the NPS η_t can be seen as a succession of intervals alternately in the upper invariant regime and in the lower regime (i.e., vacant). The interfaces between these intervals perform like independent coalescing Brownian motions, each coalescence corresponding to the disappearance of a particular interval. That is, we might expect the NPS to behave like a one-dimensional voter model in which intervals of 1's and 0's compete, though in our case the intervals of 1's would not be intervals of fully occupied sites but rather intervals in the "upper regime."

Various problems present themselves: First, the initial distribution could be far from an alternation of intervals in the upper regime and intervals completely vacant; second, we have to be more precise about the evolution of finite intervals in the "upper regime"; and third, this picture cannot give us what we want. The one-dimensional voter model is symmetric between 0's and 1's and if we start with a nontrivial translation invariant system, the system will converge in distribution to a nontrivial mixture of all 1's and all 0's.

The first problem is not too difficult: First, note that we are dealing with an attractive system and we are trying to show convergence to the upper invariant distribution; second, if we run the process for any amount of time, the process will have the possibility of forming large fully occupied intervals. This follows from the irreducibility of the finite systems. We can then remove all particles that do not belong to a fully occupied interval of length N say. In fact we can suppose that after a fixed time our configuration consists of disjoint occupied intervals of length precisely N.

The second problem presents the bulk of the work of this paper. Our main tool, which was not available to previous workers in the area, is the method of estimating the spectral gap of reversible Markov chains developed by Jerrum and Sinclair (1990); see Diaconis and Stroock (1991) for a clear exposition. This technique was used in Mountford and Sweet (1998), where a useful comparison finite state Markov chain was introduced. This enables us to clumsily reprove the result of Schinazi (1992) (in a dramatically reduced setting) in a way that can be used for finite intervals. It also enables us to establish good regeneration properties of finite, semiinfinite and infinite NPSs.

The third objection prompts us to add to our picture: In Section 5 we detail how the attractiveness of the system gives a slight drift outward for the boundaries of an interval in the "upper regime." This enables us to "grow" new "upper regime" intervals and tips the process toward the "upper regime."

The paper runs as follows. Section 1 establishes some crude bounds for the deviations of the rightmost particle for one-sided processes starting in equilibrium and starting with full occupancy to the left of the origin and full vacancy to the right. These inequalities are built on in Section 2, which gives the first application of the comparison processes in Mountford and Sweet (1998) and various coupling arguments to obtain good large deviations estimates for one-sided processes. Section 3 uses more coupling to give a central limit theorem for the rightmost particle and to give a measure of closeness to normality for the edge fluctuations of a semiinfinite process in equilibrium. Section 4 applies results of Section 3 to finite NPSs and establishes a technical "regeneration" result. It is also shown that for a sufficiently "rich" finite system the two edge fluctuations can essentially be thought of as being those of two independent semiinfinite processes. Section 5 gives the key technical result for "growing new upper equilibrium" intervals. Section 6 introduces an (unfortunately) complicated system of rules for upper equilibrium intervals; those intervals which fail are to be killed. Good bounds are given for the various probabilities of an interval being killed. The following section establishes that these killing rules do not handicap the upper regime intervals too much. Section 8 shows that upper regime intervals must eventually dominate. The proof of Theorem 1 is completed in Section 9.

Throughout the paper we adopt the convention that nonspecific constants may change their value from line to line or even from one side of an inequality to another so that we may write, for example, for $x \geq 0$, $C(x^2+1)e^{-cx} \leq Ce^{-cx}$.

1. We consider some simple inequalities for a one-sided NPS, η_t with $\eta_0 \equiv 0$ on $(0,\infty)$. Let $r_t = \sup\{x : \eta_t(x) = 1\}$. Let $\mu = \sum_n n\beta(n)$.

LEMMA 1.1. *For $\eta_0 \equiv 0$ on $(0,\infty)$, r_t satisfies*

$$P\Bigl(\sup_{s\leq t} r_s \geq 2\mu t\Bigr) \leq \frac{C}{t^{(k-3)/2}}$$

for t large.

PROOF. By attractiveness we need only consider a NPS with all deaths suppressed and with $\eta_0 \equiv 1$ on $(-\infty,0]$. We can w.l.o.g. take t to be equal to n, an integer. For this modified process it is immediate that $E[r_t] = t\mu$. It is almost immediate that $E[r_1^{k-2}] < \infty$; r_n is equal in distribution to $\sum_{j=1}^n r_1^j$, the sum of n independent copies of r_1. We suppose further (taking the worst case) that $k-2$ is an odd integer. Then

$$E\bigl[\bigl(r_n - E(r_n)\bigr)^{k-3}\bigr] \leq C_{k-3} n^{(k-3)/2}.$$

So, by Chebyshev, $P(r_n \geq 2\mu n)$ is less than or equal to

$$P(|r_n - E[r_n]| \geq \mu n) \leq \left(\frac{1}{\mu n}\right)^{k-3} C_{k-3} n^{(k-3)/2} \leq \frac{C}{n^{(k-3)/2}}. \qquad \square$$

The corresponding inequality for $\inf_{s \leq t} r_s$ over configurations with $\eta_0(0) = 1$, is false. However, we have, by reversibility [and attractiveness for part (ii)], the following lemma.

LEMMA 1.2. (i) *For η_0 distributed according to renewal measure supported on $(-\infty, 0]$ and null on $(0, \infty)$ we have*

$$P\left(\sup_{s \leq t} |r_s| \geq 2\mu t\right) \leq \frac{C}{t^{(k-3)/2}}.$$

(ii) *Starting from $\eta_0 \equiv 1$ on $(-\infty, 0]$, we have*

$$P\left(\sup_{s \leq t} |r_s| \geq 2\mu t\right) \leq \frac{C}{t^{(k-3)/2}}.$$

PROOF. (i) The position of the rightmost particle of η_s is less than the sum of all jumps to the right of the rightmost particle by time s, which in turn is less than the sum of all jumps to the right by time t. However, this latter quantity is treated in Lemma 1.1. We thus have

$$P\left(\sup_{s \leq t} r_s \geq 2\mu t\right) \leq \frac{C}{t^{(k-3)/2}},$$

where C is the constant of Lemma 1.1. The time reversal of η as seen from the rightmost particle is equal in distribution to η. From this we see that, in equilibrium, the sum of the magnitude of jumps leftward of the rightmost particle over the time interval $[0, t]$ is equal in distribution to the sum of the magnitudes of the rightward jumps of the rightmost particle. Therefore as we have that $-r_s$ is dominated by the sum of the magnitudes of leftward jumps in time interval $[0, s]$, we have

$$P\left(\inf_{s \leq t} r_s \leq -2\mu t\right) \leq \frac{C}{t^{(k-3)/2}}.$$

The result follows with $C = 2C$.

(ii) We have already seen in Lemma 1.1 that in this case

$$P\left(\sup_{s \leq t} r_s \geq 2\mu t\right) \leq \frac{C}{t^{(k-3)/2}}.$$

On the other hand by attractiveness we have that

$$P\left(\inf_{s \leq t} r_s \leq -2\mu t\right)$$

is less than the corresponding probability for the rightmost particle of a process started in equilibrium. Thus by (i) above we have

$$P\Big(\inf_{s\le t} r_s \le -2\mu t\Big) \le \frac{C}{t^{(k-3)/2}}. \qquad \square$$

We now introduce the notion of a *gap* for a configuration η. This is a vacant interval. We say that η has a gap of length b in interval I (or originating in I) if for some interval J of length b we have $\eta \equiv 0$ on J and J intersects I.

A natural question is at what rate will a one-sided process on $(-\infty, 0]$ feel the effects of its infinite tail. We gain some crude bounds on this (which can later be refined). Consider $\eta^{\mathbf{1}}$, the NPS starting fully occupied on $(-\infty, 0]$, completely vacant on $(0, \infty)$. Let $\eta^{n^{9/4}}$ be the (finite) NPS starting with the occupied sites being precisely those in interval $[-n^{9/4}, 0]$. We assume that the processes are generated by the same Harris system and therefore that, for all times t, $\eta_t^{n^{9/4}} \le \eta_t^{\mathbf{1}}$. Let $d_t = \sup\{x : \eta_t^{\mathbf{1}}(x) \ne \eta_t^{n^{9/4}}(x)\}$. Necessarily we must have $\eta_t^{\mathbf{1}}(d_t) = 1$, $\eta_t^{n^{9/4}}(d_t) = 0$.

LEMMA 1.3. $P(\sup_{s\le n^2} d_s \ge -2n^{9/4}/3) \le Cn^5/n^{(k-2)/4}$.

PROOF. We majorize d_s by a nondecreasing process D_s, where D_s is a jump process, initially at $-n^{9/4} - 1$, that jumps forward by x units at time t if and only if:

(i) $\eta_{t-}^{\mathbf{1}} \equiv 0$ on $(D_{t-}, D_{t-} + x]$;

and also:

(ii) there is an attempted birth at site $D_t + x$ (i.e., if y is the distance to the right of site x until there is an occupied site in $\eta_t^{\mathbf{1}}$, then we have $t \in B_{D_t + x}$ and $U_t \le \frac{\beta(x)\beta(y)}{\beta(x+y)}/M$).

By assumption $(***)$, we have that D_s jumps at rate bounded by V, uniformly over all joint configurations. We easily see that $\eta_s^{n^{9/4}}$ and $\eta_s^{\mathbf{1}}$ agree on random space interval (D_s, ∞) so

$$\Big\{\sup_{s\le n^2} d_s \ge -2n^{9/4}/3\Big\} \subset B(1) \cup B(2),$$

where

$$B(1) = \{\exists t \le n^2 \text{ for which } \eta_t^{\mathbf{1}} \text{ has an } n^{1/4}/6V \text{ gap in } [-n^{9/4}, -2n^{9/4}/3]\},$$

$$B(2) = \{D_s \text{ has } \ge 2Vn^2 \text{ jumps in } [0, n^2]\}.$$

This corresponds to the fact that if D_{n^2} is large, then either D made a large number of jumps or some of the jumps were large.

The jumps of D_s occur at a rate always bounded by V [recall our choice of V in $(***)$]. Accordingly the probability of event $B(2)$ is bounded by the probability that a Poisson random variable of mean Vn^2 exceeds $2Vn^2$. By Lemma 0.2 with $\alpha = 3/4$, say, we have $P(B(2)) \le e^{-Cn}$.

To bound $P(B(1))$ we introduce a comparison one-sided process η, generated by the same Harris system as $\eta^{\mathbf{1}}$ but with initial configuration equal to a renewal process supported on $(-\infty, 0]$. We denote its rightmost particle's position at time t by r_t. By attractiveness, $\eta_s \le \eta_s^{\mathbf{1}}$ $\forall s$. Therefore to bound the probability of $B(1)$ it is enough to bound the probability of

$$B(1)' = \{\exists t \le n^2 \text{ for which } \eta_t \text{ has an } n^{1/4}/6V \text{ gap in } [-n^{9/4}, -2n^{9/4}/3]\}.$$

This event is easily seen to be contained in the union

$$\begin{aligned} &B(1a) \cup B(1b) \\ &\quad = \{\exists t \le n^2 \text{ for which } \eta_t \text{ has an } n^{1/4}/6V \text{ gap starting in } [r_t - 2n^{9/4}, r_t]\} \\ &\qquad\qquad \cup \left\{ \sup_{s \le n^2} |r_s| \ge n^{9/4}/3 \right\}. \end{aligned}$$

By Lemma 1.2, the second event, $B(1b)$ above, has probability bounded above by C/n^{k-3}. By invariance the process η_t seen from r_t is always a one-sided renewal process in distribution. From elementary theory of i.i.d. random variables, the probability a renewal sequence with a particle at the origin has an $n^{1/4}/6V$ gap in the interval $[-2n^{9/4}, 0]$ is bounded by $Cn^3/n^{(k-2)/4}$. Therefore

$$Y = \int_0^{n^2+1} I_{\eta_t \text{ has an } n^{1/4} \text{ gap in } [r_t - 2n^{9/4}, r_t]} \, dt$$

has expectation bounded by $C(n^2+1)n^3/n^{(k-2)/4}$. Furthermore should such a gap appear it will remain in this condition for time 1 with probability at least $e^{-(V+1)}$. Thus by Chebyshev we have

$$P\big(B(1a)\big) \le e^{(1+V)} Cn^5/n^{(k-2)/4}.$$

Thus we have

$$\begin{aligned} P\Big(\sup_{s \le n^2} d_s \ge -2n^{9/4}/3\Big) &\le P\big(B(1a)\big) + P\big(B(1b)\big) + P\big(B(2)\big) \\ &\le Cn^5/n^{(k-2)/4}. \end{aligned}$$ □

In the same way we can prove the following lemma.

LEMMA 1.4. *Let η, η' be two one-sided NPSs supported on $(-\infty, 0]$ at time 0. Suppose also that η is in one-sided equilibrium and that $\eta_0 \equiv \eta'_0$ on $[-n^{9/4}, 0]$. Then outside probability $Cn^5/n^{(k-2)/4}$,*

$$\eta_s \equiv \eta'_s \quad \text{on } \big[-\tfrac{2}{3}n^{9/4}, \infty\big) \qquad \forall 0 \le s \le n^2.$$

2. The basic coupling. In dealing with infinite NPSs we find it useful to introduce three comparison NPS-like continuous Markov chains (hereafter MC's) on $\{0,1\}^{[-n,n]}$. These chains were discussed in Mountford and Sweet (1998) and spectral bounds calculated for them:

1. For Z_t^n, 1's are fixed at $-n$ and n; otherwise the sites flip according to NPS flip rates; $Z_t^{n,\mathbf{1}}$ is Z_t^n with $Z_0^n \equiv 1$ on $[-n,n]$.
2. For Y_t^n, 1's are fixed at $-n$ and n, 0's are fixed on $(-n,-n+n^{1/3})$ and $(n-n^{1/3},n)$ but otherwise sites flip according to NPS rates.
3. Process X_t^n is the same as Y_t^n except that the Markov chain is not allowed to hit the configuration which is null on $(-n,n)$. We tacitly assume that $X_0^n \not\equiv \mathbf{0}$ on $(-n,n)$.

If η_0 is an infinite NPS and η_t evolves according to a Harris system H, then we can use the same Harris system to generate versions of the above MC's on $\{0,1\}^{[-n,n]}$.

The comparison involving process Z^n with η (both using the same Harris system) is the most straightforward: if

$$Z_0^n|_{(-n,n)} \geq \eta_0|_{(-n,n)}$$

(where $\eta|_A$ denotes configuration η restricted to subset A), then from attractiveness we have immediately that the following holds:

(a) $\forall t,\ \eta_t|_{[-n,n]} \leq Z_t^n|_{[-n,n]} \leq Z_t^{n,\mathbf{1}}|_{[-n,n]}$.

The comparison involving Y^n is not so straightforward in that for Y^n we are not simply fixing 1's but both 0's and 1's. The large imposed intervals of 0's for process Y^n "should" result in a process "below" η; however, any domination may break down when η itself is vacant on interval $(-n,-n+n^{1/3})$ or $(n-n^{1/3},n)$. However, if we define $\tau = \inf\{s>0 : \eta_s \equiv 0 \text{ on either } (-n,-n+n^{1/3}) \text{ or } (n-n^{1/3},n)\}$ and if $Y_0^n|_{(-n+n^{1/3},n-n^{1/3})} \leq \eta_0|_{(-n+n^{1/3},n-n^{1/3})}$, then the following holds:

(b) $\forall t \leq \tau,\ \eta_t|_{(-n,n)} \geq Y_t^n|_{(-n,n)}$.

The comparison involving X^n is even more involved in that X^n may jump over η on $(-n,n)$ by virtue of the fact that deaths of X^n particles may be suppressed. We define $\sigma = \inf\{s : Y_s^n \equiv 0 \text{ on } (-n,n)\}$. We have if

$$Y_0^n|_{(-n+n^{1/3},n-n^{1/3})} = X_0^n|_{(-n+n^{1/3},n-n^{1/3})} \leq \eta_0|_{(-n+n^{1/3},n-n^{1/3})},$$

then the following holds:

(c) $\forall t \leq \tau \wedge \sigma,\ \eta_t|_{(-n,n)} \geq X_t^n|_{(-n,n)}$.

Thus Z^n always approximates η from above on $(-n,n)$; Y^n approximates η from below until a big gap for η appears; X^n approximates η from below until either a big η-gap appears or Y^n becomes vacant on $(-n,n)$.

The reason for introducing process X^n as well as Y^n is that process X^n (like Z^n) has a spectral gap of size greater than or equal to c/n^2 some $c > 0$, whereas the gap for Y^n is typically much smaller than this [see Mountford and Sweet (1998) for details]. We introduce notation P^{ξ, V^n} to refer to probabilities for process $V_n (= X^n, Y^n$ or $Z^n)$ starting from configuration ξ. We say a configuration ξ on $\{0,1\}^{[-n,n]}$ is *bad* if

$$P^{\xi', Y^n}(\sigma \leq n^8) \geq n^{-k/6},$$

where ξ' is equal to ξ except within $n^{1/3}$ of the endpoints $n, -n$ where ξ' is vacant.

The following lemmas are easy to establish. See Mountford and Sweet (1998).

LEMMA 2.1. *Let π^n be the equilibrium probability of X^n and let $\pi^{n,Y}$ be that for Y^n. Then for $\xi \in \{0,1\}^{[-n,n]} \setminus \{\mathbf{0}\}$,*

$$0 \leq \pi^n(\xi) - \pi^{n,Y}(\xi) \leq C\pi^n(\xi)/n^{k/3}.$$

LEMMA 2.2. *Let η_t be a right semiinfinite NPS in equilibrium. Let r_t be the rightmost particle. The chance that either:*

(i) *for $t \leq T$ there is a gap of size $n^{1/3}$ within S of r_t;*
(ii) *for $t \leq T$ $\eta_t \circ \theta_{-r_t+x}|_{[-n,n]}$ is bad for $n \leq x \leq S$;*

is bounded by $3(V+1)(T+1)(S+1)n \cdot n^{-k/6}$.

LEMMA 2.3. *Under equilibrium, the probability that X^n has a gap of size $n/5$ or larger is bounded by $C/n^{k/2}$ for C not depending on n. Under equilibrium, the probability that Y^n hits $\mathbf{0}$ in time n^8 is bounded by $K/n^{k/3}$.*

We now get comparisons between the stationary probabilities of $\pi^{n,1}$, the equilibrium probability for Z^n, and π^n, the equilibrium probability for X^n. Define $\pi^{l,r}$ to be the renewal measure restricted to $[l,r]$ and conditioned to have 1's at l and r. (So $\pi^{n,1} = \pi^{-n,n}$.)

LEMMA 2.4. *For $x \in [-4n/5, 4n/5]$,*

$$0 \leq \left|\pi^{n,1}(\{\xi : \xi(x) = 1\}) - \pi^n(\{\xi : \xi(x) = 1\})\right| \leq C/n^{k/3}.$$

PROOF. It follows from attractiveness that $\pi^{n,1}$ stochastically dominates $\pi^{n,Y}$. So the upper bound for $\pi^n(\{\xi : \xi(x) = 1\}) - \pi^{n,1}(\{\xi : \xi(x) = 1\})$ follows from Lemma 2.1. It also follows from attractiveness that, for $-n \leq l \leq r \leq n$, the restriction of $\pi^{n,1}$ to $[l,r]$ is stochastically dominated by $\pi^{l,r}$. Given $\eta \in \{0,1\}^{[-n,n]}$, define L to equal $\inf\{y > -n : \eta(y) = 1\}$ and R to be $\sup\{y < n :$

$\eta(y)=1\}$. It follows from the Markov property of renewal measures that, for $x\in[-4/5n,4/5n]$,

$$\begin{aligned}\pi^n(\{\eta:\eta(x)=1\}) &\geq \sum_{l\leq -4/5n, r\geq 4/5n}\pi^{l,r}(\{\eta:\eta(x)=1\})\pi^n(\{L=l,R=r\})\\ &\geq \pi^{n,1}(\{\eta:\eta(x)=1\})-2\pi^n(\{L>-4/5n\}).\end{aligned}$$

The result now follows from Lemma 2.3. $\square$

For the following note that for some $\nu\in(0,\infty)$ and not depending on n we have for all n and $\xi\in\{0,1\}^{[-n,n]}\setminus\mathbf{0}$ that $\pi^n(\{\xi\}),\pi^{n,1}(\{\xi\})\geq e^{-\nu n}$.

COROLLARY 2.1. *For X_0^n arbitrary in $\{0,1\}^{[-n,n]}\setminus\mathbf{0}$, if X_t^n and $Z_t^{n,\mathbf{1}}$ are derived from the same Harris system, then, outside probability $Cn/n^{k/3}$,*

$$Z_{n^4}^{n,\mathbf{1}}(x)=X_{n^4}^n(x)\qquad\forall x\in[-4n/5,4n/5].$$

PROOF. First consider the auxiliary process $Z^{n^4/2}$, which is derived from the same Harris system as the other two processes, whose dynamics are those of $Z^{n,\mathbf{1}}$ but which starts at time $n^4/2$ at full occupancy. By attractiveness of NPS we have that, for all $t\geq n^4/2$, $Z_t^{n^4/2}\geq Z_t^{n,\mathbf{1}}$. In particular $Z_{n^4}^{n^4/2}\geq Z_{n^4}^{n,\mathbf{1}}$. However, also by the spectral gaps for these Markov chains we have that

$$\begin{aligned}P\left(Z_{n^4}^{n^4/2}\neq Z_{n^4}^{n,\mathbf{1}}\right)&\leq\sum_x P\left(Z_{n^4}^{n^4/2}(x)\neq Z_{n^4}^{n,\mathbf{1}}(x)\right)\\ &\leq Kn\left(e^{\nu n}e^{-(c/n^2)n^4/2}\right)^{1/2}\\ &\leq Ke^{-n}.\end{aligned}$$

Thus we may compare instead process $Z^{n^4/2}$ with X^n. Now by the usual spectral gap arguments applied to process X^n, we have that the Radon–Nykodym derivative of the distribution of $X_{n^4/2}^n$ with respect to the equilibrium distribution of Y^n is bounded by 2 for n large. Thus by Lemma 2.3 we have that the chance that process X^n at some time in interval $[n^4/2,n^4]$ has a jump to all 0's blocked is bounded by $K/n^{k/3}$. So outside this probability we have that $X_{n^4}^n\leq Z_{n^4}^{n^4/2}$. It remains to bound the probability that $X_{n^4}^n<Z_{n^4}^{n^4/2}$.

However,

$$\begin{aligned}&P\left(Z_{n^4}^{n^4/2}(x)=1,X_{n^4}^n(x)=0\right)\\ &\quad\leq\left|P\left(Z_{n^4}^{n^4/2}(x)=1\right)-P\left(X_{n^4}^n(x)=1\right)\right|+P\left(Z_{n^4}^{n^4/2}(x)=0,X_{n^4}^n(x)=1\right)\\ &\quad\leq C'/n^{k/3}\qquad\text{for some }C',\end{aligned}$$

by standard spectral theory and Lemma 2.4. The result follows by summing over the (of order n) x. $\square$

From similar (but easier) arguments one has the following corollary:

COROLLARY 2.2. *For Z_0^n arbitrary in $\{0,1\}^{[-n,n]}$, if Z_t^n and $Z_t^{n,\mathbf{1}}$ are derived from the same Harris system, then outside probability Ke^{-n},*

$$Z_{n^{7/2}}^{n,\mathbf{1}}(x) = Z_{n^{7/2}}^n(x) \qquad \forall x \in [-n,n].$$

COROLLARY 2.3. *For η any infinite or semiinfinite NPS let $\eta^{\mathbf{1}}$ be a NPS starting from full occupancy and generated by the same Harris systems as η and let Y^n be the Markov chain described at the start of the section with Y_0^n agreeing with η_0 on interval $(-n+n^{1/3}, n-n^{1/3})$. If:*

(i) $\sigma = \inf\{t : \eta_t \equiv 0$ *on* $[-n, -n+n^{1/3}]$ *or* $[n-n^{1/3}, n]\}$;
(ii) $\tau = \inf\{t : Y_t^n \equiv 0$ *on* $(-n,n)\}$;

then

$$P\big(\eta_{n^4}^{\mathbf{1}}(x) = \eta_{n^4}(x),\ \forall x \in (-4n/5, 4n/5)\big) \ge 1 - \frac{Cn}{n^{k/3}} - P(\tau \le n^4) - P(\sigma \le n^4).$$

The following result is a key to the paper but as important are the techniques employed in the proof which are used in Section 4. Recall $\mu = \sum_n n\beta(n)$.

LEMMA 2.5 (A first coupling). *Consider two processes* (*generated by the same Harris system*):

(a) η_t *where* η_0 *is in* (*semiinfinite*) *equilibrium*, *supported on* $(-\infty, 0]$ *and vacant on* $(0,\infty)$;

(b) $\eta^{\mathbf{1}}$ *started from all sites on* $(-\infty, -4\mu n^{4/5}]$ *occupied and vacant on* $(-4\mu n^{4/5}, \infty)$.

Then outside of probability $(Cn^{13/4}/n^{k/30})$, $r_t \ge r_t^{\mathbf{1}}$ $\forall t \in [0, n^2]$, *where*

$$r_t = \sup\{x : \eta_t(x) = 1\},$$
$$r_t^{\mathbf{1}} = \sup\{x : \eta_t^{\mathbf{1}}(x) = 1\}.$$

Also $\eta_t^{\mathbf{1}} \le \eta_t$ *on* $[-n^{9/4}/2, \infty)$ *for* $n^{4/5} \le t \le n^2$.

PROOF. It is clear that we need only prove the inequality for n large. Lemma 1.2 implies that, outside of probability bounded by $Kn^{2(k-3)/5}$, on the time interval $[0, n^{4/5}]$,

$$r_s \ge -2\mu n^{4/5}, \quad r_s^{\mathbf{1}} \le -2\mu n^{4/5}.$$

This deals with the conclusions of the lemma in the case $t \in [0, n^{4/5}]$. We introduce the process γ, run with the same Harris system, where

$$\gamma_0(x) = I_{x \in [-n^{9/4} - 4\mu n^{4/5}, -4\mu n^{4/5}]}.$$

Since by Lemma 1.3 we have (outside probability $Kn^5/n^{(k-2)/4}$) that

$$\forall 0 \le t \le n^2, \qquad \eta_t^{\mathbf{1}} \le \gamma_t \quad \text{on } \left[-\tfrac{2}{3}n^{9/4}, \infty\right),$$

our lemma will follow if we can show that (outside of probability $Cn^{13/4}/n^{k/30}$) $\gamma_{n^{4/5}} \le \eta_{n^{4/5}}$.

Now note that by Lemma 1.1 (applied to the leftmost particle of γ, l^γ, as well as to the rightmost, r^γ), we have that, outside of probability bounded by $Kn^{2(k-3)/5}$, for $t \in [0, n^{4/5}]$, γ_t is vacant outside interval $[-n^{9/4} - 6\mu n^{4/5}, -2\mu n^{4/5}]$. Divide the interval $[-n^{9/4} - 6\mu n^{4/5}, -2\mu n^{4/5}]$ into m ($\le Kn^{9/4-1/5}$) equal (w.l.o.g.) intervals of length $2n^{1/5} + 1$, disjoint except at the endpoints, $I_0, I_1, \dots, I_m$. Now let $J_0, J_1, \dots, J_{m+1}$ be disjoint (except at the endpoints) intervals of length $2n^{1/5} + 1$ such that the left endpoint of I_i is the center of J_i and the right endpoint of I_m is the center of J_{m+1}.

Let us define Z_t^i to be NPS on $\{0, 1\}^{I_i}$ with 1's fixed at the endpoints of I_i and such that $Z_0^i \equiv 1$ on I_i, and define Y_t^i to be the NPS on $\{0, 1\}^{I_i}$ with 1's fixed at the endpoints of I_i, 0's fixed within $n^{1/15}$ of the endpoints and (subject to the above) $Y_0^i \equiv \eta_0$ on I_i. Let X^i be the process on $\{0, 1\}^{I_i}$ which has $X_0^i = Y_0^i$ and which evolves like Y^i except that it is forbidden to hit all zeros on the interior of I_i.

Then we have the following:

Outside of probability $K(2n^{9/4} + 1)(n^{4/5} + 1)n^{1/5}n^{-k/30}$ [by Lemma 2.2(i)], there is no $n^{1/15}$ gap of η_t for $0 \le t \le n^{4/5}$, within $2n^{9/4}$ of r_t.

Outside of probability $Kn^{-2(k-3)/5}$ (by Lemma 1.1),

$$\begin{aligned}
|r_t| &\le 2\mu n^{4/5} && \forall 0 \le t \le n^{4/5}, \\
r_t^\gamma &\le -2\mu n^{4/5} && \forall t \le n^{4/5}, \\
l_t^\gamma &\ge -n^{9/4} - 6\mu n^{4/5} && \forall t \le n^{4/5}.
\end{aligned}$$

Outside of probability $K(2n^{9/4}+1)n^{1/5}n^{-k/30} + Kn^{9/4-1/5}n^{-k/30}$ [by Lemma 2.2(ii)],

$$\forall i, \forall 0 \le t \le n^{4/5}, \quad Y_t^i \text{ is not identically zero on the interior of } I_i.$$

Outside of probability $n^{(9/4)-(1/5)}n^{-(k-3)/15}$ (by Corollary 2.1),

$$\forall i, \forall x \text{ in the central } 4/5 \text{ of } I_i, \qquad X_{n^{1/4}}^i(x) = Z_{n^{1/4}}^i(x).$$

The sum of these probabilities is bounded by $Kn^{13/4}/n^{k/30}$.

For the x's that are in the central four-fifths of an I_i, we have (outside of the sum of the above probabilities) that

$$\eta_{n^{4/5}}^i(x) \ge Y_{n^{4/5}}^i(x) = X_{n^{4/5}}^i = Z_{n^{4/5}}^i(x) \ge \gamma_{n^{4/5}}(x).$$

Repeating this argument with the J's replacing the I's we get, outside of probability $(Cn^{13/4}/n^{k/30})$,

$$\eta_{n^{4/5}}(x) \geq \gamma_{n^{4/5}}(x) \qquad \forall x.$$

This completes the proof that (outside of the contracted probability), at time $n^{4/5}$ and therefore (by attractiveness) for all $t \geq n^{4/5}$,

$$\gamma \leq \eta. \qquad \square$$

In the same way (but using Lemma 1.4 instead of Lemma 1.3) we obtain the following lemma.

LEMMA 2.6. *Consider two semiinfinite right NPSs in equilibrium*, η, η', *so that both are generated by same Harris system*:

(a) η_0 *is in* (*semiinfinite*) *equilibrium supported on* $(-\infty, 0]$;
(b) η' *is supported on* $(-\infty, -4\mu n^{4/5}]$;
(c) η_0 *and* η'_0 *are independently distributed.*

Then, *outside of probability* $(Cn^{13/4}/n^{k/30})$, $r_t \geq r'_t$ $\forall t \in [0, n^2]$, *where*

$$r_t = \sup\{x : \eta_t(x) = 1\},$$
$$r'_t = \sup\{x : \eta'_t(x) = 1\}.$$

Also $\eta'_t \leq \eta_t$ *on* $[-n^{9/4}/2, \infty)$ *for* $n^{4/5} \leq t \leq n^2$.

We will be (two sections hence) getting further couplings but first we use these lemmas to obtain some bounds on the probabilities of large deviations for r_t and similar processes. For now we give the following corollary.

COROLLARY 2.4. *Let* $\eta_0^{\mathbf{1}}(x) = I_{x \leq 0}$. *Let* $r_t^{\mathbf{1}} = \sup\{x : \eta_t^{\mathbf{1}}(x) = 1\}$. *Then as* $t \to \infty$,

$$P\left(\sup_{s \leq t} r_s^{\mathbf{1}} \geq \lambda\sqrt{t}\right) \to 2\Phi(-\lambda),$$

where Φ *is the distribution function for a standard normal random variable*.

PROOF. We need only treat t of the form n^2. Let r_t^b be the rightmost particle of an equilibrium NPS which is initially supported on $(-\infty, 0]$ and is run by the same Harris system as $\eta^{\mathbf{1}}$. From attractiveness we have immediately that

$$P\left(\sup_{s \leq t} r_s^{\mathbf{1}} \geq \lambda\sqrt{t}\right) \geq P\left(\sup_{s \leq t} r_s^{b} \geq \lambda\sqrt{t}\right).$$

From Schinazi (1992) and the reflection principle we have that

$$P\left(\sup_{s \leq t} r_s^{b} \geq \lambda\sqrt{t}\right) \to 2\Phi(-\lambda),$$

as n becomes large.

It remains to achieve an upper bound. Let r_t^u be the rightmost particle of an equilibrium NPS which is initially supported on $(-\infty, 4\mu n^{4/5}]$ and is run by the same Harris system as $\eta^{\mathbf{1}}$. Lemma 2.5 yields that

$$P\left(\sup_{s\le t} r_s^{\mathbf{1}} \ge \lambda\sqrt{t}\right) \le P\left(\sup_{s\le t} r_s^b \ge \lambda\sqrt{t}\right) + Cn^{13/4}/n^{k/30}.$$

However, again via [Schinazi (1992)] and the reflection principle, as n becomes large the right-hand side converges to $2\Phi(-\lambda)$. $\square$

3. The purpose of this section is to prove large deviations bounds on the behaviour of $\max\{r_s : 0 \le s \le 2^n\}$ for a right semiinfinite NPS supported in equilibrium on $(-\infty, 0]$, either in equilibrium or starting from full occupancy on this interval.

Our first tool is:

LEMMA 3.1. *Let $\eta_t^{\mathbf{1}}$ be a semiinfinite NPS, with $\eta_0^{\mathbf{1}}$ equal to* 1 *on* $(-\infty, 0]$ *and equal to* 0 *elsewhere. Let its rightmost particle at time t be denoted by $r_t^{\mathbf{1}}$. For n sufficiently large we have*

$$P\left(\sup_{s\le 2^n} r_s^{\mathbf{1}} \ge n2^{n/2}\right) \le C\left(\frac{1}{2}\right)^{n/2} \qquad \textit{for } n \textit{ large},$$

$$P\left(\sup_{s\le 2^n} r_s^{\mathbf{1}} \ge n^2 2^{n/2}/4\right) \le \frac{K2^n}{2^{n/2(k-2)}} \qquad \textit{for } n \textit{ large}.$$

PROOF. From Corollary 2.4, we have that, for t large enough,

$$P\left(\sup_{s\le t} r_s^{\mathbf{1}} \ge 2\sqrt{t}\right) \le \frac{1}{16}.$$

Now we use a simple Hoffman–Jørgensen argument: if we denote by τ_m the first time that $r_s^{\mathbf{1}}$ is greater than or equal to $\sqrt{t}(2^m + (2^{m-1} - 1)/t^{1/3})$, then for $m > 1$ we have $\{\tau_m \le t\} \subset \{\tau_{m-1} \le t\}$ and that either

$$r_{\tau_{m-1}} \le \sqrt{t}\left(2^{m-1} + \frac{2^{m-2} - 1}{t^{1/3}}\right) + \frac{\sqrt{t}}{t^{1/3}}$$

or

$$\left\{\exists 0 \le s \le t : r_s - r_{s-} \ge \frac{\sqrt{t}}{t^{1/3}}\right\}.$$

(Here r_{s-} is the limit of r as s is approached from the left.) The probability of the latter event is bounded by $t/(t^{1/2}/t^{1/3})^{k-2}$ for t large by assumption $(**)$. By attractiveness and the strong Markov property applied at time τ_1 we have

$$
\begin{aligned}
P\left(\sup_{s\le t} r_s^{\mathbf{1}} \ge \sqrt{t}\left(4+\frac{1}{t^{1/3}}\right)\right) &\le \left(\frac{1}{16}\right)^2 + t\left(\frac{t^{1/2}}{t^{1/3}}\right)^{-(k-2)} \\
&\le \left(\frac{1}{16}\right)^2 + 2t\left(\frac{t^{1/2}}{t^{1/3}}\right)^{-(k-2)} \\
\Rightarrow P\left(\sup_{s\le t} r_s^{\mathbf{1}} \ge \sqrt{t}\left(8+\frac{3}{t^{1/3}}\right)\right) &\le \left(\left(\frac{1}{16}\right)^2 + 2t^{-(k/6-2)}\right)^2 + t^{-(k/6-2)} \\
&\le \left(\frac{1}{16}\right)^4 + 2t^{-(k/6-2)}
\end{aligned}
$$

[where for the last inequality we used $(x+y)^2 = x^2 + (2x+y)y \le x^2 + y/2$ if $2x+y \le 1/2$]. This proves, after iteration of the argument, the inequality

$$
P\left(\sup_{s\le t} r_s^{\mathbf{1}} \ge \sqrt{t}\left(2^m + \frac{2^{m-1}-1}{t^{1/3}}\right)\right) \le \left(\frac{1}{16}\right)^{2^{m-1}} + 2t^{-(k/6-2)};
$$

thus (with $t = 2^n$, $m = \lfloor \log_2(n) \rfloor - 1$, where $\lfloor\ \rfloor$ represents the integer part of), we have for n large enough

$$
P\left(\sup_{s\le 2^n} r_s^1 \ge n2^{n/2}\right) \le P\left(\sup_{s\le 2^n} r_s^{\mathbf{1}} \ge 2^{n/2}\left(2^m + \frac{2^{m-1}-1}{2^{n/3}}\right)\right) \le 2\left(\frac{1}{2}\right)^{n/2}
$$

(recall k is large). For the second inequality, we first define

$$
\text{for } m \ge 0 \quad \sigma_m = \inf\{t : r_t^{\mathbf{1}} \ge n2^{n/2}3^m\}.
$$

Repeating the Hoffman–Jørgensen argument we have (on the event $\{\sigma_m \le 2^n\}$), either

$$
r_{\sigma_{m-1}}^{\mathbf{1}} \le 2n2^{n/2}3^{m-1}
$$

or

$$
\exists 0 \le s \le 2^n: \qquad r_s - r_{s-} \ge n2^{n/2}3^{m-1}.
$$

Using the strong Markov property at σ_{m-1} as before, we conclude that

$$
P(\sigma_m < 2^n) \le \big(P(\sigma_{m-1} < 2^n)\big)^2 + 2^n(n2^{n/2}3^{m-1})^{-(k-2)}.
$$

Provided that n was fixed sufficiently large we have for all m that

$$
P(\sigma_{m-1} < 2^n) > (2^n 2^{-n(k-2)/2})^{2/3} \quad \Rightarrow \quad P(\sigma_m < 2^n) \le \big(P(\sigma_{m-1} < 2^n)\big)^{3/2}.
$$

However, it is easy to see that (again provided that n was fixed large) for $m = \log_2(n)/2$ that

$$\left(P(\sigma_0 < 2^n)\right)^{(3/2)^m} \le 2^n (n2^{n/2}3^{m-1})^{-(k-2)}.$$

We conclude that for some $m \le \log_2(n)/2$ we have

$$P(\sigma_m < 2^n) \le (2^n 2^{-n(k-2)/2})^{2/3}.$$

We then consider for example σ_{m+1} and conclude

$$P\left(\sup_{s\le 2^n} r_s^1 \ge \frac{n^2 2^{n/2}}{4}\right) \le \frac{K2^n}{2^{n/2(k-2)}} \qquad \text{for } K < \infty \text{ not depending on } n. \quad \square$$

From this, attractiveness and reversibility we obtain:

PROPOSITION 3.1. *Let η_0 be in semiinfinite equilibrium with $r_0 = 0$:*

$$P\left(\sup_{s\le 2^n} |r_s| \ge 2n2^{n/2}\right) \le C\left(\tfrac{1}{2}\right)^{n/2} \qquad \textit{for } n \textit{ large};$$

$$P\left(\sup_{s\le 2^n} |r_s| \ge \frac{n^2}{2}2^{n/2}\right) \le \frac{K2^n}{2^{n/2(k-2)}} \qquad \textit{for } n \textit{ large}.$$

PROOF. The inequalities

$$P\left(\sup_{s\le 2^n} r_s \ge n2^{n/2}\right) \le C\left(\tfrac{1}{2}\right)^{n/2} \qquad \text{for } n \text{ large},$$

$$P\left(\sup_{s\le 2^n} r_s \ge \frac{n^2}{4}2^{n/2}\right) \le \frac{K2^n}{2^{n/2(k-2)}} \qquad \text{for } n \text{ large}$$

follow directly from Lemma 3.1 and attractiveness. By reversibility we also have

$$P(r_{2^n} \le -n2^{n/2}) \le C\left(\tfrac{1}{2}\right)^{n/2} \qquad \text{for } n \text{ large},$$

$$P\left(r_{2^n} \le -\frac{n^2}{4}2^{n/2}\right) \le \frac{K2^n}{2^{n/2(k-2)}} \qquad \text{for } n \text{ large}.$$

Now let us define stopping time τ by $\tau = \inf\{t : r_t \le -2n2^{n/2}\}$. By Lemma 3.1 and attractiveness (or simply by Lemma 2.5 and Schinazi's theorem) and the strong Markov property we have

$$P(r_{2^n} > -n2^{n/2} | F_\tau) \le C\left(\tfrac{1}{2}\right)^{n/2}$$

on $\{\tau < 2^n\}$, where F_τ is the σ-field generated by the Harris system Poisson processes up to stopping time τ.

Therefore

$$P\Big(\inf_{s\le 2^n} r_s \le -2n2^{n/2}\Big) \le P(r_{2^n} \le -n2^{n/2}) + C\big(\tfrac{1}{2}\big)^{n/2},$$

from which it easily follows that $P(\inf_{s\le 2^n} r_s \le -2n2^{n/2}) \le C(1/2)^{n/2}$. We conclude

$$P\Big(\sup_{s\le 2^n} |r_s| \ge 2n2^{n/2}\Big) \le C\big(\tfrac{1}{2}\big)^{n/2}.$$

The second inequality follows in an entirely similar manner. □

We can now "improve" Lemma 2.6:

PROPOSITION 3.2. *Consider two semiinfinite right NPSs in equilibrium η, η' so that both are generated by same Harris system*:

(a) η_0 *is supported on* $(-\infty, 0]$;
(b) η'_0 *is supported on* $(-\infty, -2^m]$.

Let the rightmost occupied site at time t for these processes be respectively r_t, r'_t. Then, outside of probability $(C2^{41m/8}/2^{km/15})$, $r_t \ge r'_t$ $\forall t \in [0, 2^{3m}]$ *and*

$$\forall 2^{8m/5} \le t \le 2^{3m}, \qquad \eta_t \ge \eta'_t$$

on $[2^{-25m/8}/2, \infty)$.

PROOF. Our approach follows that for Lemma 2.5 closely, so we will pass quickly over the areas which are essentially the same and stress the differences.

Divide interval $[-22^{-25m/8}, -2^m/2]$ into (w.l.o.g.) intervals $I_0, I_1, \dots, I_v$ of length $22^{2m/5}+1$, disjoint except for the endpoints. Take $J_0, J_1, \dots, J_{v+1}$ so that, for $i \le v$, the midpoint of J_i is the left endpoint of I_i and so that the midpoint of J_{v+1} is the right endpoint of I_v.

Let us define Z^i_t to be NPS on $\{0,1\}^{I_i}$ with 1's fixed at the endpoints of I_i and such that $Z^i_0 \equiv 1$ on I_i; define Y^i_t to be the NPS on $\{0,1\}^{I_i}$ with 1's fixed at the endpoints of I_i, 0's fixed within $2^{2m/15}$ of the endpoints and (subject to the above) $Y^i_0 \equiv \eta_0$ on I_i. Let X^i be the process on $\{0,1\}^{I_i}$ which has $X^i_0 = Y^i_0$ and which evolves like Y^i except that it is forbidden to hit all zeros on the interior of I_i. Let $Y^{i\prime}, X^{i\prime}$ be the processes defined when η is replaced by η'.

Then outside of probability $K2^{41m/8}2^{-mk/15}$ we have for the x's that are in the central four-fifths of an I_i,

$$\eta_{2^{8m/5}}(x) \ge Y^i_{2^{8m/5}}(x) = X^i_{2^{8m/5}} = Z^i_{2^{8m/5}}(x) \ge \eta_{2^{8m/5}}(x),$$

$$\eta'_{2^{8m/5}}(x) \ge Y'^i_{2^{8m/5}}(x) = X'^i_{2^{8m/5}} = Z^i_{2^{8m/5}}(x) \ge \eta'_{2^{8m/5}}(x)$$

and so

$$\eta_{2^{8m/5}}(x) = \eta'_{2^{8m/5}}(x).$$

Repeating this argument with the J's replacing the I's we get, outside of the above probability,

$$\eta_{2^{8m/5}} \leq \eta'_{2^{8m/5}} \qquad \text{on } [-22^{25m/8}, \infty).$$

We now argue as in Lemma 1.3. We define the bad set B to be the union of the following unlikely events:

$$\Big\{ \sup_{t \leq 2^{3m}} |r_t - r_0| \geq 2^{2m} \Big\},$$

$$\Big\{ \sup_{t \leq 2^{3m}} |r'_t - r'_0| \geq 2^{2m} \Big\},$$

$$\{\text{for } s \leq 2^{3m}, \eta'_s \text{ has a } 2^{m/8}/6V \text{ gap in } [r'_s - 32^{25m/8}, r'_s]\}.$$

By Proposition 3.1 and Lemma 2.2 we have that $P(B) \leq K2^{41m/8}2^{-mk/15}$.

Now define nondecreasing process D_s, $s \geq 2^{8m/5}$, by $D_0 = -22^{25m/8}$ and, at time t, D jumps forward by x units if and only if:

(i) $\eta'_{t-} \equiv 0$ on $(D_{t-}, D_{t-} + x]$, and also

(ii) there is an attempted birth at site $D_t + x$ (i.e., if y is the distance to the right of site x until there is an occupied site in η'_t, then we have $t \in B_{D_t+x}$ and $U_t \leq \frac{\beta(x)\beta(y)}{\beta(x+y)}/M$).

Then we have that the following hold (until $D_s = r'_s$) on event B^c:

1. $\eta_s \geq \eta'_s$ on (D_s, ∞);
2. D jumps forward by at most $2^{m/8}/6V$;
3. D jumps forward at a rate bounded by V.

Thus, if

$$\eta_{2^{8m/5}} \geq \eta'_{2^{8m/5}} \qquad \text{on } [-22^{25m/8}, \infty),$$

then

$$\forall 2^{8m/5} \leq t \leq 2^{3m}, \qquad \eta_t \geq \eta'_t \quad \text{on } [-2^{25m/8}, \infty),$$

unless either B occurs or D has at least $2V2^{3m}$ jumps in time interval $[2^{8m/5}, 2^{3m}]$. The result follows from the bounds on the tails of Poisson random variables as supplied by Lemma 0.2. $\square$

PROPOSITION 3.3. *Let η_s be a semiinfinite NPS supported on $(-\infty, 0]$, either in equilibrium on this interval or in full occupancy. For all $1 \leq l < k - 1$ there exists a constant C_l so that, for all t sufficiently large, $E[(|r_t|/\sqrt{t})^l] \leq C_l < \infty$.*

PROOF. We treat the case of full occupancy, leaving the equilibrium case to the reader. Fix g so that $l < g < k-1$. Choose $x_0 > 4$ so that

$$2\int_{x_0}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du < \frac{1}{2x_0^g}.$$

Now by Schinazi's result (or Corollary 2.4) we have that there exists t_0 so that for all $t \ge t_0$ we have

$$P\Bigl(\sup_{s\le t} r_s \ge x_0\sqrt{t}\Bigr) \le \frac{1}{x_0^g},$$

and [by assumption $(**)$]

$$\forall y \ge 4\sqrt{t}, \qquad \sum_{z\ge y} \beta(z) < \frac{1}{y^g}.$$

Employing our Hoffman–Jørgensen argument we have

$$\begin{aligned} P\Bigl(\sup_{s\le t} r_s \ge 3x_0\sqrt{t}\Bigr) &\le \Bigl(\frac{1}{x_0^g}\Bigr)^2 + t(x_0\sqrt{t})^{-g} \\ &\le \frac{1}{(3x_0)^g}\Bigl(\frac{3^g}{x_0^g} + \frac{1}{t^{(g-2)/2}}\Bigr) \le \frac{1}{(3x_0)^g}. \end{aligned}$$

Applying this argument repeatedly, we find for all $x \ge x_0$ that

$$P(r_t \ge x\sqrt{t}) \le \frac{3^g}{x^g}.$$

However, we have

$$P(r_t \le -x\sqrt{t}) \le P(r_t \ge x\sqrt{t})$$

from reversibility so we are done. $\square$

COROLLARY 3.1. *There exists constant C so that, for a NPS beginning in equilibrium supported on* $(-\infty, 0]$,

$$\begin{aligned} 1 - C\bigl(2^{-n/6} + (2^{41n/24}2^{-nk/45})^{1-3/k}\bigr) & \\ \le \frac{E[r_{2^n}^2]}{2^n} \le 1 + C\bigl(2^{-n/6} + (2^{41n/24}2^{-nk/45})^{1-3/k}\bigr).& \end{aligned}$$

PROOF. The proof for upper and lower bounds is essentially the same so we will only consider the left-hand inequality:

$$E[r_{2^{n+1}}^2] = E[(r_{2^n} + r_{2^{n+1}} - r_{2^n})^2],$$

but if $r'_t = r_{2^{n+1}} - r_{2^n}$, then

$$E[r^2_{2^{n+1}}] = E[(r_{2^n} + r'_{2^n})^2] = 2E[r^2_{2^n}] + 2E\big[r_{2^n} E^{\eta_{2^n}}[r_{2^n}]\big]$$
$$= 2E[r^2_{2^n}] + 2E\big[r_{2^n} E^{\eta_{2^n}}[r_{2^n}] I_{r_{2^n}>0}\big] + 2E\big[r_{2^n} E^{\eta_{2^n}}[r_{2^n}] I_{r_{2^n}<0}\big]$$

so

$$\frac{E[r^2_{2^{n+1}}]}{2^{n+1}} \le \frac{E[r^2_{2^n}]}{2^n} + \frac{E[r_{2^n} E^{\eta_{2^n}}[r_{2^n}] I_{r_{2^n}>0}]}{2^n} + \frac{E[r_{2^n} E^{\eta_{2^n}}[r_{2^n}] I_{r_{2^n}<0}]}{2^n}.$$

We analyze the term $E[r_{2^n} E^{\eta_{2^n}}[r_{2^n}] I_{r_{2^n}>0}]$; the third term is handled in a manner entirely similar. Introduce a comparison NPS $\gamma_t, t \ge 2^n$, which is generated by the same Harris system as η on $t \ge 2^n$, which at time 2^n is in equilibrium supported on $(-\infty, r_{2^n} + 2^{n/3}]$ but whose distribution at time 2^n is otherwise independent of η_{2^n}. Let A be the event that $r_{2^{n+1}} > r^\gamma_{2^{n+1}}$, where r^γ denotes the rightmost particle of process γ. (By Proposition 3.2 we have that the probability of event A is bounded by $C2^{41n/24}/2^{nk/45}$.) We have

$$\begin{aligned} E\big[r_{2^n} E^{\eta_{2^n}}[r_{2^n}] I_{r_{2^n}>0}\big] &\le E\big[r_{2^n}[r^\gamma_{2^{n+1}} - r_{2^n}] I_{r_{2^n}>0}\big] \\ &\quad + E\big[r_{2^n}[r_{2^{n+1}} - r_{2^n}] I_{r_{2^n}>0} I_A\big] \\ &\quad - E\big[r_{2^n}[r^\gamma_{2^{n+1}} - r_{2^n}] I_{r_{2^n}>0} I_A\big]. \end{aligned}$$

The term $E[r_{2^n}[r^\gamma_{2^{n+1}} - r_{2^n}] I_{r_{2^n}>0}]$ is actually equal to $E[r_{2^n} I_{r_{2^n}>0} 2^{n/3}]$, by the conditional independence of γ_{2^n} and the independent increments property of the Poisson processes, and thus bounded by $K2^{5n/6}$.

By Hölder's inequality we have

$$\begin{aligned} &E\big[r_{2^n} E^{\eta_{2^n}}[r_{2^n}] I_{r_{2^n}>0} I_A\big] \\ &\quad \le \big(E[(r_{2^n})^{k/3}(r_{2^{n+1}} - r_{2^n})^{k/3}]\big)^{3/k} P(A)^{1-3/k} \\ &\quad \le \big(E[(r_{2^n})^{2k/3}]\big)^{3/2k}\big(E[(r_{2^{n+1}} - r_{2^n})^{2k/3}]\big)^{3/2k} P(A)^{1-3/k}, \end{aligned}$$

by Cauchy–Schwarz. By Propositions 3.2 and 3.3 we have that this last bound is less than $C(2^n)(2^{41n/24}2^{-nk/45})^{1-3/k}$. We similarly bound $E[r_{2^n}(r^\gamma_{2^{n+1}} - r_{2^n}) \times I_{r_{2^n}>0} I_A]$.

So

$$\frac{E[r^2_{2^{n+1}}]}{2^{n+1}} \le \frac{E[r^2_{2^n}]}{2^n} + C2^{-n/6} + C(2^{41n/24}2^{-nk/45})^{1-3/k}.$$

Iterating and using the limit value of $E[r^2_t]/t$ we have

$$\begin{aligned} \frac{E[r^2_{2^n}]}{2^n} &\ge 1 - \sum_{m \ge n}\big(C2^{-m/6} + C(2^{41m/24}2^{-mk/45})^{1-3/k}\big) \\ &\ge 1 - C2^{-n/6} + C(2^{41n/24}2^{-nk/45})^{1-3/k}. \end{aligned}$$ □

PROPOSITION 3.4. *Let η be a semiinfinite NPS in equilibrium on* $(-\infty, 0]$. *Then there exist independent semiinfinite NPSs in equilibrium on* $(-\infty, 0]$, $\eta^1, \dots, \eta^{2^{2n/7}}$, *such that, with probability* $1 - C2^{45n/14}/2^{4nk/105}$,

$$r_{2^{2n}} \geq \sum_{i=1}^{2^{2n/7}} r^i_{2^{12n/7}} - 2^{6n/7}.$$

There exist independent semiinfinite NPSs on $(-\infty, 0]$, $\eta^1, \dots, \eta^{2^{2n/7}}$, *such that, with probability at least* $1 - C2^{45n/14}/2^{4nk/105}$, $r_{2^{2n}} \leq \sum_{i=1}^{2^{2n/7}} r^i_{2^{12n/7}} + 2^{6n/7}$.

PROOF. Since η is in equilibrium, for all $i = 0, 1, \dots, 2^{2n/7}$, we have (by Proposition 3.2) that, outside of probability $C2^{41n/14}/2^{4nk/105}$,

$$r_{(i+1)2^{12n/7}} - r_{i2^{12n/7}} \geq r^{i'}_{(i+1)2^{12n/7}} - r^{i'}_{i2^{12n/7}} - 2^{4n/7},$$

where $r^{i'}$, is the rightmost particle for process $\eta^{i'}$, the one sided NPS starting at time $i2^{12n/7}$ with $\eta^{i'}_{i2^{12n/7}}$ supported on $(-\infty, r_{i2^{12n/7}} - 2^{4n/7}]$ but otherwise independent of $F_{i2^{12n/7}}$ and in equilibrium on $(-\infty, r_{i2^{12n/7}} - 2^{4n/7}]$, and run with the same Harris system as η on time interval $[i2^{12n/7}, (i+1)2^{12n/7}]$. The first assertion is proven by taking η^i_s to be $\eta^i_{i2^{12n/7}+s}$. The second follows similarly. $\square$

Let $r^i_{2^{12n/7}}$ be independent copies of $r_{2^{12n/7}}$ (for a NPS initially supported on $(-\infty, 0]$ in equilibrium). It is easily seen, by standard embedding methods, that we can find a standard Brownian motion, B (possibly with enlarged filtration) and stopping times $\tau_i \leq \tau_{i+1}$, $i = 0, 1, 2, \dots$, so that the following hold:

1. $\tau_0 = 0$;
2. for $i \geq 1$, $B(\tau_i)$ is equal in distribution to $\sum_{j=1}^{i} r^j_{2^{12n/7}}$;
3. the random variables $\tau_{i+1} - \tau_i$ are independent identically distributed random variables with [for $p \leq (k-3)/2$],

$$E[(\tau_{i-1} - \tau_i)^p] = C_p E[(r_{2^{12n/7}})^{2p}].$$

Therefore using Proposition 3.3 we have that, for even $p \leq (k-2)/2$,

$$E\left[\left(\frac{\tau_{2^{2n/7}} - E[\tau_{2^{2n/7}}]}{2^{2n}}\right)^p\right] = E\left[\left(\sum_1^{2^{2n/7}} \frac{\tau_i - \tau_{i-1} - E[\tau_1]}{2^{12n/7}}\right)^p\right]\left(\frac{1}{2^{2n/7}}\right)^p.$$

By Proposition 3.3, this is less than

$$H_p 2^{np/7}\left(\frac{1}{2^{2n/7}}\right)^p \leq H_p \frac{1}{2^{pn/7}}.$$

(Here H_p is a constant not depending on n.)

Thus for any even integer $p \leq (k-2)/2$ we have by Chebyshev that

$$P\big(\big|\tau_{2^{2n/7}} - E[\tau_{2^{2n/7}}]\big| \geq 2^{2n-n/14}\big) \leq H_p \frac{1}{2^{np/14}}.$$

By taking p to be greater than 28 (recall $k \geq 100$), this last term is bounded by 2^{-2n} for n large. We therefore obtain after division by $\sqrt{E[\tau_{2^{2n/7}}]}$ that, for any $c \geq 0$,

$$\begin{aligned} P&\left(\left|\frac{\sum_{i=1}^{2^{2n/7}} r^i_{2^{12n/7}}}{\sqrt{E[\tau_{2^{2n/7}}]}} - \frac{B(E[\tau_{2^{2n/7}}])}{\sqrt{E[\tau_{2^{2n/7}}]}}\right| \geq c\right) \\ &\leq 2^{-2n} + P\left(\sup_{|t-1| \leq K2^{-n/14}} |B(t) - B(1)| \geq c\right), \end{aligned}$$

where $K = 2^{2n}/E[\tau_{2^{n/2}}]$. By Proposition 3.3, $K \leq 2$ for n large and we have that, for large n,

$$P\left(\left|\frac{\sum_{i=1}^{2^{2n/7}} r^i_{2^{12n/7}}}{\sqrt{E[\tau_{2^{2n/7}}]}} - \frac{B(E[\tau_{2^{2n/7}}])}{\sqrt{E[\tau_{2^{2n/7}}]}}\right| \geq 2^{-n/30}\right) \leq 2 \times 2^{-2n}.$$

The above and Proposition 3.4 yield the existence of standard normal random variables Z_1, Z_2 so that for large n, outside of probability $K2^{45n/14}/\, 2^{4nk/105} + 4 \times 2^{-2n}$, we have

$$\frac{r_{2^{2n}}}{\sqrt{E[\tau_{2^{2n/7}}]}} < 2^{-n/30} + Z_1, \qquad \frac{r_{2^{2n}}}{\sqrt{E[\tau_{2^{2n/7}}]}} > -2^{-n/30} + Z_2.$$

Clearly, outside of probability $K2^{45n/14}/2^{4nk/105} + 4 \times 2^{-2n}$, we have $Z_1 + 2 \times 2^{-n/30} \geq Z_2$.

By Cauchy–Schwarz, we have that, for any event A,

$$E[(Z_2 - Z_1)I_A] \leq \big(P(A)\big)^{1/2}\big(E[(Z_2 - Z_1)^2]\big)^{1/2} \leq 2\big(P(A)\big)^{1/2}.$$

Therefore we have

$$\begin{aligned} 2 \times 2^{-n/30} &= E[Z_2 + 2 \times 2^{-n/30} - Z_1] \\ &\geq E[(Z_2 + 2 \times 2^{-n/30} - Z_1)I_{Z_2 > Z_1 + 2^{-n/60}}] \\ &\quad - 2(K2^{45n/14}/2^{4nk/105} + 4 \times 2^{-2n})^{1/2}. \end{aligned}$$

So, by Chebyshev, we have

$$P(Z_2 > Z_1 + 2^{-n/60}) \leq 2^{n/60}\big(2 \times 2^{-n/30} + (K2^{45n/14}/2^{4nk/105} + 4 \times 2^{-2n})^{1/2}\big).$$

Therefore, putting this all together we have the following proposition.

PROPOSITION 3.5. *For n large there is a standard normal r.v. Z so that*

$$\left|\frac{r_{2^{2n}}}{\sqrt{E[\tau_{2^{2n/7}}]}} - Z\right| \leq 2^{-n/60}$$

outside of a set of probability

$$2^{n/60}(3 \times 2^{-n/30} + (K2^{45n/14}/2^{4nk/105})^{1/2} + 4 \times 2^{-2n})$$

for K not depending on n.

4. This section is unfortunately heavily technical and repetitive. It serves to assemble many results which will be used in later sections. The basic approach is always to argue (as in Section 2) that if a NPS does not have large gaps over a certain interval and time period and has a reasonable initial configuration, and if the time period is large, then at the end of it the NPS should look like renewal measure on the spatial interval.

As part of this program we wish to introduce "finite NPSs in equilibrium" and record some "regenerations results." We have already abused definitions by talking of one-sided NPSs "in equilibrium." We take this one step further by introducing "finite NPSs in equilibrium." As has already been noted, finite NPSs must die out so the term is nonsense. Nevertheless we believe the suggestiveness of the term outweighs this consideration.

A finite NPS in equilibrium $(\eta_t)_{t\geq T}$, supported on (spatial) interval $[x, y]$, is *generated* by one-sided equilibrium NPSs η^R, η^L over (time) interval $[S, T]$ if the following hold:

1. η_S^L is supported on $[x, \infty)$;
2. η_S^R is supported on $(-\infty, y]$;
3. $(\eta_t^L, \eta_t^R)_{t\geq S}$ are generated by the same Harris system;
4. $\eta_T \equiv \eta_T^L$ on $(-\infty, \frac{x+y}{2})$, $\eta_T \equiv \eta_T^R$ on $[\frac{x+y}{2}, \infty)$;
5. $(\eta_t)_{t\geq T}$ is generated by the same Harris system as $(\eta_t^L, \eta_t^R)_{t\geq S}$.

Note that η_T may have occupied sites outside of interval $[x, y]$ and that sites x, y may or may not be occupied. This is in contrast to the useage of "η is supported" for semi infinite processes.

The following result shows that, for "sensible" choices of time interval $[S, T]$ given interval length $y - x$, the definition of η is not so arbitrary.

LEMMA 4.1. *Let η^R be a one-sided NPS in equilibrium supported by $(-\infty, N]$ at time 0 and η^L a one-sided NPS in equilibrium supported by $[0, \infty)$ at time 0. Let η_0^R, η_0^L be independent but let the same Harris system generate the two processes. Outside of probability $C(N\log^p(N)+1)(2N+1)(N\log^p(N))^{1/4}/(N\log^p(N))^{k/24} \leq KN^{5/2}/N^{k/24}$, we have that, at time $N\log^p(N)$ for $|p| \leq 7$, η^R is equal to η^L on $[N/10, 9N/10]$. The constant K may be taken independent of p.*

PROOF. As usual we only need treat the case where N is large. The proof closely follows that of Lemma 2.5 and Proposition 3.2. We denote the rightmost particle of η_t^R by r_t and the leftmost particle of η_t^L by l_t. First note that by Proposition 3.1 (which applies equally to η^L) we have that, outside probability $KN\log^p(N)/(N\log^p(N))^{(k-2)/2}$,

(i)

$$\sup_{0\le t\le N\log^p(N)} \{|r_t-r_0|, |l_t-l_0|\} \le N^{2/3}.$$

We also have by Lemma 2.2(i) (which again applies to η^L) that, outside probability $K(N\log^p(N)+1)(2N+1)(N\log^p(N))^{1/4}/(N\log^p(N))^{k/24}$:

(ii) $\nexists 0 \le t \le N\log^p(N)$ so that η_t^R has an $(N\log^p(N))^{1/12}$ gap in $[r_t-2N, r_t]$;

(iii) $\nexists 0 \le t \le N\log^p(N)$ so that η_t^L has an $(N\log^p(N))^{1/12}$ gap in $[l_t, l_t+2N]$;

and that by Lemma 2.2(ii) (outside of the same probability):

(iv) $\nexists x \in [-N, N-(N\log^p(N))^{1/4})$ so that

$$\eta_0^R \circ \theta_x\big|_{[-(N\log^p(N))^{1/4},(N\log^p(N))^{1/4}]} \text{ is bad;}$$

(v) $\nexists x \in [(N\log^p(N))^{1/4}, 2N]$ so that

$$\eta_0^L \circ \theta_x\big|_{[-(N\log^p(N))^{1/4},(N\log^p(N))^{1/4}]} \text{ is bad.}$$

Let B (for bad) be the union of the complements of events 1–5. The above may be summarized by the bound

$$P(B) \le K\big(N\log^p(N)+1)(2N+1)(N\log^p(N)\big)^{1/4}/\big(N\log^p(N)\big)^{k/24}.$$

We divide the interval $[N/10, 9N/10]$ into disjoint (except at the endpoints) intervals of length $2(N\log^p(N))^{1/4}+1, \dots, I_1, I_2, \dots, I_V$, where $V \le N$. As with Lemma 2.5 we also take intervals $J_0, J_1, \dots, J_V$ of the same length so that, for $i < V$, J_i has midpoint equal to the left endpoint of I_{i+1}, while J_V has midpoint equal to the right endpoint of I_V. As with Lemma 2.5 we introduce processes $Y^{i,L}, Y^{i,R}, Z^i$ on $\{0,1\}^{I_i}$ and $Y^{i,L'}, Y^{i,R'}, Z^{i,'}$ on $\{0,1\}^{J_i}$, where:

(a) Z^i is the finite nearest particle system (generated by the Harris system for η^R, η^L) which has 1's fixed at the endpoints of I_i and initially has all sites occupied;

(b) $Y^{i,L}, Y^{i,R}$ are the finite NPSs (again generated by the given Harris system) which have 1's fixed at the endpoints of I_i, 0's fixed within $(N\log^p(N))^{1/12}$ of the endpoints of I_i and (subject to this) $Y_0^{i,L} = \eta_0^L$ on I_i, $Y_0^{i,R} = \eta_0^R$ on I_i.

Similarly for the primed processes with I_i replaced by J_i. Then as with Lemma 2.5, by Lemma 2.2 and Corollary 2.1, we have that, for all sites x in the middle four-fifths of I_i,

$$Y^{i,L}_{N\log^p(N)}(x) = Y^{i,R}_{N\log^p(N)}(x) = Z^i_{N\log^p(N)}(x),$$

outside of probability

$$\frac{KN}{(N\log^p(N))^{1/4}}\frac{1}{(N\log^p(N))^{(k-3)/12}} + K\big(N\log^p(N)+1\big)(2N+1)\big(N\log^p(N)\big)^{1/4}\big(N\log^p(N)\big)^{k/24}.$$

However, on the event B we have that, for such x,

$$Y^{i,L}_{N\log^p(N)}(x) \le \eta^L_{N\log^p(N)}(x) \le Z^i_{N\log^p(N)}(x)$$

and

$$Y^{i,R}_{N\log^p(N)}(x) \le \eta^R_{N\log^p(N)}(x) \le Z^i_{N\log^p(N)}(x).$$

Similarly for sites in the middle four-fifths of intervals J_i. We deduce that outside of the contracted probability

$$\eta^R_{N\log^p(N)}(x) = \eta^L_{N\log^p(N)}(x) \qquad \forall x \in [N/10, 9N/10]. \qquad \square$$

Using essentially the same proof we may show the following lemma.

LEMMA 4.2. *Let η be a finite NPS with rightmost particle at time t, r_t and leftmost l_t and so that $r_0 - l_0 = N$. Let $B(p)$ ($|p| \le 7$) be the union of the following events*:

(i) $\sup_{0\le t\le N\log^p(N)}\{|r_t - r_0| + |l_t - l_0|\} \le N^{2/3}/4$;

(ii) *there does not exist $t \in [0, N\log^p(N)]$ so that η_t has an $(N\log^p(N))^{1/12}$ gap on $[l_t, r_t]$*;

(iii) $\forall l_0 + (N\log^p(N))^{1/4} \le x \le r_0 - (N\log^p(N))^{1/4}$,

$$\eta_0 \circ \theta_x|_{[-N\log^p(N))^{1/4}, N\log^p(N))^{1/4}]} \text{ is good.}$$

Then, outside of probability $P(B) + KN^{5/2}/N^{k/24}$, we have

$$\eta_{N\log^p(N)}(x) = \eta^{\mathbf{1}}_{N\log^p(N)}(x) \quad \text{for } x \in [l_0 + N^{2/3}/2, r_0 - N^{2/3}/2],$$

where $\eta^{\mathbf{1}}$ is the NPS run with η's Harris system and starting from full occupancy.

The following result is crucial as it enables one to transfer arguments and results for one-sided processes to arguments and results for large finite systems.

LEMMA 4.3. *Let η be a NPS on $\{0,1\}^{[-n,n]}$ with 1's fixed at $-n, n$. Let η' be a NPS on $\{0,1\}^{[-n,n]}$ with 1's fixed at $-n, n$ and 0's fixed at $(-n, -n+n^{1/3})$, $(n - n^{1/3}, n)$. Suppose both are generated by the same Harris process and η_0, η'_0 are derived from a renewal process γ on $\mathbb{Z}$ (or a renewal process γ conditioned to* *have* $\gamma(x)=1$ *for some fixed* x *with* $|x|\geq n$) *as follows*:

(i) $\eta_0\equiv\gamma$ *on* $(-n,n)$;
(ii) $\eta'_0\equiv\gamma$ *on* $(-n+n^{1/3},n-n^{1/3})$.

Then

$$P\big(\nexists t\leq n^8 \text{ such that } \eta_t(x)>\eta'_t(x) \text{ for } x\in(-4n/5,4n/5)\big)\leq\frac{c}{n^{k/3-10}}.$$

PROOF. The proofs are the same for the two cases of γ so we simply treat the first.

We observe that by attractiveness $\eta'_t\leq\eta_t$ $\forall t$ and that by the strong Markov property (for the coupled processes) if, for any x, $\eta_t(x)=1$, $\eta'_t(x)=0$, then this state will persist for time 1 with probability at least e^{-m} for $m=1+\beta(1)^2/\beta(2)$.

Therefore, using these two facts, we deduce that

$$\begin{aligned}&P\bigg(\nexists t\leq n^8 \text{ such that } \eta_t(x)>\eta'_t(x)\text{for}|x|\leq\frac{4n}{5}\bigg)\\&\qquad\leq e^m\int_0^{n^8+1}\sum_{|x|\leq 4n/5}P\big(\eta_t(x)=1\big)-P\big(\eta'_t(x)=1\big)\,dt.\end{aligned}$$

The equilibrium measure $\pi^{[-n,n]}$ for η is renewal measure on $\{0,1\}^{[-n,n]}$ conditioned to have 1's at sites $-n,n$. Therefore this distribution is stochastically above renewal measure restricted to interval $[-n,n]$, which is the distribution of η_0. Thus η_0 is stochastically below measure $\pi^{[-n,n]}$ and so η_t is also stochastically below $\pi^{[-n,n]}$ for each $t\geq 0$. That is, $\forall x,t$, $P(\eta_t(x)=1)\leq\pi^{[-n,n]}\{\eta:\eta(x)=1\}$; in particular this is true for every $t\in[0,n^8+1]$ and $x\in(-4n/5,4n/5)$.

Also conditional on γ being nonidentically zero on both $(-n,-n+n^{1/3})$ and $(n-n^{1/3},n)$, η'_0 is distributed above η'_t's natural equilibrium measure $\pi^{[-n,n]'}$ and so, $\forall x,t$,

$$P\big(\eta'_t(x)=1\big)\geq\pi^{[-n,n]'}(\{\eta:\eta(x)=1\})-2P\big(\gamma\equiv 0 \text{ on } (0,n^{1/3})\big)$$

and so the integral on the right-hand side of (†) is bounded by

$$\begin{aligned}&Kn^9P\big(\gamma\equiv 0\text{ on }(0,n^{1/3})\big)\\&\qquad+Kn^8\sum_{x\in(-4n/5,4n/5)}\pi^{[-n,n]}(\{\eta:\eta(x)=1\})-\pi^{[-n,n]'}(\{\eta':\eta'(x)=1\})\\&\quad\leq\frac{Kn^9}{n^{(k-2)/3}}+\frac{Cnn^8}{n^{(k-2)/3}}<\frac{Cn^{10}}{n^{k/3}}.\end{aligned}$$

by Lemmas 2.4 and 2.1. □

Lemma 4.3 enables us to relate the behavior of finite NPSs in equilibrium over a time interval of order N^2 to that of semiinfinite processes in equilibrium.

LEMMA 4.4. *Fix p with $|p| \leq 7$. Let η be a finite NPS in equilibrium on $[0, N]$ generated by η^R, η^L on time interval $[0, N\log^p(N)]$. Then, outside of probability $KN^{7/2}/N^{k/24}$, we have, for $N\log^p(N) \leq t \leq N^2/\log^5(N)$,*

$$\eta_t|_{[N/5,\infty)} = \eta_t^R|_{[N/5,\infty)}$$

and

$$\eta_t|_{(-\infty,4N/5]} = \eta_t^L|_{(-\infty,4N/5]}.$$

PROOF. As usual we need only concern ourselves with large N. We already know from Lemma 4.1 that, outside probability $KN^{5/2}/N^{k/24}$, $\eta^R_{N\log^p(N)} \equiv \eta^L_{N\log^p(N)}$ on $[N/10, 9N/10]$. By definition we have $\eta_{N\log^p(N)} \equiv \eta^R_{N\log^p(N)}$ on $[N/2, \infty)$ and $\eta_{N\log^p(N)} \equiv \eta^L_{N\log^p(N)}$ on $[-\infty, N/2)$. Therefore we have, outside probability $KN^{5/2}/N^{k/24}$, that $\eta_{N\log^p(N)}|_{[N/10,\infty)} = \eta^R_{N\log^p(N)}|_{[N/10,\infty)}$ and $\eta_{N\log^p(N)}|_{(-\infty,9N/10]} = \eta^L_{N\log^p(N)}|_{(-\infty,9N/10]}$. Let us denote this event by A.

We choose intervals I, J of length $2N^{1/4} + 1$ so that the endpoint of I is $N/5$ and the midpoint of J is $4N/5$. As with Lemma 4.1, for $H \in \{I, J\}$, we define processes $Y^{H,L}, Y^{H,R}$ to be finite NPSs on $\{0,1\}^H$, where 1's are fixed at the endpoints of H, 0's are fixed within $N^{1/12}$ of the endpoints but otherwise the processes evolve as NPSs generated by our Harris system. For $k = R, L, H \in \{I, J\}$, $Y_0^{H,k} \equiv \eta_0^k$ on H, subject to the above constraints.

For $k = R, L$ we have processes $Z^{H,k}$ which are NPSs on $\{0,1\}^H$ run with the same Harris system, with 1's fixed at the endpoints and with (subject to this) $\eta_0^k \equiv Z_0^{H,k}$ on H.

By Lemma 4.3 we have, for $k \in \{R, L\}$, $H \in \{I, J\}$, for all $0 \leq t \leq N^2$ and all x in the middle four-fifths of H, outside probability $KN^{5/2}/N^{k/12}$, that

$$Y_t^{H,k}(x) = Z_t^{H,k}(x) \tag{1}$$

for $H \in \{I, J\}$; but also, by Corollary 2.2, we have that, outside probability $Ce^{-N^{1/4}}$, for $H \in \{I, J\}$,

$$Z^{H,\mathbf{1}}_{N\log^p(N)} \equiv Z^{H,R}_{N\log^p(N)} \equiv Z^{H,L}_{N\log^p(N)}, \tag{2}$$

where $Z^{H,\mathbf{1}}$ is the finite NPS on $\{0,1\}^H$ with 1's fixed at the endpoints and so that initially all sites are occupied. Thus we have $Z_t^{H,R} \equiv Z_t^{H,L} \forall t \geq N\log^p(N)$ outside of this probability. Let B be the union of events (1) and (2) above. Then $P(B^c) \leq KN^{5/2}/N^{k/24}$ and on event B we have for $H \in \{I, J\}$ that, for all $N\log^p(N) \leq t \leq N^2$ and all x in the middle four-fifths of H,

$$Y_t^{H,R}(x) = Z_t^{H,R}(x) = Z_t^{H,L}(x) = Y_t^{H,L}(x).$$

Let event C be the union of the following events:

$$\nexists 0 \le t \le N^2/\log^5(N) \text{ so that } |r_t - N| \ge N/20, \text{ or } |l_t| \ge N/20;$$

$$\nexists t \in [0, N^2] \text{ so that } \eta^R \text{ has an } N^{1/12} \text{ gap in } [r_t - 2N, r_t];$$

$$\nexists t \in [0, N^2] \text{ so that } \eta^L \text{ has a } N^{1/12} \text{ gap in } [l_t, l_t + 2N].$$

Then by Lemma 2.2 and Proposition 3.1 we have $P(C^c) \le KN^2 2NN^{1/4}/N^{-k/24} \le KN^{7/2}/N^{-k/24}$. By attractiveness we have that, on event C, $Y^{H,k} \le \eta^k \le Z^{H,k}$ on interval $H \in \{I, J\}$. We conclude that, on event $A \cap B \cap C$, $\eta_t^R \equiv \eta_t^L$ on the middle four-fifths of H for $N \log^p(N) \le t \le N^2/\log^5(N)$.

We claim that a.s.

$$\eta_t \equiv \eta_t^R \quad \text{on } [N/5, \infty) \text{ for } N\log^p(N) \le t \le N^2/\log^5(N)$$

and

$$\eta_t \equiv \eta_t^L \quad \text{on } (-\infty, 4N/5] \text{ for } N\log^p(N) \le t \le N^2/\log^5(N)$$

on event $A \cap B \cap C$, an event of probability greater than $1 - KN^{7/2}/N^{k/24}$.

We first note that, since Harris death points unambiguously cause deaths at occupied sites, the time

$$\tau = \inf\{t \ge N\log^p(N) : \eta_t \ne \eta^R \text{ on } [N/5, \infty) \text{ or } \eta_t \ne \eta^L \text{ on } [-\infty, 4N/5)\}$$

must occur at a potential birth point for the Harris system. Similarly

$$\sigma = \inf\{t \ge N\log^p(N) : \eta_t \text{ has an } N^{1/4} \text{ gap in } I \cup [N/5, 4N/5] \cup J\}$$

can only occur at a death point of the Harris system. Therefore we have that a.s. $\tau \ne \sigma$. We will see that no other relation is possible if $\tau < N^2/\log^5(N)$.

First note that for $t < \tau$ we have that $\eta_t^R \equiv \eta_t$ on $[N/5, \infty)$ and so (on event $A \cap B \cap C$) η_t has no $N^{1/4}$ gap on interval $[N/5, 9N/10]$. Similarly η_t has no $N^{1/4}$ gap on $[N/10, 4N/5]$ (if $A \cap B \cap C$ holds). We deduce that a.s. $\sigma \ge \tau$. However conversely for $N\log^p(N) \le t < \sigma$ we have that, on interval I,

$$Y_t^{I,L} \le \eta_t \le Z^{I,L}$$

and on J,

$$Y_t^{J,R} \le \eta_t \le Z^{J,R}.$$

Therefore for such t (on event $A \cap B \cap C$) we have that, for x in the middle four-fifths of I and J,

$$\eta_t(x) = \eta_t^R(x) = \eta_t^L(x).$$

But if immediately prior to $\tau, \eta, \eta^R, \eta^L$ have no $N^{1/4}$ gaps on $[N/10, 9N/10]$, then at time τ we must have that either there exists x in the middle four-fifths of I so that

$$\eta_\tau^R(x) \neq \eta_\tau(x)$$

or there exists x in the middle four-fifths of J so that

$$\eta_\tau^L(x) \neq \eta_\tau(x).$$

However, as we have seen on event $A \cap B \cap C$ this is impossible before time σ. This contradiction implies that $\forall N \log^p(N) \leq t \leq N^2/\log^5(N)$,

$$\eta_t|_{[(N/5),\infty)} = \eta_t^R|_{[(N/5),\infty)} \quad \text{and} \quad \eta_t|_{(-\infty,(4N/5)]} = \eta_t^L|_{(-\infty,(4N/5)]},$$

outside of probability $KN^{7/2}/N^{k/24}$. $\square$

We are now ready to begin establishing results for "finite edges processes in equilibrium." Lemma 4.4 lets us regard finite NPSs as close to semiinfinite NPSs in equilibrium which in turn are locally (at appropriate places) close to equilibrium NPSs. The following lemma makes this a little more concrete.

LEMMA 4.5. *Let f be a fixed cylinder function and let δ be a constant strictly greater than 0. There exists $N_0 = N_0(f, \delta)$ so that, for all $N \geq N_0$, if η^N is a finite NPS, in equilibrium on $[0, N]$, generated by $\eta^{R,N}, \eta^{L,N}$ over time interval $[0, N \log^p(N)]$ with $|p| > 7$ and if x^N satisfies*

$$N - x^N \leq N/\log(N)$$

and T^N satisfies

$$N \log^p(N) \leq T^N \leq \frac{N^2}{\log^{40}(N)},$$

then

$$|E[f(\eta_{T^N} \circ \theta_{x^N})] - \langle \mathrm{Ren}(\beta), f\rangle| \leq \delta.$$

PROOF. Either

$$\frac{N}{3} \leq x^N \leq N - \frac{N}{\log(N)} \quad \text{or} \quad \frac{N}{\log(N)} \leq x^N \leq \frac{2N}{3},$$

or both. We assume without loss of generality that the former holds. By Lemma 4.4 it will suffice to show

$$E[f(\eta_{T^N}^{R,N} \circ \theta_{x^N})] - \langle \mathrm{Ren}(\beta), f\rangle \to 0.$$

We introduce infinite NPS η' run with the same Harris system as $\eta^{R,N}$ and in equilibrium. By the meaning of equilibrium, we have

$$E[f(\eta'_{T^N} \circ \theta_{x^N})] = \langle \mathrm{Ren}(\beta), f\rangle.$$

Thus to show the lemma it will suffice to show that with probability tending to 1 as $N \to \infty$, $\eta'_{T^N} = \eta^{R,N}_{T^N}$ on $[x^N - m, x^N + m]$, where $[-m, m]$ is the support of cylinder function f.

We argue as in the previous lemma. Let I be the interval, centred at x, of length $2(T^N)^{1/4} + 1$. Let Z^I_t be the finite NPS on $\{0, 1\}^I$ with 1's fixed at the endpoint of I and starting from full occupancy at time 0.

Let $Y^I(Y^{I\prime})$ be the finite NPSs on $\{0, 1\}^I$ with 1's fixed at the endpoints, 0's fixed within $(T^N)^{1/12}$ of the endpoints but otherwise $Y^I_0 = \eta^{R,N}_0 (Y^{I\prime}_0 = \eta'_0)$. Then we have with probability tending to 1 as $N \to \infty$ that, for all y in the central half of I,

$$Y^I_{T^N}(y) \le \eta^{R,N}_{T^N}(y) \le Z^I_{T^N}(y),$$

$$Y^{I\prime}_{T^N}(y) \le \eta'_{T^N}(y) \le Z^I_{T^N}(y)$$

and

$$Y^{I\prime}_{T^N}(y) = Y^I_{T^N}(y) = Z^I_{T^N}(y).$$

We conclude that as $N \to \infty$, for $y \in [x^N - m, x^N + m]$,

$$\eta^{R,N}_{T^N}(y) = \eta'_{T^N}(y).$$

This proves the lemma. □

COROLLARY 4.1. *Let η be a finite NPS in equilibrium on $[0, N]$ generated by η^R, η^L during time interval $[0, N\log^p(N)]$ $(|p| \le 7)$. Let T be a fixed, nonrandom time on time interval $[N\log^p(N), N^2/\log^5(N) - N\log^q(N)]$ for $|q| \le 7$, and let $(\eta^{\mathbf{1},T}_t)_{t\ge T}$ be the one-sided NPS generated by the same Harris system as η such that*

$$\eta^{\mathbf{1},T}_T(x) = I_{x \le r_T - N^{3/5}/5}.$$

Then

$$\eta^{\mathbf{1},T}_{T+N\log^q(N)} \le \eta_{T+N\log^q(N)} \quad \text{on } [N/5, \infty),$$

outside of probability $KN^5/N^{k/12}$.

COROLLARY 4.2. *Let η be a finite NPS in equilibrium on $[0, N]$ generated by η^R, η^L during time interval $[0, N\log^p(N)]$ $(|p| \le 7)$. Then, outside of probability $KNN^{3c}N^{-ck/2} + KN^{7/2}/N^{k/24}$, for all t satisfying $N\log^p(N) \le t \le N^2/\log^5(N)$, there is no N^c gap within η_t. Equally the chance that for some t satisfying $N\log^p(N) \le t \le N^2/\log^5(N)$ and for some x within the interval of support of η_t and at least N^c away from its boundary we have $\eta_t \circ \theta_x|_{[-N^c,N^c]}$ is bad is bounded by $KNN^cN^{-ck/6} + KN^{7/2}/N^{k/24}$.*

PROOF. Let $N\log^p(N) \le t \le N^2/\log^5(N)$. By Lemma 4.4, the probability that there is an N^c gap "within" η_t is bounded by $KN^{7/2}/N^{k/24} + P(\eta_t^R$ has an N^c gap within $[r_t - 2N, r_t]) + P(\eta_t^L$ has an N^c gap within $[l_t, l_t + 2N]) + P(\sup_{t\le N^2/\log^5(N)}\{|r_t - r_0| + |l_t|\} \ge N/2)$. The result now follows from Lemma 2.2(i).

The second part of the corollary follows similarly [with Lemma 2.2(i) replaced by Lemma 2.2(ii)]. $\square$

From this corollary we immediately deduce (arguing as in Lemma 4.2) the following proposition.

PROPOSITION 4.1 (Regeneration proposition).. *Let p, q satisfy $0 \le p, |q|, |q\prime| \le 7$. Let η be a finite process in equilibrium on $[0, N]$ generated over time interval $[0, N/\log^p(N)]$. Let l_t^η (resp. r_t^η) be the position of the leftmost (resp. rightmost) particle of η at time t for $t \ge N/\log^p(N)$. Let T be a nonrandom time in the interval $[N(\log^{|q|}(N) + \log^p(N)), N^2/\log^5(N)]$. Let γ_t^L, γ_t^R for $t \ge T - N\log^{|q|}(N)$ be two one-sided NPSs in equilibrium. At time $T - N\log^{|q|}(N)$, γ^L will be supported on $[l^\eta_{T-N\log^{|q|}(N)} + N^{2/3}\log^{q'}(N),\ \infty]$ and γ^R will be supported on $(-\infty, r^\eta_{T-N\log^{|q|}(N)} - N^{2/3}\log^{q'}(N)]$, but otherwise $\gamma^R_{T-N\log^{|q|}(N)}$ and $\gamma^L_{T-N\log^{|q|}(N)}$ are independent of each other and of η at this time. Let γ^R, γ^L evolve according to the same Harris system as generated η. Then outside of probability $KN^{7/2}/N^{k/24}$ we have that the finite NPS generated by γ^R, γ^L over time interval $[T - N\log^{|q|}(N), T]$ is dominated by η.*

The importance of this proposition is that it enables us to replace periodically finite NPSs in equilibrium by finite NPSs whose initial support is close to the region of occupancy of η at the time but which are otherwise independent. The price is that a slight shrinkage is involved and that (with very small probability) there could be a breakdown.

We also have the following proposition which states that the motion of the rightmost particle for a semiinfinite NPS in equilibrium does not depend on the generating Harris system far from the rightmost site.

PROPOSITION 4.2. *Let η be a one-sided NPS in equilibrium (initially supported on $(-\infty, 0]$); let η' be a (possibly finite) NPS with $\eta_0|_{[-3N^{2/3},\infty)} = \eta'_0|_{[-3N^{2/3},\infty)}$. Suppose that η (resp. η') are generated by Harris systems H (resp. H') so that*

$$H|_{[-3N^{2/3},\infty)} = H'|_{[-3N^{2/3},\infty)}.$$

Then

$$P\big(\eta_t|_{[-N^{2/3},\infty)} = \eta'_t|_{[-N^{2/3},\infty)}\ \forall 0 \le t \le N^{16/3} \wedge \tau\big) \ge 1 - KN^{20/3}/n^{2k/9}$$

for $\tau = \inf\{s : \eta'_s \equiv 0$ *on* $[-3N^{2/3}, -3N^{2/3} + N^{2/9}]$ *or* $\eta_s \equiv 0$ *on* $[-3N^{2/3}, -3N^{2/3} + N^{2/9}]$ *or* $\eta_s \equiv 0$ *on* $[-N^{2/3} - N^{2/9}, -N^{2/3}]\}$.

PROOF. Consider the finite NPSs on $\{0, 1\}^{[-3N^{2/3}, -N^{2/3}]}$, Z and Y generated by H (and therefore by H'), where Z has 1's fixed at endpoints $-3N^{2/3}$, $-N^{2/3}$, while Y has 1's fixed at these endpoints but also 0's fixed within $N^{2/9}$ of the endpoints. Initially both coincide with η_0 (and therefore η') subject to these constraints (as in the preceding lemma).

Let B be the union of the following events:

1. Y_t has an $N^{2/3}/2$ gap for some $0 \le t \le N^{16/3}$;
2. $\exists 0 \le t \le N^{16/3}, -\frac{14}{5}N^{2/3} \le x \le -\frac{6}{5}N^{2/3}$ so that $Z_t(x) > Y_t(x)$.

The probability of event 2 occurring but not event 1 is, by preceding lemma, bounded by $KN^9/N^{2k-/9}$. Given the large value of k we have that $P(B) \le KN^{5/3}/N^{(k-1)/18}$.

We suppose that B does not occur. Let $\tau' = \tau \wedge \inf\{\eta'_s \equiv 0$ on $[-N^{2/3} - N^{2/9}, -N^{2/3}]\}$. Now as all processes on interval $[-3N^{2/3}, -N^{2/3}]$ are generated by the same Harris system we have that, for all times $t \le \tau' \wedge N^{16/3}$ and x in the interval $[-3N^{2/3}, -N^{2/3}]$,

$$Y_t(x) \le \eta_t(x), \eta'_t(x) \le Z_t(x).$$

As B does not occur we have

$$Y_t(x) = Z_t(x) \quad \text{for } -\frac{14}{5}N^{2/3} \le x \le -\frac{6}{5}N^{2/3}.$$

This implies that for the relevant t and x that $\eta_t(x) = \eta'_t(x)$. Since (by the nonoccurrence of event 1 above) there will always be an occupied site for η (and therefore η') in the interval $[-\frac{14}{5}N^{2/3}, -\frac{6}{5}N^{2/3}]$, we cannot have the spontaneous appearance of discrepancies of the processes to the right of $-\frac{6}{5}N^{2/3}$ before time $\tau\prime$. That is (on event B),

$$\tau' \wedge N^{16/3} \le \sigma = \inf\bigl\{t : \eta_t \ne \eta'_t \text{ on } \bigl[-\tfrac{6}{5}N^{2/3}, \infty\bigr)\bigr\}.$$

We are done since the first time we have a discrepancy between η and η' on an interval must correspond to a potential birth time for our Harris system while τ must correspond to a death time for the Harris system and thus we must have

$$\sigma > \inf\bigl\{t : \eta'_t \text{ is vacant on } [-N^{2/3} - N^{2/9}, -N^{2/3}]\bigr\}. \qquad \square$$

Proposition 4.2 and Lemma 4.4 yield:

COROLLARY 4.3. *Let one-sided equilibrium processes* η^R, η^L *generate* η *on* $[0, N]$ *at time* $N \log^p(N)$. *Let* $\eta^{R'}$ *be a right one-sided NPS such that* $\eta_0^{R'} = \eta_0^R$ *and* $\eta^{R'}$ *is generated by a Harris system* H' *such that* $H|_{[2N/3,\infty)} = H'|_{[2N/3,\infty)}$.

Similarly, let $\eta^{L'}$ *be a left one-sided process such that* $\eta_0^{L'} = \eta_0^L$ *and such that* $\eta^{L'}$ *is generated by* H'', *where* $H''|_{(-\infty,N/3]} = H|_{(-\infty,N/3]}$. *Then outside probability* $KN^{7/2}/N^{k/24}$ *we have* $\eta_t = \eta_t^{R'}$ *on* $[4N/5,\infty)$ *and* $\eta_t = \eta_t^{L'}$ *on* $(-\infty, N/5]$ *for* $N\log^p(N) \le t \le N^2/log^5(N)$.

REMARK. Of course $\eta^{R'}$ and $\eta^{L'}$ may be taken to be independent. This result says that edge fluctuations of a finite NPS in equilibrium are essentially independent.

5. The purpose of this section is to establish a weak convergence result for the behavior between boundaries of multiple intervals alternating between vacancy and the upper regime.

For a configuration γ, we say that it is L-*good* on interval I if

there are no $L^{1/3}$ gaps for γ within interval I

and

for every x, so that $[x-L, x+L] \subset I$, we have $\gamma \circ \theta_x|_{[-L,L]}$ is good.

Obviously for a process γ_t when we write γ_t is L-good on I_t for $t \in [S,T]$ we mean that, simultaneously for all $t \in [S,T]$, the configuration γ_t is L-good for interval I_t.

We wish to show:

THEOREM 5.1. *Let finite NPS* ξ^N *be such that* $\xi_0^N \equiv 0$ *on* $(-\infty,-\lambda N) \cup (0,N) \cup ((1+\kappa)N,\infty)$; ξ_0^N *is "in equilibrium" on* $[-\lambda N, 0]$ *and on* $[N,(1+\kappa)N]$ *where* $\kappa,\lambda > 0$. *There are associated processes* $l_t^{N,\gamma}, r_t^{N,\gamma}, l_t^N = \inf\{x : \xi_t^N(x) = 1\}$ *and* $r_t^N = \sup\{x : \xi_t^N(x) = 1\}$ *and random variable* τ^N, *so that for* τ *defined below we have for* $t \le \tau^N$ *and* $v \ge 18/k$ *that* ξ_t^N *is* N^v-*good on* $[l_t^N, l_t^{N,\gamma}] \cup [r_t^{N,\gamma}, r^N]$ *and*

$$\left(\frac{l^N_{N^2t}}{N}, \frac{l^{N,\gamma}_{N^2t}}{N}, \frac{r^{N,\gamma}_{N^2t}}{N}, \frac{r^N_{N^2t}}{N}\right)_{t\le\tau_N/N^2} \to (B_t^1, X_t^1, X_t^2, B_t^2)_{t\le\tau},$$

where $(B_0^1, X_0^1, X_0^2, B_0^2) = (-\lambda, 0, 1, 1+\kappa)$ *and* (B^1, B^2) *is a standard two-dimensional Brownian motion independent of* (X^1, X^2) *and where*

$$X_t^1 = W_t^1 + \mu t \quad \text{and} \quad X_t^2 = W_t^2 - \mu t \quad \text{for } X^2 > X^1$$

and $\tau = 1 \wedge \inf\{t : |B_t^1 + \lambda| > \frac{\lambda}{3}$ *or* $|B_t^2 - (1+\kappa)| > \frac{\kappa}{3}$ *or* $|X_t^1| > \frac{\lambda}{3} \wedge \frac{1}{3}$ *or* $|X_t^2 - 1| > \frac{\kappa}{3} \wedge \frac{1}{3}\}$. *The random variable* τ^N *is the analogous stopping time for* ξ^N; $\tau^N = \inf\{t : |l_t^N + \lambda N| > N\lambda/3$ *or* $|r_t^N - (1+\kappa)N| > \kappa/3$ *or* $|l_t^{N,\gamma}| > (\lambda/3 \wedge 1/3)N$ *or* $|r_t^{N,\gamma}| > (\kappa/3 \wedge 1/3)N\}$.

To establish this we first consider a simpler model which is semiinfinite. Let ξ_t^N be a semiinfinite process on $\{0,1\}^{\mathbb{Z}}$, ξ_0^N is Renewal measure on $(-\infty, 0]$, conditioned to have a 1 at site 0, 0 on $(0, \infty)$,

$$c^N(x, \xi^N) = \begin{cases} 1, & \text{if } \xi^N(x) = 1, \\ \beta(\ell, r), & \text{if } r < \infty, \xi^N(x) = 0, \\ (1 + 1/N)\beta(\ell), & \text{if } r = \infty, \xi^N(x) = 0. \end{cases}$$

Let r_t^N denote the position of the rightmost particle of ξ_t^N.

REMARK. Note that for this perturbed process, ξ^N, the process seen from the rightmost particle is in equilibrium and as such the bounds of Lemma 2.2 apply to the configurations of these processes as seen from the extreme particle. It should also be noted, however, that this new process is neither reversible nor attractive. However, we may couple together ξ^N and η, a one-sided semiinfinite process, so that $\xi_0^N = \eta_0$ and $\forall t \geq 0$, $\eta_t \leq \xi_t^N$.

In the following we will consider both the regular equilibrium semiinfinite NPS and our perturbed semiinfinite NPS described above. For probabilities and expectations for the former we will use P or the suffix P; for the latter we will use Q. Let Y be the number of jumps to the right of the rightmost particle in time interval $[0, N^{3/2}]$. Under Q, Y is Poisson$(N^{3/2} + N^{1/2})$, under P it is Poisson$(N^{3/2})$. Obviously the two distributions are very close in absolute variation norm. Let us consider under probability Q, the random quantity

$$X^N = r_{N^{3/2}}^N I_{|Y - N^{3/2}| \leq N^{5/6}}.$$

I wish to show:

LEMMA 5.1. *The quantity X^N satisfies the following*:

(i) $E^Q[X^N] = \mu N^{1/2} + O(e^{-gN^{1/6}})$ *for* $\mu = \sum n\beta(n)n$ *and some positive* g *not depending on* N;

(ii) $E^Q[(X^N)^2] = N^{3/2}(1 - O(1/N^{1/10}))$; *and*

(iii) $E^Q[(X^N)^4] \leq KN^3$ *for some* K *not depending on* N.

REMARK. From the proof it will be clear that the conclusions remain valid if time $N^{3/2}$ is replaced by some fixed, nonrandom time equal to $N^{3/2}(1 + o(1))$.

PROOF OF LEMMA 5.1. We first treat (i):

$$E^Q[X^N] = E^Q[r_{N^{3/2}}^N] - E^Q[r_{N^{3/2}}^N I_{Y \leq N^{3/2} - N^{5/6}}] - E^Q[r_{N^{3/2}}^N I_{Y \geq N^{3/2} + N^{5/6}}].$$

The first term on the right-hand side is precisely equal to $\mu N^{1/2}$ for $\mu = \sum n\beta(n)$ so it only remains to bound the absolute value of the other two right-hand-side terms appropriately. The second term on the right-hand side is bounded in magnitude by

$$E^Q[|r^N_{N^{3/2}}|I_{Y\le N^{3/2}-N^{5/6}}] = \sum_{i<N^{3/2}-N^{5/6}} E^Q[|r^N_{N^{3/2}}| \mid Y=i]Q(Y=i).$$

However,

$$E^Q[|r^N_{N^{3/2}}| \mid Y=i] = E^P[|r^N_{N^{3/2}}| \mid Y=i]$$

for all i while for i's in this range

$$\frac{Q(Y=i)}{P(Y=i)} = \frac{e^{-(N^{3/2}+N^{1/2})}(N^{3/2}+N^{1/2})^i}{e^{-N^{3/2}}(N^{3/2})^i} = e^{-N^{1/2}}\left(1+\frac{1}{N}\right)^i \le 1.$$

So we have

$$E^Q[|r^N_{N^{3/2}}|I_{Y\le N^{3/2}-N^{5/6}}] \le E^P[|r^N_{N^{3/2}}|I_{Y\le N^{3/2}-N^{5/6}}].$$

This latter bound is, by Cauchy–Schwarz, bounded by

$$(E^P[|r^N_{N^{3/2}}|^2])^{1/2}\big(P(Y\le N^{3/2}-N^{5/6})\big)^{1/2} < Ke^{-cN^{1/6}}$$

for large N by Proposition 3.3 and Lemma 0.2. For the third term we split it up as

$$E^Q\Big[|r^N_{N^{3/2}}|I_{r^N_{N^{3/2}}>0}I_{Y\ge N^{3/2}+N^{5/6}}\Big] + E^Q\Big[|r^N_{N^{3/2}}|I_{r^N_{N^{3/2}}<0}I_{Y\ge N^{3/2}+N^{5/6}}\Big].$$

The second term can be taken care of by the observation above that ξ^N can be coupled with η, an equilibrium semiinfinite NPS, initially supported on $(-\infty,0]$, with $\eta_t \le \xi^N_t$ $\forall t\ge 0$ and so, in particular, $\sup\{x:\eta_t(x)=1\} = r_t \le r^N_t$ for all t. It is easy to see by Cauchy–Schwarz that

$$\begin{aligned}
&E^Q\Big[|r^N_{N^{3/2}}|I_{r^N_{N^{3/2}}<0}I_{Y\ge N^{3/2}+N^{5/6}}\Big]\\
&\quad\le E^Q\Big[|r_{N^{3/2}}|I_{r^N_{N^{3/2}}<0}I_{Y\ge N^{3/2}+N^{5/6}}\Big]\\
&\quad\le (E^Q[r^2_{N^{3/2}}])^{1/2}\big(Q(\{Y\ge N^{3/2}+N^{5/6}\})\big)^{1/2} \le Ce^{-cN^{1/6}}
\end{aligned}$$

by Proposition 3.3 and Lemma 0.2.

By considering a NPS with no deaths, we have that

$$E^Q[|r^N_{N^{3/2}}|I_{r^N_{N^{3/2}}>0} \mid Y] \le \mu Y.$$

Together these two results (and the above bounds for tails of Poisson random variables) yield

$$E^Q[|r^N_{N^{3/2}}|I_{Y\ge N+N^{5/6}}] \le Ce^{-gN^{1/6}}.$$

For (ii) we note that, by Proposition 3.3 and Cauchy–Schwarz,

$$E[(r^N_{N^{3/2}})^2 I_{|Y-N^{3/2}|>N^{5/6}}] < e^{-gN^{1/6}}.$$

Thus

$$\sum_{|i-N|\le N^{5/6}} E^P\big[(r^N_{N^{3/2}})^2 \mid Y=i\big]P(Y=i) = \left(1-o\left(\frac{1}{N^{1/6}}\right)\right),$$

but it is easily seen that for i in this range $\big|\frac{P(Y=i)}{Q(Y=i)}-1\big| \le \frac{1}{N^{1/10}}$. This yields (ii). Part (iii) follows in the same way. □

Lemma 5.1, the Berry–Esseen theorem and standard weak convergence arguments [see Durrett (1996) page 126], show the following:

LEMMA 5.2. *Let Z^N_i be i.i.d. random variables with distribution equal to $r^N_{N^{3/2}}$ for a perturbed NPS initially in equilibrium, supported on $(-\infty, 0]$;*

$$X^N_t = \frac{1}{N}\sum_{i\le\sqrt{N}t} Z^N_i$$

tends in distribution as N tends to infinity to

$$X_0 = 0,$$
$$X_t = W_t + \mu t,$$

where W is a standard Brownian motion.

LEMMA 5.3. *For $T \le 2N^{3/2}$ we have for process ξ^N, started in equilibrium on $(-\infty, 0]$ with rightmost particle r^N, that*

$$Q\left(\sup_{t\le T}|r^N_t| \ge \log^2(T)\sqrt{T}\right) \le CT/T^{(k-1)/2}.$$

PROOF. Since we can couple ξ^N and a one-sided NPS in equilibrium η, so that $\eta_t \le \xi^N_t$ for all t, by Proposition 3.1, we need only show

$$Q\left(\sup_{t\le T} r^N_t \ge \log^2(T)\sqrt{T}\right) \le CT/T^{(k-1)/2}.$$

Let random variable Y be the number of rightward jumps by the rightmost particle of ξ^N in the time interval $[0, T]$; Y is a Poisson$(T(1+1/N))$ random variable. Accordingly, by Lemma 0.2, we have that $Q(|Y-T| \ge T^{7/12}) \le CT/T^{(k-1)/2}$.

On the other hand we have that if $p_\lambda(j)$ is the probability that a Poisson(λ) random variable is equal to j, then

$$\sup_{|j-T|\le T^{7/12}}\left|\frac{p_T(j)}{p_{T(1+1/N)}(j)} - 1\right| = 1 + o(1) \qquad \text{as } T \to \infty.$$

Therefore we have that

$$\begin{aligned}
&Q\Big(\sup_{t\le T} r_t^N \ge \log^2(T)\sqrt{T}\Big)\\
&\qquad \le CT/T^{(k-1)/2} + \sum_{|j-T|\le T^{7/12}} p_{T(1+1/N)}(j)P\Big(\sup_{t\le T} r_t^N \ge \log^2(T)\sqrt{T}|Y=j\Big)\\
&\qquad \le CT/T^{(k-1)/2} + 2\sum_{|j-T|\le T^{7/12}} p_T(j)P\Big(\sup_{t\le T} r_t \ge \log^2(T)\sqrt{T}|Y=j\Big),
\end{aligned}$$

where r_t is the position of the rightmost particle of a one-sided NPS started from $(-\infty, 0]$ in equilibrium and, again, Y is the number of rightward jumps made by its rightmost particle. By Proposition 3.1, this latter sum is bounded by $CT/T^{(k-1)/2}$. We are done. $\square$

COROLLARY 5.1. *Let ξ_0^N be in equilibrium supported on $(-\infty, 0]$. Let $\xi_t^{N,R}$ (R denoting restriction) be the process with the same initial configuration, run with the same Harris system but so that no births outside $(-\infty, \log^2(T)\sqrt{T}]$ are permitted*;

$$\forall 0 \le t \le T, \qquad \xi_t^N = \xi_t^{N,R},$$

outside probability $CT/T^{(k-1)/2}$.

We now check that a major tool in analyzing semiinfinite NPSs is still in force for our perturbed NPSs.

LEMMA 5.4. *There exists N_0 so that, for all $N \ge N_0$, if ξ^N is a perturbed NPS as above and H is an interval of length bounded by N, then, for all $t \ge 0$,*

$$\xi_t^N \le Z_t^{H,\mathbf{1}} \quad \text{on } H,$$

where $Z^{H,\mathbf{1}}$ be the finite NPS on $\{0,1\}^H$ with 1's fixed at the endpoints of H, generated by the same Harris system as ξ^N and starting from full occupancy.

PROOF. It will not be true in general that for all H, of arbitrary size, and all t that $Z_t^{H,\mathbf{1}} \ge \xi_t^N$ on H; however, it will be true if H is of small length. By our assumption $(**)$, on $\beta(n)/\beta(n+1)$ we have for N large that

$$\frac{\beta(N-1)}{\beta(N)} \ge \left(1 + \frac{1}{N}\right).$$

Therefore there exists N_0 so that for all $N \ge N_0$ and all $n \le N$ we have

$$\frac{\beta(n-1)}{\beta(n)} \ge \frac{\beta(N-1)}{\beta(N)} \ge \left(1 + \frac{1}{N}\right).$$

To show our result, we must simply consider the flip rate for ξ_t^N at a site $x \in H$, vacant for ξ_t^N and check that if x is also vacant for $Z_t^{H,\mathbf{1}}$ and $\xi_t^N \leq Z^{H,\mathbf{1}}$ on H, then the flip rate for $Z^{H,\mathbf{1}}$ exceeds that for ξ^N. If site x is to the left of the rightmost particle of ξ^N, then this is immediate. Accordingly we suppose that x, is l sites to the right of r_t^N, the rightmost particle of ξ_t^N. Let the distance to the left of x to a $Z^{H,\mathbf{1}}$-occupied site be l' (necessarily at most l); to the right let the distance be r.

Then the flip rate at x for process ξ^N is

$$\left(1+\frac{1}{N}\right)\beta(l),$$

while for $Z^{H,\mathbf{1}}$ it is

$$\frac{\beta(l')\beta(r')}{\beta(l'+r')} \geq \frac{\beta(l)\beta(r')}{\beta(l'+r')} \geq \frac{\beta(l)\beta(l'+r'-1)}{\beta(l'+r')}.$$

Now $l'+r'$ is at most N, so by attractiveness

$$\frac{\beta(l'+r'-1)}{\beta(l'+r')} \geq \frac{\beta(N-1)}{\beta(N)}.$$

By our choice of N_0 and our assumption that $N \geq N_0$,

$$\frac{\beta(N-1)}{\beta(N)} \geq \left(1+\frac{1}{N}\right).$$

Therefore we have that, for all intervals H of length bounded above by N, $Z_t^{H,\mathbf{1}} \geq \xi_t^N \ \forall t \geq 0$. $\square$

LEMMA 5.5. *Let $\xi^{N,A}, \xi^{N,B}$ be two perturbed NPSs in equilibrium and generated by the same Harris system. Suppose that $\xi_0^{N,A}$ is supported on $(-\infty, 0]$ and $\xi_0^{N,B}$ is supported on $(-\infty, -N^{1/3}]$. Then, outside of probability $KN^{53/20}/\,N^{k/40}$,*

$$\xi_t^{N,A} \geq \xi_t^{N,B} \qquad \text{on } [-N, \infty),$$

for $t \in [N^{3/5}, N^{3/5}+N^{3/2}]$.

PROOF. The proof follows that of Lemma 2.5 closely; a slight difference is the lack of attractiveness, which requires some care. We denote the rightmost particles at time t by $r_t^{N,A}$ and $r_t^{N,B}$, respectively. Let G be the event that, for all $t \in [0, N^{3/2}+N^{3/5}]$,

$$\xi_t^{N,A} \text{ is } N^{3/20}\text{-good on } [r_t^{N,A}-3N, r_t^{N,A}] \quad \text{and} \quad |r_t^{N,A}| \leq N.$$

By Lemma 2.2 (applied to perturbed NPSs) and Lemma 5.3 we have $P(G) \geq 1 - KNN^{3/2}N^{3/20}/N^{k/20}$.

For any interval H, $Z^{H,\mathbf{1}}$ will be (as before) the finite NPS on $\{0,1\}^H$, starting from full occupancy and generated from the Harris system for η.

We now proceed as in Proposition 3.2. We choose $\leq KN^{1-3/20}$ equal disjoint intervals $I_1, I_2, \dots, I_R$ of length $2N^{3/20}+1$ covering $[-3N, -N^{1/3}/3]$ and then we choose intervals $J_0, J_1, \dots, J_R$ (of equal length) as in Lemma 2.5 or Proposition 3.2. As in the proofs of these lemmas we argue that if (as will be the case on event G), for all $H \in \bigcup I_i \cup \bigcup J_i$,

$$\xi_0^{N,A} \circ \theta_H|_{[-N^{3/20}, N^{3/20}]} \text{ is good}$$

(where θ_H is the shift that centers H on the origin), then outside of probability bounded by $KN^{1-3/20}/N^{k/40}$ we have that, for all sites x in the middle four-fifths of H,

$$\xi_{N^{3/5}}^{N,A}(x) = Z_{N^{3/5}}^{H,\mathbf{1}}(x).$$

By Lemma 5.4 this equality for all H implies that, outside of probability $KNN^{3/2}N^{3/20}/N^{k/40} + KN^{1-3/20}/N^{k/40}$,

$$\xi_{N^{3/5}}^{N,A}(x) \geq \xi_{N^{3/5}}^{N,B}(x) \qquad \forall x \in [-3N, -N^{1/3}/3].$$

By Lemma 5.3 (this time applied to the motion of $r^{N,B}$), we have that, outside a further set of probability bounded by $KN^{3/5}/N^{3(k-1)/10}$,

$$\xi_{N^{3/5}}^{N,A}(x) \geq \xi_{N^{3/5}}^{N,B}(x) \qquad \forall x \in [-3N, \infty].$$

To summarize, outside of a set of probability $KNN^{3/5}/N^{(k-1)/20}$, the above inequality holds. We now wish to consider times $t \in (N^{3/5}, N^{3/2} + N^{3/5}]$.

We argue as in Lemma 1.3. Define $D_t = \sup\{x : \xi_t^{N,B}(x) > \xi_t^{N,A}(x)\} \wedge -2N$. (So we have shown that, with high probability as N tends to infinity, $D_0 = -2N$.) We examine how D can jump forward at time t.

We will treat the case where $D_{t-} < r_{t-}^{N,B}$: D can only jump forward to a site x at time t if $\xi^{N,B}$ has a birth at x at time t while $\xi_{t-}^{N,A}(x) = \xi_t^{N,A}(x) = 0$. Given that the flip rates for perturbed NPSs are the same as NPSs except at sites to the right of the rightmost site, we have that D cannot jump forward to x if there exists $D_{t-} < y < x < z$ with $\xi_{t-}^{N,B}(y) = \xi_{t-}^{N,B}(z) = 1$ [and so, by definition of D, $\xi_{t-}^{N,A}(y) = \xi_{t-}^{N,A}(z) = 1$]. This leaves only two possibilities:

1. $D_{t-} < r_{t-}^{N,B} < x$;
2. $D_{t-} < x < r_{t-}^{N,B}$ and $\xi_{t-}^{N,B} \equiv 0$ on $(D_{t-}, x]$.

We will show that case 1 is not possible if event G occurs and that, while case 2 can occur if event G also occurs, case 2 is restricted to $x < D_{t-} + N^{1/20}$.

We first treat case 1. We can assume that $\xi_{t-}^{N,A}(x) = 0$. There are two possibilities. If $r_{t-}^{N,A} < x$, then the flip rate at x at $t-$ for $\xi^{N,A}$ exceeds that for

$\xi^{N,B}$ and D jumping to site x at time t is impossible. If $x < r_{t-}^{N,A}$ and B occurs, then at time $t-$, there exist y, z so that $x \in (y, z)$ and $\xi_{t-}^{N,A}(y) = \xi_{t-}^{N,A}(z) = 1$ and $z - y < N^{1/20}$. Since $D_{t-} < r_{t-}^{N,B}$ we have, from the definition of D, that $\xi_{t-}^{N,A}(r_{t-}^{N,B}) = 1$, so we can take y to be at least $r_{t-}^{N,B}$. Thus the flip rate at time $t-$ at x for $\xi^{N,A}$ is at least

$$\frac{\beta(x-y)\beta(z-x)}{\beta(z-y)} \geq \frac{\beta(x - r_{t-}^{N,B})\beta(N^{1/20}-1)}{\beta(N^{1/20})} \geq \left(1 + \frac{1}{N}\right)\beta(x - r_{t-}^{N,B})$$

for N large. So again a jump of D to x is impossible.

For case 2, by a similar argument, we have that if B occurs, then for $x > D_{t-} + N^{1/20}$ the flip rate for $\xi^{N,A}$ must exceed that for $\xi^{N,B}$.

Thus we have that if event G occurs, then D_t can only jump forward by at most $N^{1/20}$. Its jump rate forward is bounded uniformly by V so we have, for all $t \in [N^{3/5}, N^{3/2} + N^{3/5}]$,

$$\xi_t^{N,A} \geq \xi_t^{N,B} \quad \text{on } [-N, \infty)$$

outside of probability

$$\frac{KNN^{3/5}}{N^{(k-1)/20}} + P\left(\inf_{t \leq N^{3/2}+N^{3/5}} \left\{r_t^{N,B} < -\frac{N}{2}\right\}\right) + P\left(X > \frac{N^{19/20}}{3}\right)$$

for X a Poisson random variable with mean $VN^{3/5}$. We are done, by Lemma 0.2. □

PROPOSITION 5.1. *Let ξ^N be a perturbed NPS in equilibrium so that ξ_0^N is supported on $(-\infty, 0]$. Let r_t^N be the site of the rightmost site at time t. Then, as $N \to \infty$,*

$$\left\{\frac{r_{N^2 t}^N}{N}\right\}_{0 \leq t \leq 1} \Rightarrow \{X_t\}_{0 \leq t \leq 1},$$

where X_t is the diffusion starting at 0 at time 0,

$$X_t = W_t + \mu t$$

for W a standard Brownian motion and $\mu = \sum_n n\beta(n)$.

PROOF. First we get a lower estimate on r_t^N for $0 \leq t \leq N^2$.

Define a process $\xi_t^{N\prime}$ with rightmost particle $r_t^{N\prime}$ as follows:

(a) for $0 \leq t < N^{3/2}$, $\xi_t^{N\prime} = \xi_t^N$ (and so $r_t^{N\prime} = r_t^N$);

(b) for $i \geq 1$ and $iN^{3/2} \leq t < (i+1)N^{3/2}$, $\xi_t^{N\prime} = \xi_t^{N,i}$, where $\xi_t^{N,i}$, $t \geq iN^{3/2}$, is a perturbed NPS in equilibrium generated by the same Harris system as ξ^N so that $\xi_{N^{3/2}}^{N,i}$ is supported on $(-\infty, r_{iN^{3/2}}^N - N^{1/3}]$ but is otherwise independent of $\xi_{iN^{3/2}}^N$.

We apply Lemma 5.5 repeatedly at times $iN^{3/2}$. This and the large deviations bounds for the behavior of rightmost particles provided by Lemma 5.3 entail that, with probability tending to 1 as N tends to infinity,

$$r_t^{N\prime} \le r_t^N \qquad \forall 0 \le t \le N^2.$$

We similarly introduce $\xi_T^{N\prime\prime}, r_t^{N\prime\prime}$:

(a) for $0 \le t \le N^{3/2}$, $\xi_t^{N\prime\prime} = \xi_t^N$ (and so $r_t^{N\prime\prime} = r_t^N$);

(b) for $i \ge 1$ and $iN^{3/2} \le t < (i+1)N^{3/2}$, $\xi_t^{N\prime} = \xi_t^{N,i\prime}$, where $\xi_t^{N,i\prime}$, $t \ge iN^{3/2}$, is a perturbed NPS in equilibrium generated by the same Harris system as ξ^N so that $\xi_{N^{3/2}}^{N,i\prime}$ is supported on $(-\infty, r_{iN^{3/2}}^N + N^{1/3}]$ but is otherwise independent of $\xi_{iN^{3/2}}^N$.

Then analogously one has that with probability tending to 1 as N tends to infinity

$$r_t^{N\prime\prime} \ge r_t^N \qquad \forall 0 \le t \le N^2.$$

However, by Lemma 5.2 one has that

$$X_t^{N\prime} = \sum_{i \le t\sqrt{N}} \frac{1}{N}\Bigl(r_{(i+1)N^{3/2}-}^{N\prime} - r_{iN^{3/2}}^{N\prime}\Bigr) \Rightarrow X_t.$$

Now since Lemma 5.3 easily implies that

$$\sup_i \sup_{t \in [iN^{3/2},(i+1)N^{3/2})} \frac{|r_t^{N\prime} - r_{iN^{3/2}}^{N\prime}|}{N} \to 0$$

in probability we have that $Y_t^N, 0 \le t \le 1$, converges in distribution to X_t where, for $s \in [iN^{3/2}, (i+1)N^{3/2})$,

$$Y_{s/N^2}^N = \sum_{j<i} \frac{1}{N}\Bigl(r_{(j+1)N^{3/2}-}^{N\prime} - r_{jN^{3/2}}^{N\prime}\Bigr) + (r_s^{N\prime} - r_{iN^{3/2}}^{N\prime}).$$

Similarly process $Z_t^N, 0 \le t \le 1$, tends in distribution to X_t where, for $s \in [iN^{3/2}, (i+1)N^{3/2})$,

$$Z_{s/N^2}^N = \sum_{j<i} \frac{1}{N}\Bigl(r_{(j+1)N^{3/2}-}^{N\prime\prime} - r_{jN^{3/2}}^{N\prime\prime}\Bigr) + (r_s^{N\prime\prime} - r_{iN^{3/2}}^{N\prime\prime}).$$

But we have that, with probability tending to 1 as N tends to infinity, for all $0 \le t \le 1$,

$$Y_t^N - \sqrt{N}N^{1/3}/N \le c r_{N^2 t}^N/N \le Z_t^N + \sqrt{N}N^{1/3}/N.$$

The result follows. $\square$

From this result we obtain by simple coupling the following corollary.

COROLLARY 5.2. *Let* ${}^N\eta_0$ *be a NPS with initial distribution equal to that of renewal measure conditioned to have* ${}^N\eta_0(0) = {}^N\eta_0(N) = 1 - {}^N\eta_0(i)\ \forall 0 < i < N$. *Then there exist processes* ${}^{N,L}\xi_t$, ${}^{N,R}\xi$ *so that the following hold*:

(i) ${}^{N,L}\xi_t$ *is a perturbed NPS in equilibrium initially supported on* $(-\infty, 0]$ *with rightmost particle* ℓ_t^N *at time t*;

(ii) ${}^{N,R}\xi_t$ *is a perturbed NPS in equilibrium initially supported on* $[N, \infty)$ *with leftmost particle* r_t^N *at time t*;

(iii) *for*

$$\tau^N = \inf\{iN^{3/2} : |r^N_{iN^{3/2}} - N| \geq N/2\} \wedge \inf\{iN^{3/2} : |l^N_{iN^{3/2}}| \geq N/2\} \wedge N^2$$

$$\left(\frac{l^N_{N^2t\wedge\tau^N}}{N}, \frac{r^N_{N^2t\wedge\tau^N}}{N}\right) \Rightarrow (X^1_{t\wedge\tau}, X^2_{t\wedge\tau}),$$

where $X_0^1 = 0 = 1 - X_0^2$ *and* $X_t^1 = W_t^1 + \mu t$ *and* $X_t^2 = W_t^2 - \mu t$ *for* W^i *independent standard Brownian motions*; $\tau = \inf\{t : |X_t^i - X_0^i| \geq 1/2$ *for* $i = 1$ *or* $2\} \wedge 1$.

With probability tending to 1 as N tends to infinity ${}^{N,L}\xi_t$ *and* ${}^{N,R}\xi$ *are dominated by* ${}^N\eta_t$ *for all* $t \leq \tau^N$.

We do not give a complete proof of Theorem 5.1 since this would involve a large amount of repetition and requires no really new ideas.

Sketch of Proof of Theorem 5.1. Introduce four processes run with H (the same Harris system as generates η), $\eta^{N,i}$, $i = 1, 2, 3$, by taking $\eta_0^{N,i}$ to be mutually independent, with the following:

1. $\eta^{N,1}$ is a semiinfinite NPS in equilibrium with initial support on $[-\lambda N + N^{1/3}, \infty)$;
2. $\eta^{N,2}$ is a perturbed NPS in equilibrium, initially supported on $(-\infty, -N^{1/3}]$;
3. $\eta^{N,3}$ is a perturbed NPS in equilibrium, initially supported on $[N + N^{1/3}, \infty)$;
4. $\eta^{N,4}$ is a semiinfinite NPS in equilibrium with initial support on $(-\infty, (1 + \kappa)N - N^{1/3}]$.

We define the following:

1. l_t^N is the position of the leftmost particle of $\eta^{N,1}$;
2. $l_t^{N,\gamma}$ is the position of the rightmost particle of $\eta^{N,2}$;
3. $r_t^{N,\gamma}$ is the position of the leftmost particle of $\eta^{N,3}$;
4. r_t^N is the position of the rightmost particle of $\eta^{N,4}$;
5. $\tau^N = \inf\{t : |l_t^N + \lambda N| > N\lambda/3$ or $|r_t^N - (1+\kappa)N| > \kappa/3$ or $|l_t^{N,\gamma}| > (\lambda/3 \wedge 1/3)N$ or $|r_t^{N,\gamma}| > (\kappa/3 \wedge 1/3)N\}$.

We then show that with high probability as $N \to \infty$, for all $t \le \tau^N$,

1. η_t dominates $\eta_t^{N,1}$ on $(-\infty, -\frac{3}{5}\lambda N]$;
2. η_t dominates $\eta_t^{N,2}$ on $[-\frac{2}{5}\lambda N, \infty)$;
3. η_t dominates $\eta_t^{N,3}$ on $(-\infty, (1+\frac{2}{5}\kappa)N]$;
4. η_t dominates $\eta_t^{N,4}$ on $[(1+\frac{3}{5}\kappa)N, \infty)$.

From this it follows easily via Lemma 2.2 that η_t is N^v-good on $[l_t^N, l_t^{N,\gamma}]$, $[r_t^{N,\gamma}, r_t^N]$ for all $t \le \tau^N$ with probability tending to 1 as $N \Rightarrow \infty$.

To achieve the necessary independence to apply Corollary 5.2 and Schinazi's theorem we introduce Harris systems H^i, $i = 1, 2, 3, 4$, mutually independent and independent of H, the Harris system generating η. We then consider the following independent processes:

1. $\eta^{N,1,\prime}$, a NPS with $\eta_0^{N,1,\prime} = \eta_0^{N,1}$ and generated by the Harris system equal to H on $(-\infty, -\lambda N/2]$; equal to H^1 elsewhere;
2. $\eta^{N,2,\prime}$, a perturbed NPS with $\eta_0^{N,2,\prime} = \eta_0^{N,2}$ and generated by the Harris system equal to H on $(-\lambda N/2, N/2]$; equal to H^2 elsewhere;
3. $\eta^{N,3,\prime}$, a perturbed NPS with $\eta_0^{N,3,\prime} = \eta_0^{N,3}$ and generated by the Harris system equal to H on $(N/2, N(1+\kappa/2)]$; equal to H^3 elsewhere;
4. $\eta^{N,4,\prime}$, a NPS with $\eta_0^{N,4,\prime} = \eta_0^{N,4}$ and generated by the Harris system equal to H on $(N(1+\kappa/2), \infty)$; equal to H^4 elsewhere.

These processes are clearly independent but using Proposition 4.2 we can show that with probability tending to 1 as $N \Rightarrow \infty$ we have that

$$\begin{aligned}
\eta_t^{N,1,\prime} &= \eta^{N,1} \quad \text{on } \left(-\infty, -\tfrac{2}{5}\lambda N\right],\\
\eta_t^{N,2,\prime} &= \eta^{N,2} \quad \text{on } \left[-\tfrac{3}{5}\lambda N, \infty\right),\\
\eta_t^{N,3,\prime} &= \eta^{N,3} \quad \text{on } \left(-\infty, \left(1+\tfrac{2}{5}\kappa\right)N\right],\\
\eta_t^{N,4,\prime} &= \eta^{N,4} \quad \text{on } \left[\left(1+\tfrac{3}{5}\kappa\right)N, \infty\right).
\end{aligned}$$

Thus we obtain the desired weak convergence for

$$\left(\frac{l_{N^2t}^N}{N}, \frac{l_{N^2t}^{N,\gamma}}{N}, \frac{r_{N^2t}^{N,\gamma}}{N}, \frac{r_{N^2t}^N}{N}\right)_{t \le \tau_N}.$$

6. After three, regrettably, highly technical sections we are now ready to start employing the accrued results toward a proof of Theorem 1. Recall from the first remark following the statement of Theorem 1 in the Introduction that it will suffice to show that, for fixed $\delta > 0$ and fixed, increasing, cylinder f with $f(\mathbf{0}) = 0$,

$E[f(\eta_t)] \geq \langle \text{Ren}(\beta), f \rangle - \delta$ for t large. We consider δ fixed now. We choose ε small but positive (in a way that depends on δ and will be specified in Section 9. Accordingly from now on ε will be considered fixed; N is a large number to be fully specified in the following work. It will depend on the ε chosen. We consider the evolution of finite intervals. By attractiveness we may (and shall) consider that the process η_t begins at time 2^{2N} and that at this time the occupied sites are precisely equal to $\bigcup_z I^z$, a collection of disjoint intervals of length exactly equal to 2^N; N is not yet precisely specified but will be very large. We will detail processes $I_t^z : t \geq 2^{2N}$ with $I_0^z = I^z$. The processes will be called upper intervals, since at all times t the process η_t will be, roughly speaking, in upper equilibrium on I_t^z, though by no means fully occupied. Given interval I^z, we will denote by x_t^z, y_t^z the left and right endpoints of I^z at time t. Of key relevance will be $R = \{t_i^n = 2^{2n} + 3i2^{2n}/n^{100} : n \geq N,\, i = 0, 1, \ldots, n^{100}\}$, the collection of "regeneration times" of the intervals (of course $t_{n^{100}}^n = t_0^{n+1}$). We denote by τ_n^z the first time in R that $|I^z|$ is greater than 2^n. Our first control rules for the evolution of intervals are as follows:

CONDITION 1. Interval I^z is killed at time τ_n^z if

$$\tau_n^z - \tau_{n-1}^z < 2^{2n}/n.$$

CONDITION 2. Interval I^z is killed at time $\tau_{n-1}^z + n2^{2n}$ if

$$\tau_n^z \geq \tau_{n-1}^z + n2^{2n}.$$

CONDITION 3. Interval I^z is killed at time $t \in R$ if

$$t > \tau_n^z \quad \text{and} \quad |I_t^z| < 2^n/n^4.$$

Conditions 1–3 ensure that up to a small power of n, all intervals alive at time t_i^n have size of order 2^n. More specifically we have:

LEMMA 6.1. *There exists a constant K not depending on n so that if interval I is still alive at time $t_i^n = 2^{2n} + 3i2^{2n}/n^{100}$, then $2^n/(Kn^5) \leq |I_{t_i^n}| \leq Kn2^n$.*

We will also specify a small growth condition:

CONDITION 4. Interval I^z is killed at time $t_i^n \in R$, $i > 0$, if for some $s \in (t_{i-1}^n, t_i^n]$,

$$\left| x_s^z - x_{t_{i-1}^n}^z \right| \geq 4 \times 2^n/n^{48} \quad \text{or} \quad \left| y_s^z - y_{t_{i-1}^n}^z \right| \geq 4 \times 2^n/n^{48}$$

(we have yet to specify the dynamics of x^z, y^z).

We also need a medium growth condition:

CONDITION 5. Interval I^z is killed at time $t_i^n \in R$, $i > 0$, if for some t_j^n, $j < i$,

$$\left|x^z_{t^n_j} - x^z_{t^n_i}\right| \geq 2^n/n^6 \quad \text{or} \quad \left|y^z_{t^n_j} - y^z_{t^n_i}\right| \geq 2^n/n^6 \quad \text{but} \quad t^n_i - t^n_j < 2^{2n}/n^{17}.$$

In addition to these conditions on the growth of the endpoints of I we will require that within the interval I no bad exceptional configurations may evolve that would make regeneration difficult.

The evolution of an interval I^z in the time interval $[t^n_{i-1}, t^n_i]$ and in particular the change in its endpoints from time t^n_{i-1} to t^n_i will depend on whether the interval I^z is *attached* at both sides, at a single side or at no side to other intervals. Accordingly we must specify these terms. Initially all intervals are *isolated* (meaning nonattached) at either side. Interval I^z becomes *attached* to interval I^w at time $t^n_i \in R$ if at this time $|I^z \cap I^w|$ is at least $2^n/n^{47}$. At this point the common points are divided equally between the two intervals so that they are disjoint intervals. Thereafter they will remain attached until one or both intervals die. They will also remain disjoint. Given the size consequences of the growth constraints, it follows that an interval can be attached to at most one interval at either side.

We first suppose that at time t^n_{i-1} interval I^z is isolated on both sides (implicitly we assume that I^z exists at time t^n_{i-1}). Associated with I^z at this time will be a finite NPS $\eta^{z,n,i-1}$ which is generated by the Harris system and which is dominated by η, our original process. The endpoints of the support of $\eta^{z,n,i-1}$ will be denoted by $X^{z,n,i-1}$ and $Y^{z,n,i-1}$, respectively. If I^z is isolated then these endpoints will coincide with x^z, y^z over time interval $[t^n_{i-1} + 2^n, t^n_i]$ but not for I^z attached.

We now define our corresponding process $\eta^{z,n,i}_t$ defined for $t \geq t^n_{i-1} + 2^n$. Introduce semiinfinite equilibrium processes $\gamma^{z,n,i,l}_t$, $\gamma^{z,n,i,r}_t$, $t \geq t^n_{i-1}$, as follows:

1. at time t^n_{i-1}, $\gamma^{z,n,i,l}$, $\gamma^{z,n,i,r}$ are independent of η and each other given that they are supported on $[x^z_{t^n_{i-1}} + 2^{2n/3}, \infty)$ and $(\infty, y^z_{t^n_{i-1}} - 2^{2n/3}]$, respectively;
2. the processes evolve according to H, the Harris system for η.

Process $\eta^{z,n,i}_t t \geq t^n_{i-1} + 2^n$ is the finite equilibrium NPS generated by the two semiinfinite NPSs over the time interval $[t^n_{i-1}, t^n_{i-1} + 2^n]$.

Our interval I_z will have boundaries given by those of $\eta^{z,n,i}$ on the time interval $[t^n_{i-1} + 2^n, t^n_i]$. On time interval $[t^n_{i-1}, t^n_{i-1} + 2^n]$ it will have boundary given by the rightmost particle of $\gamma^{z,n,i,r}$ and the leftmost particle of $\gamma^{z,n,i,l}$. Accordingly our intervals will die if they violate the growth stipulations given in Conditions 1–5.

In addition we kill our interval I^z:

(A) at time $t_{i-1}^n + 2^n$ if $\eta^{z,n,i-1}$ does not dominate $\eta^{z,n,i}$;
(B) at time t_i^n if, for some $u \in [x^z_{t_i^n} + 2^{n/4}, y^z_{t_i^n} - 2^{n/4}]$, $\eta^{z,n,i} \circ \theta_u|_{[-2^{n/4},2^{n/4}]}$ is bad;
(C) at time t_i^n if the conditional probability that a $2^{n/12}$ gap appears within $3\eta^{z,n,i}$ or that the boundaries change by $2^{2n/3}/4$ during time interval $[t_i^n, t_i^n + 2^n]$ is less than $2^{-nk/48}$;
(D) at time t_i^n if the boundaries of $\eta^{z,n,i-1}$ have changed by $2^n/n^{48}$ in time interval $[t_{i-1}^n + 2^n, t_i^n]$.

These conditions will be called the regeneration conditions.

We now detail the differences if, at time t_{i-1}^n, I^z is attached to another interval. In this case the difference is that our associated finite process $\eta^{z,n,i}$ will not define the boundaries of I^z on the time interval $[t_{i-1}^n + 2^n, t_i^n]$. In this case as before $\eta^{z,n,i}$ is generated by two semiinfinite processes $\gamma_t^{z,n,i,l}$, $\gamma_t^{z,n,i,r}$, $t \geq t_{i-1}^n$; the difference is that they are supported on $(-\infty, Y^{z,n,i-1} - 2^{2n/3}]$ and on $[X^{z,n,i-1} + 2^{2n/3}, \infty)$, where $Y^{z,n,i-1}$ is equal to $y^z_{t_{i-1}^n}$ if I^z is not attached to the right and equal to $y^z_{t_{i-1}^n} + 122^n/n^{48}$ if I^z is attached to the right at time t_{i-1}^n, and similarly for $X^{z,n,i-1}$. During time interval $[t_{i-1}^n, t_i^n]$ the boundary between two attached intervals fluctuates as an independent symmetric simple random walk of rate 1.

The natural analogues of the extinction rules (A)–(D) are enforced in this case: Thus for (B) we consider sites u within the interval of $\eta^{z,n,i}$ but $2^{n/4}$ away from the boundary. It should be noted that we kill interval I^z either because of extreme fluctuations in its boundary or because of extreme fluctuations in the boundary of $\eta^{z,n,i}$. For (A) we now kill our interval if $\cup \eta^{u,n,i-1}$ does not dominate $\eta^{z,n,i}$, where the union is taken over u equal to z or such that I^u is attached to I^z.

The next order of business is to get bounds on the various probabilities of an interval being killed in interval $(t_{i-1}^n, t_i^n]$ because of stipulations (A)–(D).

PROPOSITION 6.1. *Conditional on I^z being alive at time t_{i-1}^n, the chance that the interval is killed in interval $(t_{i-1}^n, t_i^n]$ due to failing stipulations* (A)–(D) *is bounded by $K 2^{13n/12}/2^{-nk/48}$.*

PROOF. All cases are essentially the same so we will simply treat the case that at time t_{i-1}^n interval I^z is isolated on the left and attached on the right to interval I^w. First we consider (A), the event that $\eta^{z,n,i-1} \cup \eta^{w,n,i-1}$ does not dominate $\eta^{z,n,i}$ at time $t_{i-1}^n + 2^n$. By Lemma 4.2 and Corollary 4.2 this probability is dominated by $K/2^{nk/48}$ and no further discussion is required.

We now deal with (B). By Corollary 4.2 this is bounded by $K 2^n 2^{-nk/24}$.

For (C) we note, by Lemma 2.2 and Proposition 3.1, that the chance that, for a right one-sided process in equilibrium, in time interval $[0, 2^n]$ either a $2^{n/12}$ gap appears within $Kn^2 2^n$ of the rightmost particle or the rightmost particle fluctuates by more than $n^2 2^{n/2}$ is bounded by $n^2 2^n 2^n 2^{-kn/24}$. Thus the chance that the conditional probability of this at time 0 is at least $K 2^{13n/12} 2^{-nk/48}$ is bounded by $K 2^{13n/12} 2^{-nk/48}$. By Lemma 4.4 this bound extends to cover the chance of corresponding boundary fluctuations or gap appearances for $\eta^{z,n,i}$ over the time interval in question. We similarly treat (D). $\square$

So far in this section we have listed our rules for the evolution of an interval in the upper regime and bounded the probability of a regeneration failing. In the remainder of the section we examine the chance of an interval dying because of failing the growth stipulations given by Conditions 1, 2, 4 and 5. We also wish to show that as the time or interval length becomes large most of the conditions become "irrelevant" except for the condition that the interval is killed if its size falls to below $2^n/n^4$ after achieving length 2^n. We also wish to show that, after achieving length 2^n, an interval is essentially as likely to achieve size 2^{n+1} as to disappear before gaining this size.

Already we have good bounds on the failure of an interval to "regenerate" from $2^{2n} + 3i 2^{2n}/n^{100}$ to $2^{2n} + 3(i+1) 2^{2n}/n^{100}$. Let these be called n-order regenerations. By Proposition 6.1 the chance of such a regeneration failing is bounded by $K 2^{13n/12} 2^{-nk/48}$. Suppose that our interval I^z achieves size 2^n at time τ_n^z, after an r-order regeneration. By our size constraints on intervals (Conditions 1–3) we must have that $r \in [n - 6\log(n), n + 2\log(n)]$. Now we define stopping times $S_n^0, S_n^1, \dots$ by taking S_n^0 to be τ_n^z and $S_n^i = \inf\{t > S_n^{i-1} : t$ is a regeneration time and $t - S_n^{i-1} \geq 2^{2n}\}$; it is easily checked via Corollary 3.1 that

$$P\big(\tau_{n+1}^z \text{ or } \tau_D^z \in (S_n^{i-1}, S_n^i] \mid \tau_D^z, \tau_{n+1}^z > S_n^{i-1}\big) > d > 0,$$

where d does not depend on n and τ_D^z denotes the death time of I^z.

This and the strong Markov property immediately yield:

LEMMA 6.2. *Let τ_i^z be as above. Then*

$$P(n 2^{2n}/2 < \tau_{n+1}^z - \tau_n^z < \infty \mid F_{\tau_n^z}) \leq c^n,$$

for some $c < 1$ not depending on n.

We now consider the probability that after τ_n^z the interval I^z is killed because it grows too quickly, that is, because $\tau_{n+1}^z \leq \tau_n^z + 2^{2n}/n$. For simplicity only, we will suppose that $[\tau_n^z, \tau_n^z + 2^{2n}/n] \in [2^{2m}, 2^{2(m+1)}]$ for some m. As already noted, τ_n^z must be less than $3n2^n$. Therefore $|I^z_{\tau_n^z}| < 2^n + 7 \times 2^n/n^{47}$. For $\tau_{n+1}^z - \tau_n^z \leq 2^{2n}/n$ we must have that either

$$x^z_{\tau^z_{n+1}} - x^z_{\tau^z_n} \geq \frac{2^{n+1} - (2^n + (7 \times 2^n)/n^{47})}{2}$$

or

$$y^z_{\tau^z_{n+1}} - y^z_{\tau^z_n} \geq \frac{2^{n+1} - (2^n + (7 \times 2^n)/n^{47})}{2},$$

or both. (Recall x^z and y^z denote respectively the left and right endpoints of I^z.) So, by symmetry, we have

$$P(\tau^z_{n+1} - \tau^z_n \leq 2^{2n}/n \mid F_{\tau^z_n}) \leq 2P\Biggl(\sup_{v\,:\,\tau^z_n < t^m_v < \tau^z_n + \frac{2^{2n}}{n} + \frac{32^{2m}}{m^{100}}} \sum_{j\,:\,t^m_j = \tau^z_n}^{v} (x^z_{t^m_{j+1}} - x^z_{t^m_j}) \geq \frac{2^{n+1} - (2^n + 7 \times 2^n/n^{47})}{2}\Biggr).$$

Now given $F_{t^m_j}$, $(x^z_{t^m_{j+1}} - x^z_{t^m_j})$ is either stochastically less than $r_{3 \times 2^{2m}/m^{100}}$ for a one-sided equilibrium NPS supported on $(-\infty, 0]$ at time 0 or stochastically less than $S_{3 \times 2^{2m}/m^{100}}$ for S a rate-1 symmetric nearest neighbor random walk. Thus we have that

$$\begin{aligned} &P(\tau^z_{n+1} - \tau^z_n \leq 2^{2n}/n) \\ &\quad \leq 2P\Biggl(\sup_{1 \leq v \leq \frac{2^{2n}/n}{2^{2m}/m^{100}} + 1} \sum_{j=1}^{v} r^j_{2^{2m}/m^{100}} \geq \frac{2^{n+1} - (2^n + (7 \times 2^n)/n^{47})}{4}\Biggr) \\ &\qquad + P\Biggl(\sup_{s \leq 2^{2n}/n + 2^{2m}/m^{100}} S_s \geq \frac{2^{n+1} - (2^n + (7 \times 2^n)/n^{47})}{4}\Biggr), \end{aligned}$$

for S a rate-1 symmetric nearest neighbor random walk and r^i independent copies of r. Then it is easily seen by arguments of Proposition 3.1 that this probability is bounded by $(1/2)^n$ for n large. We have proven:

LEMMA 6.3. *Let τ^z_n be as above. Then*

$$P(\tau^z_{n+1} - \tau^z_n \leq 2^{2n}/n \mid F_{\tau^z_n}) \leq C(1/2)^n.$$

We similarly prove:

LEMMA 6.4. *Given $\tau^z_D > 2^{2n}$, the probability that for some $i, j \in 0, 1, \ldots, n^{100}$, $j - i < n^{81}$, the rightmost edge of $I^z_{2^{2n} + 3i2^{2n}/n^{100}}$ is less than the rightmost edge $I_{2^{2n} + 3j2^{2n}/n^{100}} - 2^n/3n^9$ is bounded by $C(1/2)^n$.*

Now we consider the probability that an interval attains length 2^{n+1} given that it attains length 2^n. By our growth conditions $\tau_n^z \in [2^{2n}/n, 2 \times 2^{2n} n]$. Suppose $\tau_n^z = t_i^m$. Then we must have $n - \log_2(4n)/2 \le m \le n + \log_2(n)/2$. By our growth condition, we must have $2^n \le |I^z_{\tau_n^z}| \le 2^n + 4 \times 2^m/m^{48} \le 2^n + 2^n/n^{47}$. Since the overshoot of 2^n is bounded by $2^n/n^{47} \ll 2^n/n^4$ by growth stipulation of Condition 4 and Lemma 6.1, we immediately have that the probability that $\tau_{n+1}^z < \infty$ given that $\tau_n^z < \infty$ is less than $1/2$. We wish to show that it is close to $1/2$.

We couple our interval with an evolving interval which has the same increments as our interval when our interval is not killed but whose interval increments will be independently generated after our interval is killed and where the increments for an r-order regeneration are conditioned to be less than $4 \times 2^r/r^{48}$. Our interval will start with the same length as I^z at time τ_n^z. Then after N regenerations the comparison interval will have length

$$S_N = |I|_{\tau_n^z} + \sum_{i=1}^{N} X_i - d_i,$$

where X_i are independent mean-zero random variables bounded by $22^n/n^{46}$. The d_i are equal to $2 \times 2^{2m/3}$ when i corresponds to an m-order regeneration and are thus bounded by $4n2^{2n/3}$ for $i \le n^{102}$. Let τ be the first time our comparison interval has value greater than or equal to 2^{n+1} or less than $2^n/n^4$.

LEMMA 6.5. $P(S_\tau \ge 2^{n+1}) \ge 1/2 - 2/n^4$.

This result and the bounds on our real interval being killed before either attaining length 2^{n+1} or having its length decline to below $2^n/n^4$ immediately yield the following corollary.

COROLLARY 6.1. *For an interval I^z, $P(\tau_{n+1}^z < \infty \mid \tau_n^z < \infty) \ge 1/2 - K/n^4$ for some K not depending on n. There exists $K \in (0, \infty)$ so that, for $n \ge N$ and all z,*

$$\frac{1}{K} 2^{N-n} \le P(\tau_n^z < \infty) \le K 2^{N-n}.$$

COROLLARY 6.2. *There exists $K \in (0, \infty)$ so that, for $n \ge N$ and all z, $1/K \le 2^{n-N} P(I^z_{2^{2n}}$ is alive$) \le K$.*

PROOF. It is easy to check (via Corollary 3.1 and bounds on killing probabilities given in Lemmas 6.2–6.5 and Proposition 6.1) that, for some strictly positive d not depending on n,

$$P\big(\{\tau_n^z < \infty\} \cap \{I^z \text{ does not die in } [\tau_n^z, \tau_n^z + 2^{2n}]\}\big) \ge d 2^{N-n}.$$

So the lower bound follows easily. On the other hand,

$$P(I^z\text{is alive at time } 2^{2n}) = \sum_{m=N}^{\infty} P(\tau_m^z \le 2^{2n} < \tau_{m+1}^z \wedge \tau_D^z)$$

$$\le \sum_{m=N}^{n-C} P(\tau_m^z \le 2^{2n} < \tau_{m+1}^z \wedge \tau_D^z) + K2^C 2^{N-n}$$

for C fixed but large, by Corollary 6.1. Now $\sum_{m=N}^{n-C} P(\tau_m^z \le 2^{2n} < \tau_{m+1}^z \wedge \tau_D^z)$ is less than $P(\sum_{m=N}^{n-C}(\tau_{m+1}^z \wedge \tau_D^z - \tau_m^z)_+ \ge 2^{2n})$. In turn this is bounded by $\sum_{i=0}^{n-C-N} P((\tau_{n-C-i+1}^z \wedge \tau_D^z - \tau_{n-C-i}^z)_+ \ge 2^{2n}2^{-i-1})$. However, by the arguments giving Lemma 6.2 we have that there exists a $d < 1$ not depending on n or C so that this is bounded by $\sum_{i=0} K2^{N-n+i} d^{2^{2C}2^i} \le 2^{N-n}$ if C were fixed sufficiently large. $\square$

PROOF OF LEMMA 6.5. Consider the martingale $M_N = |I|_{\tau_n^I} + \sum_{i=1}^N X_i = S_N + \sum_{i=1}^N d_i$ for $1 \le N \le n^{102}$. Then for all N, $E[M_{N\wedge\tau}] = E[M_0] \ge 2^n$. On $\{\tau > N\}$, $M_{N\wedge\tau} \le 2^{n+1} + N4n2^{2n/3} < 2^{n+1}(1 + 1/n^6)$. On $\{\tau \le N, S_\tau \ge 2^{n+1}\}$, we have similarly that $M_{N\wedge\tau} \le 2^{n+1} + 2^n/n^{46} + N4n2^{2n/3} \le 2^{n+1}(1 + 1/n^6)$, while on $\{\tau \ge N,\ S_\tau \le 2^n/n^4\}$, we have that $M_{N\wedge\tau} \le 2^n/n^4 + N4n2^{2n/3} \le 2^n(1/n^4 + 1/n^6)$. Thus we have

$$\begin{aligned} 2^n &\le M_0 \le E[M_{N\wedge\tau}] \\ &\le P(\tau \le N, S_\tau \ge 2^{n+1}]\left(2^{n+1}\left(1 + \frac{1}{n^6}\right)\right) \\ &\quad + P(\tau \le N, S_\tau \le 2^n/n^4]\left(2^n\left(\frac{1}{n^4} + \frac{1}{n^6}\right)\right) + P(\tau > N)\left(2^{n+1}\left(1 + \frac{1}{n^6}\right)\right). \end{aligned}$$

However, by the argument of Lemma 6.2, we have $P(\tau > N) \le c^n$. The result now follows from routine manipulations. $\square$

7. In this section we consider the effect of our killing rules and regeneration corrections on the density of points in $\cup_z I^z$ at time t. The object is to show that this density is essentially the same as the initial density; that is, the picture is close to our "voter model" picture discussed in the Introduction.

As we move from time t_{i-1}^n to time t_i^n the size of I_z (assuming that it is alive at t_{i-1}^n) changes to

$$0 \text{ if the interval dies in } [t_{i-1}^n, t_i^n]$$

and otherwise to

$$|I^z_{t^n_{i-1}}| + \xi^{z,n,i,l} + \xi^{z,n,i,r} - 2^{2n/3}\Big(I_{I^z_{t^n_{i-1}} \text{ is left isolated}} + I_{I^z_{t^n_{i-1}} \text{ is right isolated}}\Big) \\ + C^{z,n,i,l} + C^{z,n,i,r},$$

where, for event A, I_A represents the usual indicator function (we apologize for the dual use of I) and the ξ are either edge fluctuations of equilibrium one-sided NPSs or fluctuations of a simple symmetric random walk (provided both are less than $4 \times 2^n/n^{48}$).

Here $C^{z,n,i,l}$ represents the loss of sites to I^z if I^z becomes attached at the left at time t^n_i; similarly for $C^{z,n,i,r}$.

Thus we can write $|I^z_{t^n_i}|$ as a martingale minus a decreasing process which (of course) stops decreasing once I^z dies (that is to say $|I^z|$ becomes zero). For the rest of this section we try to bound this decreasing part. Since it is decreasing we consider its expectation at ∞; this is equal to

$$\text{(A)} \qquad \sum_{n=N}\sum_{i=0}^{n^{100}-1} 2^{2n/3}\Big(I_{I^z_{t^n_{i-1}} \text{ is left isolated}} + I_{I^z_{t^n_{i-1}} \text{ is right isolated}}\Big) I_{I^z_{t^n_i} \text{ is alive}}$$

$$\text{(B)} \qquad + \sum_{n=N}\sum_{i=0}^{n^{100}-1} (C^{z,n,i,l} + C^{z,n,i,r}) I_{I^z_{t^n_i} \text{ is alive}}$$

$$\text{(C)} \qquad + \sum_{n=N}\sum_{i=0}^{n^{100}-1} L(n,i) I_{I^z \text{ dies at time } t^n_i},$$

where $L(n,i)$ is the loss resulting from the death of interval I^z at time t^n_i. Note that positive and negative large deviations of interval edges can be thought of as cancelling each other out so that $L(n,i) \le Kn2^n + 8 \times 2^n/n^{48} \le Kn2^n$.

We first bound (A). A simple bound for (A) is $\sum_{n \ge N} 2^{2n/3} n^{100} I_{\{I^z_{2^{2n}} \text{is alive}\}}$. By Corollary 6.2 this is bounded in expectation by

$$K \sum_{n \ge N} n^{100} 2^{2n/3} 2^{N-n} = K 2^N \sum_{n \ge N} n^{100} 2^{-n/3}.$$

If N was chosen sufficiently large, then this is less than $\varepsilon 2^N/10$.

For (B) note first that by the growth stipulation of Condition 4 we have that $C^{z,n,i,l}$ is less than $2^n/n^{47}$ and likewise for $C^{z,n,i,r}$. Second, it should be noted that the growth stipulation of Condition 5 ensures that if $|j - j'| \le n^{83}$, then $C^{z,n,j,l}$ and $C^{z,n,j',l}$ cannot both be nonzero. Thus $\sum_{j=0}^{n^{100}} C^{z,n,j,l} \le n^{17} 2^n/n^{47} = 2^n/n^{30}$. Therefore $E[(\mathrm{B})] \le K \sum_{n \ge N} 2^n/n^{30} 2^{N-n}$. Again provided N was fixed sufficiently high this is bounded by $\varepsilon 2^N/10$.

It remains to treat (C). As has already been noted, $L(n,i)$ can be at most $Kn2^n$. So (C) is bounded by

$$K\sum_{n\geq N} n2^n I_{\{I^z \text{ dies due to failure to regenerate at } t_i^n \text{ for some } i\}}$$
$$+\,K\sum_{n\geq N} n2^n I_{\{I^z \text{ fails Condition } 1,2,4 \text{ and } 5 \text{ for } t\in R\cap[2^{2n},2^{2(n+1)}]\}}$$
$$+\,K\sum_{n\geq N} \frac{2^n}{n^4} I_{\{I^z \text{ fails Condition 3 after } \tau_n^z \text{ before } \tau_{n+1}^z\}}.$$

By Proposition 6.1 and Corollary 6.2 the expectation of the first term is bounded by

$$K\sum_{n\geq N} n2^n 2^{N-n}2^{13n/12}2^{-nk/48} \leq \frac{\varepsilon 2^N}{10}$$

for N sufficiently large. Lemmas 6.1–6.3 and Corollary 6.2 similarly bound the expectation of the second term. Lemma 6.4 bounds the third term by $\sum_{n\geq N}(2^n/n^4)2^{N-n} \leq \varepsilon 2^N/10$ for N large.

Let us call the intervals I^z the original upper equilibrium intervals in contrast to the new or created upper equilibrium intervals that we will describe in the next section.

We conclude from the above bounds:

PROPOSITION 7.1. *If $\rho_t, t\geq 2^{2N}$, is the density of sites in an original upper equilibrium interval at time t, then for all such t we have $\rho_t \geq (1-\varepsilon)\rho_{2^{2N}}$.*

We equally could say that the expectation of site loss for any original interval is bounded by $2^N\varepsilon$.

8. In this section we introduce our mechanism for creating upper equilibrium intervals and then prove that the intervals in the upper regime become dominant as $t\to\infty$. Recall we have fixed ε very small but positive.

We need to show that the density of vacant intervals goes to zero. We consider various types of vacant interval. First there are big vacant intervals, where by big, at time t_i^n, we will mean large relative to 2^n. Lemma 8.1 shows this density is small. Then there are vacant intervals corresponding to growths of intervals originally vacant at time 2^{2n} of "reasonable" size at time t_i^n (again relative to size 2^n). Proposition 8.1 below will show that if the density of such points in such intervals is significant at time 2^{2n} then the density of "upper regime" points must significantly increase over interval $(2^{2n}, 2^{2(n+1)}]$. Accordingly this density must decrease to zero. The density of points in vacant intervals created by the deaths of "upper regime" intervals (including those created, as will be detailed) is essentially bounded by Proposition 7.1. This will leave us with showing that the density of small intervals tends to zero. This is done in Theorem 8.1.

By a simple but tedious 2-moment argument, we have:

LEMMA 8.1. *Given ε there is an M, greater than or equal to $1/\varepsilon$, so that for all $n \geq N$ the chance that there is no I^z alive at time 2^{2n}, of size 2^n or more at this time and contained in $[0, M2^n]$, is less than $\varepsilon^2/6$.*

SKETCH OF PROOF. We associate with each I_z two Gaussian processes $W_t^{z,x}$ and $W_t^{z,y}$, $t \in R$, the set of regeneration times. We let $W_{2^{2N}}^{z,x} = x_{2^{2N}}^z$, $W_{2^{2N}}^{z,y} = y_{2^{2N}}^z$, and so that, for $t_i^n \in R$, $W_{t_i^n}^{z,x} - W_{t_{i-1}^n}^{z,x}$, $W_{t_i^n}^{z,y} - W_{t_{i-1}^n}^{z,y}$ are independent Gaussian random variables so that (see Proposition 3.5)

$$\left|\left(W_{t_i^n}^{z,x} - W_{t_{i-1}^n}^{z,x}\right) - \left(x_{t_i^n}^z - x_{t_{i-1}^n}^z\right)\right| < K2^{-n/60}\frac{2^n}{n^{50}},$$

outside of probability $2^{n/60}(3.2^{-n/30} + (K2^{45n/14}/2^{4nk/105})^{1/2})$.

We may also choose these Gaussian increments so that we have independence for intervals that are separated by at least $2^n/n^{47}$ at time t_i^n.

We kill $W^{z,x}, W^{z,y}$ if $W_t^{z,y} - W_t^{z,x} < 2^n/n^3$ for $t > \tau^n = \inf\{t \in R : W_t^{z,y} - W_t^{z,x} \geq 2^n\}$.

For z such that $I_{2^{2N}}^z$ is contained in $((j-1/2)2^n, (j+1/2)2^n)$ for even j, we let event $A(z)$ be that

$$W^{z,x}, W^{z,y} \text{ are still alive at time } 2^{2n}$$

and

$$W_{2^{2n}}^{z,x} \leq (j-1/2)2^n, \quad W_{2^{2n}}^{z,y} \geq (j+1/2)2^n$$
$$\text{for all } t (\in R) \leq 2^{2n},\ W_t^{z,x}, W_t^{z,y} \in \big((j-2/3)2^n, (j+2/3)2^n\big).$$

Then we have $P(A(n)) \geq c2^{N-n}$ and for such j we have $P(A(z) \cap A(z')) \leq P(A(z))P(A(z'))$. So given ε, N we can choose $M = M(\varepsilon, N)$ so that

$$P\Bigg(\bigcup_{z:\, I_{2^{2N}}^z \in [0, M2^n]} A(z)\Bigg) \geq 1 - \frac{\varepsilon^2}{12}.$$

We then note that if we had fixed N sufficiently large, then

$$P\Bigg(\bigcup_{z:\, I_{2^{2N}}^z \in [0, M2^n]} A(z) \Big\backslash \bigcup_{z:\, I_{2^{2N}}^z \in [0, M2^n]} B(z)\Bigg) \leq \frac{\varepsilon^2}{12},$$

where $B(z)$ is the event

$$\{\text{at time } 2^{2n},\ I^z \text{ is alive, of length at least } 2^n \text{ and contained in the interval } \big((j-2/3)2^n, (j+2/3)2^n\big)\}.$$

This has the following immediate corollary:

COROLLARY 8.1. *Let M be as in Lemma* 8.1. *For all n the chance that the origin is contained in a vacant interval of length $2M2^{2n}$ or greater is bounded by $\varepsilon^2/3$.*

The importance of Corollary 8.1 for this paper is that it means we do not have to worry about very big vacant intervals in showing that for large time the density of points in vacant intervals is small.

Before describing how "upper equilibrium" intervals are created we need to develop a tagging system for vacant intervals and to introduce a distinction between original vacant intervals and vacant intervals that arise from the killing of "upper equilibrium" intervals. We have described the system of (possibly slightly overlapping) intervals in the upper regime, their killing and creation. We now introduce a system of labeling for the vacant intervals. We will divide these intervals into those that correspond to an initial vacant interval and those that correspond to a vacant interval created by the killing of an upper regime interval. Initially we take as our intervals those natural maximal occupied intervals of our "initial" configuration $\eta_{2^{2N}}$. We will stipulate that immediately after every regeneration time t all abutting vacant intervals coalesce into one large vacant interval which is taken to correspond to one of the previous intervals, the other intervals being deemed to be killed off. The "surviving interval is chosen in proportion to the respective intervals. More concretely, if at regeneration time t a maximal collection of vacant abutting intervals is $V_t^1, V_t^2, \ldots, V_t^j$, then we randomly (and independently of all other interval choices and of the Harris system) choose k equal to $i \in \{1, 2, \ldots, j\}$ with probability

$$\frac{|V_t^i|}{\sum |V_t^h|}.$$

Then at $t+$ all intervals V^h for $h \neq k$ are deemed to die, while interval V^k expands from V_t^k to $V_{t+}^k = \bigcup_l V_t^l$. This step is called *consolidation*.

We now detail the evolution of a null interval J from $t_i^n + = 2^{2n} + 3i2^{2n}/n^{100}$ to $t_{i+1}^n = 2^{2n} + 3(i+1)2^{2n}/n^{100}$. Necessarily at time t_i^n+ the interval is abutted by two upper intervals. If during the coming interval these two intervals do not die, then at time t_{i+1}^n the vacant interval endpoints are defined via the endpoints of these two upper intervals. However, if, say, the right abutting interval dies during this interval then the new vacant interval is given by the neighbor of the rightmost site of the surviving left interval and the initial leftmost site of the dead upper interval at time t_i^n minus $4 \times 2^n/n^{48}$. If this interval is null, then this interval is deemed to have died. We have similar rules for when the left upper interval dies or when both surrounding intervals die. As a consequence for vacant intervals that are alive at time t_i^n death of surrounding upper intervals is never a good thing. On the other hand this "redistribution" of sites from preexisting vacant intervals to newly created vacant intervals only constitutes an additional small fraction of the number of sites given to the new interval by the destroyed upper interval.

We now detail our mechanism for the creation of intervals in upper equilibrium. Suppose that, at time 2^{2n}, V^k is a vacant interval corresponding to an original vacant interval of size 2^n in the interval $[\varepsilon 2^n, 2M2^n]$. Match it with the first original interval of size 2^n to the right and with the first original interval of size 2^n to the left. Denote these intervals by $I^{z(k,l)}$ and $I^{z(k,r)}$ respectively. If either such interval is further than $M2^n$ away from V^k at this time we say that the match is lost. If not we say the match holds. It is easy to see that with probability at least $p(\varepsilon, M) > 0$ we have that at time 22^{2n} we have that $I^{z(k,l)}$, $I^{z(k,r)}$ abut V^k and that all three intervals have size at least 2^n and at V^k has size at most 22^n.

Now, given this situation, consider (in addition to the regular edge fluctuations of the occupied intervals), the subordinate processes $l_t^{N,\gamma}, r_t^{N,\gamma}$ so that (as described in Theorem 5.1) if

$$X_t^N = \frac{r_{tN^2}^{N,\gamma} - l_{tN^2}^{N,\gamma}}{N},$$

then "locally" $X_t^N \to X_t$, where

$$X(t) = \sqrt{2}W(t) - 2\mu t$$

and W is a Brownian motion.

It follows, using Theorem 5.1, that we can find some $g, d, q > 0$ (not depending on n) so that with probability at least q we have at time $22^{2n} + d2^{2n}$ (assumed to be t_j^n) that there exists $x^{k,j,n}$ such that:

1. $x_{t_j^n}^{z(k,r)} - 2g2^n < x^{k,j,n} < x_{t_j^n}^{z(k,r)} - g2^n$;
2. $y_{t_j^n}^{z(k,l)} + 2g2^n < x^{k,j,n}$;
3. if we consider η restricted to interval $[x^{k,j,n}, x_{t_j^n}^{z(k,r)}]$ as an interval attached to $I^{z(k,r)}$ to the right and unattached to the left, then the conditional probability of successful regeneration for this interval and of interval $I^{z(k,r)}$ at time $22^{2n} + d2^{2n}$ exceeds the lower bound of Proposition 6.1.

If these conditions are satisfied, then the interval $[x^{k,j,n}, x_{t_j^n}^{z(k,r)}]$ has been created at time t_j^n. Thereafter the system evolves as if the newly created interval is an isolated interval. (So another vacant interval has been created between the new interval and $I^{z(k,r)}$.) If these conditions are not satisfied, we forget about creating an upper equilibrium interval from V^k and the intervals evolve without the extra mechanism. We have shown:

PROPOSITION 8.1. *There exists $h = h(\varepsilon, M)$ so that, given a vacant interval V^k matched at time 2^{2n}, with probability at least h a new upper equilibrium interval is created during $[2^{2n}, 2^{2(n+1)}]$ of length at least $g2^n$, where g is as chosen above.*

It is immediate that the bounds on original intervals being killed in Proposition 6.1 still hold. It is also clear that while this added mechanism for creating new particles may result in additional sites of $I^{z(k,r)}$ being lost due to an extra attachment of $I^{z(k,r)}$ the previous bounds on this loss still hold.

Proposition 7.1 can be applied to the created intervals (the slight difference resulting from starting from initial length approximately $c2^n$ rather than exactly 2^n notwithstanding). We thus have:

PROPOSITION 8.2. *The density of sites inside an interval in upper equilibrium (whether original or created) at time t_i^n is at least $1 - \varepsilon$ times*

$$\Bigl(\rho_{2^{2n}} + \sum_{t_k^m < t_i^n} \rho(m)\Bigr),$$

where $\rho(m)$ is the density of sites in upper equilibrium sites created during $[2^{2m}, 2^{2(m+1)}]$ at the time of creation.

Finally Proposition 8.1 yields the following:

PROPOSITION 8.3. *If at time 2^{2n} the density of sites inside original vacant intervals of length between $\varepsilon 2^n$ and $2M2^n$ is at least f, then $\rho(n) \geq fgh/(2M)$.*

THEOREM 8.1. *The density of sites in an upper invariant interval is eventually more than $1 - 6\varepsilon$.*

PROOF. From Proposition 7.1 (and then Proposition 8.2) it follows that the density of sites in a created vacant interval is bounded by 2ε. By Lemma 8.1 the density of sites in an original vacant interval of size at least $2M\sqrt{t_i^n}$ is less than ε. Propositions 8.2 and 8.3 force the density of sites at time t_i^n having size in between $\varepsilon 2^n$ and $2M2^n$ to go to zero. Thus it remains only to consider the density of sites in an original vacant intervals of size less than $\varepsilon\sqrt{t_i^n}$. To show this it suffices to show that the density of distinct original vacant intervals is eventually less than $\frac{1}{4}2^{-n}$ at time 2^{2n}.

Let the density of original intervals (not sites in original intervals) at time 2^{2n} be $d(n)$. It is easily seen that, provided ε was fixed sufficiently small (in a way not depending on N), a vacant interval alive at time 2^{2n} of length less than $\varepsilon 2^n$ has chance at least $\frac{3}{4}$ of dying before time $2^{2(n+1)}$. Also the density of original intervals of length at least $\varepsilon 2^n$ is for large n less than $\frac{2}{3M}2^{-n}$ by the above discussion. Thus

$$d(n+1) \leq \left(d(n) - \frac{2}{3M}2^{-n}\right) \cdot \frac{1}{4} + \frac{2}{3M}2^{-n}.$$

Therefore if $d(n) \geq \frac{6}{M}2^{-n}$, then $d(n+1) \leq d(n)(1 - \frac{5}{6} \cdot \frac{3}{4}) \leq d(n)\frac{3}{8}$. It follows that $2^n d(n)$ decreases with n by a factor of $2 \cdot \frac{3}{8} = \frac{3}{4}$, until $d(n)2^n \leq \frac{6}{M}$. Thus we have eventually that $d(n) \leq \frac{12}{M}2^{-n}$. This completes the proof. $\square$

9. Proof of Theorem 1. All that remains is to show that the eventual dominance of "upper equilibrium" intervals over vacant intervals translates into the desired convergence in distribution. This part of the proof is close to the argument for convergence employed in Mountford and Sweet (1998) but is given fully here for completeness. As already noted, to show the theorem it will suffice to show that $E[f(\eta_t)]$ tends to $\langle \mathrm{Ren}(\beta), f \rangle$ for f an increasing cylinder function so that $f(\mathbf{0})$ is equal to zero. Without loss of generality we will assume that $\|f\|_\infty \leq 1$. Let its support be $[-m, m]$.

Since for any such function f we have

$$\limsup_{t \to \infty} E[f(\eta_t)] \leq \langle \mathrm{Ren}(\beta), f \rangle,$$

it will suffice to show that, for fixed f as above and $\delta > 0$,

$$E[f(\eta_t)] \geq \langle \mathrm{Ren}(\beta), f \rangle - \delta.$$

Accordingly let us fix such f, δ. By Proposition 8.1 we may fix ε, N so that, with these parameters defining our comparison process, the density of "upper equilibrium" intervals exceeds $1 - \delta/10$ for all large times.

Let us fix t so large that if $t \in [2^{2v}, 2^{2(v+1)}]$ the density of "upper equilibrium" intervals at time $t - 12\, 3\frac{2^{2v}}{v^{100}}$ and subsequent times exceeds $1 - \delta/10$. Eventually of course we will let t go off to infinity. Choose t_i^n so that $t_i^n + 3\frac{2^{2n}}{n^{100}} \leq t \leq t_i^n + 2 \times 3\frac{2^{2n}}{n^{100}}$. By slight abuse of notation let the two points in R following t_i^n be denoted by t_{i+1}^n and t_{i+2}^n, even though i may equal n^{100} or $n^{100} - 1$.

Consider the following events:

(A) At time t_i^n, $[-m, m]$ is within $9\frac{2^n}{n^8}$ of a vacant interval.
(B) At times t_{i+1}^n or t_{i+2}^n, $[-m, m]$ intersects a vacant interval.

At this time all such intervals have length at least $K2^n/n^5$ by Lemma 6.1. So the density of points in an upper interval but within $m + 9\frac{2^n}{n^8}$ of a vacant interval is bounded by $\frac{m+9(2^n/n^8)}{K2^n/n^5}$. Thus for t large the chance of event (A) above occurring is bounded by $\frac{2}{10}\delta$, and similarly for event (B).

Event (B) was introduced because, by Condition 4 introduced in Section 6, if event (A) above does not occur but the upper interval in which the origin is within at time t_i^n dies at t_{i+1}^n or t_{i+2}^n, then event (B) must occur.

Suppose that event A does not occur and that at time t_i^n the origin is contained within I^z, at least $9\frac{2^n}{n^8}$ from the boundary. Let $(\gamma_T)_{t\geq t_i^n+2^n}$ be the finite NPS generated by $\gamma^{z,n,i,r}$, $\gamma^{z,n,i,l}$ over time interval $[t_i^n, t_i^n+2^n]$. (Recall the definitions in Section 6.)

As remarked above, if at time $t_i^n+2^n$ (and therefore for all subsequent times) η does not dominate γ, then event (B) must occur. By Lemma 4.5 (and the growth conditions to which I^z is subject), if (A) does not occur (and n is sufficiently large) then

$$E[f(\gamma_t)] \geq \langle \mathrm{Ren}(\beta), f\rangle - \delta/10.$$

Putting these observations together we have for large t that

$$\begin{aligned} E[f(\eta_t)] &\geq E[f(\eta_t)I_{A^c\cap B^c}] - P(A) - P(B) \\ &\geq E[f(\gamma_t)I_{A^c\cap B^c}] - P(A) - P(B) \\ &\geq E[f(\gamma_t)] - 2P(A) - 2P(B), \end{aligned}$$

which by our assumption on t being large is at least $\langle \mathrm{Ren}(\beta), f\rangle - \delta$.

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