TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 157, June 1971

THE CONVERGENCE WITH VANISHING VISCOSITY OF NONSTATIONARY NAVIER-STOKES FLOW TO IDEAL FLOW IN R_3

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Abstract. It is shown here that a unique solution to the Navier-Stokes equations exists in R_3 for a small time interval independent of the viscosity and that the solutions for varying viscosities converge uniformly to a function that is a solution to the equations for ideal flow in R_3. The existence of the solutions is shown by transforming the Navier-Stokes equations to an equivalent system solvable by applying fixed point methods with estimates derived from using semigroup theory.

Introduction. We wish to find a solution, local in time, to the Cauchy problem for the Navier-Stokes equations for viscous incompressible flow in R_3 and show that the solutions of the Navier-Stokes equations for various viscosities converge, as the viscosity goes to zero, to a function that is a solution to the Euler equations for an ideal (inviscid) fluid.

The Navier-Stokes equations are

$$\text{(E}'\text{)}\qquad \partial v/\partial t+(v\cdot \operatorname{grad})\, v-\nu\Delta v = -\operatorname{grad} P+B, \qquad \nabla\cdot v = 0,$$

with constraints

$$\lim_{|x|\to\infty} v(x,t) = 0 \quad\text{and}\quad v(x,0) = C(x),$$

where $x=(x_1, x_2, x_3)$ is a point in R^3; t is in some time interval $[0, T]$; the velociy $v(x, t)=(v_1(x, t), v_2(x, t), v_3(x, t))$; the pressure is $P(x, t)$; the force is $B(x, t) =(B_1(x, t), B_2(x, t), B_3(x, t))$; and the constant $\nu>0$ is the viscosity (the coefficient of kinematic viscosity).

The Euler equations for ideal flow differ from the Navier-Stokes equations (E′) only in that the viscosity term $\nu\Delta v$ does not occur in the Euler equations.

Uniqueness and existence of a solution to the Navier-Stokes equations in R^3 has been shown for both bounded and unbounded domains: in both cases existence has been shown only for a sufficiently small time interval. The first results are those of C. W. Oseen [11] and Jean Leray [8]. The time interval where the solution is shown to exist must be small enough to satisfy a condition of form $T\leqq K\nu$, where K is an appropriate constant and ν is the viscosity. Thus the length of the time interval

Received by the editors January 28, 1970 and, in revised form, June 25, 1970.
AMS 1968 *subject classifications*. Primary 7635, 7646; Secondary 3536, 4750.
Key words and phrases. Navier-Stokes equations, viscous flow, Euler equations, ideal flow, potential theory, semigroups of operators, Sobolev space, nonlinear partial differential equations.

goes to zero (see [8, p. 223]) and will not allow us to consider the convergence of these solutions to the solution for ideal flow as the viscosity ν goes to zero. Later techniques of solution share this problem (see [1, pp. 142, 173]). Existence and uniqueness of a solution to the Euler equations for ideal flow, again for a sufficiently small time interval, was shown in R^3 by Leon Lichtenstein [9, p. 422] and on compact manifolds with boundary by Ebin and Marsden [15]. The existence, global in time, of "weak solutions" to the Navier-Stokes system was shown by Hopf [6], but satisfactory uniqueness results have not been found as yet. O. A. Ladyženskaja's recent book [1] provides an excellent survey of the various methods used for the solution of the Navier-Stokes equations and calls attention to the problem we consider in this paper [1, p. 6].

Convergence of viscous planar flow to ideal planar flow as the viscosity goes to zero was shown independently by McGrath [10] and Golovkin [5] with no restriction on the time interval of solution. Marsden has recently shown the existence for a short time (independent of viscosity) of viscous flow and its convergence to ideal flow on compact Riemannian manifolds without boundary using a technique suggested by V. Arnol'd [16]. We use an approach similar to that of McGrath and use techniques developed by Kato and Fujita [3], [4]. The result in this paper for R^3 differs from that of McGrath (for planar flow) in that we can demonstrate the existence of a unique classical solution to the Euler equations for ideal flow in R^3 by showing that the limit of solutions of the Navier-Stokes equations for various viscosities exists as the viscosity goes to zero, for a small but nontrivial time interval, and the limit function is a solution to the Euler equations for ideal flow. We call attention to the paper of Judovič [7] where he shows that the solution to the Euler equations for any domain in the plane is the limit of certain functions that are solutions of equations similar to the Navier-Stokes equations, but with a different form of boundary condition.

In §§I, II, and III we solve equations (E) derived by formally computing the curl of the Navier-Stokes equations (E′):

$$\text{(E)}\quad \begin{aligned} &\text{(a)} \quad \partial w/\partial t+(v\cdot \text{grad})\, w-(w\cdot \text{grad})\, v-\nu\Delta w = \nabla\times B \equiv b.\\ &\text{(b)} \quad w(x,0) = \nabla\times C(x) \equiv a(x).\\ &\text{(c)} \quad \nabla\times v = w;\ \nabla\cdot v = 0.\\ &\text{(d)} \quad \lim_{|x|\to\infty} v(x,t) = 0.\end{aligned}$$

In solving the auxiliary problem (E) we use the following version of the Schauder fixed point theorem: Let S be a closed convex subset of a Banach space X and let F be a continuous operator on S such that $F(S)$ is contained in S and $F(S)$ is a relatively compact subset of X. Then there is a "fixed point" y in S, i.e. $F(y)=y$.

In §I we show that for any w in an appropriate class of functions there is a function $v=F_1(w)$ that solves (E)(c) and (d).

In §II, for $v=F_1(w)$, we show that there is a solution, denoted $F_2(v)$, to equations (E)(a) and (b) for any time interval.

In §III it is shown that the function $F_2(F_1(\cdot))$ maps a closed convex set of functions in a Banach space continuously into itself and satisfies the conditions of the Schauder fixed point theorem, provided we restrict ourselves to a sufficiently small time interval which is, however, independent of the viscosity. The fixed point w is then shown to give a classical solution to (E) and then (E′).

In §IV we show that the solution (w, v) to (E) converges, with shrinking viscosity, to functions that give rise to a function that is a solution to the Euler equations for ideal flow.

The author is indebted to Professor Tosio Kato for his many helpful suggestions and comments during the preparation of this paper.

0. **Preliminary results and definitions.** The following notational conventions will be used:

f, g and h are scalar-valued functions over R^3 or $Q_T = R^3 \times [0, T]$.

$B, C, a, b, p, q, u, v, w$ are vector-valued functions over R^3 or Q_T. For such functions, say w, we define $|w(x)|^2 = \sum_{i=1}^3 |w_i(x)|^2$.

Constants are denoted K_i and do not depend on the viscosity ν. The symbol K denotes a constant used during a proof and K may take different values during the same proof.

For $f \in L_1(R^3)$, the Fourier transform of f is

$$F(f)(x) = (2\pi)^{-3/2} \int_{R^3} e^{-ix\cdot z} f(z)\, dz$$

with the inverse Fourier transform of f denoted $F^{-1}(f)$. By taking the limit-in-mean, we can define the Fourier transform on $L_2(R^3)$, and, if $(\ ,\ )_{L_2}$ denotes the inner product in Hilbert space $L_2(R^3)$, we have

$$\|f\|_{L_2}^2 \equiv (f, f)_{L_2} = (F(f), F(f))_{L_2}$$

and

$$(f_1, f_2)_{L_2} = (F(f_1), F(f_2))_{L_2}.$$

For $n \geqq 0$, the space H^n is the completion of C_0^∞ functions (infinitely differentiable functions with compact support in R^3) in the metric derived from norm

$$\|f\|_{H^n} = \|F(f)(z)(1+|z|^2)^{n/2}\|_{L_2}$$

which, for n an integer, is equivalent to the norm whose square is $\|f\|^2 = \sum_{|e|\leqq n} \|D_x^e f\|_{L_2}^2$ where $e = (e_1, e_2, e_3)$; e_i are integers $\geqq 0$; $|e| = e_1 + e_2 + e_3$ and $D_x^e f = (\partial/\partial x_1)^{e_1} (\partial/\partial x_2)^{e_2} (\partial/\partial x_3)^{e_3} f$.

H_σ^n is the subspace of H^n of all vector functions u with $\nabla \cdot u = 0$.

All explicit $D_x^e f$ are in L_2 and are understood as distribution derivatives. We note that $F(D_x^e f)(x) = (ix)^e F(f)(x)$, where $(ix)^e = (ix_1)^{e_1} (ix_2)^{e_2} (ix_3)^{e_3}$.

The following spaces will be used; h may be a vector-valued or scalar-valued function.

(i) $C_T=\{h(x,t) \mid h$ is continuous and bounded in $Q_T\}$ with norm $\|h\|_{C_T} = \sup_{(x,t)\in Q_T} |h(x,t)|$.

(ii) $C_T^\delta=\{h(x,t) \mid D_x^e h(x,t)\in C_T$ for all e satisfying $|e|\leqq \delta$ and $D_x^e h$ is Hölder-continuous in x with exponent $\delta-[\delta]$, uniformly for $(x,t)\in Q_T$ if $|e|=[\delta]\}$ where $[\delta]$ is the largest integral part of $\delta \geqq 0$.

We use the norm whose square is

$$\|h\|^2_{C_T^\delta} = \sum_{|e|\leqq[\delta]} \|D_x^e h\|^2_{C_T} + \sum_{|e|=[\delta]} H_\delta(D_x^e h)^2$$

where

$$H_\delta(f) = \sup_{x,x^1\in R^3;\, t\in[0,T]} \frac{|f(x,t)-f(x^1,t)|}{|x-x^1|^{\delta-[\delta]}}.$$

(iii) $C(T,H)=\{h(x,t) \mid h(\ ,t)\in H$; the mapping $h\colon [0,T]\to H$ is continuous$\}$ with norm $\|h\|_{C(T,H)}=\sup_{t\in[0,T]}\|h\|_H$ where H is a Hilbert space, usually H^n or H^n_σ. For notational convenience, we drop the σ when subscripting the norm.

(iv) $C_0^\delta=\{h\in C_T^\delta \mid h$ has compact support in R^3, uniformly in $t\in[0,T]\}$.

C_0^∞ is dense in $C(T,H^n)$. Where we consider t only in an interval $[\varepsilon,T]$ with $\varepsilon>0$, we use analogous classes of functions $C_{[\varepsilon,T]}$, $C^\delta_{[\varepsilon,T]}$ and $C([\varepsilon,T],H)$. Where t is omitted or fixed, we use similarly defined classes of functions C, C^δ and C_0^δ.

$(\ ,\ )_H$ denotes the inner product in a Hilbert space H. In any equation involving an inner product, the subscript H will be used: subsequent inner products are assumed to be of the same kind unless the notation is changed.

LEMMA 0.1. *If $f\in C(T,H^n)$ where n is an integer, we can assume $f\in C_T^{n-2+\delta}$ for any δ with $0\leqq\delta<\frac{1}{2}$ and there is a constant $K_{n,\delta}$ depending only on n and δ such that*

$$\|f\|_{C_T^{n-2+\delta}} \leqq K_{n,\delta}\|f\|_{C(T,H^n)}.$$

Proof. See [14, p. 221] for a proof for bounded domains. The proof for R^3 by use of Fourier transforms is somewhat easier.

LEMMA 0.2. *Let f_1 and f_2 be scalar functions over R^3.*

(i) *If $f_1\in C$ and $f_2\in L_2$, then $f_1f_2\in L_2$ and*

$$\|f_1f_2\|_{L_2} \leqq \|f_1\|_C\|f_2\|_{L_2}.$$

(ii) *If $f_1\in H^1$ and $f_2\in H^1$, then $f_1f_2\in L_2$ and*

$$\|f_1f_2\|_{L_2} \leqq \|f_1\|_{H^1}\|f_2\|_{H^1}.$$

(iii) *If $f_1\in C$; $D_x^e f_1\in L_2$, $|e|=1$; $\lim_{|x|\to\infty} f_1(x)=0$, then $f_1\in L_6$, $\|f_1\|_{L_6}\leqq\|\nabla f_1\|_{L_2}$ where $\|\nabla f_1\|^2_{L_2}=\sum_{|e|=1}\|D_x^e f_1\|^2_{L_2}$, and if $f_2\in H^1$, then $f_1f_2\in L_2$ and*

$$\|f_1f_2\|_{L_2} \leqq \|\nabla f_1\|_{L_2}\|f_2\|_{H^1}.$$

(iv) *If $f\in L_6$ and $\nabla f\in H^1$, then f can be taken as a locally Hölder-continuous function with exponent $\delta<\frac{1}{2}$. There is a constant $K_1(N,\delta)$, depending on δ and N, such that, in any ball $B(0,N)$,*

$$\|f\|_{C^\delta(B(0,N))} \leqq K_1(N,\delta)(\|f\|_{L_6}+\|\nabla f\|_{H^1})$$

where $\|f\|_{C^\delta(B(0,N))}$ *is defined as in space* C^δ *over* $B(0, N)$ *instead of* R^3; $B(x, L) = \{y \in R^3 \mid |x-y| < L\}$.

Proof. (i) is immediate.

(ii) We can improve an inequality from [1, p. 12] to obtain, for all $f \in C_0^\infty$, $\|f\|_{L_4} \leqq \|f\|_{H^1}$. The result follows from a density argument applied to

$$\left(\int f_1^2 f_2^2\, dx\right)^2 \leqq \int f_1^4\, dx \int f_2^4\, dx \leqq \|f_1\|_{H^1}^4 \|f_2\|_{H^1}^4.$$

(iii) From [1, p. 12] we get, with some improvements, $\|f\|_{L_6} \leqq \|\nabla f\|_{L_2}$ if $f \in C_0^\infty$. Working in space $L_2 \times L_2 \times L_2$ of vector-valued functions whose components are in L_2 (denoted L_2 here), we let D be the closure of $\{\nabla g \mid g \in C_0^\infty\}$ in this space. Since $\partial f_1/\partial x_i \in L_2$, $i = 1, 2, 3$, we can find unique $w \in D$, $u \in L_2 \ominus D$ such that $\nabla f_1 = u + w$. Since $w \in D$, there is a sequence $\{g_i\} \subset C_0^\infty$ such that $\nabla g_i \to w$ in L_2 as $i \to \infty$. The inequality above holds for g_i, so there is some $g \in L_6$ such that $g_i \to g$ and $w \in D$ is the distribution gradient of g. Thus, for $f \in C_0^\infty$,

$$0 = (u, \nabla f)_{L_2} = (\nabla f_1 - w, \nabla f) = (f_1 - g, \Delta f).$$

So $f_1 - g$ must be harmonic in R^3; but $f_1 \in C$; $\lim_{|x| \to \infty} f_1(x) = 0$ and $g \in L_6$, which can only occur if $f_1 - g = 0$, by standard results concerning harmonic functions. Thus

$$\|f_1\|_{L_6} = \|g\|_{L_6} \leqq \|\nabla g\|_{L_2} = \|w\|_{L_2} \leqq \|\nabla f_1\|_{L_2}.$$

Now, using Hölder's inequality, we compute

$$\|f_1 f_2\|_{L_2}^2 \leqq \left(\int f_1^6\, dx\right)^{1/3} \left(\int (f_2^2)^{3/2}\, dx\right)^{2/3}$$

$$\leqq \|\nabla f_1\|_{L_2}^2 \|f_2\|_{L_2}^{2/3} \|f_2\|_{L_4}^{4/3} \leqq \|\nabla f_1\|_{L_2}^2 \|f_2\|_{H^1}^2$$

using the inequality $\|f_2\|_{L_4} \leqq \|f_2\|_{H^1}$.

(iv) This result can easily be obtained by multiplying f by a function $g \in C_0^\infty$ that equals 1 on $B(0, N)$, using Lemma 0.1 on fg, and the result

$$\int_{B(0,N)} f^2\, dx \leqq K\|f\|_{L_6}^2 \qquad (K \text{ depends on } N).$$

Lemma 0.3. *If a set of functions* $S \subset L_2$ *has a uniform Hölder constant* M *and exponent* δ *and if, for any* $\varepsilon > 0$, *there is an* N_ε *such that*

$$\int_{R^3 - B(0,N_\varepsilon)} |f|^2\, dx < \varepsilon \quad \textit{for all } f \in S,$$

then for any ε^1, *there is an* N_{ε^1} *such that* $|x| > N_{\varepsilon^1}$ *implies that* $|f(x)| < \varepsilon^1$ *for all* $f \in S$.

Proof. Suppose that S has the properties postulated in the lemma and there is some $f_1 \in S$ and $x_1 \in R^3 - B(0, N+1)$ with

$$|f_1(x_1)| \geqq L \equiv (6\varepsilon(3+\delta)(3+2\delta)(8\pi\delta^2)^{-1} M^{3/\delta})^{\delta/(2\delta+3)}.$$

Then $|f_1(x)| \geqq L - M|x-x_1|^\delta$ and

$$\int_{R^3 - B(O,N)} |f_1|^2 \, dx \geqq \int_{B(x_1,(L/M)^{1/\delta})} (L - M|x-x_1|^\delta)^2 \, dx$$

$$= 4\pi \int_0^{(L/M)^{1/\delta}} (L^2 - 2LMr^\delta + M^2 r^{2\delta}) r^2 \, dr = 2\varepsilon,$$

which contradicts the assumption that $f_1 \in S$. Hence $|f(x)| \leqq L$, all $f \in S$, all $x \in R^3 - B(0, N+1)$ and the conclusion is immediate from this.

I. In §I we show that if $w \in H^2_\sigma$ then there is a unique $v \in C^{1+\delta} \cap L_6$ such that $\nabla \times v = w$ and $\lim_{|x| \to \infty} |v(x)| = 0$. We use potential theory to construct v.

LEMMA 1.1. *For any scalar function $f \in C^\delta \cap L_2$ we can define a linear operator*

$$G(f)(x) = \int_{R^3} \left(\frac{1}{|x-y|} - \frac{1}{|y|} \right) f(y) \, dy.$$

$G(f)$ is twice continuously differentiable, $\Delta G(f) = -4\pi f$ and $\|\nabla G(f)\|_C \leqq 4\pi(\|f\|_C + \|f\|_{L_2})$.

Proof. For any $x \in R^3$, we can choose $z \in R^3$ such that $|z - x| < 1$. Then

$$G(f)(x) = \int_{R^3} (|x-y|^{-1} - |y|^{-1}) f(y) \, dy$$

$$= \int_{B(z,2)} + \int_{B(0,1)} + \int_{R^3 - B(z,2) - B(0,1)}$$

$$\equiv I_1 + I_2 + I_3.$$

By potential theory arguments (see [2, p. 249]) I_1 and I_2 exist and are twice continuously differentiable;

$$\frac{\partial I}{\partial x_j} = \int_B \frac{\partial}{\partial x_j} \frac{1}{|x-y|} f(y) \, dy \quad \text{for both } I_1 \text{ and } I_2.$$

By Schwarz inequality and $|\,|y| - |x-y|\,| \leqq |x|$,

$$|I_3|^2 \leqq \|f\|^2_{L_2} \int_{R^3 - B(z,2) - B(0,1)} \left(\frac{|y| - |x-y|}{|y|\;\;|x-y|} \right)^2 dy$$

$$\leqq \|f\|^2_{L_2} |x|^2 \int_{R^3 - B - B} |y|^{-2} |x-y|^{-2} \, dy \leqq K \|f\|^2_{L_2} |x|^2.$$

This is sufficient for I_3 to exist; showing that it is sufficiently differentiable uses an argument similar to the following reasoning concerning the desired inequality:

$$|\nabla G(f)(x)| \leqq \int_{R^3} |\nabla(|x-y|^{-1})| \, |f(y)| \, dy$$

$$= \int_{R^3} |x-y|^{-2} |f(y)| \, dy$$

$$\leqq \int_{R^3 - B(x,1)} |x-y|^{-2} |f(y)| \, dy + \|f\|_C \cdot 4\pi \int_0^1 dr$$

$$\leqq 4\pi(\|f\|_{L_2} + \|f\|_C).$$

Finally, from potential theory (see [2, p. 249]), $\Delta G(f) = -4\pi f$.

LEMMA 1.2. *Let* $w \in C^{\delta} \cap H_{\sigma}^{0}$. *Then if* $G(w) = (G(w_1), G(w_2), G(w_3))$ (*see Lemma* 1.1),

(i) $\lim_{|x| \to \infty} |\partial G(w)/\partial x_i| = 0$, *and*

(ii) $\nabla \cdot G(w) = 0$.

Proof. (i) The absolute value of a component of $\partial G(w)(x)/\partial x_i$ is

$$\left| \int_{R^3} ((x_i - y_i)|x-y|^{-3}) w_j(y)\, dy \right| \leqq \int_{R^3} |x-y|^{-2} |w_j(y)|\, dy.$$

Hence it suffices to show that $f \in C^{\delta} \cap L_2$ implies

$$\lim_{|x| \to \infty} \int_{R^3} \frac{1}{|x-y|^2} f(y)\, dy = 0.$$

Using Lemma 0.3 with $S = \{f\}$, for any $\varepsilon > 0$ we can find N such that $\int_{R^3 - B(0,N)} |f|^2\, dx < \varepsilon^2$ and $|f(x)| < \varepsilon$ if $x \in R^3 - B(0, N)$. Suppose $|x| > N+1$. Write

$$\int_{R^3} |x-y|^{-2} f(y)\, dy = \int_{R^3 - B(0,N)} + \int_{B(0,N)} \equiv I_1 + I_2,$$

$$\begin{aligned} |I_1| &\leqq \int_{R^3 - B(0,N) - B(x,1)} + \int_{B(x,1)} \\ &\leqq \left(\int_{R^3 - B(x,1)} |x-y|^{-4}\, dy \right)^{1/2} \left(\int_{R^3 - B(0,N)} f^2\, dy \right)^{1/2} + K \sup_{y \in B(x,1)} |f(y)| \leqq K\varepsilon, \end{aligned}$$

$$\begin{aligned} |I_2|^2 &\leqq \|f\|_{L_2}^2 \int_{B(0,N)} |x-y|^{-4}\, dy \\ &\leqq \|f\|_{L_2}^2 (|x| - N)^{-1/2} \int_{R^3 - B(x,1)} |x-y|^{-7/2}\, dy \\ &\leqq K(|x| - N)^{-1/2} \|f\|_{L_2}^2 \to 0 \quad \text{as } |x| \to \infty. \end{aligned}$$

These suffice to establish (i).

(ii) The result $\nabla \cdot G(w) = 0$ follows from a conventional procedure involving approximating $|x-y|^{-1}$ in $B(x, 2N) - B(x, \delta/2)$ by a C_0^{∞} function $f_{N,\delta}(|x-y|)$ equal to $|x-y|^{-1}$ in $B(x, N) - B(x, \delta)$. Then, for $f_{N,\delta}$, $\nabla \cdot w = 0$ implies

$$\sum \frac{\partial}{\partial x_i} \int f_{N,\delta}(|x-y|) w_i(y)\, dy = -(\nabla f_{N,\delta}(|x-y|), w)_{L_2} = 0.$$

THEOREM 1.1. *If* $w \in H_{\sigma}^{n}$ $(n \geqq 2)$ *we can define a linear map* F_1:

$$F_1(w)(x) = (4\pi)^{-1} \int_{R^3} |x-y|^{-3} (x-y) \times w(y)\, dy = (4\pi)^{-1} \nabla \times G(w)$$

with the following properties:

(a) $F_1(w) \in C^{1+\delta} \cap L_6$ *for any* $\delta < \frac{1}{2}$ *and* $\partial F_1(w)/\partial x \in H^n$,

(b) $\nabla \times F_1(w) = w$,

(c) $\nabla \cdot F_1(w)=0$,

(d) $\lim_{|x|\to\infty} F_1(w)(x)=0$.

$F_1(w)$ is the unique vector with properties (a) *through* (d).

For a vector-valued function $u=(u_1, u_2, u_3)$, let $u_{i,x_j}=\partial u_i/\partial x_j$, and $u_{,x}$ be the array (u_{i,x_j}) and

$$\|u_{,x}\|^2 = \sum_{i,j=1,2,3} \|u_{i,x_j}\|^2.$$

The following inequalities hold:

(i) $\|F_1(w)\|_{L_6} \leqq K_2\|w\|_{L_2}$,

(ii) $\|F_1(w)\|_C \leqq K_3\|w\|_{H^2}$,

(iii) $\|(F_1(w))_{,x}\|_{H^m} \leqq \|w\|_{H^m}$, $m=0, 1, 2, \ldots, n$.

Proof. Since $w \in H^n$, where $n \geqq 2$, Lemma 0.1 shows that $w \in C^\delta$ for any $\delta < \frac{1}{2}$. Hence Lemmas 1.1 and 1.2 are valid and $F_1(w)$ exists. Using these lemmas,

$$\nabla \times F_1(w) = \nabla \times ((4\pi)^{-1}\nabla \times G(w)) = (4\pi)^{-1}(-\Delta G(w) + \nabla(\nabla \cdot G(w))) = w,$$

where $\Delta G(w) = (\Delta G(w_1), \Delta G(w_2), \Delta G(w_3))$. $\lim_{|x|\to\infty} F_1(w)=0$ by Lemma 1.2(i).

To establish uniqueness: If v_1 and v_2 both have properties (a) through (d), then $\nabla \times (v_1 - v_2)=0$ and $\nabla \cdot (v_1 - v_2)=0$, so there is a potential function f with $v_1 - v_2 = \nabla f$ and $0=\nabla \cdot (v_1 - v_2)=\Delta f$, i.e. f is harmonic. But v_1 and v_2 are small near ∞, so f, harmonic, can only be constant; hence $v_1 = v_2$.

F_1 is clearly linear; inequality (ii) follows easily from the inequality of Lemma 1.1 and Lemma 0.1.

To establish the remaining results, we first show that, if $v=F_1(w)$, then

$$v_{,x_i} = -F^{-1}(|z|^{-2}z_i(z \times F(w)(z))),$$

where F is the Fourier transform. Let $u=-(4\pi)^{-1}G(w)$. Then $\Delta u = w$ and $\nabla \times u = -v$. Denote

$$-F^{-1}(|z|^{-2}z_i(z \times F(w)(z)))$$

by p_i; note that $p_i \in H^n$. Then if $q \in C_0^\infty$,

$$\begin{aligned}(p_i, \Delta q)_{L_2} &= (-|z|^{-2}z_i(z \times F(w)(z)), |z|^2F(q)(z)) \\ &= -(F(w)(z), z_i(z \times F(q)(z))) = (F(\Delta u), F((\nabla \times q)_{,x_i})) \\ &= (\Delta u, (\nabla \times q)_{,x_i}) = (u, \Delta(\nabla \times q)_{,x_i}) = -((\nabla \times u)_{,x_i}, \Delta q) \\ &= (v_{,x_i}, \Delta q).\end{aligned}$$

Hence $v_{,x_i} - p_i$ is harmonic; $v_{,x_i}$ and p_i are both continuous and bounded by Lemmas 1.1 and 0.1. So $v_{,x_i} = p_i + \text{constant}$. Now $p_i \in H^2$, so Lemma 0.1 implies that $p_i \in C^\delta \cap L_2$ and Lemma 0.3 implies that p_i is uniformly small outside a sufficiently large ball in R^3. Since v is also small uniformly for x large, the mean-value theorem implies that for at least some large x_0, $v_{,x}(x_0)$ is also small; hence the "constant" must be 0 and $v_{,x_i}=p_i$. Thus (iii) follows:

$$\begin{aligned}\|v_{,x}\|_{H^n}^2 &= \sum_i \|F(v_{,x_i})(z)(1+|z|^2)^{n/2}\|_{L_2}^2 \\ &= \|\,|z|^{-2}(|z|\;|z\times F(w)(z)|)(1+|z|^2)^{n/2}\|_{L_2}^2 \\ &\leqq \|F(w)(1+|z|^2)^{n/2}\|_{L_2}^2 = \|w\|_{H^n}^2.\end{aligned}$$

Inequality (i) now can easily be derived using Lemma 0.2(iii); $v=F_1(w)\in C^{1+\delta}$ since $v_{,x}\in H^2$ and Lemma 0.1 holds.

II. In §II, for fixed v, we wish to find a solution to

$$\text{(E)}\qquad \begin{aligned}&\text{(a)}\quad \partial w/\partial t+(v\cdot\text{grad})\,w-(w\cdot\text{grad})\,v-\nu\Delta w = b,\\ &\text{(b)}\quad w(x,0) = a(x).\end{aligned}$$

Equations (E)(a) and (b) provide an example of more general parabolic equations of form

$$\text{(Q)}\qquad \begin{aligned}&\text{(a)}\quad dw/dt+(P(t)+A)w = b,\\ &\text{(b)}\quad w(0) = a,\end{aligned}$$

where A is selfadjoint and independent of time and $P(t)$ is a time dependent linear "perturbation of lower order." The following theorem provides a solution to equations of form (Q). ($D(\cdot)$ denotes the domain of operator.)

THEOREM 2.1. *Let A be a selfadjoint operator in a Hilbert space H and suppose that $A\geqq d>0$. Let $P(t)$ be a time-dependent linear operator with $D(P(t))\supset D(A^\delta)$ for some $\delta\geqq 0$ and all $t\in[0,T]$. Suppose that if $w\in D(A^\delta)$, then $P(t)w\in C(T,H)$ and*

$$\|P(t)w\|_{C(T,H)} \leqq K\|A^\delta w\|_H \qquad (K \text{ is some constant}).$$

(I) *If $a\in D(A^\delta)$ and $b\in C(T,H)$, then there is a generalized solution $w(t)\in C(T,H)$ to* (Q) *which satisfies*

(Q¹)

(a) $w(t) = e^{-tA}a+\int_0^t e^{-(t-s)A}(-P(s)w(s)+b(s))\,ds$

where e^{-tA} is the semigroup generated by $-A$; and

(b) $w(t)\in D(A^\delta)$, $t\in[0,T]$, $A^\delta w(t)\in C(T,H)$ *and*

$$\lim_{t\downarrow 0}\|A^\delta(w(t)-a)\|_H = 0.$$

(II) *If $a\in D(A^{\delta+\mu})$ for some $\mu>0$, b is Hölder continuous in t as a $C(T,H)$ function, and $P(t)$ also satisfies*

$$\|P(t_1)u-P(t_2)u\|_H \leqq K'|t_1-t_2|^\mu\|A^\delta u\|_H$$

for some constants, K', $\mu>0$, and all $u\in D(A^\delta)$, then $w(t)$ is also a solution of (Q) *in the sense that dw/dt exists in $C([\varepsilon,T],H)$ for any $\varepsilon>0$; $w(t)\in D(A)$ for $t>0$ and* (Q) *is satisfied in H for all $t>0$.*

The following lemma is needed for the proof of Theorem 2.1; the dependence of estimates on the positive constant ν is crucial for later results.

LEMMA 2.1. *If A is a positive selfadjoint operator in Hilbert space H, $A \geqq d > 0$, then $-A$ is the generator of a contraction semigroup e^{-tA} and*

(i) $e^{-tA}e^{-sA} = e^{-(t+s)A}$, $t, s > 0$; $Ae^{-tA} \supset e^{-tA}A$; $\|e^{-tA}\| \leqq 1$.

(ii) $e^{-tA} \to I$ *strongly as* $t \to 0$.

(iii) e^{-tA} *maps* H *into* $D(A)$ $(t > 0)$.

(iv) $d(e^{-tA})/dt = -Ae^{-tA}$ *exists* $(t > 0)$.

(v) $e^{-tA}w \in C(T, H)$ *for all* $w \in H$.

(vi) *If* $a \in H$ *and* $b(s)$ *is Hölder-continuous as a* $C(T, H)$ *function, then*

$$w(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}b(s)\,ds$$

solves

(a) $dw/dt + Aw = b$,

(b) $w(0) = a$,

in the sense that $w(t) \in D(A)$, $t > 0$ *and* dw/dt *and* $Aw(t)$ *exist and* (a) *is true in* $C([\varepsilon, T], H)$ *for* $\varepsilon > 0$ *and* $\lim_{t\downarrow 0} \|w(t) - a\|_H = 0$.

(vii) *If* $0 \leqq \delta < 1$, $\|A^\delta u\|_H \geqq d^\delta \|u\|_H$ *for all* $u \in D(A^\delta)$.

(viii) $\|A^\delta e^{-tA}\| \leqq t^{-\delta}(\delta/e)^\delta$.

(ix) $\|(e^{-hA} - 1)A^{-\delta}\| \leqq h^\delta((1-\delta)/e)^{1-\delta}\delta^{-1}$.

(x) *For any* $u \in C(T, H)$,

$$\left\| \int_0^t A^\delta e^{-(t-s)\nu A}u(s)\,ds \right\|_H \leqq t^{1-\delta}\nu^{-\delta}(\delta/e)^\delta(1-\delta)^{-1}\|u\|_{C(T,H)}.$$

(xi) *If* $w(t) = \int_0^t e^{-(t-s)\nu A}u(s)\,ds$, *where* $u(s) \in C(T, H)$, *then, for any* $0 \leqq \delta < 1$ *and* $0 < \mu < 1 - \delta$, $A^\delta w(t) \in C(T, H)$ *and*

$$\|A^\delta(w(t+h) - w(t))\|_H \leqq h^\mu \nu^{-\delta}((t/\mu)^{1-(\delta+\mu)} + (1-\delta)^{-1}h^{1-(\delta+\mu)})\|u\|_{C(t+h,H)}.$$

Proof. The results (i) through (viii) are well known; for example, see [12, p. 231 ff.] and [13].

(ix) $(e^{-hA} - 1) = \int_0^h (d/dt)e^{-tA}\,dt = \int_0^h -Ae^{-tA}\,dt$ so

$$(e^{-hA} - 1)A^{-\delta} = \int_0^h -A^{1-\delta}e^{-tA}\,dt$$

and the result follows from integrating inequality (viii).

(x) can be easily shown by integrating (viii) directly.

(xi) $w(t+h) - w(t) = (e^{-h\nu A} - 1)\int_0^t e^{-(t-s)\nu A}u(s)\,ds + \int_t^{t+h} e^{-(t+h-s)\nu A}u(s)\,ds$ so

$$\|A^\delta(w(t+h) - w(t))\|_H \leqq \|(e^{-h\nu A} - 1)A^{-\mu}\| \left\| \int_0^t A^{\delta+\mu}e^{-(t-s)\nu A}u(s)\,ds \right\| + \left\| \int_t^{t+h} A^\delta e^{-(t+h-s)\nu A}u(s)\,ds \right\|.$$

The result follows easily from the previous estimates.

Proof of Theorem 2.1.

Part I. We solve the integral equation (Q^1)(a) by an approximation technique, with

$$u_0(t) = e^{-tA}a+\int_0^t e^{-(t-s)A}b(s)\,ds$$

and

$$u_n(t) = -\int_0^t e^{-(t-s)A}P(s)u_{n-1}(s)\,ds.$$

Now $u_0(t)$ exists in $C(T, H)$ by Lemma 2.1(xi) (for $\delta=0$) which also allows us to assert that

$$A^\delta u_0(t) = e^{-tA}A^\delta a+\int_0^t A^\delta e^{-(t-s)A}b(s)\,ds$$

exists in $C(T, H)$ and so, by the assumptions of the theorem, $u_0(t)\in D(P(t))$. Assume that $u_{n-1}(t)$ and $A^\delta u_{n-1}(t)$ exist in $C(T, H)$ so that $u_{n-1}(t)\in D(P(t))$. We note that for any $w(t)\in D(A^\delta)$, $t\in[0, T]$,

$$\|P(t_1)w(t_1)-P(t_2)w(t_2)\|_H \leqq \|\{P(t_1)-P(t_2)\}w(t_1)\|_H+K\|A^\delta(w(t_1)-w(t_2))\|_H \tag{2.1}$$

from the properties assumed of P. Hence $P(t)u_{n-1}(t)\in C(T, H)$. Then, by Lemma 2.1(xi), $u_n(t)$ and $A^\delta u_n(t)$ exist in $C(T, H)$. The inequalities

$$\|A^\delta u_0\|_{C(t,H)} \leqq \|A^\delta a\|_H+t^{1-\delta}(1-\delta)^{-1}\|b\|_{C(t,H)}$$

and

$$\begin{aligned}\|A^\delta u_n\|_{C(t,H)} &\leqq t^{1-\delta}(1-\delta)^{-1}\|Pu_{n-1}\|_{C(t,H)}\\ &\leqq t^{1-\delta}(1-\delta)^{-1}K\|A^\delta u_{n-1}\|_{C(t,H)}\end{aligned}$$

follow from Lemma 2.1(x) and the restrictions on P. Thus

$$\sum_{i=0}^n \|A^\delta u_i\|_{C(t,H)} \leqq (\|A^\delta a\|_H+t^{1-\delta}(1-\delta)^{-1}\|b\|_{C(t,H)})\sum_{i=0}^n (t^{1-\delta}(1-\delta)^{-1}K)^i$$

which converges if t satisfies $t^{1-\delta}(1-\delta)^{-1}K<1$. Note that this restriction on t is independent of the initial data and $b(t)$. If T_0 satisfies this restriction, then the series $(A^\delta w)_n\equiv\sum_{i=0}^n A^\delta u_i$ converges in $C(T_0, H)$ and since $d^\delta\|u_i\|_H\leqq\|A^\delta u_i\|_H$, $w_n \equiv\sum_{i=0}^n u_i$ will converge also to some function $w\in C(T_0, H)$. Since A^δ is closed, $\lim_{n\to\infty}(A^\delta w)_n=A^\delta w$ and hence $w\in D(P(t))$, and

$$\begin{aligned} w(t) &= u_0(t)+\lim_{n\to\infty}\sum_{i=1}^n u_i(t)\\ &= u_0(t)-\lim_{n\to\infty}\int_0^t e^{-(t-s)A}P(s)\sum_{i=0}^n u_i(s)\,ds\\ &= u_0(t)-\int_0^t e^{-(t-s)A}P(s)w(s)\,ds.\end{aligned}$$

Now $w(T_0)\in D(A^\delta)$; hence this process can be continued to another t-interval $[T_0, 2T_0]$ of the same length etc. since the requirement for convergence of the approximation is independent of the initial data. So $w(t)$ satisfying the integral equation exists for any interval $[0, T]$ where $b(t)$ and $P(t)$ are defined. We note that

$$A^\delta w(t) - A^\delta a = (e^{-tA}-1)A^\delta a + \int_0^t A^\delta e^{-(t-s)A}(-Pw+b)\,ds$$

and the proper convergence to the initial value follows from Lemma 2.1(ii) and (x).

Part II. To show the second part of the result, we need only show that $P(s)w(s)$ is Hölder-continuous in s by Lemma 2.1(vi). Inequality (2.1) and the additional assumptions on P reduce this to showing that $A^\delta w(s)$ is Hölder-continuous. First

$$e^{-(s+h)A}A^\delta a - e^{-sA}A^\delta a = ((e^{-hA}-1)A^{-\mu})(e^{-sA}A^{\delta+\mu}a)$$

and inequalities (viii) and (ix) of Lemma 2.1 show that $A^\delta e^{-sA}a$ is Hölder-continuous if $a \in D(A^{\delta+\mu})$, as is assumed.

Then Lemma 2.1(xi) establishes that

$$A^\delta\left(\int_0^t e^{-(t-s)A}(-P(s)w(s)+b(s))\,ds\right)$$

is Hölder-continuous in t also.

Let A' be the closure of $(1-\Delta)$ on Hilbert space H^2; then $D(A')=H^4$. Now it is clear that for sufficiently smooth q, $\nabla\cdot(A'q)=A'(\nabla\cdot q)$ where A' is regarded as an operator on both scalar- and vector-valued functions. Hence $A=(A'$ restricted to $H^2_\sigma)$ is an operator in this subspace with domain H^4_σ, and νA is the selfadjoint operator in Hilbert space H^2_σ for use with Theorem 2.1 to show the existence of solutions of equations (E).

Formally define

$$P_{v,\nu}(t)w = (v\cdot\text{grad})w - (w\cdot\text{grad})v - \nu w.$$

Recalling that $\nabla\cdot v=0$, for smooth u with $\nabla\cdot u=0$,

$$\begin{aligned}\nabla\cdot P_{v,\nu}u &= \nabla\cdot((v\cdot\text{grad})u) - \nabla\cdot((u\cdot\text{grad})v) - \nu\nabla\cdot u\\ &= \frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial x_j}u^i + (v\cdot\text{grad})(\nabla\cdot u) - \frac{\partial}{\partial x_j}u^i\frac{\partial v_j}{\partial x_i} - (u\cdot\text{grad})(\nabla\cdot v) + 0 = 0.\end{aligned}$$

Thus we can define $P_{v,\nu}(t)$ in H^2_σ. For use in Theorem 2.1 we need the following estimates.

Note that, from Lemma 0.1, there is a constant K_7 such that, for suitable q, $\|q_{,x}\|_C \leqq K_7\|q_{,x}\|_{H^2}$ and $\|q\|_C \leqq K_7\|q\|_{H^2}$.

LEMMA 2.2. *If* $v\in C_T$, $\nabla\cdot v=0$, $\lim_{|x|\to\infty} v(x,t)=0$ *uniformly in* $t\in[0,T]$ *and* $v_{,x}\in C(T,H^2)$, *then, for suitable constants* K_4, K_5 *and* K_6, *if* $w\in D(A^{1/2})=H^3_\sigma$,

(i) $\|(v\cdot\text{grad})w\|_{C(T,H^2)} \leqq K_4(\|v_{,x}\|_{C(T,H^2)} + \|v\|_{C_T})\|A^{1/2}w\|_{H^2_\sigma}$,

(ii) $\|(w\cdot\text{grad})v\|_{C(T,H^2)} \leqq K_5\|v_{,x}\|_{C(T,H^2)}\|w\|_{H^2_\sigma}$,

(iii) $\|(v\cdot\text{grad})w\|_{C(T,H^1)} \leqq K_6(\|v_{,x}\|_{C(T,H^2)} + \|v\|_{C_T})\|w\|_{H^2_\sigma}$.

Proof. For appropriate p and q,

$$\begin{aligned}\|(p\cdot\text{grad})q\|_{H_2} &= \|(1-\Delta)((p\cdot\text{grad})q)\|_{L_2}\\ &\leqq \|(p\cdot\text{grad})q\|_{L_2}+\|\Delta((p\cdot\text{grad})q)\|_{L_2}\end{aligned}$$

and

$$\Delta((p\cdot\text{grad})q) = (\Delta p\cdot\text{grad})q+2\sum_i (p_{,x_i}\cdot\text{grad})q_{,x_i}+(p\cdot\text{grad})\Delta q.$$

Proof of (i).

$$\begin{aligned}\|(p\cdot\text{grad})q\|_{L_2} &\leqq \|p\|_C\|q_{,x}\|_{L_2},\\ \|\Delta((p\cdot\text{grad})q)\|_{L_2} &\leqq K\|p_{,x}\|_{H^2}\|q_{,x}\|_{H^1}+K\|p_{,x}\|_C\|q_{,x}\|_{H^1}+\|p\|_C\|\Delta q_{,x}\|_{L_2},\\ \|q_{,x}\|^2_{H_2} = \sum_{i,j}\|q_{i,x_j}\|^2_{H_2} &\leqq \int(1+|z|^2)^3|F(q)|^2\,dz = \|A'^{1/2}q\|^2_{H^2}.\end{aligned}$$

Lemma 0.2 gives $\|p_{,x}\|_C\leqq K_7\|p_{,x}\|_{H^2}$; $\|q_{,x}\|_{L_2}\leqq\|q_{,x}\|_{H^2}$ and $\|\Delta(q_{,x})\|_{L_2}\leqq\|q_{,x}\|_{H^2}$: these inequalities combine to give (i).

Proof of (ii).

$$\begin{aligned}\|(p\cdot\text{grad})q\|_{L_2} &\leqq \|p\|_{L_2}\|q_{,x}\|_C \leqq K_7\|p\|_{L_2}\|q_{,x}\|_{H^2},\\ \|\Delta((p\cdot\text{grad})q)\|_{L_2} &\leqq K_7\|\Delta p\|_{L_2}\|q_{,x}\|_{H^2}+K\|p\|_{H^2}\|q_{,x}\|_{H^2}+K_7\|p\|_{H^2}\|q_{,x}\|_{H^2}.\end{aligned}$$

These combine to give (ii). Note that $2K_7<K_5$.

Proof of (iii).

$$\begin{aligned}\|(p\cdot\text{grad})q\|^2_{H^1} &= ((p\cdot\text{grad})q,(1-\Delta)((p\cdot\text{grad})q))_{L_2}\\ &\leqq \|p\|^2_C\|q_{,x}\|^2_{L_2}+\|p\|_C\|q_{,x}\|_{L_2}(K\|\Delta p\|_{H^1}\|q_{,x}\|_{H^1}+\|p_{,x}\|_C\|q\|_{H^2})\\ &\quad+((p\cdot\text{grad})q,(p\cdot\text{grad})\Delta q)_{L_2}.\end{aligned}$$

Now, since $\nabla\cdot v=0$ and p has the role of v here, $\nabla\cdot p=0$ and

$$\begin{aligned}((p\cdot\text{grad})q,(p\cdot\text{grad})\Delta q)_{L_2} &= (p_jq_{i,x_j},(p_k\Delta q_i)_{,x_k})\\ &= -((p_jq_{i,x_j})_{,x_k},p_k\Delta q_i)\\ &\leqq K\|p\|^2_C\|q\|^2_{H^2}+K\|p_{,x}\|_C\|q_{,x}\|_{L_2}\|p\|_C\|q\|_{H^2}.\end{aligned}$$

The inequalities combine using Lemmas 0.1 and 0.2 to give (iii).

Theorem 2.2. *For any* $[0,T]$ *where* b *and* v *are defined, if* $a\in H^3_\sigma$, $b\in C(T,H^2_\sigma)$ *and* $v\in C_T$, $\lim_{|x|\to\infty}v(x,t)=0$ *uniformly in* $t\in[0,T]$, $v_{,x}\in C(T,H^2)$, $\nabla\cdot v=0$ *then*

$$w(t) = e^{-t\nu A}a+\int_0^t e^{-(t-s)\nu A}(-(v\cdot\text{grad})\,w+(w\cdot\text{grad})\,v+\nu w+b)\,ds \tag{2.2}$$

has a solution $w(t)\in C(T,H^2_\sigma)$ *with* $A^{1/2}w(t)\in C(T,H^2_\sigma)$ *also and*

$$\lim_{t\downarrow 0}\|A^{1/2}w(t)-A^{1/2}a\|_{H^2_\sigma} = 0.$$

Proof. The inequalities for $P_{v,\nu}$ and the continuity required by Theorem 2.1 follow from Lemma 2.2, the linearity of expressions like $(v\cdot\text{grad})\,w$ in both v and w, and the properties assumed of v. Clearly $D(A^{1/2})\subset D(P_{v,\nu})$. Hence Theorem 2.1(I) gives the existence of solutions $w\in C(T,H^2_\sigma)$ of (2.2) with the proper requirements.

III. For $u \in C(T, H^2_\sigma)$, we can form $F_1(u)=v$ satisfying the requirements of Theorem 1.1 and construct $w(t)$, the solution of Theorem 2.2 to

$$w(t) = e^{-t\nu A}a+\int_0^t e^{-(t-s)\nu A}(-(v\cdot\text{grad})w+(w\cdot\text{grad})r+\nu w+b)\,ds \tag{2.2}$$

which we call $F_2(v)=F_2(F_1(u))$. We wish to establish that the mapping $F_2(F_1(\ ))\colon u\to w$ maps a closed convex set S of a Banach space continuously into a relatively compact subset of S and use Schauder's fixed point theorem to find a solution to (E)(a), (b), (c) and (d). The most important estimate giving t-independence of the viscosity ν comes from the following lemma.

LEMMA 3.1. *In any Hilbert space H, if A is a positive selfadjoint operator, $q(t) \in C(T, H)$, $a \in H$ and*

$$w(t) = e^{-t\nu A}a+\int_0^t e^{-(t-s)\nu A}q(s)\,ds$$

then

$$\|w(t)\|_H^2 \leqq \|a\|_H^2+2\int_0^t (q(s), w(s))_H\,ds.$$

Proof. Suppose that $q(s)$ is Hölder-continuous in s. Then, from Lemma 2.1(vi), dw/dt exists in $C([\varepsilon, T], H)$, $\varepsilon>0$; $w \in D(A)$ $(t>0)$ and $dw/dt+\nu Aw=q(t)$ in H; $w(t) \to a$ strongly in H as $t \to 0$. So

$$(dw/dt, w)_H+\nu(Aw, w)_H = (q(t), w(t))_H.$$

Now $\nu(Aw, w)_H=\nu\|A^{1/2}w\|_H^2\geqq 0$, so integrating the inequality gives

$$\tfrac{1}{2}(\|w(t)\|_H^2-\|a\|_H^2) = \frac{1}{2}\int_0^t \frac{d}{ds}\|w\|_H^2\,ds \leqq \int_0^t (q(s), w(s))_H\,ds.$$

By approximating strongly continuous $q(s)$ by Hölder-continuous functions, we can easily obtain the result for $q(s) \in C(T, H)$.

LEMMA 3.2. *If $w \in C(T, H^3)$, then there is a constant K_8 such that*

$$|((v\cdot\text{grad})w, w)_{H^2}| \leqq K_8\|v_{,x}\|_{H^2}\|w\|_{H^2}^2.$$

Proof. If $w \in C_0^\infty$, then, since $\nabla\cdot v=0$,

$$\begin{aligned}((v\cdot\text{grad})w, w)_{H^2} &= ((1-\Delta)(v\cdot\text{grad})w, (1-\Delta)w)_{L_2}\\ &= ((v\cdot\text{grad})w, w)-((v\cdot\text{grad})w, \Delta w)-(\Delta(v\cdot\text{grad})w, (1-\Delta)w)\\ &= \frac{1}{2}\int \nabla\cdot(|w|^2v)\,dx-\left(\frac{\partial}{\partial x_j}(v_jw_i), \Delta w_i\right)-((\Delta v\cdot\text{grad})w, (1-\Delta)w)\\ &\quad+\frac{1}{2}\int \nabla\cdot(|\Delta w|^2v)\,dx-\left(\frac{\partial}{\partial x_j}(v_j\Delta w_i), w_i\right)\\ &\quad-2\left(\frac{\partial v_j}{\partial x_K}\frac{\partial^2 w_i}{\partial x_j\partial x_K}, (1-\Delta)w_i\right).\end{aligned}$$

Now

$$\frac{1}{2}\int \nabla\cdot(|w|^2v)\,dx = 0 = \frac{1}{2}\int \nabla\cdot(|\Delta w|^2 v)\,dx$$

and

$$\left(\frac{\partial}{\partial x_j}(v_jw_i),\ \Delta w_i\right) = -\left(\frac{\partial}{\partial x_j}(v_j\Delta w_i),\ w_i\right),$$

so four terms of the expression drop out, and we have

$$\begin{aligned}|((v\cdot\text{grad})w,\ w)_{H^2}| &\leqq \|(\Delta v\cdot\text{grad})w\|_{L_2}\|(1-\Delta)w\|_{L_2}\\ &\quad + K\left\|\frac{\partial v_j}{\partial x_K}\,\frac{\partial^2 w_i}{\partial x_j\,\partial x_K}\right\|_{L_2}\cdot\|(1-\Delta)w\|_{L_2}\\ &\leqq (K\|v_{,x}\|_{H^2}\|w\|_{H^2}+K\|v_{,x}\|_C\|w\|_{H^2})\|w\|_{H^2}.\end{aligned}$$

The result follows from a density argument for $w\in C(T, H^3)$ and the inequality $\|v_{,x}\|_C \leqq K_7\|v_{,x}\|_{H^2}$.

LEMMA 3.3. *Let $K_9=K_5+K_8$. Suppose that $a\in H^3_\sigma$ and $b\in C(T, H^2)$. Choose T such that*

$$4T(\nu+K_9(2\|a\|^2_{H^2}+K_9^{-1}\|b\|_{C(T,H^2)})^{1/2}) < 1$$

and with T so restricted, let

$$M^2 = 2\|a\|^2_{H^2}+K_9^{-1}\|b\|_{C(T,H^2)}.$$

Define $S=\{w\in C(T, H^2_\sigma)\mid \|w\|_{C(T,H^2)}\leqq M\}$. Then $F_2(F_1(S))\subset S$.

Proof. By Theorem 1.1, if $u\in S$, then $F_1(u)$ exists. Since $u\in C(T, H^2)$, $F_1(u)\in C_T$ and $(F_1(u))_{,x}\in C(T, H^2)$ by inequalities (ii) and (iii) of Theorem 1.1. Lemma 0.3 and a study of Lemma 1.2 shows that $\lim_{|x|\to\infty} F_1(u)=0$ uniformly in t; hence we can apply Theorem 2.2 to obtain $w(t)=F_2(F_1(u))$ satisfying (2.2) with $w\in C(T, H^3_\sigma)$. By Lemmas 2.2, 3.1 and 3.2

$$\begin{aligned}\|w(t)\|^2_{H^2} &\leqq \|a\|^2_{H^2}+2\int_0^t(-(v\cdot\text{grad})w+(w\cdot\text{grad})v+\nu w+b,\ w)_{H^2}\,ds\\ &\leqq \|a\|^2_{H^2}+2t\,\|w\|^2_{C(t,H^2)}((K_5+K_8)\|v_{,x}\|_{C(t,H^2)}+\nu)+2t\,\|b\|_{C(t,H^2)}\|w\|_{C(t,H^2)}.\end{aligned}$$

Now $t\leqq T$ and $\|v_{,x}\|_{C(t,H^2)}=\|F_1(u)_{,x}\|_{C(t,H^2)}\leqq\|u\|_{C(t,H^2)}\leqq M$; hence

$$2t((K_5+K_8)\|v_{,x}\|_{C(t,H^2)}+\nu) \leqq 2T(K_9M+\nu) < \tfrac{1}{2}$$

and we can rearrange the inequality to

$$\begin{aligned}\tfrac{1}{2}\|w\|^2_{C(t,H)} &\leqq \|a\|^2_{H^2}+2t\|b\|_{C(t,H^2)}\|w\|_{C(t,H^2)}\\ &\leqq \|a\|^2_{H^2}+(2K_9M)^{-1}\|b\|_{C(t,H^2)}\|w\|_{C(t,H^2)}.\end{aligned}$$

This quadratic expression in $\|w\|_{C(t,H^2)}$ can hold only if

$$\begin{aligned}\|w\|_{C(T,H^2)} &\leqq \tfrac{1}{2}(\|b\|_{C(T,H^2)}(K_9M)^{-1}+((K_9M)^{-2}\|b\|^2_{C(T,H^2)}+8\|a\|^2_{H^2})^{1/2})\\ &= M \quad \text{which establishes that } F_2(F_1(S))\subset S.\end{aligned}$$

LEMMA 3.4. *$S=\{w \in C(T, H^2_\sigma) \mid \|w\|_{C(T,H^2)} \leqq M\}$ is a closed convex set in $C(T, L_2)$.*

Proof. Convexity is immediate. Suppose $w_n \to w$ in $C(T, L_2)$ as $n \to \infty$ with $w_n \in S$. Then $w \in C(T, L_2)$, $F(w)$ exists and

$$\left(\int_{B(0,N)} (1+|z|^2)^2 |F(w)|^2 \, dz\right)^{1/2} \leqq (1+N^2)\|F(w_n)-F(w)\|_{L_2} + \|F(w_n)\|_{H^2} \leqq \varepsilon + M$$

for n sufficiently large. This is true for any N; so $w \in C(T, H^2)$ and $\|w\|_{C(T,H^2)} \leqq M$. Also $(w, \nabla f)_{L_2} = \lim_{n\to\infty} (w_n, \nabla f) = 0$ for smooth f since $w_n \in S$. Hence $w \in S$.

LEMMA 3.5. *$F_2(F_1(\ ))\colon S \to S$ is continuous in the $C(T, L_2)$ topology.*

Proof. Suppose $u_n \in S_i$ and $u_n \to u_0 \in S$ in $C(T, L_2)$. Let $v_i = F_1(u_i)$, $w_i = F_2(v_i) = F_2(F_1(u_i))$, $i = 0, 1, 2, \ldots$. We wish to use Lemma 3.1 with Hilbert space L_2; thus formally we must deal with the closure A^1 of $1-\Delta$ regarded as an operator on L_2. However, if $q \in H^2_\sigma \subset L_2$, then $A^1 q = Aq$, $e^{-tA^1} q = e^{-tA} q$ etc., so we can use our previous notation without difficulty. Now

$$w_i(t) = e^{-t\nu A} a + \int_0^t e^{-(t-s)\nu A}(-(v_i \cdot \text{grad})w_i + (w_i \cdot \text{grad})v_i + \nu w_i + b) \, ds$$

so by using Lemma 3.1 on the representation of $w_i - w_0$ and the fact that $((v_i \cdot \text{grad})(w_i - w_0), w_i - w_0)_{L_2} = 0$, we obtain

$$\|w_i(t) - w_0(t)\|^2_{L_2} \leqq 2 \int_0^t (-((v_i - v_0)\cdot \text{grad})w_0 + ((w_i - w_0)\cdot \text{grad})v_i + (w_0 \cdot \text{grad})(v_i - v_0), w_i - w_0)_{L_2} + \nu \|w_i - w_0\|^2_{L_2}) \, ds.$$

The following inequalities hold by Lemmas 0.2, 2.2 and Theorem 1.1:

$$\|((v_i - v_0)\cdot \text{grad})w_0\|_{C(T,L_2)} \leqq K\|v_{i,x} - v_{0,x}\|_{C(T,L_2)} \|w_{0,x}\|_{C(T,H^1)},$$

$$\|((w_i - w_0)\cdot \text{grad})v_i\|_{C(T,L_2)} \leqq \|v_{i,x}\|_{C_T} \cdot \|w_i - w_0\|_{C(T,L_2)},$$

$$\|(w_0 \cdot \text{grad})(v_i - v_0)\|_{C(T,L_2)} \leqq \|w_0\|_{C_T} \cdot \|v_{i,x} - v_{0,x}\|_{C(T,L_2)},$$

$$\|v_{i,x} - v_{0,x}\|_{C(T,L_2)} \leqq \|u_i - u_0\|_{C(T,L_2)},$$

$$\|v_{i,x}\|_{C_T} \leqq K_7 \|v_{i,x}\|_{C(T,H^2)} \leqq K_7 \|u_i\|_{C(T,H^2)} \leqq K_7 M \leqq K_9 M.$$

Hence using Schwarz' inequality

$$\|w_i - w_0\|^2_{C(T,L_2)} \leqq 2T(K_9 M + \nu)\|w_i - w_0\|^2_{C(T,L_2)} + \|w_i - w_0\|_{C(T,L_2)} 2T(K\|w_{0,x}\|_{C(T,H^1)} + \|w_0\|_{C_T})\|u_i - u_0\|_{C(T,L_2)}.$$

Since $4T(K_9 M + \nu) < 1$, we easily obtain $\|w_i - w_0\|_{C(T,L_2)} \leqq K\|u_i - u_0\|_{C(T,L_2)}$ and the mapping is therefore continuous.

To show that $F_2(F_1(S))$ is relatively compact in $C(T, L_2)$ we first show that it is an equicontinuous set of functions and then, with a somewhat intricate argument, show that the functions are uniformly small near infinity and thus we can use the Arzela-Ascoli theorem for compactness.

LEMMA 3.6. *For any* $w \in F_2(F_1(S))$,

(i) $\|w(t+h)-w(t)\|_{H^2} \leqq (h/\nu^2)^{1/4}K_{10}$,

(ii) $\|w(t+h)-w(t)\|_{H^1} \leqq h^{1/2}K_{11}$,

where K_{10} *and* K_{11} *are independent of* ν *and depend only on* M.

Proof. If $w \in F_2(F_1(S))$, then, for some $u \in S$, $w = F_2(F_1(u)) = F_2(v)$ and

$$w(t) = e^{-t\nu A}a + \int_0^t e^{-(t-s)\nu A}(-(v\cdot \operatorname{grad})w)\,ds + \int_0^t e^{-(t-s)\nu A}((w\cdot\operatorname{grad})v+\nu w+b)\,ds$$
$$= w^1(t)+w^2(t)+w^3(t).$$

$w^1(t+h)-w^1(t) = (e^{-h\nu A}-1)A^{-1/2}e^{-t\nu A}A^{1/2}a$, so

$$\|w^1(t+h)-w^1(t)\|_{H^1} \leqq \|w^1(t+h)-w^1(t)\|_{H^2} \leqq K(h\nu)^{1/2}\|A^{1/2}a\|_{H^2}$$

by Lemma 2.1(ix). Lemma 2.1(xi) applied to $w^3(t)$ (with $\delta=0$) gives

$$\begin{aligned}\|w^3(t+h)-w^3(t)\|_{H^1} &\leqq \|w^3(t+h)-w^3(t)\|_{H^2}\\ &\leqq Kh^{1/2}(\|b\|_{C(T,H^2)}+\nu\|w\|_{C(T,H^2)}+\|w\|_{C(T,H^2)}\|v_{,x}\|_{C_T})\\ &\leqq Kh^{1/2}(\|b\|_{C(T,H^2)}+\nu M+K_3M^2).\end{aligned}$$

Now $A^{-1/2}w^2(t) = -\int_0^t e^{-(t-s)\nu A}A^{-1/2}(v\cdot\operatorname{grad})w\,ds$, so by Lemma 2.1(xi) with $\delta=\frac{1}{2}$, $\mu=\frac{1}{4}$, we get

$$\begin{aligned}\|w^2(t+h)-w^2(t)\|_{H^2} &= \|A^{1/2}(A^{-1/2}w^2(t+h)-A^{-1/2}w^2(t))\|_{H^2}\\ &\leqq h^{1/4}\nu^{-1/2}K\|A^{-1/2}(v\cdot\operatorname{grad})w\|_{H^2}\\ &\equiv h^{1/4}\nu^{-1/2}K\cdot\|(v\cdot\operatorname{grad})w\|_{H^1}\\ &\leqq h^{1/4}\nu^{-1/2}K\cdot K_6M^2(1+K_3) \quad \text{by Lemma 2.2(iii).}\end{aligned}$$

This establishes (i).

Lemma 2.1 will hold for $1-\Delta$ as an operator on H^1, and the result (ii) for $\|w^2(t+h)-w^2(t)\|_{H^1}$ independent of ν follows immediately from Lemma 2.1(xi) applied with $\delta=0$ to H^1.

LEMMA 3.7. $F_2(F_1(S))$ *is an equicontinuous set of functions for fixed* $\nu>0$.

Proof. S is uniformly Hölder-continuous in x with exponent δ, $\delta<\frac{1}{2}$ by Lemma 0.1. $F_2(F_1(S))$ is uniformly Hölder-continuous in t with exponent $\frac{1}{4}$ by Lemmas 0.1 and 3.6.

LEMMA 3.8. *For any* $\varepsilon>0$, *there is an* N_ε *such that if* $w \in F_2(F_1(S))$ *then*

$$\sup_{t\in[0,T]}\int_{R^3-B(0,N_\varepsilon)}|w(t)|^2\,dx < \varepsilon^2.$$

N_ε *is independent of* $\nu>0$.

Proof. To obtain the result independent of ν, we wish to use Lemma 3.1. To this end, we note that if $w(t) = e^{-t\nu A}a+\int_0^t e^{-(t-s)\nu A}q(s)\,ds$ with $q\in C(T, H^2)$ then, for any C^∞ scalar function $f(x)$ bounded through its 4th derivatives,

$$w(t)f = e^{-t\nu A}af+\int_0^t e^{-(t-s)\nu A}(qf-\nu w\Delta f-2\nu(\nabla f\cdot\operatorname{grad})w)\,ds. \tag{3.1}$$

To establish this: if q is Hölder-continuous, then $dw/dt+\nu Aw=q$; hence, since $A=1-\Delta$, $d(wf)/dt+\nu A(wf)=qf-\nu w\Delta f-2\nu(\nabla f\cdot \text{grad})w$ and the integral representation of wf follows from semigroup theory. If q is only in $C(T, H^2)$, we can approximate q with Hölder-continuous functions and still obtain the integral equation representation for wf. Then let $f(r)\in C^\infty(0,\infty)$; $0\leqq f(r)\leqq 1$; $f(r)=0$, $0\leqq r\leqq 1$; $f(r)=1$, $2\leqq r<\infty$; and use $f_N(x)=f(|x|/N)$ in the representation (3.1) with $q=(-(v\cdot\text{grad})w+(w\cdot\text{grad})v+\nu w+b)$ together with Lemma 3.1 applied $H=L_2$. Inequalities in Lemmas 0.2, 2.2, and Theorem 1.1, together with the restriction $4T(K_9M+\nu)<1$ eventually yield an inequality of the form

$$\|f_Nw\|_{C(T,L_2)} \leqq K\|f_Na\|_{L_2}+K\|f_Nb\|_{C(T,L_2)}+K/N$$

where the third constant depends on M. Now

$$\sup_{t\in[0,T]}\int_{R^3-B(0,2N)}|w(t)|^2\,dx \leqq \|f_Nw\|^2_{C(T,L_2)} \leqq \sup_{t\in[0,T]}\int_{R^3-B(0,N)}|w(t)|^2\,dx$$

and similar inequalities for $\|f_Na\|_{L_2}$ and $\|f_Nb\|_{C(T,L_2)}$ establish the result.

LEMMA 3.9. *$F_2(F_1(S))$ is relatively compact in the topology $C(T,L_2)$ for fixed $\nu>0$.*

Proof. Since, by Lemma 3.7, $F_2(F_1(S))$ is equicontinuous, for any sequence $\{w_i\}$ by the Arzela-Ascoli theorem we can choose a subsequence that converges uniformly in $[0,T]\times B(0,1)$, a further subsequence that converges in $[0,T]\times B(0,2)$ and so forth; thus we can find a "diagonal" sequence $\{w_j\}$ that converges pointwise in $[0,T]\times R^3=Q_T$ and uniformly in any $[0,T]\times B(0,N)$ to some continuous w. Then

$$\|w(t)\|_{L_2(B(0,N))} \leqq \|w(t)-w_j(t)\|_{L_2(B(0,N))}+\|w_j(t)\|_{L_2} \leqq \varepsilon+M$$

if j is large, for any N, gives $w(t)\in L_2$. Similar reasoning with Lemma 3.6 shows that $w(t)\in C(T,L_2)$. Finally

$$\begin{aligned}\|w_i(t)-w(t)\|^2_{L_2} &\leqq \|w_i(t)-w(t)\|^2_{L_2(B(0,N))}+2(\|w(t)\|^2_{L_2(R^3-B(0,N))}+\|w_i(t)\|^2_{L_2(R^3-B(0,N))})\\ &\leqq \varepsilon\end{aligned}$$

for N large and $i\geqq i_0$ also large, uniformly in $t\in[0,T]$, by Lemma 3.8, and this gives convergence in $C(T,L_2)$.

THEOREM 3.1. *If $C\in C$; $\lim_{|x|\to\infty} C(x)=0$; $\nabla\cdot C=0$; $a\equiv\nabla\times C\in H_\sigma^{3+\delta}$; B is a continuous function on Q_T; $B_{,x}\in C(T,H^2)$; $b\equiv\nabla\times B$ is continuous in t as a $C(T,H_\sigma^2)$ function and T satisfies*

$$4T(\nu+K_9(2\|a\|^2_{H^2}+K_9^{-1}\|b\|_{C(T,H^2)})^{1/2}) < 1,$$

then there exist unique functions w, v, and P (P is unique up to an arbitrary function of t) such that

(E)
(a) $\partial w/\partial t+(v\cdot\text{grad})w-(w\cdot\text{grad})v-\nu\Delta w = b$,
(b) $w(x,0) = a(x)$,
(c) $\nabla\times v = w$; $\nabla\cdot v = 0$,
(d) $\lim_{|x|\to\infty} v(x,t) = 0$ *uniformly in* $t\in[0,T]$,

and

$$(\mathrm{E}^1)\qquad \begin{aligned}&\text{(a) } \partial v/\partial t+(v\cdot \operatorname{grad})v-\nu\Delta v = -\operatorname{grad} P+B,\\ &\text{(b) } v(x,0) = C(x),\end{aligned}$$

with $w \in C(T, H^3_\sigma) \cap C^{2+\mu}([\varepsilon, T]\times R^3) \cap C^{1+\mu}_T$; $\partial w/\partial t \in C^\mu([\varepsilon, T]\times R^3)$ *and* $v\in C^{3+\mu}([\varepsilon,T]\times R^3)\cap C^{2+\mu}_T$; $v_{,x}\in C([\varepsilon,T],H^4)\cap C(T,H^3)$; $\partial v/\partial t\in C^{1+\mu}([\varepsilon,T\times R^3)$ *and* $P_{,x}\in C^{1+\mu}([\varepsilon,T]\times R^3)$ *for any* $\varepsilon>0$ *and* $0<\mu<\frac{1}{2}$.

(E)(a) *and* (E^1)(a) *are satisfied in the classical sense*;

$$\|v(t)-C\|_{C^{2+\mu}};\qquad \|w(t)-a\|_{C^{1+\mu}}\quad \textit{and}\quad \|w(t)-a\|_{H^3}$$

all $\to 0$ *as* $t\to 0$.

Proof. First we prove the results for equations (E).

By the preceding lemmas of §III, Schauder's fixed point theorem can be applied to find some w such that $w=F_2(F_1(w))$. Let $v=F_1(w)$. Theorem 2.2 gives $w\in C(T, H^3_\sigma)$ and $\lim_{t\to 0}\|w(t)-a\|_{H^3}=0$. (E)(c) and (d) are satisfied by Theorem 1.1. To show that (a) is satisfied (Theorem 2.2 gives only the integral equation (2.2)) we must establish the necessary inequality for $P_{v,v}$ to use part (II) of Theorem 2.1. For this we must show that if $u\in H^3_\sigma=D(A^{1/2})$ and $v=F_1(w)$, then there are some constants K and $\mu>0$ such that

$$\begin{aligned}\|((v(t_1)\cdot \operatorname{grad})u-(u\cdot\operatorname{grad})v(t_1))-((v(t_2)\cdot\operatorname{grad})u-(u\cdot\operatorname{grad})v(t_2))\|_{H^2}&\\ \leqq K|t_1-t_2|^\mu\|A^{1/2}u\|_{H^2}.&\end{aligned}$$

First

$$\|((v(t_1)-v(t_2))\cdot\operatorname{grad})u\|_{H^2}\leqq K_4(\|v_{,x}(t_1)-v_{,x}(t_2)\|_{H^2}+\|v(t_1)-v(t_2)\|_C)\|A^{1/2}u\|_{H^2}$$

and

$$\|(u\cdot\operatorname{grad})(v(t_1)-v(t_2))\|_{H^2}\leqq K_5\|v_{,x}(t_1)-v_{,x}(t_2)\|_{H^2}\|u\|_{H^2}$$

by Lemma 2.2. Theorem 1.1 gives

$$\|v(t_1)-v(t_2)\|_C=\|F_1(w(t_1)-w(t_2))\|_C\leqq K_3\|w(t_1)-w(t_2)\|_{H^2}$$

and

$$\|v_{,x}(t_1)-v_{,x}(t_2)\|_{H^2}=\|(F_1(w(t_1)-w(t_2)))_{,x}\|_{H^2}\leqq\|w(t_1)-w(t_2)\|_{H^2}.$$

But w is Hölder-continuous in t by Lemma 3.6. Hence (a) is satisfied as in Theorem 2.1(II) and the smoothness of v and w is given by Lemma 0.1 applied with Theorems 1.1 and 2.1; thus the solution is classical.

For equations (E^1), we need to establish the differentiability of v with respect to t. We have established that $\partial w/\partial t\in C([\varepsilon,T];H^2)$. Hence we can form $v^1(t)\equiv F_1(\partial w/\partial t)$ for any $t>0$. Then, using the linearity of F_1 and Theorem 1.1, $v^1(t)=\partial v/\partial t$ and $v^1(t)_{,x}=(\partial v/\partial t)_{,x}\in C([\varepsilon,T],H^2)$ by the following reasoning:

$$\begin{aligned}&\|(v(t+\Delta t)-v(t))/\Delta t-v\,(t)\|_C\\ &\qquad=\|(1/\Delta t)F_1(w(t+\Delta t)-w(t))-F_1(\partial w/\partial t)\|_C\\ &\qquad\leqq K_3\|(1/\Delta t)(w(t+\Delta t)-w(t))-\partial w/\partial t\|_{H^2}\to 0\quad\text{as } \Delta t\to 0.\end{aligned}$$

Similarly $v^1(t)_{,x}=(\partial v/\partial t)_{,x}$. The smoothness of v then follows from results for (E). Thus if $\partial v/\partial t+(v\cdot\text{grad})v-\nu\Delta v-B=u$, (E)(a) gives $\nabla\times u=0$ for $t>0$ and $u_{,x}\in C([\varepsilon, T]; H^2)$, which, by Lemma 0.1 gives $u\in C^{1+\mu}([\varepsilon, T]\times R^3)$. Thus there is a function P, unique to a function of t, such that $u=\nabla(-P)$ and

$$P_{,x}\in C^{1+\mu}([\varepsilon, T]\times R^3).$$

Now $C=F_1(a)$, so

$$\|v(t)-C\|_C = \|F_1(w(t)-a)\|_C \leqq K_3\|w(t)-a\|_{H^2}$$

and

$$\|v_{,x}(t)-a_{,x}\|_{C^{1+\mu}} \leqq K_{3,\mu}\|w(t)-a\|_{H^3}$$

implies by the result for (E) that $\|v(t)-C\|_{C^{2+\mu}}\to 0$ as $t\to 0$.

We must establish uniqueness. Suppose that (w^1, v^1) and (w^2, v^2) are both solutions of (E). Then

$$w^i(t) = e^{-t\nu A}a+\int_0^t e^{-(t-s)\nu A}(-(v^i\cdot\text{grad})w^i+(w^i\cdot\text{grad})v^i+\nu w^i+b)\,ds.$$

Applying Lemma 3.1 with $H=L_2$ to the integral representation of $w^1(t)-w^2(t)$ and recalling that $((v^1\cdot\text{grad})(w^1-w^2), w^1-w^2)_{L_2}=0$, we obtain the inequality

$$\begin{aligned}\|w^1(t)-w^2(t)\|_{L_2}^2 \leqq{}& 2\int_0^t \|w^1(s)-w^2(s)\|_{L_2}\|((v^1(s)-v^2(s))\cdot\text{grad})w^2(s)\|_{L_2}\,ds\\ &+2\int_0^t \|w^1-w^2\|_{L_2}(\|(w^1\cdot\text{grad})(v^1-v^2)\|_{L_2}\\ &\qquad+\|((w^1-w^2)\cdot\text{grad})v^2\|_{L_2}+\nu\|w^1-w^2\|_{L_2})\,ds.\end{aligned}$$

The uniqueness statement of Theorem 1.1 gives $v^i=F_1(w^i)$. Hence Lemma 0.2(iii) provides

$$\|((v^1-v^2)\cdot\text{grad})w^2\|_{L_2} \leqq K\|v^1_{,x}-v^2_{,x}\|_{L_2}\|w^2\|_{H^2} \leqq K\|w^1-w^2\|_{L_2}\|w^2\|_{H^2}.$$

Similar inequalities concerning the remaining terms yield an inequality of form $\|w^1(t)-w^2(t)\|_{L_2}^2\leqq K\int_0^t \|w^1(s)-w^2(s)\|^2\,ds$ which can only hold if $w^1=w^2$; hence $v^1=F(w^1)=F(w^2)=v^2$ also.

(E^1) will have a unique solution also, since if (v, P) solves (E^1) then $(\nabla\times v, v)$ solves (E).

IV. In §III we established the existence, for any viscosity $\nu>0$, of a solution (v^ν, P^ν) to the Navier-Stokes equations. In §IV we show that v^ν converges, as the viscosity ν goes to zero, to a function v that gives the solution to the Euler equations for the flow of an ideal fluid in R^3.

The Euler equations are

$$\begin{aligned}&\partial v/\partial t+(v\cdot\text{grad})v = -\text{grad}\,P+B,\\ &\nabla\cdot v = 0 \quad\text{with constraints}\\ &\lim_{|x|\to\infty} v(x, t) = 0 \quad\text{and}\quad v(x, 0) = C(x).\end{aligned}\tag{4.1}$$

By formally computing the curl of equation (4.1), we obtain the system

$$
(4.2)\quad
\begin{aligned}
&\text{(i)} && \partial v/\partial t+(v\cdot\text{grad})w-(w\cdot\text{grad})v = \nabla\times B \equiv b,\\
&\text{(ii)} && w(x,0) = \nabla\times C(x) \equiv a(x),\\
&\text{(iii)} && \nabla\times v = w;\ \nabla\cdot v = 0,\\
&\text{(iv)} && \lim_{|x|\to\infty} v(x,t) = 0.
\end{aligned}
$$

Assume for this section that $0<\nu\leqq\nu_0$, T satisfies

$$
(4.3)\qquad 4T(\nu_0+K_9(2\|a\|_{H^2}^2+K_9^{-1}\|b\|_{C(T,H^2)})^{1/2}) < 1
$$

and $M^2=2\|a\|_{H^2}^2+K_9^{-1}\|b\|_{C(T,H^2)}$ where $a\equiv\nabla\times C$ and $b\equiv\nabla\times B$. Use these restrictions and functions to obtain the results of §III for various viscosities ν.

LEMMA 4.1. *If* (w^ν, v^ν) *is the solution of* (E) *with viscosity* $\nu\leqq\nu_0$ *of Theorem* 3.1, *then there is a function* $w\in C(T, H_\sigma^2)$; $\|w\|_{C(T,H^2)}\leqq M$ *such that if* $v=F_1(w)$, *then*

(i) $\|w^\nu-w\|_{C(T,L_2)}\leqq 8TM\nu$,

(ii) $\|v^\nu-v\|_{C(T,L_6)}\leqq 8TMK_2\nu$,

(iii) $\|v^\nu{}_{,x}-v{}_{,x}\|_{C(T,L_2)}\leqq 8TM\nu$.

Proof. Let (w^i, v^i) be solutions of (E) with viscosities ν_i of Theorem 3.1. Using the notation $A=1-\Delta$, Theorem 3.1 gives

$$
(4.4)\quad
\begin{aligned}
d(w^i-w^j)/dt+\nu_iA(w^i-w^j) &= p\\
&\equiv (\nu_j-\nu_i)(Aw^j-w^i)+\nu_j(w^i-w^j)+((w^i-w^j)\cdot\text{grad})v^i\\
&\quad+(w^j\cdot\text{grad})(v^i-v^j)+((v^j-v^i)\cdot\text{grad})w^i+(v^j\cdot\text{grad})(w^j-w^i).
\end{aligned}
$$

The initial value of $w^i-w^j=a-a=0$, so we can use Lemma 3.1 with Hilbert space L_2 and the result $((v^j\cdot\text{grad})(w^j-w^i),\, w^j-w^i)_{L_2}=0$ to obtain

$$
\begin{aligned}
\|w^i-w^j\|_{C(t,L_2)} &\leqq 2t\,|\nu_j-\nu_i|(\|Aw^j\|_{C(t,L_2)}+\|w^i\|_{C(t,L_2)})+2t(\nu_j+\|v^i{}_{,x}\|_{C_t})\|w^i-w^j\|_{C(t,L_2)}\\
&\quad+2t(\|w^j\|_{C_t}\cdot\|v^i{}_{,x}-v^j{}_{,x}\|_{C(t,L_2)}+\|((v^j-v^i)\cdot\text{grad})w^i\|_{C(t,L_2)}).
\end{aligned}
$$

Now (see proof of Lemma 3.2)

$$
\|((v^j-v^i)\cdot\text{grad})w^i\|_{C(t,L_2)} \leqq K_8\|v^j{}_{,x}-v^i{}_{,x}\|_{C(t,L_2)}\|w^i{}_{,x}\|_{C(t,H^1)}
$$

and

$$
\|v^i{}_{,x}-v^j{}_{,x}\|_{C(t,L_2)} = \|F_1(w^i-w^j)\,{}_{,x}\|_{C(t,L_2)} \leqq \|w^i-w^j\|_{C(t,L_2)}.
$$

So

$$
\|w^i-w^j\|_{C(t,L_2)} \leqq 4tM|\nu_j-\nu_i|+2t(\nu_j+(2K_7+K_8)M)\|w^i-w^j\|_{C(t,L_2)},
$$

since both w^i and w^j are bounded in $C(t, H^2)$ by M. Then

$$
(4.5)\qquad \|w^i-w^j\|_{C(T,L_2)} \leqq 8TM|\nu_i-\nu_j|
$$

follows from $2t(\nu_j+(2K_7+K_8)M)\leqq 2T(\nu_0+K_9M)<\frac{1}{2}$. Both w^i and $w^j\in S$ where

$$
S = \{w\in C(T, H_\sigma^2)\mid \|w\|_{C(T,H^2)}\leqq M\}
$$

which is closed in $C(T, L_2)$ by Lemma 3.4; hence $w \in S$ and $v = F_1(w)$ exist and the inequalities follow from Theorem 1.1 and (4.5).

THEOREM 4.1. *If $C \in C$, $\lim_{|x|\to\infty} C(x) = 0$, $\nabla \cdot C = 0$, $a \equiv \nabla \times C \in H^{3+\delta}$ and B is a continuous function such that $B_{,x} \in C(T, H^2)$ and $b \equiv \nabla \times B$ is Hölder-continuous in t as a $C(T, H^2)$ function, T satisfies*

$$4T(\nu_0 + K_9(2\|\nabla \times C\|_{H^2}^2 + K_9^{-1}\|\nabla \times B\|_{C(T,H^2)})^{1/2}) < 1$$

then there are unique functions w, v and P (P is unique up to an arbitrary function of t) in $[0, T]$ such that

$$(4.2)\quad \begin{aligned} &\text{(i)} && \partial w/\partial t + (v \cdot \operatorname{grad})w - (w \cdot \operatorname{grad})v = b \\ &\text{(ii)} && w(x, 0) = a(x), \\ &\text{(iii)} && \nabla \times v = w;\ \nabla \cdot v = 0, \\ &\text{(iv)} && \lim_{|x|\to\infty} v(x, t) = 0 \quad \text{uniformly in } t \in [0, T], \end{aligned}$$

and

$$(4.1)\quad \begin{aligned} &\text{(i)} && \partial v/\partial t + (v \cdot \operatorname{grad})v = -\operatorname{grad} P + B, \\ &\text{(ii)} && v(x, 0) = C(x), \end{aligned}$$

with $w \in C(T, H^2)$; $\partial w/\partial t \in C(T, H^1)$; $v \in C_T^{1+\mu}$; $v_{,x} \in C(T, H^2)$; $\partial v/\partial t \in C^\mu$ (locally in $x \in R^3$ uniformly in $t \in [0, T]$); $\partial v/\partial t \in C(T, L_6)$; $\partial v_{,x}/\partial t \in C(T, H^1)$; $\partial(\operatorname{grad} P)/\partial t \in C(T, H^1)$ for $\mu < \frac{1}{2}$.

$\|w(t) - a\|_{H^1}$; $\|v(t) - C\|_C$; $\|v(t) - C\|_{L_6}$ and $\|v_{,x}(t) - C_{,x}\|_{H^1}$ all go to zero as $t \to 0$.

If (v^ν, P^ν) is the solution to the Navier-Stokes flow of Theorem 3.1 with $\nu < \nu_0$, then $\lim_{\nu\to 0} \|v^\nu - v\|_{C_T} = 0$; $\|v^\nu - v\|_{C(T,L_6)} \leqq 8TMK_4\nu$ and $\|v^\nu_{,x} - v_{,x}\|_{C(T,L_2)} \leqq 8TM\nu$.

Proof. We first show that (w, v) of Lemma 4.1 is a "weak solution" of (4.2)(i). Now

$$(4.6)\quad \begin{aligned} &\|(w^\nu \cdot \operatorname{grad})v^\nu - (w \cdot \operatorname{grad})v\|_{C(T,L_2)} \\ &\qquad \leqq \|((w^\nu - w) \cdot \operatorname{grad})v^\nu\|_{C(T,L_2)} + \|(w \cdot \operatorname{grad})(v^\nu - v)\|_{C(T,L_2)} \\ &\qquad \leqq \|w^\nu - w\|_{C(T,L_2)}\|v^\nu_{,x}\|_{C_T} + \|w\|_{C_T}\|v^\nu_{,x} - v_{,x}\|_{C(T,L_2)} \\ &\qquad \to 0 \quad \text{as } \nu \to 0 \end{aligned}$$

by Lemma 4.1 and the uniform (in ν) boundedness of $\|v^\nu_{,x}\|_{C_T}$ ($w^\nu \in S$ for all $\nu > 0$). For $p(x) \in C_0^\infty$,

$$((v^\nu \cdot \operatorname{grad})w^\nu - (v \cdot \operatorname{grad})w, p)_{L_2} = (((v^\nu - v) \cdot \operatorname{grad})w^\nu, p)_{L_2} + ((v \cdot \operatorname{grad})(w^\nu - w), p)_{L_2}.$$

$$\|((v^\nu - v) \cdot \operatorname{grad})w^\nu\|_{L_2} \leqq K\|v^\nu_{,x} - v_{,x}\|_{L_2}\|w^\nu\|_{H^2} \to 0 \quad \text{as } \nu \to 0$$

uniformly in t by Lemmas 0.2(iii) and 4.1.

$$\begin{aligned} |((v \cdot \operatorname{grad})(w^\nu - w), p)_{L_2}| &= |(\partial(v_j(w^\nu - w))/\partial x_j, p)_{L_2}| \\ &= |(v_j(w^\nu - w), \partial p/\partial x_j)_{L_2}| \\ &\leqq K\|v\|_{C_T} \cdot \|w^\nu - w\|_{L_2}\|p_{,x}\|_{L_2} \to 0 \quad \text{uniformly in } t \in [0, T]. \end{aligned}$$

Hence

$$((v^\nu \cdot \mathrm{grad})w^\nu, p)_{L_2} \to ((v \cdot \mathrm{grad})w, p)_{L_2} \quad \text{as } \nu \to 0. \tag{4.7}$$

From Theorem 3.1, $\partial w^\nu/\partial t = b + (w^\nu \cdot \mathrm{grad})v^\nu - (v^\nu \cdot \mathrm{grad})w^\nu + \nu\Delta w^\nu$ so, for $p \in C_0^\infty$,

$$\begin{aligned} \int_0^t (b+(w^\nu \cdot \mathrm{grad})v^\nu - (v^\nu \cdot \mathrm{grad})w^\nu, p)_{L_2}\, ds + \nu \int_0^t (\Delta w^\nu, p)_{L_2}\, ds \\ = \int_0^t \frac{d}{ds}(w^\nu, p)_{L_2}\, ds = (w^\nu(t) - a, p)_{L_2}. \end{aligned} \tag{4.8}$$

Inequalities (4.6), (4.7), $|(\Delta w^\nu, p)_{L_2}| \leqq \|w^\nu\|_{H^2}\|p\|_C$, and Lemma 4.1 then give

$$(w(t)-a, p)_{L_2} = \int_0^t (b+(w\cdot \mathrm{grad})v - (v \cdot \mathrm{grad})w, p)_{L_2}\, ds. \tag{4.9}$$

Let $q(t) \equiv b + (w \cdot \mathrm{grad})v - (v \cdot \mathrm{grad})w \in C(T, H^1)$. If $u(t) = a + \int_0^t q(s)\, ds$, then $u(t)$ and $du/dt = q(t) \in C(T, H^1)$ and $\|u(t) - a\|_{H^1} \to 0$. Hence

$$(u(t)-a, p)_{L_2} = \int_0^t (q, p)_{L_2}\, ds = (w(t)-a, p)_{L_2},$$

which can only occur if $u = w$, which establishes (4.2)(i) and (ii). (4.2)(iii) is true since $v = F_1(w)$. $\lim_{|x|\to\infty} v(x, t) = 0$ uniformly in $t \in [0, T]$ follows from Lemmas 0.3, 3.8 and a study of the proof of Lemma 1.2(i).

To show differentiability in t of v, let $q_h(t) = (q(t+h) - q(t))/h$ for any function q. Then by the results of Theorem 1.1 and the linearity of F_1,

$$\|(v_{,x})_{h_1}(t) - (v_{,x})_{h_2}(t)\|_{H^1} \leqq \|w_{h_1}(t) - w_{h_2}(t)\|_{H^1}.$$

Since $\partial w/\partial t$ exists in $C(T, H^1)$, $\partial(v_{,x})/\partial t$ will exist in H^1 for any $t \in [0, T]$: strong continuity in t follows from a similar inequality and the strong continuity of $\partial w/\partial t$. Parallel reasoning and inequality (i) of Theorem 1.1 shows that $\partial v/\partial t$ exists in $C(T, L_6)$. Lemma 0.2(iv) then gives $\partial v/\partial t \in C^\lambda([0, T] \times B(0, N))$, $\lambda < \frac{1}{2}$, for any N. Thus we can write (4.2)(i) as $\nabla \times q = 0$, where $q = \partial v/\partial t + (v \cdot \mathrm{grad})v - B \in C^\lambda$ (locally). Define $P(x, t) = -\int_\Gamma q(x, t)\, d\sigma$ where Γ is any smooth path from 0 to x. If q is sufficiently smooth, the condition $\nabla \times q = 0$ guarantees that $P(x, t)$ is defined independent of choice of path Γ; by use of mollifier theory we can construct smooth approximations to $q \in C^\lambda$ (locally) preserving the property $\nabla \times q = 0$ and easily obtain this result for q only in C^λ (locally). Then

$$\partial v/\partial t + (v \cdot \mathrm{grad})v = -\mathrm{grad}\, P + B$$

and the statements concerning the smoothness of v follow from the results for system (4.2).

The properties postulated for C give $C = F_1(\nabla \times C)$, so

$$\|v(t) - C\|_{L_6} = \|F_1(w)(t) - F_1(\nabla \times C)\|_{L_6} \leqq K_2\|w(t) - \nabla \times C\|_{L_2} \to 0 \quad \text{as } t \to 0.$$

Similar reasoning and Theorem 1.1 gives

$$\|v_{,x}(t) - C_{,x}\|_{H^1} \leqq \|w(t) - \nabla \times C\|_{H^1} \to 0 \quad \text{as } t \to 0.$$

Lemma 0.7(iv) then gives $\|v(t)-C\|_{C(B(0,N))}\to 0$ as $t\to 0$; $v(t)$ is uniformly (in t) small near ∞; C is small near ∞; hence $\|v(t)-C\|_C\to 0$ as $t\to 0$.

To show uniqueness: If (w^1, v^1) and (w^2, v^2) are both solutions of (4.2), then $v^i=F(w^i)$ by the uniqueness statement of Theorem 1.1. Also

$$(d(w^1-w^2)/dt,\ w^1-w^2)_{L_2}$$
$$= (((w^1-w^2)\cdot\text{grad})v^1,\ w^1-w^2)_{L_2}+((w^2\cdot\text{grad})(v^1-v^2)+((v^2-v^1)\cdot\text{grad})w^1 + (v^2\cdot\text{grad})(w^2-w^1),\ w^1-w^2)_{L_2}.$$

Since $w^1(0)-w^2(0)=a-a=0$ and $(v^2\cdot\text{grad}\,(w^2-w^1),\ w^1-w^2)_{L_2}=0$ we can use estimates similar to those of the uniqueness proof of Theorem 3.1 to obtain an inequality of form $\|w^1(t)-w^2(t)\|^2\leqq K\int_0^t \|w^1(s)-w^2(s)\|^2\,ds$, which can only occur if $w^1=w^2$; hence $v^1=F_1(w^1)=F_1(w^2)=v^2$ also. (v, P) is a unique solution of (4.1) since $(\nabla\times v, v)$ is a unique solution of (4.2).

Lemma 4.1 gives the convergence of v^ν to v except for the result $\|v^\nu-v\|_{C_T}\to 0$ as $\nu\to 0$. We prove this by contradiction. Suppose, for some $\varepsilon>0$, there is a sequence of ν_i with associated $v_i\equiv v^{\nu_i}$ such that $\|v_i-v\|_{C_T}>\varepsilon$. Let $w_i=F_2(v_i)$; $w_i\in F_2(F_1(S))$, hence Lemma 3.8, Lemma 0.3 and Lemma 1.2 show that v_i is small near ∞ uniformly in t and *independent of* ν. Hence $\|v_i-v\|_{C_T}>\varepsilon$ occurs only within some ball $B(0, N)$; i.e., $\|v_i-v\|_{C([0,T]\times B(0,N))}>\varepsilon$. Lemma 0.1(iv), $\|v_i\|_{C(T,L_6)}\leqq K_2\|w_i\|_{C(T,L_2)}\leqq K_2M$ and $\|(v_i)_{,x}\|_{C(T,H^1)}\leqq B$ show that $\{v_i\}$ are equicontinuous in $x\in B(0,N)$. The same inequality can be used with Lemma 3.6(ii) to establish that $\{v_i\}$ are equicontinuous in t (*independent* of ν). Hence there exists, by the Arzela-Ascoli theorem, a subsequence $\{v_j\}$ and $v'\in C([0,T]\times B(0,N))$ such that $\|v_j-v'\|_{C([0,T]\times B(0,N))}\to 0$ as $j\to\infty$; hence $\|v_j-v'\|_{C([0,T],L_6(B(0,N)))}\to 0$ as $j\to\infty$ also. But $v_j\to v$ in $C(T, L_6)$, which implies that $v=v'$ and contradicts the assumption that $\|v_i-v\|_{C([0,T]\times B(0,N))}>\varepsilon$. Thus $\|v^\nu-v\|_{C_T}\to 0$ as $\nu\to 0$.

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