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Dedicated to Richard Rado on his sixty-fifth birthday.

Let $\pi_n : A_1A_2 \cdots A_r$ be a polygon of real order n in real projective n-space L_n, $r \geqq n+3$, with the vertices $A_1, A_2, \cdots, A_r$. If H_{n-1} be a hyperplane for which $A_i \notin H_{n-1}$, $1 \leqq i \leqq r$, let $H(\pi_n)$ be the convex full of the set $\{A_1, A_2, \cdots, A_r\}$ defined in the affine space $L_n \backslash H_{n-1}$. This paper gives a classification of the combinatorial types of the sets $H(\pi_n)$ for a fixed π_n.

The restriction $r \geqq n + 3$ is necessary because the classification is obtained from the properties of the polygon π_n associated with $H(\pi_n)$. Such a polygon is determined uniquely by its vertices if and only if $r \geqq n + 3$ [2]. If $r = n + 3$ and the points $A_1, A_2, \cdots, A_{n+3}$ are in general position there is exactly one polygon π_n with these vertices [1]. Thus the set of all $H(\pi_n)$, $r = n + 3$, for which no vertex of π_n an interior point of $H(\pi_n)$ is also the set of convex polytopes with $n + 3$ vertices in general position. Consequently the invariants which characterize the sets $H(\pi_n)$ can be used to characterize the convex polytopes with $n+3$ vertices in general position. The method used here is different from that used by M. A. Perles in his solution [3] of this problem.

The work is divided into three sections. The first contains definitions and known or easily proved results dealing with the polygons π_n. The second section develops theorems involving the convex hulls of the polygon vertices which are used to obtain the characterizing invariants in the final section. Except for the case in which H_{n-1} intersects π_n in n points the sets are characterized by a cycle of the type used in combinatorial analysis [4].

1. Preliminaries.

1.1. The subspace of the real projective n-space L_n spanned by the points or point sets $A, B, \cdots$ is denoted by $[A, B, \cdots]$.

1.2. If $A_1, A_2, \cdots, A_r$, $r \geqq n + 3$, are distinct points of L_n, π: $A_1A_2 \cdots A_r$ denotes a *closed* polygon with the sides A_iA_{i+1}, $1 \leqq i \leqq r$, $(A_{r+s} = A_s)$ and vertices $A_1, A_2, \cdots, A_r$.

An (open) segment α: $A_iA_{i+1} \cdots A_{i+h}$, $0 \leqq h \leqq r - 1$, is called an *arc* of π of length h.

1.3. If any one hyperplane L_{n-1}, $A_i \notin L_{n-1}$, $1 \leqq i \leqq r$, contains an even (odd) number of points of π then any hyperplane which does not contain any vertex of π also contains an even (odd) number of points of π. This property is known as the *parity* of π.

1.4. An *intersection point* of a hyperplane L_{n-1} and a polygon π is a point of $L_{n-1} \cap \pi$ which is either a vertex of π or the only point of a side of π within L_{n-1}.

A closed polygon in L_n for which no hyperplane contains more than n intersection points is said to have order n.

Such polygons will be denoted by the symbols π_n, σ_n.

A consequence of the order condition is that the vertices of a polygon π_n are in general position.

If P_i be an interior point of a side A_iA_{i+1}, $1 \leqq i \leqq n$, of a polygon π_n then a hyperplane L_{n-1} exists for which $[P_i, P_2, \cdots, P_n] \subseteqq L_{n-1}$. It follows from the order of π_n that $L_{n-1} \cap \pi_n$ is exactly the point set $\{P_1, P_2, \cdots, P_n\}$ and $L_{n-1} = [P_1, P_2, \cdots, P_n]$. Hence, by the parity of π_n, every hyperplane which does not contain any vertex of π_n intersects it in an even (odd) number of points if n is even (odd).

1.5. If, for $n > 1$, $A_{i-1}A_{i+1}$ is the line segment in L_n which together with the arc $A_{i-1}A_iA_{i+1}$ of a polygon π_n: $A_1A_2 \cdots A_r$ forms an even triangle then the lines in the plane $[A_{i-1}, A_i, A_{i+1}]$ which contain A_i but do not contain an interior point of $A_{i-1}A_{i+1}$ are defined to be the *tangents* of π_n at A_i. A tangent at A_i is denoted by the symbol $L(A_i)$. The following result is a simple consequence of the order of π_n.

1.6. *If, for $n > 1$, L_{n-1} is the projection of L_n from a vertex A_k of the polygon π_n: $A_1A_2 \cdots A_r$, $A'_iA'_{i+1}$, $i \neq k$, $i + 1 \neq k$, that of the side A_iA_{i+1} and $A'_{k+1}A'_{k-1}$ that of the set of all the tangents $L(A_k)$, then the polygon $A'_1A'_2 \cdots A'_{k-1}A'_{k+1} \cdots A'_r$ has order $n - 1$ in L_{n-1}.*

We shall call the polygon π_{n-1}: $A'_1A'_2 \cdots A'_{k-1}A'_{k+1} \cdots A'_r$ the projection of π_n from A_k.

1.7. If A is a point of hyperplane T_{n-1} of a projective (affine) space $L_n(R_n)$, $n > 1$, then if L_{n-1}, T_{n-2}, are the projections of $L_n(R_n)$, T_{n-1} respectively, from A the affine space $L_{n-1}\backslash T_{n-2}$ is said to be an *affine projection* of $L_n(R_n)$ from A.

1.8. Throughout this paper a fixed hyperplane H_{n-1}, $A_i \notin H_{n-1}$, $1 \leqq i \leqq r$, will be associated with each polygon π_n: $A_1A_2 \cdots A_r$. The convex hull of the vertices of π_n defined in the affine space $L_n \backslash H_{n-1}$ is denoted by the symbol $H(\pi_n)$.

A segment A_pA_q is said to be finite if $H_{n-1} \cap A_pA_q = \varnothing$ and otherwise infinite.

1.9. *Hypothesis*: A_k *is a vertex of a polygon* π_n: $A_1A_2 \cdots A_r$, $n>1$, *on the boundary of the convex hull* $H(\pi_n)$ *defined in* $L_n \backslash H_{n-1}$.

T_{n-1} *is a hyperplane which contains exactly the one vertex* A_k *of* π_n *and supports* $H(\pi_n)$.

If π_{n-1}: $A'_1A'_2 \cdots A'_{k-1}A'_{k+1} \cdots A'_r$ *is the projection of* π_n *and* T_{n-2} *that of* T_{n-1} *from* A_k $H(\pi_{n-1})$ *is defined in the affine projection* $L_{n-1} \backslash T_{n-2}$ *of* L_n *from* A_k.

Conclusion: (1) $A'_{k-1}A'_{k+1}$ *is finite in* $L_{n-1} \backslash T_{n-2}$ *if and only if exactly one of the two sides of the arc* $A_{k-1}A_kA_{k+1}$ *of* π_n *is finite in* $L_n \backslash H_{n-1}$; *each other side* $A'_iA'_{i+1}$ *of* π_{n-1} *is finite if and only if the corresponding side* A_iA_{i+1} *of* π_n *is finite.*

(2) *A hyperplane* $[A'_{i_2}, A'_{i_3}, \cdots, A'_{i_n}]$ *of* $L_{n-1} \backslash T_{n-2}$ *supports* $H(\pi_{n-1})$ *if and only if the hyperplane* $[A_k, A_{i_2}, \cdots, A_{i_n}]$ *supports* $H(\pi_n)$.

Proof. If Q_{n-1} be a hyperplane which supports $H(\pi_n)$ let A_p, A_q be distinct vertices of π_n for which $A_p, A_q \notin Q_{n-1}$. Then a segment A_pA_q is finite if and only if $Q_{n-1} \cap A_pA_q = \varnothing$. But, by the definition 1.8, A_pA_q is finite if and only if $H_{n-1} \cap A_pA_q = \varnothing$. Hence $Q_{n-1} \cap A_pA_q = \varnothing$ if and only if $H_{n-1} \cap A_pA_q = \varnothing$.

If Q_{n-1} is specialized to be T_{n-1} it follows that a side A_iA_{i+1}, $i \neq k$, $i+1 \neq k$, of π_n is finite if and only if $T_{n-1} \cap A_iA_{i+1} = \varnothing$. Let $A'_iA'_{i+1}$, T_{n-2} be the projections of A_iA_{i+1}, T_{n-1}, respectively, from A_k. Hence, as $T_{n-2} \cap A'_iA'_{i+1} = \varnothing$ if and only if $T_{n-1} \cap A_iA_{i+1} = \varnothing$, $A'_iA'_{i+1}$ is finite in $L_{n-1} \backslash T_{n-2}$ if and only if A_iA_{i+1} is finite in $L_n \backslash H_{n-1}$. This means that the arcs $A_{k+1}A_{k+2} \cdots A_{k+r-1}$, $A'_{k+1}A'_{k+2} \cdots A'_{k+r-1}$ of π_n, π_{n-1}, respectively, both contain the same number of finite sides. π_{n-1}, because of its parity, must then contain either one infinite side more or one infinite side less than π_n. This implies that $A'_{k-1}A'_{k+1}$ is finite if and only if exactly one of the two sides of the arc $A_{k-1}A_kA_{k+1}$ of π_n is finite. (1) is now clear. (2) follows if Q_{n-1} is the hyperplane $[A_k, A_{i_2}, \cdots, A_{i_n}]$ and thus completes the proof.

1.10. A polygon π_{n-1}: $A'_1A'_2 \cdots A'_{k-1}A'_{k+1} \cdots A'_r$, constructed by projecting a polygon π_n: $A_1A_2 \cdots A_r$ and a hyperplane T_{n-1}, $A_k \in T_{n-1}$, which supports $H(\pi_n)$, from A_k, as in 1.9, is called a *normal projection* of π_n. The existence of the space T_{n-1} and the space $L_{n-1} \backslash T_{n-2}$ in which $H(\pi_{n-1})$ is defined is tacitly assumed.

2. Maximal arcs.

2.1. *An arc* $A_iA_{i+1}\cdots A_{i+h}$, $h\geqq 0$, *of a* polygon π_n: $A_1A_2\cdots A_r$ is defined to be a *maximal arc of length* h *if* $A_{i-1}A_i$, $A_{i+h}A_{i+h+1}$ are both finite and are the only finite sides of the arc $A_{i-1}A_i\cdots A_{i+h+1}$ of π_n.

A vertex of π_n within a maximal arc of positive length is called a *j-point*.

2.2. *If one vertex of an infinite side* A_iA_{i+1} *of a polygon* π_n: $A_1A_2\cdots A_r$ *is not within a hyperplane* Q_{n-1}: $[A_{i_1}, A_{i_2}, \cdots, A_{i_n}]$ *which supports* $H(\pi_n)$ *then* Q_{n-1} *contains the other vertex of* A_iA_{i+1}.

Proof. If $n=1$ $H(\pi_1)$ is the finite segment which is the complement of A_iA_{i+1} in the projective line. Hence $Q_0=A_i$ or $Q_0=A_{i+1}$. Thus the result is proved for $n=1$. If, for $n>1$, $A_i, A_{i+1}\notin Q_{n-1}$ then as Q_{n-1} supports $H(\pi_n)$ it cannot separate A_i and A_{i+1}. It must therefore intersect the infinite side A_iA_{i+1}. As this is impossible because of the order of π_n the result is established.

2.3. *If* A_i *is an interior point of the convex hull* $H(\pi_n)$ *of a polygon* π_n: $A_1A_2\cdots A_r$ *then both sides* $A_{i-1}A_i$, A_iA_{i+1} *of* π_n *are finite.*

Proof. By 1.2 $r\geqq n+3$. As the vertices of π_n are in general position a hyperplane Q_{n-1}: $[A_{i_1}, A_{i_2}, \cdots, A_{i_n}]$ exists which supports $H(\pi_n)$ and does not contain $A_{i-1}(A_{i+1})$. As A_i is in the interior of $H(\pi_n)$ $A_i\notin Q_{n-1}$. If $A_{i-1}A_i(A_iA_{i+1})$ were infinite then, by 2.2, $A_{i-1}(A_{i+1})$ would be in Q_{n-1}. This contradiction proves the result. An immediate consequence is

2.4. *A j-point of a polygon* π_n *is on the boundary of* $H(\pi_n)$,

2.5. *If, for the side* A_iA_{i+1} *of a polygon* π_n: $A_1A_2\cdots A_r$, A_i *is on the boundary of* $H(\pi_n)$ *but* A_{i+1} *is in its interior then* A_i *is a j-point.*

Proof. The result is clear for $n=1$ as the two boundary points of $H(\pi_1)$ are the endpoints of the single infinite side of π_1. If, for $n>1$, A_i is not a j-point then both sides $A_{i-1}A_i$, A_iA_{i+1} are finite. Then, as A_i is on the boundary $H(\pi_n)$, it follows from 1.9 that a projection π_{n-1}: $A'_1A'_2\cdots A'_{i-1}A'_{i+1}\cdots A'_r$ of π_n from A_i exists in an affine space $L_{n-1}\backslash T_{n-2}$ for which $A'_{i-1}A'_{i+1}$ is infinite. The j-point A'_{i+1} of π_{n-1} is, by 2.4, on the boundary of $H(\pi_{n-1})$. It follows, then, from 1.9 (2) that A_{i+1} is on the boundary of $H(\pi_n)$ contrary to the hypothesis. Hence A_i is a j-point and the result is established.

2.6. *If* π_{n-1}: $A'_1A'_2 \cdots A'_{k-1}A'_{k+1} \cdots A'_r$ *is a normal projection of a polygon* π_n: $A_1A_2 \cdots A_r$, $n > 1$, *from a j-point* A_k *then every maximal arc* α *of* π_n *of length h is projected into a maximal arc* α' *of* π_{n-1} of *length* $h-1$ *or* h *according as* $A_k \in \alpha$ *or* $A_k \notin \alpha$.

Conversely every maximal arc α' *of* π_{n-1} *is the projection of a uniquely determined maximal arc of* π_n.

Proof. If α: $A_iA_{i+1} \cdots A_{i+h}$ be a maximal arc of π_n then, by the definition 2.1, all the sides of the arc $\bar{\alpha}$: $A_{i-1}A_i \cdots A_{i+h}A_{i+h+1}$ are infinite except $A_{i-1}A_i$, $A_{i+h}A_{i+h+1}$ both of which are finite. If $A_k \notin \bar{\alpha}$ then, by 1.9, every side of the projection $A'_{i-1}A'_i \cdots A'_{i+h+1}$ is infinite except $A'_{i-1}A'_i$, $A'_{i+h}A'_{i+h+1}$ both of which are finite. Thus the projection α': $A'_iA'_{i+1} \cdots A'_{i+h}$ is a maximal arc of π_{n-1}. If $A_k = A_{i-1}(A_{i+h+1})$ then $A_{i-2}A_{i-1}(A_{i+h+1}A_{i+h+2})$ is infinite as A_k is a j-point. In this case, again by 1.9, only the sides $A'_{i-2}A'_i$, $A'_{i+h}A'_{i+h+1}(A'_{i-1}A'_i$, $A'_{i+h}A'_{i+h+2})$ of the arc $A'_{i-2}A'_i \cdots A'_{i+h+1}(A'_{i-1}A'_i \cdots A'_{i+h}A'_{i+h+2})$ are finite. Thus, as before, the projection $A'_iA'_{i+1} \cdots A'_{i+h}$ is maximal. If $A_k \in \alpha$ then the length of α is positive as it contains at least one infinite side. The projection $A'_{i-1} \cdots A'_{i+h+1}$ of $A_{i-1}A_i \cdots A_{i+h+1}$ is an arc of length $h+1$ of which only the first and last sides are finite. Hence the projection of α in this case is a maximal arc of length $h-1$. The proof of the result is now complete.

To prove the converse let α': $A'_u \cdots A'_v$ be a maximal arc of π_{n-1} and $A_uA_{u+1} \cdots A_v$ the corresponding arc of π_n. As A_k is a j-point it follows from 1.9 that every side of this latter arc is infinite. Consequently it is included in a maximal arc α: $A_jA_{j+1} \cdots A_{j+m}$ of π_n which is unique as a maximal arc is determined by any one of its vertices. As proved in the previous paragraph the projection of α is a maximal arc of π_{n-1}. As this projection includes α' which is itself maximal it must coincide with α'. The converse is thus established and the proof is complete.

2.7. *A hyperplane* Q_{n-1}: $[A_{i_1}, A_{i_2}, \cdots, A_{i_n}]$ *which supports the convex hull* $H(\pi_n)$ *of a polygon* π_n: $A_1A_2 \cdots A_r$ *must contain at least h vertices of every maximal arc of* π_n *of length h.*

Proof. If a maximal arc has length 1 the theorem coincides with 2.2 and so is already proved. We assume, then, that it is true for all maximal arcs of lengh $h-1$, $h > 1$, and proceed by induction. We can assume $n > 1$ as π_1 contains only 1 infinite side. As the side A_iA_{i+1} of a maximal arc α: $A_iA_{i+1} \cdots A_{i+h}$, $h > 1$, is infinite, at least one of the two j-points A_i, A_{i+1} is within Q_{n-1} by 2.2. If A_k is such a j-point the subscripts can be adjusted so that Q_{n-1} is

$[A_k, A_{i_2}, \cdots, A_{i_k}]$. As $n>1$ a normal projection π_{n-1}: $A'_1A'_2\cdots A'_{k-1}A'_{k+1}\cdots A'_r$ of π_n from A_k exists, by 1.9, so that if $H(\pi_{n-1})$ is defined in an affine space $L_{n-1}\backslash T_{n-2}$, the projection Q_{n-2}: $[A'_{i_2}, \cdots, A'_{i_n}]$ of Q_{n-1} supports $H(\pi_{n-1})$. By 2.6 α is projected into a maximal arc α' of π_{n-1} of length $h-1$. By applying the induction assumption to π_{n-1}, α' and Q_{n-2} it follows that Q_{n-2} contains at least $h-1$ vertices of α'. As these are projections of vertices of α within Q_{n-1} if follows that Q_{n-1} contains at least h vertices of α as $A_k \in Q_{n-1}$. The result now follows by induction.

2.8. *A hyperplane Q_{n-1}, spanned by n vertices of a polygon π_n: $A_1A_2\cdots A_r$, supports $H(\pi_n)$ if and only if*

(1) *for every arc α: $A_iA_{i+1}\cdots A_{i+h}$, $h>0$, within Q_{n-1} with $A_{i-1}, A_{i+h+1} \notin Q_{n-1}$, $p+h$ is odd where p is the number of infinite sides of the arc $A_{i-1}A_i\cdots A_{i+h+1}$,* and

(2) *Q_{n-1} contains at least h vertices of every maximal arc of π_n of length h.*

Proof. To check the result for $n=1$ let A_iA_{i+1} be the infinite side of π_1. This side is the only maximal arc of π_1 of positive length and $H(\pi_1)$ is the finite complement of A_iA_{i+1} in the projective line. Q_0 supports $H(\pi_1)$ if and only if $Q_0 = A_i$ or $Q_0 = A_{i+1}$. This is true if and only if Q_0 satisfies (2). If Q_0 satisfies (2) it also satisfies (1). Thus the result is true for $n=1$. We assume it true for polygons π_{n-1}, $n>1$, and proceed by induction.

We first assume that Q_{n-1} satisfies (1) and (2) and show that it supports $H(\pi_n)$. Let A_k be a vertex of π_n within Q_{n-1} which is a j-point if Q_{n-1} contains such a point and otherwise arbitrary. It follows from (2) that if A_k is not a j-point that π_n has only finite sides. By 2.4 and 2.5 A_k is on the boundary of $H(\pi_n)$. The subscripts may be adjusted so that Q_{n-1} may be written as $[A_k, A_{i_2}, \cdots, A_{i_n}]$. Let π_{n-1}: $A'_1A'_2\cdots A'_{k-1}A'_{k+1}\cdots A'_r$ be a normal projection of π_n from A_k where, following 1.10, $H(\pi_{n-1})$ is defined in an affine space $L_{n-1}\backslash T_{n-2}$. Let Q_{n-2} be the projection $[A'_{i_2}, A'_{i_3}, \cdots, A'_{i_n}]$ of Q_{n-1} from A_k.

To show that Q_{n-2} satisfies (1) for π_{n-1} let α': $A'_u\cdots A'_v$ be an arc of π_{n-1} within Q_{n-2} chosen so that, for the arc $\bar\alpha'$: $A'_tA'_u\cdots A'_vA'_w$ of π_{n-1}, $A'_t, A'_w \notin Q_{n-2}$. $\bar\alpha'$, α' are the projections from A_k of the arcs $\bar\alpha$: $A_tA_{t+1}\cdots A_w$, α: $A_{t+1}A_{t+2}\cdots A_{w-1}$ of π_n, respectively. $\alpha \subseteqq Q_{n-1}$ as $\alpha' \subseteqq Q_{n-2}$ while $A_t, A_w \notin Q_{n-1}$ as $A'_t, A'_w \notin Q_{n-2}$. If p be the number of infinite sides of $\bar\alpha$ and h the length of α then, as Q_{n-1} satisfies (1), $p+h$ is odd. If $A_k \notin \bar\alpha$, then, by 1.9, α' has length h while a side of $\bar\alpha'$ is finite if and only if it is a projection of a finite side of $\bar\alpha$. As $\bar\alpha'$ has, then, p infinite sides Q_{n-2} satisfies (1) for π_{n-1}. If $A_k \in \bar\alpha$, then $A_k \neq A_t$, $A_k \neq A_w$ as $A_t, A_w \notin Q_{n-1}$. By 1.9 $\bar\alpha'$ has $p-1$ or $p+1$

infinite sides according as A_k is or is not a j-point, while α' has length $h-1$. As $(p \pm 1)+h-1$ is odd, Q_{n-2} satisfies (1) in this case also.

To check that Q_{n-2} satisfies (2) let β' be a maximal arc of length $m, m>0$ of π_{n-1}. If A_k is not a j-point β' must, by 1.9, be the single infinite side $A'_{k-1}A'_{k+1}$ for, by the choice of A_k, π_n has only finite sides. As Q_{n-1} satisfies (1) at least one of A_{k-1}, A_{k+1} is within Q_{n-1} and so at least one of A'_{k-1}, A'_{k+1} is within Q_{n-2}. Hence Q_{n-2} satisfies (2). If A_k is a j-point then, by 2.6, β' is the projection of a maximal arc β of π_n of length $m+1$ or m according as $A_k \in \beta$ or $A_k \notin \beta$. As Q_{n-1} satisfies (2) this implies that Q_{n-2} contains at least m vertices of β'. Hence Q_{n-2} satisfies (2) for π_{n-1} in all cases.

If we now apply the induction assumption to Q_{n-2} and π_{n-1} it follows that Q_{n-2} supports $H(\pi_{n-1})$. Consequently Q_{n-1} supports $H(\pi_n)$ by 1.9.

If, conversely, Q_{n-1} supports $H(\pi_n)$ then, by 2.7, Q_{n-1} must satisfy (2). This implies that Q_{n-1} contains a j-point A_k unless π_n has no infinite sides in which case A_k is chosen to be an arbitrary vertex of π_n in Q_{n-1}. As above let π_{n-1}: $A'_1A'_2 \cdots A'_{k-1}A'_{k+1} \cdots A'_r$ be a normal projection of π_n from A_k and Q_{n-2}: $[A'_{i_2}, A'_{i_3}, \cdots, A'_{i_n}]$ that of Q_{n-1}. By 1.9 Q_{n-2} supports $H(\pi_{n-1})$ and so by the induction assumption Q_{n-2} satisfies (1) for π_{n-1}. We retain the previous notation and let $\bar{\alpha}$: $A_tA_{t+1}\cdots A_w$ be an arc of π_n for which the subarc α: $A_{t+1}A_{t+2}\cdots A_{w-1}$ is included in Q_{n-1} but for which $A_t, A_w \notin Q_{n-1}$.

It remains to show that α satisfies (1). Suppose first that α has exactly one vertex A_{t+1} which is also the point A_k. If A_tA_{t+1}, $A_{t+1}A_{t+2}$ were either both finite or both infinite then Q_{n-1} would contain a tangent $L(A_k)$. It would then follow from 1.6 that Q_{n-2} would contain a point of the side $A'_{k-1}A'_{k+1}$ as well as $n-1$ vertices of π_{n-1}. As this is impossible because of the order of π_{n-1}, exactly one of A_tA_{t+1}, $A_{t+1}A_{t+2}$ is finite. Hence (1) is satisfied if α is the single vertex A_k. If α is not the single vertex A_k then let α': $A'_u \cdots A'_v$, $\bar{\alpha}'$: $A'_tA'_u \cdots A'_vA'_w$ be the projections of α and $\bar{\alpha}$ from A_k. By reversing the previous argument it follows that if (1) is valid for α' and $\bar{\alpha}'$ it is also valid for α and $\bar{\alpha}$. This implies that α satisfies (1) and so that Q_{n-1} satisfies (1) for all arcs α which it contains.

The proof can now be completed by induction.

2.9. *If a polygon* π_n: $A_1A_2 \cdots A_r$ *has* n *infinite sides then,*

(1) *each vertex on the boundary of* $H(\pi_n)$ *is a* j-*point and*

(2) *the necessary and sufficient condition that a hyperplane* Q_{n-1} *spanned by* n *vertices of* π_n *support* $H(\pi_n)$ *is that it contain exactly* h *vertices of each maximal arc of* π_n *of length* h.

Proof. Let $\alpha_1, \alpha_2, \cdots, \alpha_p$ be the maximal arcs of π_n of positive length and $h_1, h_2, \cdots, h_p$ their respective lengths. As π_n has n infinite sides $h_1 + h_2 + \cdots + h_p = n$. By 2.7 Q_{n-1} contains at least h_i vertices of each arc $\alpha_i, 1 \leqq i \leqq p$. Hence as π_n has order n Q_{n-1} contains exactly h_i vertices of each of these arcs and each vertex of π_n within it is a j-point.

Therefore, to complete the proof of (2) it remains to show that any hyperplane Q_{n-1} which contains h_i vertices of each $\alpha_i, 1 \leqq i \leqq p$, supports $H(\pi_n)$. Q_{n-1} satisfies 2.8 (2). To show that it also satisfies 2.8 (1) let $\alpha: A_i A_{i+1} \cdots A_{i+h}$ be an arc of π_n within Q_{n-1} for which $A_{i-1}, A_{i+h+1} \notin Q_{n-1}$. If $\alpha \subseteqq \alpha_i$ then $h = h_i - 1$ and exactly one of $A_{i-1}A_i, A_{i+h}A_{i+h+1}$ is within α_i. The side not within α_i is finite. Hence $A_{i-1}A_i \cdots A_{i+h+1}$ contains exactly $h+1$ infinite sides. As the number $(h+1)+h$ is odd α satisfies 2.8 (1). If α is not a subarc of a maximal arc then the vertices of α are included in two consecutive maximal arcs as α cannot contain all the vertices of any maximal arc. Moreover the two maximal arcs which contain α must be separated by a single finite side because all the vertices of α are j-points. Hence both the sides $A_{i-1}A_i, A_{i+h}A_{i+h+1}$ are infinite. As before $A_{i-1}A_i \cdots A_{i+h+1}$ contains exactly $h+1$ infinite sides. Thus α satisfies 2.8 (1) in all cases and so by 2.8 Q_{n-1} supports $H(\pi_n)$. The proof is now complete.

2.10 *If a vertex of a polygon π_n is an interior point of $H(\pi_n)$ then π_n has n infinite sides.*

Proof. If $n = 1$ there is nothing to prove. To prove the result it is sufficient to show that, if a polygon π_n has less than n infinite sides, each of its vertices is on the boundary of $H(\pi_n)$. This follows from 2.5 if every side of π_n is finite. In particular this establishes the result for $n=2$. We assume it true for polygons π_{n-1}, $n>2$, and proceed by induction. As we may assume that π_n has at least one infinite side, it has at least one j-point A_k. A_k is, by 2.4, on the boundary of $H(\pi_n)$. Therefore a normal projection π_{n-1}: $A'_1A'_2 \cdots A'_{k-1}A'_{k+1} \cdots A'_r$ of π_n from A_k exists following 1.10. By 1.9 π_{n-1} has at most $n-3$ infinite sides. Consequently, by the induction assumption, every vertex of π_{n-1} is on the boundary of $H(\pi_{n-1})$. Hence, by 1.9, every vertex of π_n is within a supporting hyperplane of $H(\pi_n)$ which contains A_k and so is on the boundary of $H(\pi_n)$. The result now follows by induction.

3. Equivalence.

3.1. Two sets U: $\{A_1, A_2, \cdots, A_r\}$, V: $\{B_1, B_2, \cdots, B_r\}$ of r points,

$r \geqq n+1$, in general position in the affine subspaces R_n, $\bar{R}_n$ of L_n, respectively, are defined to be *equivalent* if a $1-1$ mapping $A_i \to f(A_i)$, $1 \leqq i \leqq r$, of the points of U onto those of V exists with the property that each hyperplane $[A_{i_1}, A_{i_2}, \cdots, A_{i_n}]$ supports the convex hull $H(U)$ in R_n if and only if $[f(A_{i_1}), f(A_{i_2}), \cdots, f(A_{i_n})]$ supports the convex hull $H(V)$ in $\bar{R}_n$.

3.2. *Hypothesis*: $A_i \to f(A_i)$, $1 \leqq i \leqq r$, *is an equivalence mapping for the sets* U: $\{A_i, A_2, \cdots, A_r\}$, V: $\{B_1, B_2, \cdots, B_r\}$ *in the affine spaces* R_n, $\bar{R}_n$, *respectively*, $n > 1$.

$T_{n-1}(\bar{T}_{n-1})$ *is a hyperplane of* $R_n(\bar{R}_n)$ *which supports* $H(U)(H(V))$ *for which* $T_{n-1} \cap U = A_k(\bar{T}_{n-1} \cap V = f(A_k))$.

L_{n-1}, T_{n-2}, U', A'_i, $i \neq k$, $(\bar{L}_{n-1}$, $\bar{T}_{n-2}$, V', $f'(A_i)$, $i \neq k$,) *are the projections of* L_n, T_{n-1}, U, $A_i(L_n$, $\bar{T}_{n-1}$, V, $f(A_i))$ *from* $A_k(f(A_k))$.

Conclusion. $A: \to f(A'_i)$, $i \neq k$, *is an equivalence mapping of* U' *onto* V' *where the convex hulls* $H(U')$, $H(V')$ *are defined in the affine spaces* $L_{n-1} \backslash T_{n-2}$, $\bar{L}_{n-1} \backslash \bar{T}_{n-2}$, *respectively*.

Proof. The points of $U'(V')$ are in general position in $L_{n-1}(\bar{L}_{n-1})$ as those of $U(V)$ are in general position in $L_n(L_n)$.

Let Q_{n-2} be a hyperplane $[A'_{i_2}, A'_{i_3}, \cdots, A'_{i_n}]$ of $L_{n-1} \backslash T_{n-2}$ which does not support $H(U')$. We show that the corresponding hyperplane $\bar{Q}_{n-2}$: $[f'(A_{i_2}), f'(A_{i_3}), \cdots, f'(A_{i_n})]$ does not support $H(V')$. A segment $A'_pA'_q$ exists in $L_{n-1} \backslash T_{n-2}$ for which A'_p, $A'_q \notin Q_{n-2}$ and $Q_{n-2} \cap A'_pA'_q \neq \varnothing$. If A_pA_q be the segment of L_n the projection of which from A_k is $A'_pA'_q$ then $T_{n-1} \cap A_pA_q = \varnothing$ as $A'_pA'_q \subseteqq L_{n-1} \backslash T_{n-2}$. Hence $A_pA_q \subseteqq R_n$ as T_{n-1} supports $H(U)$. As Q_{n-2} is the projection of the hyperplane Q_{n-1}: $[A_k, A_{i_2}, \cdots, A_{i_n}]$ from A_k, $Q_{n-1} \cap A_pA_q \neq \varnothing$. Thus Q_{n-1} does not support $H(U)$ and so, by the definition 3.1, the corresponding hyperplane $\bar{Q}_{n-1}$: $[f(A_k)$, $f(A_{i_2}), \cdots, f(A_{i_n})]$ does not support $H(V)$. This implies that B_h, B_k exist in V, B_h, $B_k \notin \bar{Q}_{n-1}$, so that $\bar{Q}_{n-1} \cap B_hB_k \neq \varnothing$. As $\bar{T}_{n-1}$ supports $H(V)$, $\bar{T}_{n-1} \cap B_hB_k = \varnothing$. Consequently $\bar{T}_{n-2} \cap B'_hB'_k = \varnothing$ and $B'_hB'_k \subseteqq \bar{L}_{n-1} \backslash \bar{T}_{n-2}$. As $\bar{Q}_{n-2}$ is the projection of $\bar{Q}_{n-1}$ from $f(A_k)$, $\bar{Q}_{n-2} \cap B'_hB'_k \neq \varnothing$ and so $\bar{Q}_{n-2}$ does not support $H(V')$.

If follows from the symmetry of the equivalence relation that if $\bar{Q}_{n-2}$ does not support $H(V')$ that Q_{n-2} does not support $H(U')$. Hence $A'_i \to f'(A_i)$, $i \neq k$, is an equivalence mapping for the sets U', V' and the proof is complete.

3.3. If U and V are the vertex sets of two polygons π_n: $A_1A_2 \cdots A_r$, σ_n: $B_1B_2 \cdots B_r$, defined in spaces $L_n \backslash H_{n-1}$, $L_n \backslash \bar{H}_{n-1}$, respectively, we say the polygons are equivalent if U and V are equivalent and write $\pi_n \sim \sigma_n$.

A vertex A_k of the polygon π_n on the boundary of $H(\pi_n)$ is in a hyperplane $[A_k, A_{i_2}, \cdots, A_{i_n}]$ which supports $H(\pi_n)$. Hence if $\pi_n \sim$

σ_n $[f(A_k), f(A_{i_2}), \cdots, f(A_{i_n})]$ supports $H(\sigma_n)$ and $f(A_k)$ is on the boundary of $H(\sigma_n)$. Therefore if A_k is a boundary point of $H(\pi_n)$ and $n > 1$ the hypothesis of 3.2 is satisfied as the vertices of π_n, σ_n are in general position. Let $f(A_k) = B_e$. If π_{n-1}: $A'_1 A'_2 \cdots A'_{k-1} A'_{k+1} \cdots A'_r$, σ_{n-1}: $B'_1 B'_2 \cdots B'_{e-1} B'_{e+1} \cdots B'_r$ are the normal projections of π_n, σ_n from A_k, B_e, respectively for which $H(\pi_{n-1})$, $H(\sigma_{n-1})$ are defined in $L_{n-1} \backslash T_{n-2}$, $\bar{L}_{n-1} \backslash \bar{T}_{n-2}$ then, by 3.2, $A'_i \rightarrow f'(A_i)$, $i \neq k$, is an equivalence mapping for the vertex sets of π_{n-1} and σ_{n-1}. In short, if $\pi_n \sim \sigma_n$, then projections π_{n-1}, σ_{n-1} exist for which $\pi_{n-1} \sim \sigma_{n-1}$.

3.4. *If π_n, σ_n are two polygons each with r sides, $r \geqq n+3$, for which $\pi_n \sim \sigma_n$, then* (1) *both polygons have the same number of infinite sides and* (2) *an equivalence mapping maps a j-point of one onto a j-point of the other.*

Proof. If $n = 1$ (1) is trivial as π_1 and σ_1 both have one infinite side. An equivalence mapping maps the endpoints of the segment $H(\pi_1)$ into the endpoints of the segment $H(\sigma_1)$ following the Definition 3.1. But these endpoints are the j-points as the convex hull is the complement of the infinite side in the projective line. Thus (2) is satisfied by an equivalence mapping if $n = 1$. We now assume the result to be true for equivalent polygons π_{n-1}, σ_{n-1}, $n > 1$, and proceed by induction.

One of the two polygons π_n, σ_n say σ_n, has at least as many infinite sides as the other. If π_n has at least one infinite side let A_k be a j-point of π_n and otherwise an arbitrary vertex. By 2.4 and 2.5 A_k is on the boundary of $H(\pi_n)$. If $B_e = f(A_k)$ let π_{n-1}: $A'_1 A'_2 \cdots A'_{k-1} A'_{k+1} \cdots A'_r$, σ_{n-1}: $B'_1 B'_2 \cdots B'_{e-1} B'_{e+1} \cdots B'_r$ be normal projections of π_n, σ_n from A_k, B_e, respectively. Then, following 3.3, $A'_i \rightarrow f'(A_i)$, $i \neq k$, is an equivalence mapping for π_{n-1} and σ_{n-1}. The induction assumption may be applied to these two polygons as they both have $r-1$ vertices and $r - 1 \geqq (n-1) + 3$. Consequently they have the same number of infinite sides. Let q be this number. Suppose first that A_k is a j-point. It follows, then, from 1.9 that π_n has $q + 1$ infinite sides and that σ_n has $q + 1$ or $q - 1$ infinite sides according as B_e is or is not a j-point. As σ_n is assumed to have at least as many infinite sides as π_n it follows that B_e is a j-point and that π_n, σ_n have the same number of infinite sides. Because of the last assertion it follows, by interchanging π_n and σ_n in the above argument, that every j-point of σ_n is the map of a j-point of π_n. Thus the result is true if π_n has at least one j-point. In the remaining case π_n has no infinite sides. It follows from 1.9 that π_{n-1} has exactly one infinite side and then that σ_n has either 0 or 2 infinite sides. As $r \geqq n + 3$ and $n \geqq 2$, σ_n has at least 5 vertices.

Consequently a vertex A_k exists so that $f(A_k)$ is not a j-point of σ_n. It now follows that σ_{n-1} has one infinite side more than σ_n. This means that σ_n has no infinite sides and consequently no j-points. Thus (1) and (2) hold for π_n and σ_n.

The result now follows by induction.

3.5. *An equivalence mapping for two polygons* π_n: $A_1A_2 \cdots A_r$, σ_n: $B_1B_2 \cdots B_r$, $r \geqq n+3$, *maps the set of the vertices of a maximal arc of one polygon onto the set of the vertices of a maximal arc of the other polygon.*

Proof. It follows from the definition 2.1 that the maximal arcs of a polygon of length 0 are those vertices of the polygon which are not j-points. The result for the maximal arcs of length 0 is now clear as, by 3.4, the vertices of π_n which are not j-points are mapped into the vertices of σ_n which are not j-points. If one of the polygons, and consequently the other, has no infinite sides the proof is complete.

If π_n has exactly one maximal arc α: $A_iA_{i+1} \cdots A_{i+h}$ of positive length h then π_n has h infinite sides and $h+1$ j-points. By 3.4 the equivalent polygon σ_n also has h infinite sides and $h+1$ j-points. As the number of j-points of any polygon is the number of maximal arcs of positive length plus the number of infinite sides it follows that σ_n has exactly one maximal arc of length h. As j-points are mapped into j-points the result follows. In particular this proves the result for $n=1$. We assume it true for polygons of order $n-1$, $n>1$, and proceed by induction.

We may assume that π_n contains at least two maximal arcs of positive length. If α: $A_iA_{i+1} \cdots A_{i+h}$ be one of these let A_k, A_{k+1} be two vertices from a second maximal arc. Let $A_i \rightarrow f(A_i)$ be an equivalence mapping for the polygons π_n and σ_n. If $B_e = f(A_k)$ then, following 3.3, normal projections π_{n-1}: $A'_1A'_2 \cdots A'_{k-1}A'_{k+1} \cdots A'_r$, σ_{n-1}: $B'_1B'_2 \cdots B'_{e-1}B'_{e+1} \cdots B'_r$ of π_n, σ_n from A_k, B_e, respectively, exist for which $A'_i \rightarrow f'(A_i)$, $i \neq k$, is a equivalence mapping for π_{n-1} and σ_{n-1}. By 2.6 the projection α': $A'_iA'_{i+1} \cdots A'_{i+h}$ of α from A_k is a maximal arc of π_{n-1} as A_k is a j-point and $A_k \notin \alpha$. As π_{n-1} has $r-1$ vertices and $r-1 \geqq (n-1)+3$, the equivalent polygons π_{n-1}, σ_{n-1} satisfy the hypothesis. It follows, then, from the induction assumption that $f'(A_i), f'(A_{i+1}), \cdots, f'(A_{i+h})$ are the vertices of a maximal arc of σ_{n-1}. By 2.6 either $f(A_i), f(A_{i+1}), \cdots, f(A_{i+h})$ are the vertices of a maximal arc of σ_n, in which case the result is proved or $f(A_k), f(A_i), \cdots, f(A_{i+h})$ are the vertices of a maximal arc of σ_n. If the latter case occurs the procedure may be repeated with the use of A_{k+1} instead of A_k.

In this case $\{f(A_{k+1})f(A_i), \cdots, (A_{i+h})\}$ would be the vertex set of a maximal arc of σ_n. As $A_i \to f(A_i)$ is a $1-1$ mapping the two sets $\{f(A_k), f(A_i), \cdots, f(A_{i+h})\}$, $\{f(A_{k+1}), f(A_i), \cdots, f(A_{i+h})\}$ are distinct. This is impossible as any single vertex within a maximal arc determines it uniquely. Hence $\{f(A_i), f(A_{i+1}), \cdots, f(A_{i+h})\}$ is the set of the vertices of a maximal arc of σ_n and the result is clear.

The proof now follows by induction.

3.6. *If a polygon* π_n: $A_1A_2 \cdots A_r$, $r \geqq n+3$, *has* n *infinite sides then a polygon* σ_n: $B_1B_2 \cdots B_r$ *is equivalent to* π_n *if and only if, for each* h, $0 \leqq h \leqq n$, π_n *and* σ_n *both have the same number of maximal arcs of length* h.

Proof. If $\pi_n \sim \sigma_n$ then, by 3.5, each of the polygons have the same number of maximal arcs of length h, $0 \leqq h \leqq n$.

If, conversely, π_n and σ_n satisfy this condition we construct a mapping of the vertices of π_n onto those of σ_n as follows. As π_n and σ_n have the same number of maximal arcs of length h, $0 \leqq h \leqq n$, an arbitrary $1-1$ correspondence can be defined between the maximal arcs of π_n of length h and those of σ_n of length h for each h, $0 \leqq h \leqq n$. After this has been done we define a $1-1$ correspondence $A_i \to f(A_i)$ which maps the vertices of each maximal arc of length h onto the set of vertices of the corresponding maximal arc of σ_n of length h.

To check that the mapping $A_i \to f(A_i)$ is an equivalence mapping let Q_{n-1}: $[A_{i_1}, A_{i_2} \cdots A_{i_n}]$ be a hyperplane which supports $H(\pi_n)$. By 2.9 (2) Q_{n-1} contains h vertices of each maximal arc of π_n of length h. By the construction of the mapping $\bar{Q}_{n-1}$: $[f(A_{i_1}), f(A_{i_2}), \cdots, f(A_{i_n})]$ contains h vertices of every maximal arc of σ_n of length h. Hence, by 2.9 (2), $\bar{Q}_{n-1}$ supports $H(\sigma_n)$. Hence $A_i \to f(A_i)$ is an equivalence mapping and the proof is complete.

3.7. *Hypothesis:* $A_i \to f(A_i)$, $1 \leqq i \leqq r$, *is an equivalence mapping for the polygons* π_n: $A_1A_2 \cdots A_r$, σ_n $B_1B_2 \cdots B_r$, $r \geqq n+3$, *both of which have less than* n *infinite sides.*

$\alpha_1, \alpha_2, \cdots, \alpha_s$ *are the maximal arcs of* π_n *arranged in the order in which they occur on* π_n.

$\{f(\alpha_j)\}$ *is the set of vertices* $f(A_i)$ *of* σ_n *for which* A_i *is a vertex of* α_j, $1 \leqq j \leqq s$.

Conclusion: The sets $\{f(\alpha_j)\}$ *occur in the order*

$$\{f(\alpha_1)\}, \{f(\alpha_2)\}, \cdots, \{f(\alpha_s)\}$$

on σ_n.

Proof. As π_n has less than n infinite sides $n > 1$. Let A_k be a j-point of π_n if π_n has at least one infinite side but otherwise an arbitrary vertex of π_n. If follows from 3.4 that $B_e = f(A_k)$ is a j-point of σ_n if and only if σ_n has at least one infinite side. Let π_{n-1}: $A'_1A'_2 \cdots A'_{k-1}A'_{k+1} \cdots A'_r$, σ_{n-1}: $B'_1B'_2 \cdots B'_{e-1}B'_{e+1} \cdots B'_r$ be normal projections of π_n, σ_n from A_k, B_e, respectively. Following 3.3 $A'_i \rightarrow f'(A_i)$, $i \neq k$, is an equivalence mapping for π_{n-1} and σ_{n-1}.

We consider the case for which π_n has no infinite sides. By 3.4 σ_n also has no infinite sides. By 1.9 $A'_{k-1}A'_{k+1}(B'_{e-1}B'_{e+1})$ is the only infinite side of $\pi_{n-1}(\sigma_{n-1})$. By 3.4 $f'(A_{k-1}), f'(A_{k+1})$ are the only j-points of σ_{n-1} and so these must be the vertices B'_{e-1}, B'_{e+1}. This implies that $f'(A_k), f'(A_{k+1})$ must be consecutive vertices of σ_{n-1}. As an equivalence mapping is a $1-1$ mapping this implies that as A_k runs monotonously through consecutive vertices of π_n that $f(A_k)$ runs monotonously through consecutive vertices of σ_n. By the definition 2.1 each maximal arc of a polygon with no infinite sides consists of a single vertex. Thus $\alpha_1, \alpha_2, \cdots, \alpha_s$ are consecutive vertices $A_i, A_{i+1}, \cdots, A_{i+r-1}$. Hence $\{f(\alpha_1)\}, \{f(\alpha_2)\}, \cdots, \{f(\alpha_s)\}$ either is a sequence $\{B_j\}, \{B_{j+1}\}, \cdots, \{B_{j+r-1}\}$ or a sequence $\{B_j\}\{B_{j-1}\}, \cdots, \{B_{j-r+1}\}$. This proves the result if π_n has no infinite sides. In particular the proof is complete if $n = 2$. We assume it to be true for polygons π_{n-1}, σ_{n-1}, $n > 2$, and proceed by induction.

In the case which remains π_n has at least one infinite side. Consequently A_k is a j-point. Therefore the maximal arc which contains A_k has at least two vertices. Hence we may choose vertices A_{i_j}, $A_{i_j} \in \alpha_j$, $1 \leqq j \leqq s$, $A_{i_i} \neq A_k$. By 3.5 each set $\{f(\alpha_j)\}$ consists of the vertices of a maximal arc of σ_n. To show that these sets are ordered on σ_n it is therefore sufficient to show that $f(A_{i_1}), f(A_{i_2}), \cdots, f(A_{i_s})$ are ordered on σ_n. As $f(A_{i_j}) \neq f(A_k) = B_e$, $1 \leqq j \leqq s$, to prove this result it is sufficient to show that the projections $f'(A_{i_1}), f'(A_{i_2}), \cdots, f'(A_{i_s})$ are ordered on σ_{n-1} as the order of the vertices of σ_{n-1} is that of the order of the corresponding vertices of σ_n.

To do this we consider the polygons π_{n-1}, σ_{n-1}. As A_k is a j-point it follows from 2.6 that the maximal arcs of π_{n-1} are the projections α'_j of α_j from A_k, $1 \leqq j \leqq s$. Again, as A_k is a j-point, it follows from 1.9 that π_{n-1} has exactly one infinite side less than π_n and so has less than $n-1$ infinite sides. Hence we may apply the induction assumption to the equivalent polygons π_{n-1}, σ_{n-1}. If $\{f'(\alpha'_j)\}$ denotes the map of set of vertices of α'_j defined by the mapping $A'_i \rightarrow f'(A_i)$, $1 \leqq j \leqq s$, then the sets $\{f'(\alpha_1)\}, \{f'(\alpha'_2)\}, \cdots, \{f'(\alpha'_s)\}$ occur in this order on σ_{n-1} as $\alpha'_1, \alpha'_2, \cdots, \alpha'_s$ are ordered on π_{n-1}. Consequently

$f'(A_{i_1}), f'(A_{i_2}), \cdots, f'(A_{i_s})$ follow in order on σ_{n-1} as no two of these points are in the same set $\{f'(\alpha'_j)\}$.

The result now follows by induction.

3.8. $C(\pi_n)$ is the cycle of the cyclically ordered sequence of 0's and 1's obtained by replacing each side A_iA_{i+1} of the set of sides $A_1A_2, A_2A_3, \cdots, A_rA_1$ of a polygon π_n: $A_1A_2 \cdots A_r$ by 0 or 1 according as A_iA_{i+1} is finite or infinite. If the vertices of a polygon π_n are written in reverse order the numbers of the corresponding cycle are written in reverse order. For this reason if a cycle is obtained by writing the numbers of another cycle in the reverse order the cycles are considered to be the same.

3.9. *Two polygons* π_n: $A_1A_2 \cdots A_r$, $B_1B_2 \cdots B_r$, $r \geqq n+3$, *both of which have less than n infinite sides are equivalent if and only if* $C(\pi_n) = C(\sigma_n)$.

Proof. If $\pi_n \sim \sigma_n$ then, by 3.7, $C(\pi_n) = C(\sigma_n)$.

If $C(\pi_n) = C(\sigma_n)$ then the subscripts of π_n: $A_1A_2 \cdots A_r$, σ_n: $B_1B_2 \cdots B_r$ can be adjusted so that A_iA_{i+1} is finite if and only if B_iB_{i+1} is finite, $1 \leqq i \leqq r$. It follows, then, from 2.8 that a hyperplane $[A_{i_1}, A_{i_2}, \cdots, A_{i_n}]$ supports $H(\pi_n)$ if and only if $[B_{i_1}, B_{i_2}, \cdots, B_{i_n}]$ supports $H(\sigma_n)$. Therefore $A_i \rightarrow B_i$ is an equivalence mapping for π_n and σ_n. Thus the result is proved.

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