The Copernican principle in compact space–times

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ABSTRACT

Copernicus realized that we are not at the centre of the Universe. A universe made finite by topological identifications introduces a new Copernican consideration: while we may not be at the geometric centre of the Universe, some galaxy could be. A finite universe also picks out a preferred frame: the frame in which the universe is smallest. Although we are not likely to be at the centre of the Universe, we must live in the preferred frame (if we are at rest with respect to the cosmological expansion). We show that the preferred topological frame must also be the comoving frame in a homogeneous and isotropic cosmological space–time. Some implications of topologically identifying time are also discussed.

Key words: relativity – cosmology: theory.

1 INTRODUCTION

Copernicus asserted that the Sun rather than the Earth is at the centre of the Solar system. Cosmologists have generalized this Copernican principle into a form of cosmological principle: that our place in the Universe is not special within the set of habitable locations (Barrow & Tipler 1986). A homogeneous and isotropic Friedmann universe with simply connected topology is generally consistent with this cosmological expression of the Copernican principle, since no spatial locations are preferred. However, a topologically finite universe introduces a new ingredient to the issue of special locations. A compact manifold will possess a geometric centre.[1] However, the geometric centre is not in any sense the location of the big bang. The big bang happened everywhere simultaneously. The geometric centre is only the most symmetrically located point in a compact space; for further discussion see the reviews on topology and cosmology by Lachieze-Rey & Luminet (1995), Starkman (1998) and Levin (2002).

In addition to a geometric centre, there is also a preferred frame. To see this, it is helpful first to consider the situation in topologically identified flat space. In a finite, flat space–time, a variant of the twin paradox arises which exposes the existence of a preferred frame (Barrow & Levin 2001; Weeks, unpublished). One twin stays on Earth. The other travels at constant velocity out into space. They both believe that the other's time dilates, so which one is actually younger when the space-travelling twin returns home and they meet again? The twin paradox in an infinite Minkowski space–time is re-

solved by recognizing the asymmetry that arises because the space traveller must stop and turn around and therefore experience accelerated motion. In a finite space–time this resolution no longer exists. The space-travelling twin can remain inertial and still return home by travelling on a periodic geodesic. If they are both inertial, which one is younger when the travelling twin passes by the stay-at-home twin? The space-travelling twin is again younger. The paradox is resolved by the existence of a preferred frame which is introduced by the finite topology. It is the frame in which the universe is seen to be smallest. In this preferred coordinate system, space is finite and clocks can be synchronized. For any observer moving at constant velocity relative to this topologically preferred frame, the finite space will appear distorted, space–time points will be identified with other space–time points, and there is no way to synchronize clocks consistently (Weeks, unpublished; Peters 1983, 1986; Lucas & Hodgson 1990; Barrow & Levin 2001).

Instead of flat space–time, consider an expanding homogeneous and isotropic Friedmann–Robertson–Walker (FRW) universe. These space–times select a different preferred frame, namely the coordinate system in which observers comove with the cosmological expansion. Only the comoving observers are inertial, since any initial peculiar velocity or rotational velocity perturbation in the cosmological fluid will die away as the universe expands so long as the pressure p and density ρ of the fluid satisfy $p < \rho/3$. This suggests that the only inertial observers are comoving observers and so, if the space is both expanding and topologically compact, the preferred topological frame must be the comoving frame. This we will show in what follows.

2 COMPACT SPACE AND A PREFERRED FRAME

To demonstrate the existence of a preferred frame, we consider again the compactification of a flat Minkowski space–time. Let our

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[1] The origin can be defined as the maximum of the injectivity radius (Thurston 1997). The only exceptions to this are the homogeneous compact spaces such as the sphere S^3, the projective plane, and the hypertorus T^3. As a result of invariance under translations, any point in the manifold could equally well be the origin.
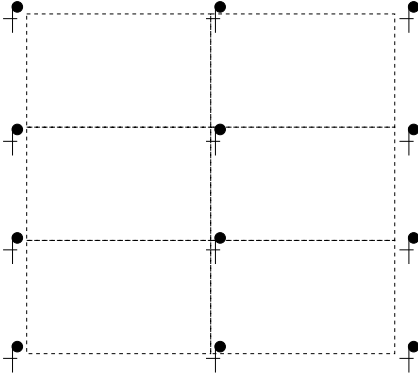
Figure 1. A two-dimensional slice through a finite volume for the preferred observer x'. A baby is born at a well-defined time in the hypertorus. There are an infinite number of repeated ghost images of the birth.

coordinate system be $x = (\tau, \boldsymbol{x})$. In a flat, static space–time,

$$ds^2 = -d\tau^2 + dx^2 = -d\tau'^2 + dx'^2, \tag{2.1}$$

where we have also introduced $x' = (\tau', \boldsymbol{x}')$, the coordinate system of an observer travelling at constant velocity relative to x, so that

$$x' = \Lambda x \tag{2.2}$$

with the Lorentz transformation

$$\Lambda = \begin{pmatrix} \cosh b & \sinh b \\ \sinh b & \cosh b \end{pmatrix}, \tag{2.3}$$

where b is a boost related to the relative velocity by $v = \tanh b$. The Lorentz transformation is a rotation in Minkowski space–time.

Let space be topologically identified into a simple hypertorus in frame x. For an observer in this frame, the topology only affects the spatial coordinates and leaves the time coordinate unchanged. We represent the topological boundary conditions by

$$x \rightarrow \gamma x, \tag{2.4}$$

where γ is an element of the discrete subgroup of the full group of isometries, $\gamma \in \Gamma$. The compact manifold is $\mathcal{M} \sim \mathcal{M}^{U}/\Gamma$, where $\mathcal{M}^{U}$ is the simply connected universal cover – in this case, flat space–time. An observer with worldline x will interpret the finite space as a simple tiling of the universal cover as in Fig. 1.

Suppose that we are in relative motion with respect to the primed frame. The topological identification appears to us as the condition

$$x' \rightarrow \Lambda \gamma x. \tag{2.5}$$

Suppressing all but one of the spatial dimensions for simplicity, we reduce the hypertorus to a circle of circumference L. The identification is effected by the simple translation $x \rightarrow \gamma x$ which is equivalent to the condition that $(\tau, x) \rightarrow (\tau, x + L)$, and equation (2.5) becomes[2]

$$\begin{pmatrix} \tau' \\ x' \end{pmatrix} \rightarrow \begin{pmatrix} \tau' + L \sinh b \\ x' + L \cosh b \end{pmatrix}. \tag{2.6}$$

One point in space and time is identified with another point in space *and time*.

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[2] We could have written γ as an explicit matrix by embedding our $(1 + 1)$-dimensional space–time in a $(2 + 1)$-dimensional space–time with the additional spatial coordinate fixed (Barrow & Levin 2001; Weeks, unpublished). For this simple flat space–time it is not necessary, although it is extremely useful when working in curved space–times.
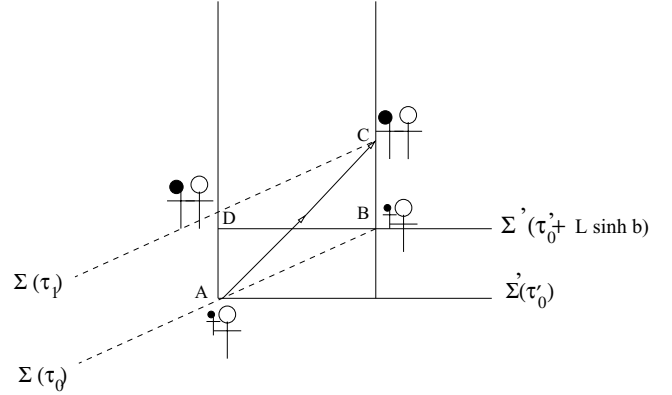

Figure 2. The space–time diagram in coordinate system x'. Σ' is the space-like hypersurface foliated by τ'. Also drawn for reference is Σ, the space-like hypersurface foliated by τ. A baby is born at space–time point A: (τ'_0, x'_0). The baby is also born at another space–time point B: $(\tau'_0 + L\sinh b, x'_0 + L\cosh b)$. A and B are topologically identified. The mother travels from A to C but can only arrive after the birth so she will never observe the paradox that the same baby was born more than once. The points C and D are topologically identified.

Notice that in the x-frame, the spatial size of the universe is L while in the x'-frame, the spatial size of the universe is $L\cosh b > L$. Of course the space–*time* interval is invariant. However, the universe appears smallest to the observer at rest with respect to the topological identification. The topology thereby selects this preferred frame. It is also the frame in which clocks can be properly synchronized.

The space–time diagram is drawn in Fig. 2. Suppose that a baby is born at space–time point (τ'_0, x'_0). The baby must also be born at $(\tau'_0 + L\sinh b, x'_0 + L\cosh b)$. That is, the baby also appears to be born at another time and place – in fact, at an infinite number of other times and places. Of course, the baby can only be born once, and all observers must agree that the baby is only born once. For instance, as shown in Fig. 2, if the mother of the child travelled near light speed from A with the intention of arriving at B to relive the birth, she would inevitably arrive late. The time it would take her to intercept the baby's worldline is $\Delta x/\tanh b = L\cosh b/\tanh b > L\sinh b$ and superluminal motion would be required to arrive in time. Still, she *is* simultaneously at space–time points A and B in Fig. 2. What time does she think it is? She might decide that it is time τ'_0 or she might decide that it is time $\tau'_0 + L\sinh b$. Regardless, she would find that it was impossible to synchronize her clocks (Weeks, unpublished; Peters 1983, 1986; Lucas & Hodgson 1990; Barrow & Levin 2001). Despite the identification of one time with another, we will never experience a unique event twice. For null and time-like observers, there is no way to travel back in time.

3 COMPACT SPACE AND THE COMOVING FRAME

Topology is of cosmological interest. We may very well live in a universe that is finite. In a finite space there is definite meaning to the question: where are we in space? We can search the cosmic background radiation (CMB) for evidence of the size, shape, and our location in space. In fact, preliminary results from the *Wilkinson Microwave Anisotropy Probe* (*WMAP*) do give hints of a finite extent to the Universe (Efstathiou 2003; Spergel et al. 2003; Tegmark & de Oliviera-Costa 2003).
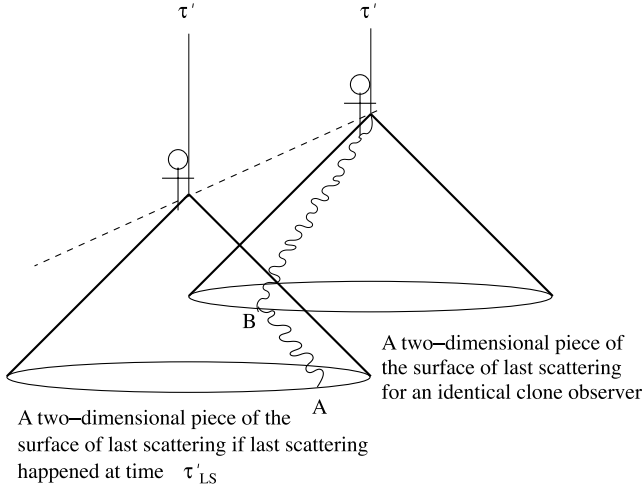
Figure 3. An observer in the x' frame looks back billions of years to the time of last scattering and assumes that last scattering happened at one well-defined time $\tau'_{\rm LS}$ only to find that this cannot be the case. Notice that the surfaces of last scattering of two clone observers do not intersect, implying no circles and no repeated images in the light that they receive. However, light emitted at A can travel to hit another point on the surface of last scattering at B, before B was emitted. Therefore the emission of light from B could not be simultaneous with the emission of light from A. Last scattering could not occur at a well-defined time in x'. This is another manifestation of the observers' inability to synchronize their clocks in this frame.

For the purposes of searching for topology in CMB maps, the topological size of the Universe would have to be smaller than or just comparable to the particle horizon to have observable consequences. Since the Universe appears to be homogeneous and isotropic within the observable horizon, research is often restricted to finite spaces which support a FRW metric over the entire space. There exists a frame in which observers at rest with respect to that frame see topological identifications of spatial sections only. Observers in motion relative to this frame will see a mixture of spatial and temporal identifications.

Notice also that if x', which is not the preferred topological frame, were also the comoving frame in an expanding universe, observers would see no patterns and no circles (Cornish, Spergel & Starkman 1998; Levin et al. 1998) in CMB maps. All directions in the sky would give a unique point on a CMB map, as illustrated in Fig. 3.

However, if the universe is expanding, then x and not x' must be the comoving frame, for if x' were the comoving frame it would follow that

$$\mathrm{d}s^2 = a(\tau')^2(-\mathrm{d}\tau'^2 + \mathrm{d}\boldsymbol{x}'^2),\tag{3.1}$$

with $a(\tau')$ the homogeneous and isotropic FRW scale factor. The scale factor would have to match the boundary condition

$$a(\tau') = a(\tau' + L \sinh b),\tag{3.2}$$

which is impossible if $a(\tau')$ evolves monotonically. By contrast, if x is the comoving frame then there are no additional constraints on $a(\tau)$ since compactification occurs on a purely space-like hypersurface. In other words, unless the comoving frame is coincident with the preferred topological frame, the local Einstein equations will not be satisfied. Still, no galaxy is truly at rest with respect to the cosmic expansion. One might wonder if any peculiar motion with respect to the comoving frame would degrade a detection of topology through patterns and circles.

This suggests that, however the Universe may have begun, if there is initially a preferred topological frame the expansion must settle down to become the comoving frame. The topology of the three large spatial dimensions, along with the topology of any extra small dimensions, is fixed in the very early Universe presumably by quantum gravitational processes. Initially the cosmos may have been very inhomogeneous and anisotropic, even chaotic. Some physical process drives the Universe to the homogeneous and isotropic state that we observe. [Interestingly, we know that the most general homogeneous anisotropic spaces which provide the modes that render the negatively curved FRW models unstable are excluded by finite topological identifications (Barrow & Kodama 2001a,b).][3] Barring inflation,[4] if the Universe really is smooth across the entire space, then one can argue heuristically that a physical smoothing mechanism should align the preferred topological frame with the comoving frame. Any smoothing that requires causal communication across the entire manifold will be restricted to communicate via modes on that manifold which are consistent with the boundary conditions. Only in the preferred topological frame will modes settle into a statistically homogeneous and isotropic form, without shearing. That frame will therefore naturally settle down to be the comoving frame.

4 COMPACT TIME

In the simplest global FRW model of the Universe it is not possible to make time compact, since that would require $a(\tau) = a(\tau + nT)$ which is inconsistent with the Einstein equations except for the recollapsing positively curved FRW metrics. Even then the condition $a(\tau) = a(\tau + nT)$ is only satisfied at one catastrophic time, namely at the singularity, so identifying time there is overkill. More generally, Geroch (1970) has shown that if space–time is globally hyperbolic, so that initial data on a space-like slice determine the entire global structure of space–time uniquely and completely, then the global topology of space–time must be $\Sigma \times R$, where Σ is the topology of any Cauchy hypersurface. If one departs from exact FRW models then the complexities of entropy production, gravitational clustering and black hole formation will inevitably make any final singularity different in structure from the initial one.[5] Therefore, when looking for topology in CMB maps, there is no obviously natural way to include compact time or Lorentz transformations.

If we suspend the restriction to expanding FRW models, we can consider compact time in a hypothetical universe. Since space and time might be on an equal footing, we could wonder if time is topologically identified at the big bang or its quantum analogue. While it is conceivable that topological conditions are set across very small spatial sections initially, it is harder to imagine how topological conditions could be fixed across large time intervals. The topology of the universe will presumably be set by quantum gravity across a Planck scale and similarly across a Planck time-span because

[3] These considerations suggest that finite topology can play some role in providing a basis for Mach's principle (King 1995).
[4] If the Universe inflates, then the topological size of the large dimensions would naturally be well beyond the observable horizon, in which case the Universe is not homogeneous and isotropic across the entire manifold.
[5] In the non-general relativistic context we are familiar with the time recurrence 'paradox' of Poincaré, but Tipler (1979, 1980) has shown that this recurrence does not occur in general relativity when gravity is attractive ($\rho + 3p > 0$). Two states of a generic positively curved universe cannot be identical or even arbitrarily close. For a shorter proof see also Galloway (1984). The initial and final singularity could be interpreted as the recurrent state but we do not expect them to be identical.

of the fundamental ambiguities in the definition and distinction of space and time in the quantum gravity era. A quantum initial state is then blown up by the expansion of the cosmos, so that three space dimensions become very large (although further space dimensions may have remained quite small). There is no known reason why only one time dimension exists (or is perceived) and string theories only pick out special dimensions of space–time, never of space and time separately, but there are persuasive anthropic reasons why (3 + 1)-dimensional space–times are needed for observers to exist (Barrow & Tipler 1986; Tegmark 1998). Would a topological cyclic time always expand so that any temporal periodicity would exceed cosmological time-scales? If entropy always increases, one might ask how the entire Universe in the future could approach a lower entropy state identical to the Universe in the past?

While it is not obvious how to generate a large compact time direction from a big bang, we can for the sake of argument assume a universe that has a compact time direction *ab initio*. The full isometry group of a (3 + 1)-dimensional Minkowski space–time includes Lorentz boosts as well as translations and rotations. There are two distinct ways in which to involve time in the compactification. First, we could identify time under a simple translation so that $\tau = \tau + nT$, where n is an integer. This is a simple closing of time into a circle. Another possibility would be to identify space–time points under boosts so that

$$x \rightarrow \Lambda x, \tag{4.1}$$

which leads to

$$\begin{pmatrix} \tau \\ x \end{pmatrix} \rightarrow \begin{pmatrix} \tau \cosh nb + x \sinh nb \\ x \sinh nb + \tau \cosh nb \end{pmatrix} \rightarrow \begin{pmatrix} \tau' \\ x' \end{pmatrix}, \tag{4.2}$$

where n is an integer. Twins would now encounter genuine factual conflicts since (τ, x) is identified with (τ', x'). They must be the same age but they have no uniquely definable ages. The result is the Misner space–time which featured prominently in discussions of the chronology projection conjecture (Misner 1967; Gott & Li 1998). Both the circular time universe and the Misner space–time support closed time-like loops.

With circular time, if the spatial sections are infinite so that space–time is topologically like a cylinder, then only observers at rest in the preferred frame will see closed time-like loops. Any observer x' moving at relative velocity v with respect to x will never be able to execute a closed time-like circuit in space–time. However, if space is identified under a translation, as with the hypertorus, then an observer could follow a closed time-like curve by executing a rational number of windings around space relative to his windings in time. Technically, this would be a set of measure zero amongst all relative windings, but the winding ratio may be necessarily rational because of the physical requirements of finite physical resolution.

One could argue that there could be no kill-your-grandmother paradox since topologically identified events are identical. If an event happened once, it would happen in exactly the same way each time the space–time point was revisited. No free will would be possible. We couldn't choose to commit the murderous act. We couldn't age. We couldn't permanently change anything. We couldn't reorganize things that started in a disordered state. We could transfer no information. And for all intents and purposes time would stand still.

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