



The correspondence between shadow and test field in a four-dimensional charged Einstein–Gauss–Bonnet black hole

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Abstract In this paper, we investigate the photon sphere, shadow radius and quasinormal modes of a four-dimensional charged Einstein–Gauss–Bonnet black hole. The perturbation of a massless scalar field in the black hole’s background is adopted. The quasinormal modes are gotten by the 6th order WKB approximation approach and shadow radius, respectively. When the value of the Gauss–Bonnet coupling constant increase, the values of the real parts of the quasinormal modes increase and those of the imaginary parts decrease. The coincidence degrees of quasinormal modes derived by the two approaches increases with the increase of the values of the Gauss–Bonnet coupling constant and multipole number. It shows the correspondence between the shadow and test field in the four-dimensional Einstein–Gauss–Bonnet–Maxwell gravity. The radii of the photon sphere and shadow increase with the decrease of the Gauss–Bonnet coupling constant.

1 Introduction

Shadows are an important feature of black holes. The research and capture of the shadows are helpful for us to understand the fundamental properties of black holes. Recently, the first image of a black hole in the center of galaxy *M87* has been captured by the Event Horizon Telescope Collaboration [1–4]. It is a dark region in a bright background, which is formed by the black hole’s accretion of photons and strong gravitational lensing effect. This work provides a strong evidence for the existence of black holes in our universe. In the aspect of theoretical research, the expressions

of shadow radii of some black holes have been gotten and the influence of the black holes’ parameters on the shapes and sizes of shadows has been found.

Quasinormal modes (QNMs), which characterize discontinuous complex frequencies, are asymptotic solutions of perturbation fields around compact objects under certain boundary conditions. Their real parts maintain oscillations and the imaginary parts determines oscillation feature (damped oscillation or undamped oscillation). As an important feature of black holes dynamics, they attract people’s attention. One of the reasons is that the dominant QNMs can be seen in the gravitational wave signals from black holes. Through the observation of the QNMs by LIGO/Virgo laboratories, astronomers and physicists found the gravitational wave signals, which indirectly verified the existence of the black holes [5–7]. Another reason involves the AdS/CFT correspondence [8–12]. It shows that QNMs of a $(D+1)$ -dimensional asymptotically anti-de Sitter spacetime are poles of the retarded Green’s function in the D -dimensional dual conformal field theory. This correspondence has been successfully applied to the research of various properties of strongly coupled quark-gluon plasmas. In addition, QNMs also play an important role in the quantization of areas of black holes. Using Bohr’s correspondence principle in the ringing frequencies, Hod quantized the area of the Schwarzschild black hole by the relationship between the real part of the QNMs and the black hole’s mass and obtained the area spacing as $4l_p^2 \ln 3$ [13]. Subsequently, Dreyer and Kunstatter quantized the Bekenstein–Hawking entropies by the loop quantum gravity and semiclassical method [14, 15], respectively. Considering the transformation between arbitrary states of a black hole, Maggiore regarded the black hole as a damped harmonic oscillator and found that its area spectrum is related to the physical frequency of the harmonic oscillator [16].

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There are various approaches to derive QNMs.¹ Different approaches produce different preciseness [18–48]. Null geodesics are useful tools to obtain QNMs. The angular velocities at the unstable null geodesics orbits of the black holes determine the real parts of the QNMs. The second derivatives of the effective potentials for radial motions are introduced to express the Lyapunov exponents in the imaginary parts. In the eikonal limit, the relation between the QNMs and null geodesics was gotten by Cardoso et al. [49]. However, the correspondence between QNMs in the eikonal limit and null geodesics does not always exist. As shown in [50, 51], this correspondence is guaranteed only for the test fields, while for the gravitational and other non-minimally coupled fields it may not be fulfilled. Shadow radii are closely related to QNMs. Through the relationship between the photon spheres and shadow radii, Jusufi expressed the real parts of the QNMs in the eikonal limit by the shadow radii [52, 53]. This shows a correspondence between the shadows and test fields in the spacetimes of the black holes. And then, he studied the influence of the perfect fluid dark matter parameter k on the QNMs by the massless scalar and electromagnetic field perturbations, and found the value of the reflecting point k_0 corresponding to the maximal values for the real parts. This work is helpful to detect indirectly the dark matter near the event horizon. Subsequently, this correspondence was verified by the 3rd order WKB approximation approach in [54, 55] and by the 13th order WKB approximation approach in [56], respectively.

In this paper, we investigate the photon sphere, shadow radius and QNMs of a four-dimensional charged Einstein–Gauss–Bonnet black hole. The QNMs are calculated by the 6th order WKB approximation approach and shadow radius, respectively. The influence of the Gauss–Bonnet coupling constant on the shadow radius and QNMs is discussed. Einstein–Gauss–Bonnet theory is a simple generalization of the Hilbert Lagrangian for high-dimensional spacetime. It was obtained by adding the Bach–Lanczos Lagrangian to the Hilbert Lagrangian. In Einstein–Gauss–Bonnet gravity, the higher-dimensional black hole solutions have been gotten. Four-dimensional black hole solutions have been a challenge until recently when the work of [57] was presented. To get a four-dimensional black hole solution, Glavan and Lin rescaled the coupling constant of the Gauss–Bonnet term, $\alpha \rightarrow \frac{\alpha}{D-4}$, and let $D \rightarrow 4$. Then they searched for the spherically symmetric vacuum solution. This four-dimensional Einstein–Gauss–Bonnet gravity theory was further improved and a consistent theory was proposed in [58]. The generalization to other black hole solutions are referred to [59–78].

The rest is organized as follows. In the next section, we investigate the photon sphere and shadow radius of the four-

dimensional charged Einstein–Gauss–Bonnet black hole. In Sect. 3, we calculate the QNMs by the 6th order WKB approximation approach and shadow radius, respectively. The effect of the Gauss–Bonnet coupling constant is discussed. The last section is devoted to our conclusions.

2 The photon sphere and shadow radius of the charged Einstein–Gauss–Bonnet black hole

The action for the Einstein–Gauss–Bonnet gravity with electromagnetic field in D -dimensional spacetime is described by

$$S = \frac{1}{16\pi} \int dx^D \sqrt{-g} \left[R - F_{\mu\nu} F^{\mu\nu} + \alpha \left(R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\beta\gamma} R^{\mu\nu\beta\gamma} \right) \right], \quad (1)$$

where α is the Gauss–Bonnet coupling constant, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell tensor. The spherically symmetric vacuum solution in four-dimensional spacetime was first gotten by Glavan and Lin. Subsequently, various solutions of black holes in the four-dimensional Einstein–Gauss–Bonnet gravity were obtained. The charged Einstein–Gauss–Bonnet black hole in the four-dimensional spacetime is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

with the gauge potential $A_\mu = -\frac{Q}{r}dt$, where

$$f(r) = 1 + \frac{r^2}{2\alpha} \left[1 \pm \sqrt{1 + 4\alpha \left(\frac{2M}{r^3} - \frac{Q^2}{r^4} \right)} \right], \quad (3)$$

M and Q are the mass and charge of the black hole, respectively. In the above equation, “+/-” are labeled as the “positive/negative” branches. When it is in the far region and the coupling constant α disappears, the Reissner–Nordström black hole is recovered by the negative branch. Therefore, we focus our attention on the negative branch. From $f(r) = 0$, the event horizon (r_+) and inner horizon (r_-) are gotten as follows

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - \alpha}. \quad (4)$$

Here, α , M and Q must obey certain conditions to avoid the disappearance of the event horizon. For convenience, we let $M = 1$ in this paper. When $0 < Q < \sqrt{3/2}$, α obeys $Q^2 - 4 - 2\sqrt{4 - 2Q^2} < \alpha < 1 - Q^2$, and there are two horizons r_\pm . When $\sqrt{3/2} < Q < \sqrt{2}$, α satisfies $Q^2 - 4 - 2\sqrt{4 - 2Q^2} < \alpha < Q^2 - 4 + 2\sqrt{4 - 2Q^2}$. In the region where $\sqrt{3/2} < Q < \sqrt{2}$ and $Q^2 - 4 - 2\sqrt{4 - 2Q^2} < \alpha < Q^2 - 4 + 2\sqrt{4 - 2Q^2}$, there is only one horizon r_+ .

¹ The detailed discussion on this aspect is referred to [17] and the references therein.

We first investigate a free photon orbiting around the equatorial orbit of the black hole (2). The photon moves along a null geodesic. Its Lagrangian is

$$\mathcal{L} = \frac{1}{2} \left[-f(r)\dot{t}^2 + \frac{1}{f(r)}\dot{r}^2 + r^2\dot{\phi}^2 \right]. \tag{5}$$

In the above equation, the dot over a symbol is the differentiation with respect to an affine parameter. From this Lagrangian, we get the generalized momenta,

$$\begin{aligned} p_t &= -f(r)\dot{t} = -E, \\ p_r &= \frac{\dot{r}}{f(r)}, \\ p_\phi &= r^2\dot{\phi} = L. \end{aligned} \tag{6}$$

Due to the Killing vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ in the spacetime, E and L are constants and denote the energy and orbital angular momentum of the photon, respectively. Using Eqs. (5), (6) and the Hamiltonian for this system [79]

$$\mathcal{H} = 2(p_\mu \dot{x}^\mu - \mathcal{L}) = -E\dot{t} + \frac{\dot{r}^2}{f(r)} + L\dot{\phi} = 0, \tag{7}$$

we obtain the equation of radial motion

$$\dot{r}^2 + V(r) = 0, \tag{8}$$

where $V(r) = -E^2 + \frac{L^2}{r^2}f(r)$ is an effective potential and determines the position of an unstable orbit. This position satisfies the condition

$$V(r) = 0, \quad \frac{\partial V(r)}{\partial r} = 0, \quad \frac{\partial^2 V(r)}{\partial r^2} < 0. \tag{9}$$

Solving the middle equation of the above equations yields four roots. But only a root corresponds to this position, and this root is the radius of the photon sphere. Using the first and third equations, we can exclude the other three roots and get the radius r_{ps} . Due to the complexity of the expression r_{ps} , we do not write it here. Its specific values are calculated and listed in Table 1. As described in [80–82], α cannot be too negative, because the metric function may not be real inside the event horizon when α is too negative. The observational constraints on the coupling parameter α was found in [84].

From Table 1, we find that the radius of the photon sphere is closely related to the charge and Gauss–Bonnet coupling constant. When the charge is fixed, the radius decreases with the increase of the value of the Gauss–Bonnet coupling constant. When the Gauss–Bonnet coupling constant is fixed, the radius increases with the decrease of the value of the charge.

A black hole shadow is an observable quantity. Its radius is calculated strictly via celestial coordinates. Through the calculation, it was found that the radius is related to the photon sphere. For a spherically symmetric black hole, the shadow

Table 1 The radii of the photon sphere and shadow under the different values of Q and α

Q	α	r_{ps}	R_{Sh}
0.1	−3.0	3.889873182	6.003984441
	−1.0	3.367024270	5.519636364
	0.8	2.536680250	4.811597083
0.4	−3.0	3.826945093	5.910587761
	−1.0	3.286378595	5.406970581
	0.8	2.364595705	4.628872985
0.7	−3.0	3.682145249	5.694974891
	−1.0	3.095251850	5.140325436
	0.5	2.225312089	4.397492519
1.0	−2.0	3.141336116	5.036599384
	−1.0	2.746568247	4.655643115
	−0.1	2.129360012	4.101527592

radius is [52]

$$R_{Sh} = \left. \frac{r}{\sqrt{f(r)}} \right|_{r=r_{ps}}. \tag{10}$$

Inserting the radius of the photon sphere into the above equation, we get the shadow radius of the charged Einstein–Gauss–Bonnet black hole. The specific expression of the shadow radius is also very complex and is not presented here. The values of the radius are obtained by using the different values of Q and α . They are listed in Table 1.

It is found from the table that the shadow radius decreases with the increase of the value of the Gauss–Bonnet coupling constant when the charge is fixed and decreases with the increase of the value of the charge when the Gauss–Bonnet coupling constant is fixed. The radii of the photon sphere and shadow increase or decrease at the same time.

3 Quasinormal modes, shadow and the correspondence with a test field

As an important feature of black holes, QNMs are one of important ways to detect black holes. A lot of work has been done on QNMs by analytical and numerical approaches. The WKB approximation approach, which was first put forward in [19], is an effective method to derive QNMs. This approach was extended to the 3rd order WKB approximation in [22] and higher order WKB approximation in [29–31, 39], respectively. The 13th order WKB approximation approach was developed in [39]. In this section, we adopt the 6th order WKB approximation approach [29–31] to derive the QNMs by a scalar field perturbation in the four-dimensional charged Einstein–Gauss–Bonnet black hole. Although the accuracy of the QNMs derived by the 13th order WKB approximation

Table 2 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 0.1$ and $\alpha = -3$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
l=1	0.232962 – 0.396877i	0.249834 – 0.118605i	–7.242%	70.115%
l=2	0.443244 – 0.121925i	0.416390 – 0.118605i	6.059%	2.723%
l=3	0.587432 – 0.121927i	0.582946 – 0.118605i	0.764%	2.725%
l=4	0.751461 – 0.120694i	0.749502 – 0.118605i	0.261%	1.731%
l=5	0.917271 – 0.119989i	0.916058 – 0.118605i	0.132%	1.153%
l=10	1.749280 – 0.118967i	1.748839 – 0.118605i	0.025%	0.304%
l=50	8.411160 – 0.118620i	8.411081 – 0.118605i	0.0009%	0.013%

Table 3 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 0.1$ and $\alpha = -1$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
l=1	0.259512 – 0.120429i	0.271757 – 0.106983i	–4.718%	11.165%
l=2	0.453456 – 0.109868i	0.452928 – 0.106983i	0.116%	2.626%
l=3	0.635009 – 0.108202i	0.634100 – 0.106983i	0.143%	1.127%
l=4	0.816074 – 0.107674i	0.815271 – 0.106983i	0.098%	0.642%
l=5	0.997130 – 0.107433i	0.996442 – 0.106983i	0.069%	0.419%
l=10	1.902680 – 0.107102i	1.902299 – 0.106983i	0.020%	0.111%
l=50	9.149230 – 0.106988i	9.149153 – 0.106983i	0.0008%	0.005%

Table 4 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 0.1$ and $\alpha = 0.8$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
l=1	0.316086 – 0.080425i	0.311747 – 0.078850i	1.373%	1.958%
l=2	0.522310 – 0.079271i	0.519578 – 0.078850i	0.523%	0.531%
l=3	0.729380 – 0.079047i	0.727409 – 0.078850i	0.270%	0.249%
l=4	0.936778 – 0.078967i	0.935240 – 0.078850i	0.164%	0.148%
l=5	1.144330 – 0.078928i	1.143072 – 0.078850i	0.110%	0.099%
l=10	2.182890 – 0.078872i	2.182228 – 0.078850i	0.030%	0.027%
l=50	10.49560 – 0.078851i	10.495476 – 0.07885i	0.001%	0.001%

approach is higher than that derived by the 6th order WKB approximation approach, the latter is enough for us to elaborate the correspondence between the QNMs in the eikonal limit and test field in the Einstein–Gauss–Bonnet–Maxwell gravity.

Let us start with a massless scalar field perturbation in the background of the metric (2). The scalar field in the curved spacetime obeys

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = 0, \tag{11}$$

where Ψ is a function of coordinates (t, r, θ, ϕ) and represents the scalar field. We adopt the following ansatz

$$\Psi = \sum_{l,m} e^{-i\omega t} \Phi(r) r^{-1} Y_{lm}(\theta, \phi). \tag{12}$$

In the above equation, ω is the perturbation frequency, $e^{-i\omega t}$ denotes the time evolution of the scalar field, and $Y_{lm}(\theta, \phi)$ is a spherical harmonics. Inserting Eqs. (2) and (12) into Eq. (11) and introducing a “tortoise” coordinate $dr_\star = \frac{dr}{f(r)}$ yield the Regge–Wheeler equation,

$$\left[\partial_{r_\star}^2 + \omega^2 - \mathcal{V}(r) \right] \Phi(r) = 0, \tag{13}$$

Table 5 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 0.4$ and $\alpha = -3$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
l=1	0.262187 - 0.414285i	0.253782 - 0.121017i	3.206%	70.789%
l=2	0.453712 - 0.122533i	0.422970 - 0.121017i	6.775%	1.237%
l=3	0.597166 - 0.124275i	0.592158 - 0.121017i	0.839%	2.622%
l=4	0.763557 - 0.123146i	0.761346 - 0.121017i	0.290%	1.729%
l=5	0.931902 - 0.122444i	0.930534 - 0.121017i	0.147%	1.165%
l=10	1.776960 - 0.121395i	1.776473 - 0.121017i	0.027%	0.311%
l=50	8.544080 - 0.121033i	8.543990 - 0.121017i	0.001%	0.013%

Table 6 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 0.4$ and $\alpha = -1$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
l=1	0.263484 - 0.124484i	0.277420 - 0.108859i	-5.289%	12.552%
l=2	0.463000 - 0.111939i	0.462366 - 0.108859i	0.137%	2.752%
l=3	0.648293 - 0.110147i	0.647313 - 0.108859i	0.151%	1.169%
l=4	0.833116 - 0.109586i	0.832259 - 0.108859i	0.103%	0.663%
l=5	1.017940 - 0.109331i	1.017205 - 0.108859i	0.072%	0.432%
l=10	1.942340 - 0.108983i	1.941938 - 0.108859i	0.021%	0.114%
l=50	9.339880 - 0.108864i	9.339796 - 0.108859i	0.0009%	0.005%

Table 7 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 0.4$ and $\alpha = 0.8$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
l=1	0.327346 - 0.0765758i	0.324053 - 0.074864i	1.006%	2.235%
l=2	0.542113 - 0.0753940i	0.540088 - 0.074864i	0.374%	0.703%
l=3	0.757579 - 0.0751335i	0.756124 - 0.074864i	0.192%	0.359%
l=4	0.973297 - 0.0750294i	0.972159 - 0.074864i	0.117%	0.220%
l=5	1.189130 - 0.0749761i	1.188194 - 0.074864i	0.079%	0.150%
l=10	2.268860 - 0.0748956i	2.268371 - 0.074864i	0.022%	0.042%
l=50	10.90990 - 0.0748655i	10.909783 - 0.074864i	0.001%	0.002%

where

$$\mathcal{V}(r) = f(r) \left[\frac{f'(r)}{r} + \frac{l(l+1)}{r^2} \right]. \tag{14}$$

The QNMs arise from the solution of the above differential equation under the specific boundary condition. The boundary condition is $r_\star \rightarrow \pm\infty$ which map the event horizon and infinity. The solution takes form $\Phi(r) \sim \exp[-i\omega(t \mp r_\star)]$ with the oscillation modes $\omega = \omega_R - i\omega_I$. Our interest is focused on the fundamental modes where $n = 0$. Since they have the least damping among the detected ringing signals and dominate the waveform of gravitational waves. Using

the WKB approximation approach, we get the modes and list them in Tables 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and 13. Here we did not calculate the modes with $l = n = 0$. The reason is that this approach does not give a satisfactory accuracy for $n \geq l$. Nevertheless, it does not mean that this fundamental modes can not be calculated. One can get the modes by other approaches.

There is a close connection between null geodesics and QNMs. It was proven in [49] that QNMs in the eikonal limit can be gotten via properties of null geodesics. In the eikonal limit, the QNMs takes form

Table 8 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 0.7$ and $\alpha = -3$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
l=1	0.380439 – 0.437918i	0.263390 – 0.127070i	30.767%	70.983%
l=2	0.476503 – 0.122625i	0.438983 – 0.127070i	7.874%	–3.625%
l=3	0.620628 – 0.129982i	0.614577 – 0.127070i	0.975%	2.240%
l=4	0.793031 – 0.129229i	0.790170 – 0.127070i	0.361%	1.671%
l=5	0.967561 – 0.128571i	0.965764 – 0.127070i	0.186%	1.167%
l=10	1.844370 – 0.127484i	1.843731 – 0.127070i	0.035%	0.325%
l=50	8.867580 – 0.127087i	8.867467 – 0.127070i	0.001%	0.013%

Table 9 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 0.7$ and $\alpha = -1$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
l=1	0.273221 – 0.136267i	0.291810 – 0.113586i	–6.804%	16.645%
l=2	0.487400 – 0.117201i	0.486351 – 0.113586i	0.205%	3.084%
l=3	0.682100 – 0.115066i	0.680891 – 0.113586i	0.176%	1.286%
l=4	0.876450 – 0.114416i	0.875431 – 0.113586i	0.116%	0.725%
l=5	1.070830 – 0.114122i	1.069971 – 0.113586i	0.080%	0.470%
l=10	2.043140 – 0.113726i	2.042672 – 0.113586i	0.023%	0.123%
l=50	9.824380 – 0.113592i	9.824281 – 0.113586i	0.001%	0.005%

Table 10 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 0.7$ and $\alpha = 0.5$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
l=1	0.343629 – 0.0803095i	0.341103 – 0.078513i	0.735%	2.237%
l=2	0.570174 – 0.0790292i	0.568506 – 0.078513i	0.293%	0.653%
l=3	0.797117 – 0.0787623i	0.795908 – 0.078513i	0.152%	0.317%
l=4	1.024250 – 0.0786622i	1.023310 – 0.078513i	0.092%	0.190%
l=5	1.251490 – 0.0786127i	1.250713 – 0.078513i	0.062%	0.127%
l=10	2.388130 – 0.0785405i	2.387724 – 0.078513i	0.017%	0.035%
l=50	11.483900 – 0.0785143i	11.483817 – 0.078513i	0.0007%	0.002%

$$\omega = \Omega L - i \left(n + \frac{1}{2} \right) \lambda, \tag{15}$$

where $\Omega = \sqrt{\frac{f'(r)}{2r}}$ is the angular velocity at the unstable null geodesic orbit, L is the angular momentum, n is the overtone number, and $\lambda = \sqrt{\frac{f(r)[2f(r)-r^2f''(r)]}{2r^2}} \Big|_{r=r_{ps}}$ is the

Lyapunov exponent that determines the instability time scale of the orbit. The connections between QNMs and photon spheres and between photon spheres and shadow radii have been found. Therefore, it is natural to conjecture that whether there is a connection between QNMs and shadow radii. This

conjecture was verified in [52]. The relation between the QNMs in the eikonal limit and shadow radii of the spherically symmetrical black holes was found. From this relation, the QNMs can be rewritten as

$$\omega = R_{Sh}^{-1} \left(l + \frac{D-3}{2} \right) - i \left(n + \frac{1}{2} \right) \lambda, \tag{16}$$

where D is the dimension of spacetime and it is equal to 4 in this paper. In the eikonal limit, the term $\frac{D-3}{2}$ in Eq. (16) is neglected. However, we do not neglect it, since the QNMs at the small multipole number is also in our discussion. Using Eq. (16), we get the QNMs of the charged Einstein-Gauss-

Table 11 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 1$ and $\alpha = -2$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
$l = 1$	0.340785 – 0.358543i	0.297820 – 0.133492i	12.608%	62.768%
$l = 2$	0.521267 – 0.136806i	0.496367 – 0.133492i	4.777%	2.422%
$l = 3$	0.699962 – 0.136866i	0.694913 – 0.133492i	0.721%	2.465%
$l = 4$	0.895992 – 0.135613i	0.893460 – 0.133492i	0.283%	1.564%
$l = 5$	1.093700 – 0.134901i	1.092007 – 0.133492i	0.155%	1.044%
$l = 10$	2.085430 – 0.133863i	2.084740 – 0.133492i	0.033%	0.277%
$l = 50$	10.026700 – 0.133507i	10.026606 – 0.133492i	0.0009%	0.011%

Table 12 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 1$ and $\alpha = -1$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
$l = 1$	0.297584 – 0.168863i	0.322190 – 0.123414i	8.564%	26.914%
$l = 2$	0.540091 – 0.128214i	0.536983 – 0.123414i	0.575%	3.744%
$l = 3$	0.753793 – 0.125369i	0.751776 – 0.123414i	0.268%	1.559%
$l = 4$	0.968111 – 0.124504i	0.966569 – 0.123414i	0.159%	0.875%
$l = 5$	1.182620 – 0.124114i	1.181362 – 0.123414i	0.107%	0.564%
$l = 10$	2.255990 – 0.123595i	2.255328 – 0.123414i	0.029%	0.146%
$l = 50$	10.84720 – 0.123422i	10.847051 – 0.123414i	0.0009%	0.006%

Table 13 The QNMs derived by the WKB approximation approach and shadow radius and the relative deviations. In the calculation, we let $Q = 1$ and $\alpha = -0.1$

	QNMs		Relative deviations	
	$\omega(WKB)$	$\omega(Sh)$	Re ω	Im ω
$l = 1$	0.371004 – 0.097845i	0.365717 – 0.097172i	1.425%	0.688%
$l = 2$	0.612625 – 0.097468i	0.609529 – 0.097172i	0.505%	0.304%
$l = 3$	0.855547 – 0.097328i	0.853341 – 0.097172i	0.258%	0.160%
$l = 4$	1.098870 – 0.097267i	1.097152 – 0.097172i	0.157%	0.098%
$l = 5$	1.342370 – 0.097236i	1.340964 – 0.097172i	0.107%	0.066%
$l = 10$	2.560760 – 0.097189i	2.560022 – 0.097172i	0.029%	0.017%
$l = 50$	12.3126 – 0.097172i	12.312486 – 0.097172i	0.0008%	0%

Bonnet black hole and list them in Table 2.1-Table 5.3. In the tables, $\omega(WKB)$ denotes the QNMs derived by the $6th$ order WKB approximation approach, and $\omega(Sh)$ denotes the QNMs gotten by Eq. (16). The relative deviations of the real (imaginary) parts of the QNMs calculated by Eq. (16) from those calculated by the WKB approximation approach are listed in the last two columns. From these tables, we find the following phenomena.

1. For the fixed charge and same multipole number, when the value of the Gauss–Bonnet coupling constant increase,

the values of the real parts of the QNMs increase and those of the imaginary parts decrease. The coincidence degrees of the QNMs derived by the two approaches increase with the increase of the value of α . For $\alpha = -3$ and $l = 1$, the degree of coincidence is the smallest. The reason may be that the value of α cannot be too negative.

2. For the fixed charge and Gauss–Bonnet coupling constant, the result derived by the WKB approximation approach shows that the values of the real parts increase and those of the imaginary parts decrease slowly when the value of the multipole number increases. However,

when $l = 1$, the imaginary parts' values are the largest and the degrees of coincidence are the smallest. When $l = 3$, the values are the second largest. The reason may be related to the condition of the accuracy for the WKB approximation approach.

3. For the fixed Gauss–Bonnet coupling constant and multipole number, when the value of the charge increases, the values of the real parts and imaginary parts calculated by the WKB approximation approach increase at the same time, but the coincidence degrees of the QNMs obtained by the two approaches decrease. For example, when $\alpha = -1$ and $l = 5$, we can find the QNMs and the degrees of coincidence for the different value of Q .
4. The values in the tables show that when the value of the multipole number is large, the QNMs gotten by the two approaches are consistent. In fact, when l is small, the QNMs are in good agreement with each other except for the values at $\alpha = -3$. This result is full in consistent with those derived in [56,83].

4 Conclusions

In this paper, we investigated the photon sphere, shadow and QNMs of the four-dimensional charged Einstein–Gauss–Bonnet black hole. The QNMs were gotten by the $6th$ order WKB approximation approach and shadow radius, respectively. When the value of the multipole number is large, the QNMs calculated by the two approaches are in good agreement, which shows the correspondence between the shadow and test field in the four-dimensional Einstein–Gauss–Bonnet–Maxwell gravity. Equation (16) is obtained under the condition of the large multipole numbers. However, we found that when the value of the multipole number is small, except for the values at $\alpha = -3$ and $l = 1$, the degrees of coincidence of the QNMs obtained by the two approaches are also in good agreement. The degrees of coincidence increase with the increases of the value of the Gauss–Bonnet coupling constant. When the value of the Gauss–Bonnet coupling constant increase, the values of the real parts of the QNMs increase and those of the imaginary parts decrease. It shows that the Gauss–Bonnet coupling constant plays an important role in the QNMs. This investigation reveals a potential relationship between the black hole shadow and gravitational wave.

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References

1. The Event Horizon Telescope Collaboration et al, First M87 event horizon telescope results. I. The shadow of the supermassive black hole, *Astrophys. J. Lett.* **875**, L1 (2019)
2. The Event Horizon Telescope Collaboration et al, First M87 event horizon telescope results. IV. Imaging the central supermassive black hole, *Astrophys. J. Lett.* **875**, L4 (2019)
3. The Event Horizon Telescope Collaboration et al, First M87 event horizon telescope results. V. Physical origin of the asymmetric ring, *Astrophys. J. Lett.* **875**, L5 (2019)
4. The Event Horizon Telescope Collaboration et al, First M87 event horizon telescope results. VI. The shadow and mass of the central black hole, *Astrophys. J. Lett.* **875**, L6 (2019)
5. B.P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett.* **116**, 061102 (2016)
6. B.P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), Tests of general relativity with GW150914. *Phys. Rev. Lett.* **116**, 221101 (2016)
7. B.P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), GW151226: Observation of gravitational waves from a 22-solar-mass binary black hole coalescence. *Phys. Rev. Lett.* **116**, 241103 (2016)
8. J.M. Maldacena, The large N limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2**, 231 (1998) (*Int. J. Theor. Phys.* **38**, 1113, 1999)
9. G.T. Horowitz, V. Hubeny, Quasinormal modes of AdS black holes and the approach to thermal equilibrium. *Phys. Rev. D* **62**, 024027 (2000)
10. D.T. Son, A.O. Starinets, Minkowski-space correlators in AdS/CFT correspondence: recipe and applications. *JHEP* **0209**, 042 (2002)
11. P. Kovtun, D.T. Son, A.O. Starinets, Viscosity in strongly interacting quantum field theories from black hole physics. *Phys. Rev. Lett.* **94**, 111601 (2005)
12. J. Morgan, A.S. Miranda, V.T. Zamchin, Electromagnetic quasinormal modes of rotating black strings and the AdS/CFT correspondence. *JHEP* **1303**, 169 (2013)
13. S. Hod, Bohr's correspondence principle and the area spectrum of quantum black holes. *Phys. Rev. Lett.* **81**, 4293 (1998)
14. O. Dreyer, Quasinormal modes, the area spectrum, and black hole entropy. *Phys. Rev. Lett.* **90**, 081301 (2003)
15. G. Kunstatler, D-dimensional black hole entropy spectrum from quasi-normal modes. *Phys. Rev. Lett.* **90**, 161301 (2003)
16. M. Maggiore, The physical interpretation of the spectrum of black hole quasinormal modes. *Phys. Rev. Lett.* **100**, 141301 (2008)
17. R.A. Konoplya, A. Zhidenko, Quasinormal modes of black holes: from astrophysics to string theory. *Rev. Mod. Phys.* **83**, 793 (2011)

18. H.J. Blome, B. Mashhoon, Quasi-normal oscillations of a Schwarzschild black hole. *Phys. Rev. A* **100**, 231 (1984)
19. B.F. Schutz, C.M. Will, Black hole normal modes: a semianalytic approach. *Astrophys. J.* **291**, L33 (1985)
20. E.W. Leaver, An analytic representation for the quasi-normal modes of Kerr black holes. *Proc. R. Soc. Lond. A* **402**, 285 (1985)
21. E.W. Leaver, Spectral decomposition of the perturbation response of the Schwarzschild geometry. *Phys. Rev. D* **34**, 384 (1986)
22. S. Iyer and C. M. Will, Black-hole normal modes: a WKB approach. I. Foundations and application of a higher-order WKB analysis of potential-barrier scattering. *Phys. Rev. D* **35**, 3621 (1987)
23. K.D. Kokkotas, B.F. Schutz, Black-hole normal modes: a WKB approach. III. The Reissner-Nordstrom black hole. *Phys. Rev. D* **37**, 3378 (1988)
24. B. Majumdar, N. Panchapakesan, Schwarzschild black-hole normal modes using the Hill determinant. *Phys. Rev. D* **40**, 2568 (1989)
25. E. Seidel, S. Iyer, Black-hole normal modes: a WKB approach. IV. Kerr black holes. *Phys. Rev. D* **41**, 374 (1990)
26. H.P. Nollert, Quasinormal modes of Schwarzschild black holes: the determination of quasinormal frequencies with very large imaginary parts. *Phys. Rev. D* **47**, 5253 (1993)
27. H.P. Nollert, Quasinormal modes: the characteristic “sound” of black holes and neutron stars. *Class. Quant. Grav.* **16**, R159 (1999)
28. G.T. Horowitz, V.E. Hubeny, Quasinormal modes of AdS black holes and the approach to thermal equilibrium. *Phys. Rev. D* **62**, 024027 (2000)
29. R.A. Konoplya, Quasinormal behavior of the D-dimensional Schwarzschild black hole and higher order WKB approach. *Phys. Rev. D* **68**, 024018 (2003)
30. R.A. Konoplya, On quasinormal modes of small Schwarzschild-anti-de-Sitter black hole. *Phys. Rev. D* **66**, 044009 (2002)
31. R.A. Konoplya, A. Zhidenko, A.F. Zinhailo, Higher order WKB formula for quasinormal modes and grey-body factors: recipes for quick and accurate calculations. *Class. Quantum Grav.* **36**, 155002 (2019)
32. H.T. Cho, Dirac quasi-normal modes in Schwarzschild black hole spacetimes. *Phys. Rev. D* **68**, 024003 (2003)
33. H.T. Cho, A.S. Cornell, J. Doukas, W. Naylor, Black hole quasinormal modes using the asymptotic iteration method. *Class. Quant. Grav.* **27**, 155004 (2010)
34. H.T. Cho, A.S. Cornell, J. Doukas, T.R. Huang, W. Naylor, A new approach to black hole quasinormal modes: a review of the asymptotic iteration method. *Adv. Math. Phys.* **2012**, 281705 (2012)
35. B. Wang, C.Y. Lin, E. Abdalla, Quasinormal modes of Reissner-Nordstrom anti-de Sitter black holes. *Phys. Lett. B* **481**, 79 (2000)
36. J.L. Jing, Neutrino quasinormal modes of the Reissner-Nordstrom black hole. *JHEP* **0512**, 005 (2005)
37. J.L. Jing, Q.Y. Pan, Quasinormal modes and second order thermodynamic phase transition for Reissner-Nordstrom black hole. *Phys. Lett. B* **660**, 13 (2008)
38. S.B. Chen, J.L. Jing, Quasinormal modes of a black hole in the deformed Hořava-Lifshitz gravity. *Phys. Lett. B* **687**, 124 (2010)
39. J. Matyjasek, M. Opala, Quasinormal modes of black holes. The improved semianalytic approach. *Phys. Rev. D* **96**, 024011 (2017)
40. E. Berti, K.D. Kokkotas, Quasinormal modes of Reissner-Nordstrom-anti-de Sitter black holes: scalar, electromagnetic and gravitational perturbations. *Phys. Rev. D* **67**, 064020 (2003)
41. J.S.F. Chan, R.B. Mann, Scalar wave fall off in asymptotically anti de Sitter backgrounds. *Phys. Rev. D* **55**, 7546 (1997)
42. F.W. Shu, Y.G. Shen, Quasinormal modes of Rarita-Schwinger field in Reissner-Nordstrom black hole. *Phys. Lett. B* **614**, 195 (2005)
43. K. Lin, W.L. Qian, A matrix method for quasinormal modes: Schwarzschild black holes in asymptotically flat and (anti-) de Sitter spacetimes. *Class. Quant. Grav.* **34**, 095004 (2017)
44. V. Cardoso, J.P.S. Lemos, Quasi-normal modes of Schwarzschild anti-de Sitter black holes: electromagnetic and gravitational perturbations. *Phys. Rev. D* **64**, 084017 (2001)
45. V. Cardoso, J.P.S. Lemos, Scalar, electromagnetic and Weyl perturbations of BTZ black holes: quasinormal modes. *Phys. Rev. D* **64**, 124015 (2001)
46. M. Rahman, S. Chakraborty, S. SenGupta, A.A. Sen, Fate of strong cosmic censorship conjecture in presence of higher spacetime dimensions. *JHEP* **1903**, 178 (2019)
47. I. Banerjee, S. Chakraborty, S. SenGupta, Silhouette of M87*: a new window to peek into the world of hidden dimensions. *Phys. Rev. D* **101**, 041301 (2020)
48. K. Kokkotas, B. Schmidt, Quasi-normal modes of stars and black hole. *Living Rev. Relat.* **2**, 2 (1999)
49. V. Cardoso, A.S. Miranda, E. Berti, H. Witek, V.T. Zanchin, Geodesic stability, Lyapunov exponents and quasinormal modes. *Phys. Rev. D* **79**, 064016 (2009)
50. R.A. Konoplya, Z. Stuchlik, Are eikonal quasinormal modes linked to the unstable circular null geodesics? *Phys. Lett. B* **771**, 597 (2017)
51. R.A. Konoplya, A.F. Zinhailo, Quasinormal modes, stability and shadows of a black hole in the 4D Einstein-Gauss-Bonnet gravity. *Eur. Phys. J. C* **80**, 1049 (2020)
52. K. Jusufi, Quasinormal modes of black holes surrounded by dark matter and their connection with the shadow radius. *Phys. Rev. D* **101**, 084055 (2020)
53. K. Jusufi, Connection between the shadow radius and quasinormal modes in rotating spacetimes. *Phys. Rev. D* **101**, 124063 (2020)
54. B. Cuadros-Melgar, R.D.B. Fontana, J. de Oliveira, Analytical correspondence between shadow radius and black hole quasinormal frequencies. *Phys. Lett. B* **811C**, 135966 (2020)
55. K. Jusufi, M. Amir, M.S. Ali, S.D. Maharaj, Quasinormal modes, shadow and greybody factors of 5D electrically charged Bardeen black holes. *Phys. Rev. D* **102**, 064020 (2020)
56. Y. Guo, Y.G. Miao, Null geodesics, quasinormal modes and the correspondence with shadows in high-dimensional Einstein-Yang-Mills spacetimes. *Phys. Rev. D* **102**, 084057 (2020)
57. D. Glavan, C.S. Lin, Einstein-Gauss-Bonnet gravity in 4-dimensional space-time. *Phys. Rev. Lett.* **124**, 081301 (2020)
58. K. Aoki, M.A. Gorji, S. Mukohyama, A consistent theory of $D \rightarrow 4$ Einstein-Gauss-Bonnet gravity. *Phys. Lett. B* **810**, 135843 (2020)
59. P.G.S. Fernandes, Charged black holes in AdS spaces in 4D Einstein Gauss-Bonnet gravity. *Phys. Lett. B* **805**, 135468 (2020)
60. S.W. Wei, Y.X. Liu, Testing the nature of Gauss-Bonnet gravity by four-dimensional rotating black hole shadow. [arXiv:2003.07769](https://arxiv.org/abs/2003.07769) [gr-qc]
61. R.A. Konoplya, A. Zhidenko, Black holes in the four-dimensional Einstein-Lovelock gravity. *Phys. Rev. D* **101**, 084038 (2020)
62. R.A. Konoplya, A. Zhidenko, 4D Einstein-Lovelock black holes: hierarchy of orders in curvature. *Phys. Lett. B* **807**, 135607 (2020)
63. H. Lu, Y. Pang, Horndeski gravity as $D \rightarrow 4$ limit of Gauss-Bonnet. *Phys. Lett. B* **809**, 135717 (2020)
64. S.G. Ghosh, R. Kumar, Generating black holes in 4D Einstein-Gauss-Bonnet gravity. *Class. Quant. Grav.* **37**, 245008 (2020)
65. R. Kumar, S.G. Ghosh, Rotating black holes in 4D Einstein-Gauss-Bonnet gravity and its shadow. *JCAP* **07**, 053 (2020)
66. B.E. Panah, K. Jafarzade, S.H. Hendi, Charged 4D Einstein-Gauss-Bonnet-AdS black holes: shadow, energy emission, deflection angle and heat engine. *Nucl. Phys. B* **961**, 115269 (2020)
67. K. Jusufi, Nonlinear magnetically charged black holes in 4D Einstein-Gauss-Bonnet gravity. *Ann. Phys.* **421**, 168285 (2020)
68. L. Ma, H. Lu, Vacua and exact solutions in Lower-D Limits of EGB. [arXiv:2004.14738](https://arxiv.org/abs/2004.14738) [gr-qc]
69. X.H. Ge, S.J. Sin, Causality of black holes in 4-dimensional Einstein-Gauss-Bonnet-Maxwell theory. *Eur. Phys. J. C* **80**, 695 (2020)

70. R.A. Hennigar, D. Kubiznak and R.B. Mann, Rotating Gauss-Bonnet BTZ black holes, *Class. Quant. Grav.* **38**, 03LT01 (2021)
71. R.A. Hennigar, D. Kubiznak, R.B. Mann, On taking the $D \rightarrow 4$ limit of Gauss-Bonnet gravity: Theory and solutions. *JHEP* **2007**, 027 (2020)
72. M.A. Cuyubamba, Stability of asymptotically de Sitter and anti-de Sitter black holes in 4D regularized Einstein-Gauss-Bonnet theory. *Phys. Dark Univ.* **31**, 100789 (2021)
73. S. Hansraj, A. Banerjee, L.N. Moodly, M.K. Jasim, Isotropic compact stars in 4D Einstein-Gauss-Bonnet gravity. *Class. Quant. Grav.* **38**, 035002 (2021)
74. Y.L. Wang, X.H. Ge, Black holes in 4D Einstein-Maxwell-Gauss-Bonnet gravity coupled with scalar fields. [arXiv:2011.08604](https://arxiv.org/abs/2011.08604) [hep-th]
75. S.A. Hosseini Mansoori, Thermodynamic geometry of the novel 4D Gauss-Bonnet AdS black hole. *Phys. Dark Univ.* **31**, 100776 (2021)
76. A. Aragon, R. Becar, P.A. Gonzalez, Y. Vasquez, Perturbative and nonperturbative quasinormal modes of 4D Einstein-Gauss-Bonnet black holes. *Eur. Phys. J. C* **80**, 773 (2020)
77. P. Wang, H.T. Yang, S.X. Ying, Thermodynamics and phase transition of a Gauss-Bonnet black hole in a cavity. *Phys. Rev. D* **101**, 064045 (2020)
78. S.X. Ying, Thermodynamics and weak cosmic censorship conjecture of 4D Gauss-Bonnet-Maxwell black holes via charged particle absorption. *Chin. Phys. C* **44**, 125101 (2020)
79. S.W. Wei, Y.X. Liu, Photon orbits and thermodynamic phase transition of d-dimensional charged AdS black holes. *Phys. Rev. D* **97**, 104027 (2018)
80. C.Y. Zhang, S.J. Zhang, P.C. Li, M.Y. Guo, Superradiance and stability of the regularized 4D charged Einstein-Gauss-Bonnet black hole. *JHEP* **08**, 105 (2020)
81. M. Guo, P.C. Li, Innermost stable circular orbit and shadow of the 4D Einstein-Gauss-Bonnet black hole. *Eur. Phys. J. C* **80**, 588 (2020)
82. C.Y. Zhang, P.C. Li, M. Guo, Greybody factor and power spectra of the Hawking radiation in the novel 4D Einstein-Gauss-Bonnet de-Sitter gravity. *Eur. Phys. J. C* **80**, 874 (2020)
83. E. Berti, K.D. Kokkotas, Quasinormal modes of Kerr-Newman black holes: coupling of electromagnetic and gravitational perturbations. *Phys. Rev. D* **71**, 124008 (2005)
84. T. Clifton, P. Carrilho, P.G.S. Fernandes, D.J. Mulryne, Observational constraints on the regularized 4D Einstein-Gauss-Bonnet theory of gravity. *Phys. Rev. D* **102**, 084005 (2020)