

The cosmic horizon

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ABSTRACT

The cosmological principle, promoting the view that the Universe is homogeneous and isotropic, is embodied within the mathematical structure of the Robertson–Walker (RW) metric. The equations derived from an application of this metric to the Einstein Field Equations describe the expansion of the Universe in terms of comoving coordinates, from which physical distances may be derived using a time-dependent expansion factor. These coordinates, however, do not explicitly reveal the properties of the cosmic space–time manifested in Birkhoff’s theorem and its corollary. In this paper, we compare two forms of the metric – written in (the traditional) comoving coordinates, and a set of observer-dependent coordinates – first for the well-known de Sitter universe containing only dark energy, and then for a newly derived form of the RW metric, for a universe with dark energy and matter. We show that Rindler’s event horizon – evident in the comoving system – coincides with what one might call the ‘curvature horizon’ appearing in the observer-dependent frame. The advantage of this dual prescription of the cosmic space–time is that with the latest *Wilkinson Microwave Anisotropy Probe* results, we now have a much better determination of the Universe’s mass-energy content, which permits us to calculate this curvature with unprecedented accuracy. We use it here to demonstrate that our observations have probed the limit beyond which the cosmic curvature prevents any signal from having ever reached us. In the case of de Sitter, where the mass-energy density is a constant, this limit is fixed for all time. For a universe with a changing density, this horizon expands until de Sitter is reached asymptotically, and then it too ceases to change.

Key words: cosmic microwave background – cosmological parameters – cosmology: observations – cosmology: theory – distance scale.

1 INTRODUCTION

The smoothness of the cosmic microwave background (CMB) provides direct evidence that the universe is isotropic about us. Invoking the Copernican principle – that we do not live in a special place – allows one to argue further that the universe should be isotropic around every point, i.e., it is also (on average) homogeneous. Standard cosmology is therefore based on the Robertson–Walker (RW) metric for a spatially homogeneous and isotropic three-dimensional space, expanding or contracting as a function of time:

$$ds^2 = c^2 dt^2 - a^2(t)[dr^2(1 - kr^2)^{-1} + r^2(\theta^2 + \sin^2 \theta d\phi^2)]. \quad (1)$$

The coordinates for this metric have been chosen so that t represents the proper time as measured by a comoving observer, $a(t)$ is the expansion factor and r is an appropriately scaled radial coordinate in the comoving frame. The geometric factor k is $+1$ for a

closed universe, 0 is for a flat, open universe, or -1 for an open universe.

Assuming that general relativity (GR) provides the correct framework for interpreting cosmological dynamics, one may then apply the RW metric to Einstein’s Field Equations (EFE) to obtain the [Friedman–Robertson–Walker (FRW)] differential equations of motion. These are the Friedman equation, written as

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{kc^2}{a^2}, \quad (2)$$

and the ‘acceleration’ equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3p). \quad (3)$$

An overdot denotes a derivative with respect to cosmic time t , and ρ and p represent the total energy density and the total pressure, respectively. A further application of the RW metric to the energy conservation equation in GR yields the final equation,

$$\dot{\rho} = -3H(\rho + p) \quad (4)$$

which, however, is not independent of equations (2) and (3).

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But written in this form, equations (2)–(4) do not explicitly reveal the spacetime curvature most elegantly inferred from the *corollary* to Birkhoff’s theorem (Birkhoff 1923). This theorem states that in a spherically symmetric spacetime, the only solution to the Einstein equations in vacuum is the Schwarzschild exterior solution. Furthermore, a spherically symmetric vacuum solution in the exterior is necessarily static. Birkhoff’s purpose was to prove that for GR, as with the Newtonian theory, the exterior gravitational field of a spherically symmetric distribution of matter is independent of interior radial pulsations.

However, what is relevant to our discussion here is not so much the Birkhoff theorem *per se*, but rather its very important corollary. The latter is a generalization of a well-known result of Newtonian theory (which also finds a direct application in electrodynamics), in which the gravitational field of a spherical shell vanishes inside the shell. The corollary to Birkhoff’s theorem states that the metric inside an empty spherical cavity, at the centre of a spherically symmetric system, must be equivalent to the flat-space Minkowski metric. Space must be flat in a spherical cavity even if the system is infinite. It matters not what the constituents of the medium outside the cavity are, as long as the medium is spherically symmetric.

If one then imagines placing a spherically symmetric mass at the centre of this cavity, according to Birkhoff’s theorem and its corollary, the metric between this mass and the edge of the cavity is necessarily of the Schwarzschild type. Thus, the worldlines linked to an observer in this region are curved relative to the centre of the cavity in a manner determined solely by the mass we have placed there. This situation may appear to contradict our assumption of isotropy, which one might naively take to mean that the spacetime curvature within the medium should cancel since the observer sees mass energy equally distributed in all directions. In fact, the observer’s worldlines are curved in every direction because, according to the corollary to Birkhoff’s theorem, only the mass energy between any given pair of points in this medium affects the path linking those points. This is why, of course, the universe does not expand at a constant rate, due to the spacetime curvature induced (in every direction) by its internal constituents.

This consequence of the corollary to Birkhoff’s theorem is so important – and critical to the discussion in this paper – that it merits restatement: the space–time curvature of a worldline linking any point in the universe to an observer a distance R away may be determined by calculating the mass energy enclosed within a sphere of radius R centred at the origin (i.e. at the location of the observer). The mass energy outside of this volume has a net zero effect on observations made within the sphere.

In this paper, we will introduce a new set of coordinates (distinct from the comoving coordinates $[r, t]$), that elicit this effect directly from the transformed metric, originally appearing in equation (1). We will show that, with the recent *Wilkinson Microwave Anisotropy Probe* results (Spergel et al. 2003), our observational limit clearly corresponds to the distance beyond which the spacetime curvature prevents any signal from ever reaching us.

2 THE ASYMPTOTIC UNIVERSE: DE SITTER’S COSMOLOGY

Since the universe appears to be expanding indefinitely, the dark energy (with constant density) will eventually dominate ρ completely. Other contributions – from radiation and (luminous and non-luminous) matter – drop off rapidly with increasing volume.

The de Sitter cosmology (de Sitter 1917) corresponds to a universe devoid of matter and radiation, but filled with a cosmolog-

ical constant, whose principal property is the equation of state (EOS) $p = -\rho$. The RW metric in this case may be written as

$$ds^2 = c^2 dt^2 - e^{2Ht} [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (5)$$

where clearly $k = 0$ and the expansion factor has the specific form $a(t) \equiv \exp(Ht)$, in terms of the Hubble constant H . (Note that, though this cosmology may represent the Universe’s terminal state, it may also have corresponded to the early inflationary phase, where it would have produced an exponentiation in size due to the expansion factor $\exp[Ht]$.)

It is not obvious from the form of this metric how the corollary to Birkhoff theorem’s may be recovered. But we must remember that t and r appearing here are the local (comoving) coordinates. To compare distances and times over large separations, one must also know $a(t)$. An alternative approach not used before (or at best only used very rarely) is to introduce the transformation relating these coordinates to those in the observer’s frame. Only then does the spacetime curvature emerge directly.

For pedagogical purposes, it may be useful here – in understanding the roles played by our two coordinate systems – to find an analogy between them and those commonly used in the case of a gravitationally collapsed object. The comoving coordinates (including the cosmic time) play the same roles as those of an observer falling freely under the influence of that object, whereas our new coordinates correspond to an ‘accelerated’ frame, like that of an observer held at a fixed spatial point in the surrounding spacetime. There is an important difference, however, in that the new set of coordinates (cT, R, θ, ϕ) we introduce below are observer-dependent. They are not universal nor do they need to be.

Let us now consider de Sitter’s metric in its *originally* published form:

$$ds^2 = c^2 dT^2 [1 - (R/R_0)^2] - dR^2 [1 - (R/R_0)^2]^{-1} - R^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (6)$$

The inspiration for this metric is clear upon considering Schwarzschild’s (vacuum) solution describing the spacetime around an enclosed, spherically symmetric object of mass M :

$$ds^2 = c^2 dT^2 [1 - 2GM/c^2 R] - dR^2 [1 - 2GM/c^2 R]^{-1} - R^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (7)$$

De Sitter’s metric describes the spacetime around a radially dependent enclosed mass $M(R)$, as we have for a uniform density ρ permeating an infinite, homogeneous medium. In that case,

$$M(R) = M(R_0)(R/R_0)^3, \quad (8)$$

and the Schwarzschild factor $[1 - (2GM/c^2 R)]$ transitions into $[1 - (R/R_0)^2]$. It should be emphasized that equation (6) implicitly contains the restriction that no mass energy outside of R should contribute to the gravitational acceleration inside of R , as required by the corollary to Birkhoff’s theorem.

For a given interval s , the observer-dependent time T clearly diverges as R approaches R_0 . This is the limiting distance beyond which the spacetime curvature prevents any signal from ever reaching us; the quantity R_0 is the only (classical) scale in the system and it should not surprise us – in view of the corollary to Birkhoff’s theorem – that it is defined as a *Schwarzschild* radius:

$$\frac{2GM(R_0)}{c^2} = R_0. \quad (9)$$

That is, R_0 is the distance at which the enclosed mass is sufficient to turn it into the Schwarzschild radius for an observer at the origin of the coordinates.

It is trivial to show that, in terms of ρ , we must have

$$R_0 = \left(\frac{3c^4}{8\pi G\rho} \right)^{1/2}, \quad (10)$$

or more simply, $R_0 = c/H$. As it should be, this is also the radius corresponding to Rindler's event horizon (Rindler 1956), which we will discuss below. The impact of R_0 , and hence the corollary to Birkhoff's theorem, may be gauged directly by considering the transformation of coordinates $(cT, R, \theta, \phi) \rightarrow (ct, r, \theta, \phi)$, for which

$$R = e^{Ht} r = e^{ct/R_0} r, \quad (11)$$

and

$$T = \ln[\exp(-2Ht) - (r/R_0)^2]^{-1/2H}. \quad (12)$$

It is clear from the definition of $a(t) = \exp(Ht)$, and the metric coefficients in equation (6), that R is the expanded (or physical) radius and T is the time corresponding to R as measured by an observer at the origin of the coordinates – not the local time measured by a comoving observer at r (which we have also called the cosmic time t). An observer's worldline must therefore always be restricted to the region $R < R_0$, that is, to radii bounded by the cosmic horizon, consistent with the corollary to Birkhoff's theorem.

3 A UNIVERSE WITH DARK ENERGY AND MATTER

But though the contribution from radiation has by now largely subsided, the Universe does contain matter, in addition to dark energy. Unfortunately, the de Sitter metric is not a good representation of this cosmology. However, the impact of a cosmic horizon should be independent of the actual metric used to describe the spacetime. The restrictions on an observer's worldlines should be set by the physical radius R_0 , beyond which no signal can reach her within a finite time, no matter what internal structure the spacetime may possess.

There now exists empirical evidence that our observation of the early universe is already close to this limit. The precision measurements (Spergel et al. 2003) of the CMB radiation indicate that the universe is extremely flat (i.e. $k = 0$). Thus, ρ is at (or very near) the 'critical' value $\rho_0 \equiv 3c^2 H_0^2 / 8\pi G$. The Hubble Space Telescope Key Project (Mould et al. 2000) on the extragalactic distance scale has measured H with unprecedented accuracy, finding a current value $H_0 \equiv H(t_0) = 71 \pm 6 \text{ km s}^{-1} \text{ Mpc}^{-1}$. (Throughout this paper, subscript '0' denotes values pertaining to the current cosmic time t_0 .) It is straightforward to show that with this H_0 , $R_0 \approx ct_0$ (specifically, 13.4 versus 13.7 billion light-years). Note that this t is the same as T for an observer at the origin. However, even though T changes with radius R (see equation 12), the difference between t and T emerges only when $R \rightarrow R_0$. Thus, as long as $R < R_0$, t and T remain close to each other. We will consider the properties of $T(R, t)$ for a universe containing both matter and dark energy shortly.

The near equality $R_0 \approx ct_0$ may be an indication that by now our worldlines are already bounded by the cosmic horizon. It is trivial to see from the FRW equations that the condition $R_0 \approx ct_0$ is equivalent to the statement that $\ddot{a} \approx 0$. Type Ia supernova data clearly do not support a decelerating universe (Perlmutter et al. 1999; Riess et al. 2004), pointing instead to a universe that is currently coasting ($\ddot{a} = 0$) or, more likely, one that has undergone periods of past deceleration and present acceleration. (This should not be confused with the Milne universe (Milne 1940), which itself expands at a constant rate, though is completely empty with $\rho = 0$; such a cosmology is not relevant to this discussion.) It is easy to write down

the appropriate RW metric when R_0 is strictly equal to ct_0 , for then $\ddot{a} = 0$, and therefore

$$ds^2 = c^2 dt^2 - (H_0 t)^2 [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)], \quad (13)$$

with an expansion factor $a(t) = H_0 t$. This is the form of the RW metric we will use in the rest of this paper, albeit understanding that R_0 may not be exactly equal to ct_0 currently. The latter would be realized if \ddot{a}/a were slightly greater than zero, as suggested by observations of Type Ia supernovae. But finding a metric appropriate for a universe containing both matter and dark energy is very difficult when $\ddot{a}/a \neq 0$, forcing us to adopt this approximation in order to extend our analysis beyond simply de Sitter. Fortunately, our results will not depend on this simplification, since we will confirm that a cosmic horizon emerges in both cases; it is a general property of cosmological models, independent of the actual spacetime.

The difficult part is to find a transformation, analogous to equations (11) and (12), that will permit us to write the metric in terms of physical coordinates (R and T). The main hurdle here is that, whereas ρ is constant in a de Sitter universe and no velocity relative to ρ is discernible (so that the four-velocity in the stress-energy tensor has just a single non-vanishing component), quite the opposite is true when ρ changes with expansion.

Recognizing that

$$R = (H_0 t) r, \quad (14)$$

we transform the RW metric (equation 13) into the form

$$ds^2 = c^2 dt^2 [1 - (R/ct)^2] + (2R/t) dt dR - dR^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (15)$$

Already one begins to see the role played by ct , which here becomes the cosmic (or 'curvature') horizon analogous to R_0 in equation (6). To diagonalize the metric and complete the transformation, we put

$$dT = \exp[-(1/2)(R/ct)^2] \{dt + dR(R/c^2 t)\} [1 - (R/ct)^2]^{-1} [(R/ct)^2 - 1]^{-1}, \quad (16)$$

which produces the final form,

$$ds^2 = c^2 dT^2 [1 - (R/ct)^2]^3 \exp(R/ct)^2 - dR^2 [1 - (R/ct)^2]^{-1} - R^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (17)$$

Unlike the situation with the de Sitter metric, however, this dT is not an exact differential. Physically, this simply means that the time difference between two spacetime points, according to the observer at the origin, depends on the path of integration. This should not be surprising given that diverse paths sample different expansion rates as seen in the observer's frame. (Contrast this with the RW metric, in which the cosmological time t is the same everywhere.)

But since dT depends on only two variables, there exists an integrating factor (call it $\tau[R, t]$) that converts dT into an exact differential. To find it, let us define the functions $A(R, t)$ and $B(R, t)$ such that equation (16) may be written in the form

$$dT = A(R, t) dt + B(R, t) dR. \quad (18)$$

In the R - t plane, curves of constant T are specified by the condition $dT = 0$, or equivalently, $dR/dt = -A/B$, which provides the simple solution

$$\chi(R, t) \equiv 2 \ln(t/t_0) + (R/ct)^2 = \chi_0. \quad (19)$$

We have chosen the (integration) constant multiplying t to yield a value $\chi_0 = 0$ for $R = 0$ at the current time. The constant χ_0 represents the value of T on the isochronal curve. In terms of χ , the integration factor $\tau(R, t)$ may then be found from the equations

$$A(R, t) = \tau(R, t) \frac{\partial \chi}{\partial t} \quad (20)$$

and

$$B(R, t) = \tau(R, t) \frac{\partial \chi}{\partial R}. \quad (21)$$

Evidently,

$$\tau(R, t) = -\frac{1}{2}t [1 - (R/ct)^2]^{-2} \times \exp[-(1/2)(R/ct)^2], \quad (22)$$

and the exact differential representing the time in this coordinate system is

$$d\chi = \frac{dT}{\tau}, \quad (23)$$

which allows us to migrate from one isochronal curve to the next.

Finally, an alternative form of the metric in equation (17) may be obtained by replacing dT with the exact differential $d\chi$, using equation (23). The result is

$$ds^2 = c^2(t d\chi)^2 [4[1 - (R/ct)^2]^{-1} - R^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (24)$$

The derivation of the metric in equations (17) and (24) is one of the principal results of this paper. As was the case with de Sitter, we see directly through its coefficients the fact that the time T diverges at a specific radius, here equal to $ct = cT(0)$ – the cosmic horizon for this particular spacetime. No worldlines are permitted with R extending beyond this limiting radius. For this reason, it is reasonable to conclude that the observed near equality $R_0 \approx ct_0$ is not a coincidence, but is instead an indication that our observational limit has reached the universe's cosmic horizon.

4 CONCLUDING REMARKS

The cosmic horizon we have discussed in this paper coincides with the event horizon derived earlier by Rindler (1956) because they represent aspects of the same spacetime, though viewed in two different coordinates systems. Rindler formalized the definition of two horizons in cosmology, both functioning as hypersurfaces in spacetime that divide ‘things’ into two separate non-null classes. For a fundamental observer A , the *event* horizon is defined to be a hypersurface that divides all events into the class in which these have been, are, or will be, observable by A , and the complementary class in which the events are forever outside of A 's power of observation. These event horizons are created by the Universe's expansion, which produces distant regions receding from A at such speeds that no causal contact can ever occur between them and the observer. Since the comoving coordinates represent the Universe's ‘freely falling’ frame, it is clear that this critical condition must therefore correspond to the radius R_0 at which the curvature in the cosmic spacetime causes T to diverge.

Rindler's second type of horizon is called a *particle* horizon. For the fundamental observer A and cosmic instant t_0 , the particle horizon is a surface in instantaneous three-space at time $t = t_0$ that divides all fundamental particles into two classes: those that have already been observed by A at time t_0 and those that have not. These horizons are therefore created by the finite propagation speed of signals that couple particles at distance d to the observer A for times $t > d/c$. The cosmic horizon R_0 is not related to the particle horizon.

An interesting property of R_0 emerges upon closer scrutiny of its value in the case of de Sitter and the alternate cosmology containing both matter and dark energy. First of all, the density ρ is constant in the former, so R_0 does not change with time. Thus, the horizon in de Sitter is fixed forever, and no events occurring beyond it

can ever be in causal contact with the observer at the origin. This is presumably the situation that emerges as the universe approaches de Sitter asymptotically, in which limit R_0 must be calculated for ρ due to dark energy alone. Second, the density ρ in the alternate metric decreases with time, and therefore R_0 correspondingly increases – i.e. $R_0(t)$ is a function of $t = T(0)$. This means, of course, that some portions of the universe that produced effects at $R < R_0(t)$ we measure at time t have by now moved beyond the limiting radius. However, the effects of gravity travel at the speed of light, so what matters in setting the structure of the Universe within the horizon at time t is the mass-energy content within R_0 . The influence of these distant regions of the Universe ended once their radius from us exceeded R_0 .

The light-travel time distance (~ 13.7 billion light-years) could only have been identified with the cosmic horizon once the data (Spergel et al. 2003) confirmed that the Universe is flat, and revealed that the density ρ is close to, or at, its ‘critical’ value. It has never been quite clear why this critical condition exists, except for the possibility that inflation in the early Universe could have led to this. But based on our analysis of the various metrics in this paper, an alternative interpretation – or perhaps simply an additional reason – for this criticality is that the Universe has now expanded long enough for the condition $R_0 \approx ct$ to have been attained.

One can easily show from the FRW equations that once R_0 has been reached, the expansion thereafter proceeds at a constant rate. But this does not necessarily mean that this dynamical state must have been present since the big bang, so it is by no means trivial to see how such a scenario would affect big bang nucleosynthesis (Burles, Nollett & Turner 2001), and structure formation in the early Universe (Springel, Frenk & White 2006).

But for the current Universe, the near-equality $R_0 \approx ct$ would produce an observable signature because it is equivalent to the condition $\ddot{a} \geq 0$. At least out to a redshift of ~ 1.8 , the Universe appears to be expanding with a slight positive acceleration (Riess et al. 2004). With ongoing observations of Type Ia supernovae, the value of \ddot{a} will continue to be refined and, with it, so too the value of $R_0 - ct$.

Clearly, more work needs to be done. There is little doubt that a cosmic horizon exists. It is required by the application of the corollary to Birkhoff's theorem to an infinite, homogeneous medium, and there is some evidence that we have already observed phenomena close to it. However, it may be that observational cosmology is not entirely consistent with the condition $R_0 \approx ct$ in the current epoch. If not, there must be some other reason for this apparent coincidence. Perhaps the assumption of an infinite, homogeneous universe is incorrect. Whatever the case may be, the answer could be even more interesting than the one we have explored here.

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