PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 4, April 1982

# THE COTYPESET OF A TORSION FREE ABELIAN GROUP OF RANK TWO

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ABSTRACT. The cotypeset (set of types of rank one factors) of a torsion free abelian group of rank two is characterized.

Let $G$ be a torsion free abelian group of finite rank, hereafter called simply a "group". The *typeset* of $G$, $\{\text{type } (x) \mid 0 \neq x \in G\}$, has been studied by several authors (e.g. [**3**, **5**, **7**, **8**, **9**, **10**]). However, the main problem: When is a set of types the typeset of a group $G$?, has not yet been solved, even for groups of rank two. The *cotypeset* of $G$, introduced in [**10**], is the set of types of all rank one factors of $G$. This set also seems to be important in the study of torsion free groups (see e.g. [**1**, **2**, **11**, **12**, **13**]). It therefore seems of interest to characterize those sets of types which are the cotypeset of a group $G$. In this paper we give necessary and sufficient conditions for a set of types to be the cotypeset of a rank two group.

Familiarity is assumed with the notions of type and characteristic (height vector) —see [**6**]. In particular, the inner type, IT, and outer type OT, of Warfield are used [**13**]. The symbols $\vee$ and $\wedge$ are used to denote the sup and inf, respectively, of collections of characteristics. If $t$ is a characteristic, $t^p$ denotes the value of $t$ at the prime $p$. If $\tau$ is a type, $\tau^p$ is called finite (infinite), if $t^p$ is finite (infinite) for some characteristic $t \in \tau$. If $\tau_1$ and $\tau_2$ are types with $\tau_1 \leq \tau_2$, then $\tau_2 - \tau_1$ is the type of $t_2 - t_1$, where $t_1 \in \tau_1$, $t_2 \in \tau_2$ are characteristics chosen so that $t_1 \leq t_2$ (note: $\infty - \infty = 0$). The type of $Z$ is denoted by $\mathbf{0}$. Finally, we call two types, $\tau_1$ and $\tau_2$, equivalent on a subset $P$ of the primes ($\tau_1 \sim \tau_2$ on $P$) if there exist characteristics $t_1 \in \tau_1$, $t_2 \in \tau_2$ such that $t_1^p = t_2^p$ for all $p \in P$.

The characterization of rank two groups by Beaumont-Wisner [**4**] is employed repeatedly. As usual, $h_G^p(x)$ denotes the $p$-height of an element $x$ in a group $G$, and $\text{type}_G(x)$ is the type of $x$ in $G$. We use $\langle\ \rangle$ ($\langle\ \rangle_*$) to denote the subgroup (pure subgroup) generated by a set of elements.

We begin with a simple lemma.

LEMMA 1. *Let $S = \{\sigma_i\}_{i \in I}$ be a set of types and $\sigma_0$ a type such that $\sigma_0 = \sigma_i \vee \sigma_j$ for all $i \neq j$ in $I$. Then*

(a) $\sigma_0 - \sigma_i \leq \sigma_j$ *for all $i \neq j$ in $I$.*

(b) $(\sigma_0 - \sigma_i) \wedge (\sigma_0 - \sigma_j) = \mathbf{0}$ *for all $i \neq j$ in $I$.*

Received by the editors January 15, 1981 and, in revised form, May 27, 1981.
1980 *Mathematics Subject Classification*. Primary 20K15.

PROOF. Obvious.

A special case of the main theorem is proved first.

THEOREM 1. *Let* $S = \{\sigma_i\}$ *be a finite or countable set of types. Then* $S$ *is the cotypeset of a group* $G$ *of rank two with* $\mathrm{IT}(G) = \mathbf{0}$ *if and only if there exists a type* $\sigma_0$ *such that*:

(1) $\sigma_0 = \sigma_i \vee \sigma_j$ *for all* $i \neq j$, *and if* $S$ *is finite*, $\sigma_0 = \sigma_i$ *for some* $i$.

(2) $\sigma_j - (\sigma_0 - \sigma_i) = \sigma_i - (\sigma_0 - \sigma_j)$ *for all* $i \neq j$.

PROOF. I. *Necessity*. Suppose $S$ is the cotypeset of a rank two group $G$ with $\mathrm{IT}(G) = \mathbf{0}$. Let $\sigma_0 = \mathrm{OT}(G)$, the outer type of $G$. Then the first claim of condition (1) is immediate. For all $i$, choose $x_i \in G$ such that $G/\langle x_i \rangle_*$ has type $\sigma_i$. If $S = \{\sigma_1, \ldots, \sigma_n\}$ is finite, choose $0 \neq x \in G$ such that $\langle x \rangle_* \neq \langle x_i \rangle_*$, $i = 1, \ldots, n$. Let $\sigma = \text{type } G/\langle x \rangle_*$. Then $\sigma \in S$ and $\sigma \vee \sigma_i = \sigma_0$ for all $i$. Thus, $\sigma = \sigma_0$.

To establish the remaining claim, for each $i$ define $\tau_i = \sigma_0 - \sigma_i$ and let $\tau_i' = $ type $x_i$. It is well known that since $G$ is rank two of inner type $\mathbf{0}$, $\tau_i' + \sigma_i = \sigma_0$. It follows that we can choose characteristics $t_i \in \tau_i$, $t_i' \in \tau_i'$ with $t_i^p = (t_i')^p$ for all primes $p$ except possibly those for which $\sigma_i^p = \sigma_0^p = \infty$. At all such $p$ we can set $t_i^p = 0$, although $(t_i')^p$ can be any nonnegative integer. Thus $t_i \leqslant t_i'$, but equality need not hold.

If $i \neq j$, let $X_i = G/\langle x_i \rangle_*$, $X_j = G/\langle x_j \rangle_*$. Then $X_i$ and $X_j$ are rank one groups such that $X_i \supseteq \langle x_j \rangle_*$, $X_j \supseteq \langle x_i \rangle_*$ under the natural embeddings. It is well known (see e.g. [**4**]) that $X_j/\langle x_i \rangle_* \cong X_i/\langle x_j \rangle_*$. The isomorphism implies that we can choose characteristics $s_i \in \sigma_i$, $s_j \in \sigma_j$ with $s_j - t_i' = s_i - t_j'$. Thus, in view of our earlier remarks, if $p$ is a prime with $s_i^p$ and $s_j^p$ finite, then $t_i^p = (t_i')^p$, $t_j^p = (t_j')^p$, and $s_j^p - t_i^p = s_i^p - t_j^p$. If $s_i^p = s_j^p = \infty$, then $t_i^p = t_j^p = 0$, so again $s_j^p - t_i^p = s_i^p - t_j^p$. Let $p$ be a prime for which $s_i^p = \infty$, $s_j^p < \infty$. Then $\infty = s_0^p = s_j^p + (t_j')^p$, so $(t_j')^p = \infty$. Thus, $0 = s_i^p - (t_j')^p = s_j^p - (t_i')^p$. Moreover, $\tau_i' \wedge \tau_j' = \mathbf{0}$, so $\tau_i' \sim \mathbf{0}$ on the set of all primes $p$ where $(\tau_j')^p = \infty$.

Since $\tau_i \leqslant \tau_i'$ the same claim holds for $\tau_i$. Thus, the equation $s_j^p - t_i^p = s_i^p - t_j^p$ holds for almost all $p$ where $s_i^p = \infty$, $s_j^p < \infty$, and where the equality fails both sides are finite. By symmetry the same statement is true for the set of primes where $s_i^p < \infty$, $s_j^p = \infty$. This establishes condition (2).

II. *Sufficiency*.

A. Choosing characteristics. Let $S = \{\sigma_i\}$ be a finite or countable set of types satisfying conditions (1) and (2). Define $\tau_i = \sigma_0 - \sigma_i$ as before. We show $\{\tau_i\}$ is *relatively disjoint*, that is, there exists a set of characteristics $t_i \in \tau_i$ such that $t_i^p \wedge t_j^p = 0$ for all primes $p$ and $i \neq j$. Suppose $t_1 \in \tau_1, \ldots, t_n \in \tau_n$ have been chosen such that:

(i) for all $p$ and $1 \leqslant i < j \leqslant n$, $t_i^p \wedge t_j^p = 0$,

(ii) $t_i^p = 0$ if $\tau_k^p = \infty$ for any $k \in Z^+$, $k \neq i$.

Condition (ii) is possible since if $\tau_k^p = \infty$ then $\sigma_0^p$ and $\sigma_i^p$ must both be infinite by Lemma 1(a) and Condition (1) of the theorem. Thus $\tau_i = \sigma_0 - \sigma_i$ has a characteristic which is 0 at all such $p$. Choose $t_{n+1}' \in \tau_{n+1}$. Note that if $(t_{n+1}')^p = \infty$ for some $p$ then $t_i^p = 0$ for $1 \leqslant i \leqslant n$. Furthermore, $\Pi_i = \{p \mid 0 < (t_{n+1}')^p \wedge t_i^p < \infty\}$ is finite for each $i$ by Lemma 1(b). Define $t_{n+1}^p = 0$ if $p \in \bigcup_{i=1}^n \Pi_i$ or if $\tau_k^p = \infty$ for some $k \in Z^+$, $k \neq n+1$; and $t_{n+1}^p = (t'_{n+1})^p$ otherwise. Then $t_{n+1} \in \tau_{n+1}$ and $t_1, \ldots, t_{n+1}$ satisfy (i) and (ii). By induction, a set $\{t_i\}$ of characteristics can be chosen with $t_i \in \tau_i$ and $t_i^p \wedge t_j^p = 0$ for all $p$, $i \neq j$.

B. Ito's construction [7]. Given a relatively disjoint set of types $\{\tau_i\}$, Ito constructs a rank two group $G$ with typeset $G = \{\tau_i\}$ as follows. Let $t_i \in \tau_i$ be a set of relatively disjoint characteristics. For each $i$, choose a rank one group $G_i$ of type $\tau_i$, and $x_i \in G_i$ whose characteristic is exactly $t_i$. Let $\{(a_i, b_i) \mid 1 \leqslant i < \infty\}$ be the set of all coprime pairs of integers with $b_i \geqslant 0$, numbered so that $\operatorname{Max}\{|a_i|, b_i\} \leqslant i$. Then if $H = \bigoplus_{i=3}^{\infty} \langle a_i x_1 + b_i x_2 - x_i \rangle$ and $G = \bigoplus_{i=1}^{\infty} G_i/H$, Ito shows that $G$ is a torsion free rank two group, and that the element $g_i = a_i \bar{x}_1 + b_i \bar{x}_2 = a_i(x_1 + H) + b_i(x_2 + H)$ has type $\tau_i$ in $G$.

Two additional facts about Ito's construction can be noted. First, he considers only the case when $\{\tau_i\}$ is infinite, the finite case having been settled in [3]. However, his construction also works if $\{\tau_i\}$ is finite. In this case, all but a finite number of the elements $\{g_i\}$ are assigned the type $\mathbf{0}$, which is necessarily in element of $\{\tau_i\}$. Second, an easy computation shows that the outer type of the rank two group constructed, $\mathrm{OT}(G)$, is the type represented by the characteristic $\bigvee_i t_i$.

Henceforth we assume $S = \{\sigma_i\}$ is a set of types satisfying conditions (1) and (2) of the theorem, $\tau_i = \sigma_0 - \sigma_i$ and $\{t_i\}$ is a relatively disjoint set of characteristics with $t_i \in \tau_i$. Let $G_1$ be the group constructed by Ito's method with typeset $G_1 = \{\tau_i\}$, and let $\sigma = \mathrm{OT}(G_1)$ with representative $s = \bigvee_i t_i$.

C. Changing $\mathrm{OT}(G_1)$. In general $\sigma \leqslant \sigma_0$, since each $\tau_i \leqslant \sigma_0$. Choose $s_0 \in \sigma_0$ with $s^p \leqslant s_0^p$ for all primes $p$. Let $P_1 = \{p \mid 0 < s_0^p - s^p < \infty$, $P_2 = \{p \mid s^p < \infty, s_0^p = \infty\}$. In this section we construct a rank two group $G \geqslant G_1$ such that:

(i) $\mathrm{IT}(G) = \mathbf{0}$,

(ii) $\mathrm{OT}(G) = \sigma_0$,

(iii) for all $g_i$, $\operatorname{type}_G(g_i) \sim \operatorname{type}_{G_1}(g_i)$ on the complement of $P_2$ (denoted $P_2^c$),

(iv) for all $g_i$ and $p \in P_2$, $h_G^p(g_i) < \infty$.

First note that if $|P_1| < \infty$ we already have $\sigma \sim \sigma_0$ on $P_2^c$. In this case we leave the $p$ height of all $g_i$ unchanged for $p \in P_2^c$. Otherwise, index the primes in $P_1$ in their natural order, say $P_1 = \{p_j\}$. It is easy to see that we can select a subsequence $(a_j, b_j)$, with all terms distinct, from our original sequence $(a_i, b_i)$ of coprime pairs of integers, such that: $\lim_{j \to \infty} j/p_j = 0$ and if $t_i^{p_j} > 0$ then

$$\det\begin{pmatrix} a_j & b_j \\ a_i & b_i \end{pmatrix} \equiv 0 \pmod{p_j}.$$

(Note that for each $p_j \in P_1$ there is at most one $t_i$ with $t_i^{p_j} > 0$.) Let

$$G_2 = \left\langle G_1, \{g_j/p_j^{n_j} \mid p_j \in P_1\} \right\rangle,$$

where $n_j = s_0^{p_j} - h_{G_1}^{p_j}(g_j)$. It is apparent that, for $p \notin P_1$, the $p$ height of any $g_i$ is unchanged in going from $G_1$ to $G_2$. Furthermore, if $h_{G_2}^{p_j}(g_i) > 0$ then

$$\det\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \equiv 0 \mod(p_j)$$

[5, part I]. But $|a_ib_j - a_jb_i| \leq 2ij$ and $\lim_{j\to\infty} j/p_j = 0$. Thus, we can have $h_{G_2}^{p_j}(g_i) > 0$ for only finitely many $p_j \in P_1$, and $\text{type}_{G_2}(g_i) = \text{type}_{G_1}(g_i)$ for all $g_i$. Since each element on $G_2$ is rationally dependent on some $g_i$, it follows that typeset $G_2 =$ typeset $G_1$. Finally, it is easy to see that the outer type of $G_1$ has been increased so that $\text{OT}(G_2) \sim \sigma_0$ on $P_2^c$.

We now consider the set of primes $P_2$, on which the outer type must be made infinite. Let $X_1 = G_2/\langle \bar{x}_1 \rangle_*$ and $X_2 = G_2/\langle \bar{x}_2 \rangle_*$. As in I (Necessity), $X_1/\langle \bar{x}_2 \rangle_* \overset{\theta}{\cong} X_2/\langle \bar{x}_1 \rangle_*$, where $\theta$ is uniquely determined by $G_2$. Let $R$ be the subring of $Q$ generated by $\{1/p \mid p \in P_2\}$. By definition of $P_2$, none of the rank one groups $X_1$, $\langle \bar{x}_1 \rangle_*$, $X_2$, $\langle \bar{x}_2 \rangle_*$ is $p$-divisible for $p \in P_2$. Thus $R \otimes_Z X_1/\langle \bar{x}_2 \rangle_* \overset{\theta'}{\cong} R \otimes_Z X_2/\langle \bar{x}_1 \rangle_*$ and the isomorphism $\theta'$ can be chosen so that

(*) $\theta'$ is an irrational $p$-adic unit on each $p$ component such that $p \in P_2$.

(**) The natural diagram commutes:

$$\begin{array}{ccc} X_1/\langle \bar{x}_2 \rangle_* & \overset{\theta}{\cong} & X_2/\langle \bar{x}_1 \rangle_* \\ \downarrow & & \downarrow \\ R \otimes_Z X_1/\langle \bar{x}_2 \rangle_* & \overset{\theta'}{\cong} & R \otimes_Z X_2/\langle \bar{x}_1 \rangle_* \end{array}$$

Let

$$G = \left\{(\alpha, \beta) \mid \alpha \in R \otimes X_1, \beta \in R \otimes X_2 \text{ and } \theta'\left(\alpha + \langle \bar{x}_2 \rangle_*\right) = \beta + \langle \bar{x}_1 \rangle_*\right\}.$$

The following facts about $G$ are easily verified:

(a) The map $\beta\bar{x}_1 + \alpha\bar{x}_2 \to (\alpha, \beta)$ gives a natural embedding of $G_2$ into $G$ (use (**)).

(b) Under this embedding $h_{G_2}^p(x) = h_G^p(x)$ for all $x \in G_2$ and $p \notin P_2$.

(c) $h_G^p(x) < \infty$ for all $0 \neq x \in G$ and $p \in P_2$ (use (*)).

(d) $\text{OT}(G)$ is infinite at all primes $p \in P_2$.

(e) $\text{OT}(G) \sim \text{OT}(G_2)$ on $P_2^c$.

(f) $\text{IT}(G) = \mathbf{0}$. (The embedding maps $\langle \bar{x}_1 \rangle_*$ and $\langle \bar{x}_2 \rangle_*$ onto distinct pure subgroups of $G$.)

In view of (a)–(f), $G$ is a group satisfying conditions (i)–(iv) stated at the beginning of this section.

D. The cotypeset of $G$. In this section we complete the proof of Theorem 1 by showing that the cotypeset of $G$ is exactly $S$. By the results of the previous section, $\text{OT}(G) = \sigma_0$ and the typeset of $G$ is $\{\tau_i'\}$ where $\tau_i' \sim \tau_i = \sigma_0 - \sigma_i$ on $P_2^c$ and $\tau_i' < \infty$ at all primes $p \in P_2$. For each $i$, let $g_i$ be an element of $G$ with type $\tau_i'$. Then, type $G/\langle g_i \rangle_* = \sigma_i'$ where $\sigma_i' + \tau_i' = \sigma_0$. Moreover, for all $i \neq j$, $\sigma_i' - \tau_j' = \sigma_j' - \tau_i'$ and $\tau_i' \wedge \tau_j' = \mathbf{0}$. We will show $\sigma_i' = \sigma_i$ for all $i$. The proof is divided into four cases.

*Case* I. $p \in P_2$. For $p \in P_2$, $\sigma_0^p$ is infinite while $\tau_i^p$ and $\tau_i'^p$ are both finite. Thus both $\sigma_i^p$ and $\sigma_i'^p$ are infinite on $P_2$.

*Case* II. $\sigma_0^p$ is finite. Let $P_3 = \{p \mid \sigma_0^p < \infty\}$. Then $\tau_i \sim \tau_i'$ on $P_3$ since $P_3 \subseteq P_2^c$. Since $\sigma_0 = \sigma_i + \tau_i = \sigma_i' + \tau_i'$, then $\sigma_i \sim \sigma_i'$ on $P_3$.

*Case* III. $\sigma_0^p = \infty$, $p \notin P_2$, $\sigma_i^p = \infty$. At these primes, $\tau_i' \sim \tau_i \sim \mathbf{0}$, and hence $\sigma_i'$ is infinite and equal to $\sigma_i$.

*Case* IV. $\sigma_0^p = \infty$, $\sigma_i^p < \infty$, $p \notin P_2$.

For each fixed $i$, this case can occur for only a finite number of primes $p$. This follows from the equations, $\sigma_i - \tau_j = \sigma_j - \tau_i$ (Condition (2)) and $\tau_i \wedge \tau_j = \mathbf{0}$ (Lemma 1(b)). Since $\tau_i$ infinite implies $\tau_i'$ infinite, then $\sigma_i'$ is finite and hence equivalent to $\sigma_i$ on this finite set of primes. Cases I–IV imply $\sigma_i = \sigma_i'$ for all $i$.

REMARK 1. In the proof of sufficiency (II) we have tacitly assumed $S = \{\sigma_i\}$ contains more than one element. If $S = \{\sigma_0\}$ is a set containing only one type, it is easy to construct a rank two group $G$, homogeneous of type $\mathbf{0}$, whose cotypeset is $S$.

REMARK 2. Theorem 1 would be a trivial consequence of §§5 and 6 of **[10]** if the results therein were true.

However, as the following example shows, the results in **[10]** are incorrect.

EXAMPLE. The following adaptation of an example of Dubois shows that conditions (1)–(4) of Proposition 4 of **[10]** are not sufficient to guarantee that a set of type-cotype pairs $\{(\tau_i, \sigma_i) \mid 1 \leqslant i < \infty\}$ is the type-cotype set of a rank two group. Let $P$ be the set of primes in $Z$ and write $P = \bigcup_{i=1}^{\infty} P_i$ as a disjoint union of countable sets $P_i$. Let $t_1$ be the characteristic defined by $t_1^p = 1$ for $p \in P_1$, $t_1^p = 0$ otherwise. For $i > 1$ define $t_i^p = 1$ for $p \in P_i$, $t_i^p = \infty$ if $p$ is the $i$th prime in $P_j$ for some $j < i$, $t_i^p = 0$ otherwise. Let $\tau_i$ be the type of $t_i$ for $1 \leqslant i < \infty$, and let $\sigma_0 = \text{type}\, Q$, $\sigma_i = \sigma_0 - \tau_i$ for $1 \leqslant i < \infty$. It is easy to check that the set $\{(\tau_i, \sigma_i) \mid 1 \leqslant i < \infty\}$ satisfies (1)–(4) of Proposition 4 of **[10]**. However, as shown by Dubois, $\{\tau_i \mid 1 \leqslant i < \infty\}$ is not the type set of any rank two group.

A final lemma precedes the Main Theorem.

LEMMA 2. *Let $S = \{\sigma_i\}$ be a finite or countable set of types. Then $S$ is the cotypeset of a rank two group $G$ if and only if there exists a type $\tau_0$ such that* (1) $\tau_0 \leqslant \sigma_i$ *for all $i$,* (2) $\{\sigma_i - \tau_0\}$ *is the cotypeset of a rank two group of inner type* $\mathbf{0}$. (*That is,* $\{\sigma_i - \tau_0\}$ *satisfies the conditions of Theorem* 1.)

PROOF. Suppose $S = \{\sigma_i\}$ is the cotypeset of a rank two group $G$. Let $\tau_0 = \mathrm{IT}(G)$. Clearly, $\tau_0 \leqslant \sigma_i$ for all $i$. As in **[10]**, $G$ can be written $G = U \otimes G_0$, where $U$ is a rank one group of type $\tau_0$ and $G_0$ is a rank two group of inner type $\mathbf{0}$. Furthermore, $G_0$ can be chosen so that $\mathrm{OT}(G_0) \sim \mathbf{0}$ on the set of primes $p$ where $\tau_0^p$ is infinite. Let $K$ be a pure rank one subgroup of $G$. Then there is an obvious isomorphism: $G/K \cong U \otimes G_0/U \otimes K_0 \cong U \otimes (G_0/K_0)$, where $K_0$ is a pure rank one subgroup of $G_0$. Conversely, if $K_0$ is a pure rank one subgroup of $G_0$, then $K = U \otimes K_0$ is a pure rank one subgroup of $G = U \otimes G_0$, and the same isomorphism holds. Furthermore, $\text{type}\, G/K = \tau_0 + \text{type}\, G_0/K_0$, and therefore, $\text{type}\, G_0/K_0 = \text{type}\, G/K - \tau_0$, in view of our choice of $G_0$. It follows that cotypeset $G_0$ is exactly $\{\sigma_i - \tau_0\}$.

Conversely, suppose there exists a type $\tau_0$ such that $\{\sigma_i - \tau_0\}$ is the cotypeset of a rank two group $G_0$ of inner type $\mathbf{0}$. If $\tau_0 < \text{type}\, Q$ let $U$ be a rank one group of type $\tau_0$. Then $G = U \otimes G_0$ is a (reduced) group $G$ with cotypeset $\{\sigma_i\}$. If $\tau_0 = \text{type}\, Q$, then $S = \{\text{type}\, Q\}$ by (1) and (2) holds trivially. As mentioned previously, in this case it is not difficult to construct the desired rank two group $G$.

THEOREM 2. *Let $S = \{\sigma_i\}$ be a finite or countable set of types. Then $S$ is the cotypeset of a rank two group $G$ if and only if there exists a type $\sigma_0$ such that $\sigma_i \vee \sigma_j = \sigma_0$ if $i \neq j$, subject to the additional requirement that $\sigma_0 \in S$ if $|S| < \infty$.*

PROOF. The condition is clearly necessary. To prove sufficiency, we produce a type $\tau_0$ such that $\{\sigma_i - \tau_0\}$ satisfies the conditions of Theorem 1 and apply Lemma 2.

In view of our earlier remarks, we may assume $|S| > 1$. Choose $s_0 \in \sigma_0$, $s_1 \in \sigma_1$ with $s_1^p \leqslant s_0^p$ for all $p$. If $\{s_1, \ldots, s_n\}$ have been chosen, choose $s_{n+1} \in \sigma_{n+1}$ with $s_{n+1}^p \leqslant s_0^p$, for all $p$, and with $s_{n+1}^p \vee s_j^p = s_0^p$, for all $p$ and $1 \leqslant j \leqslant n$. This choice is possible since $\sigma_{n+1} \vee \sigma_j = \sigma_0$, $1 \leqslant j \leqslant n$. Thus, by induction, we have a set of characteristics $s_i \in \sigma_i$ with $s_i \vee s_j = s_0$. Note that for all $p$ there is at most one $s_i$ with $s_i^p < s_0^p$. Let $t_0$ be the characteristic defined by $t_0^p = \wedge s_i^p$ and let $\tau_0$ be the type represented by $t_0$. It is straightforward to verify that $\{\sigma_i' = \sigma_i - \tau_0 \mid 1 \leqslant i < \infty\}$ satisfies conditions (1) and (2) of Theorem 1 with respect to the type $\sigma_0' = \sigma_0 - \tau_0$. The Theorem now follows from Lemma 2.

The inner type $\tau_0$ is not, in general, uniquely determined by the cotypeset $S$, due to the different possible choices for the characteristics $s_i \in \sigma_i$. A characterization of the possible inner types for a group $G$ with cotypeset $S$ is given by

THEOREM 3. *Let $S = \{\sigma_i\}$ and $\tau_0$ be as in Theorem 2. Let $S_i = \{p \mid s_i^p = t_0^p < \infty$ and $s_0^p = \infty\}$. Then a type $\tau$ with characteristic $t$ is the inner type of a rank 2 group $G$ with cotypeset $S$ if and only if for all $i$, $\tau \leqslant \sigma_i$ and $\{p \mid t^p \neq t_0^p\} \cap S_i$ is finite.*

PROOF. If $\tau$ is a type satisfying these hypotheses, it is easy to show that $\{\sigma_i - \tau\}$ satisfies the conditions of Theorem 1. Thus, as in the proof of Lemma 2, there is a rank 2 group $G$ of inner type $\tau$ and cotypeset $S$.

Conversely, if $\tau$ is the inner type of a rank 2 group with cotypeset $S$, then $\{\sigma_i - \tau\}$ satisfies the conditions of Theorem 1.

In particular, if $i \neq j$, then

$$(\dagger) \quad (\sigma_j - \tau) - [(\sigma_0 - \tau) - (\sigma_i - \tau)] = (\sigma_i - \tau) - [(\sigma_0 - \tau) - (\sigma_j - \tau)].$$

Let $P_i = S_i \cap \{p \mid t^p \neq t_0^p\}$. Then, on this set of primes the left hand side of (†) is $\sim \mathbf{0}$, while the right-hand side is $\sim (\sigma_i - \tau)$. Since, on $P_i$, $\sigma_i \sim \tau_0$, we have $\tau_0 - \tau \sim \mathbf{0}$ on this set. It follows that $P_i$ must be finite.

REMARK 3. The "dual" of Theorem 2 is false. There exists a set of types $T = \{\tau_i \mid 1 \leqslant i < \infty\}$ and a type $\tau_0$ such that $\tau_0 = \tau_i \wedge \tau_j$ for all $i \neq j$, but $T$ is not the typeset of a rank two group [**5**, Example 1].

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