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# THE COUNTING VECTOR OF A SIMPLE GAME[1]

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ABSTRACT. The counting vector of a simple game is the vector $f = (f(1), f(2), \cdots, f(n))$ where $f(i)$ is the number of winning coalitions containing the player $i$. In this paper, we show that the counting vector of a weighted majority game determines the game uniquely. With the aid of the counting vector we find an upper bound on the number of weighted majority games.

1. **Preliminaries on simple games.** A simple game is a pair $G = (N; W)$, where $N = \{1, 2, \cdots, n\}$ is a set of $n$ members and $W$ is a set of subsets of $N$. The members of $N$ are called players; subsets of $N$ are called coalitions. The elements of $W$ are called winning coalitions. A simple game is called monotone if every superset of a winning coalition is itself a winning coalition. A weighted majority game is a simple game for which there exist $n$ nonnegative numbers $w_1, w_2, \cdots, w_n$ and a positive number $q$, such that, $S$ is a winning coalition if and only if $w(S) = \sum_{i \in S} w_i \geqq q$. $w = [w_1, w_2, \cdots, w_n; q]$ is called the representation of the game. A weighted majority game is denoted by $G = (N; w)$ where $w$ is its representation.

$G$ is called constant-sum if for each coalition $S$ exactly one of the two coalitions $S$ and $N - S$ is winning.

2. **The counting vector theorem.** Given a simple game $G = (N; W)$, we denote by $f(i)$ the number of winning coalitions containing player $i$, and by $f$ the vector $f = (f(1), f(2), \cdots, f(n))$ called the counting vector of the game.

THEOREM 2.1. *Let*

$$G_1 = (N; [w^{(1)}, w_2^{(1)}, \cdots, w_n^{(1)}; q^{(1)}])$$

*and*

$$G_2 = (N; [w_1^{(2)}, w_2^{(2)}, \cdots, w_n^{(2)}; q^{(2)}])$$

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*be two weighted majority games. Their counting vectors* $f_1$ *and* $f_2$ *are equal if and only if* $G_1 = G_2$.

PROOF. Suppose that $G_1 \neq G_2$ but $f_1 = f_2$, i.e., for every $i \in N$,

$$f_1(i) = f_2(i). \tag{2.1}$$

(We may assume that $f \neq 0$ since otherwise we have the trivial game $(N; \phi)$.) Let $W_1$, $W_2$ be the sets of winning coalitions in $G_1$, $G_2$ respectively. We set $\Omega_1 = W_1 - W_2$, $\Omega_2 = W_2 - W_1$. If $W_k \subset W_p$ $(k;\ p = 1, 2,$ $k \neq p)$, then $f_k(i) \leqq f_p(i)$ for every $i$ and inequality holds for at least one player $i_0$, in contradiction to (2.1); hence,

$$\Omega_1 \neq \phi, \qquad \Omega_2 \neq \phi. \tag{2.2}$$

For each $S \in \Omega_1$ and $T \in \Omega_2$ we have

$$w^{(1)}(S) > w^{(1)}(T). \tag{2.3}$$

Summing these inequalities over $\Omega_2$ for a fixed $S \in \Omega_1$, we have

$$|\Omega_2|\, w^{(1)}(S) > \sum_{T \in \Omega_2} w^{(1)}(T) \tag{2.4}$$

where $|\Omega|$ is the number of elements of $\Omega$. Summing all the inequalities (2.4) over $\Omega_1$, we have

$$|\Omega_2| \sum_{S \in \Omega_1} w^{(1)}(S) > |\Omega_1| \sum_{T \in \Omega_2} w^{(1)}(T). \tag{2.5}$$

Let $\phi_k(i)$ be the number of coalitions in $\Omega_k$ containing player $i$. Since $\Omega_1 = W_1 - (W_1 \cap W_2)$ and $\Omega_2 = W_2 - (W_1 \cap W_2)$, $f_1(i) = f_2(i)$ implies $\phi_1(i) = \phi_2(i)$. Denoting the common value by $\phi(i)$, we have

$$\sum_{S \in \Omega_1} w^{(1)}(S) = \sum_{i \in N} \phi(i) w_i^{(1)}, \tag{2.6}$$

$$\sum_{T \in \Omega_2} w^{(1)}(T) = \sum_{i \in N} \phi(i) w_i^{(1)}. \tag{2.7}$$

Since $q^{(1)}$ is positive and $\Omega_1 \neq \phi$, $\Omega_2 \neq \phi$, we have, from (2.5), (2.6), and (2.7),

$$|\Omega_1| < |\Omega_2|. \tag{2.8}$$

Using the representation $[w_1^{(2)}, w_2^{(2)}, \cdots, w_n^{(2)}; q^{(2)}]$ of $G_2$ we have

$$|\Omega_2| < |\Omega_1| \tag{2.9}$$

in contradiction to (2.8) and hence $f_1 \neq f_2$.

Corollary 2.1. *Let $G_1 = (N; W_1)$ be a weighted majority game and $G_2 = (N; W_2)$ a simple game. If $G_1 \neq G_2$ and*

$$|W_1| \geqq |W_2|, \tag{2.10}$$

*then $f_1 \neq f_2$.*

Proof. (2.10) implies

$$|\Omega_1| \geqq |\Omega_2|. \tag{2.11}$$

If $f_1 = f_2$ then as in the proof of Theorem 2.1, $|\Omega_1| < |\Omega_2|$. Hence $f_1 \neq f_2$.

Corollary 2.2. *Let $G_1$ be a constant-sum weighted majority game and $G_2$ a constant-sum simple game. If $G_1 \neq G_2$, then $f_1 \neq f_2$.*

Proof. In any constant-sum simple game, the number of winning coalitions is $2^{n-1}$ ($n$ being the number of players), hence $|W_1| = |W_2|$. The assumptions of Corollary 2.1 are valid, hence $f_1 \neq f_2$.

## 3. An upper bound on the number of weighted majority games.

In **[1]**, J. R. Isbell raised the problem of finding an upper bound on the number of weighted majority games. No answer was given, except for the bound $2^{2^n}$, which is the number of sets of coalitions in $N$. Using the counting vector concept a smaller value is obtainable for the upper bound.

It is evident that for every $i$, $0 \leqq f(i) \leqq 2^{n-1}$. If $f(i) = 0$ for some $i$, then $N \notin W$; hence $W = \phi$, i.e., the game is the trivial game $G_0 = (N; \phi)$. For all other games $1 \leqq f(i) \leqq 2^{n-1}$. The number of vectors

$$a = (a_1, a_2, \cdots, a_n), \qquad a_i = 1, 2, \cdots, 2^{n-1}, \quad i = 1, 2, \cdots, n, \tag{3.1}$$

is $2^{n(n-1)}$. Since not every vector of the type (3.1) is a counting vector, there are less than $2^{n(n-1)}$ weighted majority $n$-person games.

Two games, $G_1 = (N; W_1)$ and $G_2 = (N; W_2)$, are said to be equivalent if there exists a permutation $\pi$ of $N$ such that

$$W_2 = \{\pi S \colon S \in W_1\} \tag{3.2}$$

where $\pi S = \{\pi(i) \colon i \in S\}$.

It is evident that this relation is an equivalence. If we identify equivalent games, $f$ is no longer a vector but an unordered set of $n$ numbers. Hence their number is less than $\binom{2^{n-1}+n-1}{n}$, the number of $n$-selections of $2^{n-1}$ given elements.

The following example shows that there exist two nonequivalent monotone games which have the same counting vector.

Example. Let $N$ be a set of seven players $\{1, 2, 3, 4, 5, 6, 7\}$. $W_1$ consists of all 4, 5, 6, and 7 player coalitions and of the following 3 players coalitions: (2, 5, 3), (3, 7, 1), (1, 4, 2), (2, 6, 7), (3, 6, 4), (1, 6, 5), and

(4, 5, 7). $W_2$ consists of all 4, 5, 6, and 7 player coalitions and of the following 3 player ones: (2, 4, 3), (3, 7, 1), (1, 5, 2), (2, 6, 7), (3, 6, 4), (1, 6, 5), and (4, 5, 7).

The two games $(N; W_1)$ and $(N; W_2)$ are monotone and it is easy to see that they have the same counting vector. However, $W_1$ contains no two disjoint 3 player coalitions while $W_2$ does, and thus they are not equivalent.

4. **The desirability relation.** Two players $i$ and $j$, in a simple game $(N; W)$ are called symmetric if for every coalition $S$ not containing $i$ nor $j$, $S \cup \{i\}$ is winning if and only if $S \cup \{j\}$ is winning; this is denoted by $i \sim j$. If for every coalition $S$ not containing $i$ nor $j$, $S \cup \{j\} \in W$ implies $S \cup \{i\} \in W$, then $i$ is said to be more desirable than $j$; this is denoted by $i \gtrsim j$:

If $i \gtrsim j$ and $j \gtrsim i$ then $i \sim j$. If $i \gtrsim j$ but $i$ and $j$ are not symmetric, we say that $i$ is strictly more desirable than $j$; this is denoted by $i \succ j$. In a weighted majority game with a representation $[w_1, w_2, \cdots, w_n; q]$, $w_1 \geqq w_j$ implies $i \gtrsim j$ and there exists a representation $[w_1^*, w_2^*, \cdots, w_n^*; q^*]$ such that $i \sim j$ implies $w_i^* = w^*$[3]. The counting vector strictly preserves the desirability relation, i.e., $f(i) > f(j)$ implies $i \succ j$ and $f(i) = f(j)$ implies $i \sim j$ [**2**]. This enables us to reduce the number of unknowns when seeking a representation of a given weighted majority game.

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