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The coupled seiberg-witten equations, vortices, and moduli spaces of stable pairs

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Abstract: We introduce coupled Seiberg-Witten equations, and we prove, using a generalized vortex equation, that, for Kaehler surfaces, the moduli space of solutions of these equations can be identified with a moduli space of holomorphic stable pairs. In the rank 1 case, one recovers Witten's result identifying the space of irreducible monopoles with a moduli space of divisors. As application, we give a short proof of the fact that a rational surface cannot be diffeomorphic to a minimal surface of general type.

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The Coupled Seiberg-Witten Equations, Vortices, and Moduli Spaces of Stable Pairs

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0 Introduction

Recently, Seiberg and Witten [W] introduced new invariants of 4-manifolds, which are defined by counting solutions of a certain non-linear differential equation.

The new invariants are expected to be equivalent to Donaldson's polynomial-invariants—at least for manifolds of simple type [KM 1]—and they have already found important applications, like e.g. in the proof of the Thom conjecture by Kronheimer and Mrowka [KM 2].

Nevertheless, the equations themselves remain somewhat mysterious, especially from a mathematical point of view.

The present paper contains our attempt to understand and to generalize the Seiberg-Witten equations by coupling them to connections in unitary vector bundles, and to relate their solutions to more familiar objects, namely stable pairs.

Fix a Spin^c-structure on a Riemannian 4-manifold X, and denote by $\Sigma^{\pm}$ the associated spinor bundles. The equations which we will study are:

$$\left\{ \begin{array}{ccc} \not{D}_{A,b}\Psi & = & 0 \\ \Gamma(F_{A,b}^{+}) & = & (\Psi\bar{\Psi})_0 \end{array} \right.$$

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This is a system of equations for a pair (A, Ψ) consisting of a unitary connection in a unitary bundle E over X, and a positive spinor $\Psi \in A^0(\Sigma^+ \otimes E)$. The symbol b denotes a connection in the determinant line bundle of the spinor bundles $\Sigma^\pm$ and $\not{D}_{A,b} : \Sigma^+ \otimes E \longrightarrow \Sigma^- \otimes E$ is the Dirac operator obtained by coupling the connection in Σ^+ defined by b (and by the Levi-Civita connection in the tangent bundle) with the variable connection A in E.

These equations specialize to the original Seiberg-Witten equations if E is a line bundle. We show that the coupled equations can be interpreted as a differential version of the generalized vortex equations [JT].

Vortex equations over Kähler manifolds have been investigated by Bradlow [B1], [B2] and by Garcia-Prada [G1], [G2]: Given a pair $(\mathcal{E}, \varphi)$ consisting of a holomorphic vector bundle with a section, the vortex equations ask for a Hermitian metric h in $\mathcal{E}$ with prescribed mean curvature: more precisely, the equations—which depend on a real parameter τ—are

$$i\Lambda F_h = \frac{1}{2}(\tau \mathrm{id}_{\mathcal{E}} - \varphi \otimes \varphi^*).$$

A solution exists if and only if the pair $(\mathcal{E}, \varphi)$ satisfies a certain stability condition (τ-stability), and the moduli space of vortices can be identified with the moduli space of τ-stable pairs. A GIT construction of the latter space has been given by Thaddeus [T] and Bertram [B] if the base manifold is a projective curve, and by Huybrechts and Lehn [HL1], [HL2] in the case of a projective variety. Other constructions have been given by Bradlow and Daskalopoulus [BD1], [BD2] in the case of a Riemann surface, and by Garcia-Prada for compact Kähler manifolds [G2]. In this connection also [BD2] is relevant. In this note we prove the following result:

Theorem 0.1 *Let (X, g) be a Kähler surface of total scalar curvature σ_g, and let Σ be the canonical* Spin^c*-structure with associated Chern connection c. Fix a unitary vector bundle E of rank r over X, and define $\mu_g(\Sigma^+ \otimes E) := \frac{\deg_g(E)}{r} + \sigma_g$.*

Then for $\mu_g < 0$, the space of solutions of the coupled Seiberg-Witten equation is isomorphic to the moduli space of stable pairs of topological type E, with parameter σ_g.

If the constant $\mu_g(\Sigma^+ \otimes E)$ is positive, one simply replaces the bundle E with $E^\vee \otimes K_X$, where K_X denotes the canonical line bundle of X (cf. Lemma 3.1).

Note that the above theorem gives a complex geometric interpretation of the moduli space of solutions of the coupled Seiberg-Witten equation associated to <u>all</u> Spin^c-structures on X: The change of the Spin^c-structure is equivalent to tensoring E with a line bundle.

Notice also that in the special case $r = 1$ one recovers Witten's result identifying the space of irreducible monopoles on a Kähler surface with the set of all divisors associated to line bundles of a fixed topological type; the stability condition which he mentions is the rank-1 version of the stable pair-condition.

Having established this correspondence, we describe some of the basic properties of the moduli spaces, and give a first application: We show that minimal surfaces of general type cannot be diffeomorphic to rational ones. This provides a short proof of one of the essential steps in Friedman and Qin's proof of the Van de Ven conjecture [FQ]. More detailed investigations and applications will appear in a later paper.

We like to thank A. Van de Ven for very helpful questions and remarks.

1 Spin^c-structures and almost canonical classes

The complex spinor group is defined as $\mathrm{Spin}^c := \mathrm{Spin} \times_{\mathbb{Z}_2} S^1$, and there are two non-split exact sequences

$$\begin{array}{c} 1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c \longrightarrow \mathrm{SO} \longrightarrow 1 \\ 1 \longrightarrow \mathrm{Spin} \longrightarrow \mathrm{Spin}^c \longrightarrow S^1 \longrightarrow 1 \end{array}$$

In dimension 4, $\mathrm{Spin}^c(4)$ can be identified with the subgroup of $\mathrm{U}(2)\times\mathrm{U}(2)$ consisting of pairs of unitary matrices with the same determinant, and one has two commutative diagrams:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathrm{Spin}(4) & \longrightarrow & \mathrm{SO}(4) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & S^1 & \longrightarrow & \mathrm{Spin}^c(4) & \longrightarrow & \mathrm{SO}(4) & \longrightarrow & 1\\
 & & \downarrow (\cdot)^2 & & {\scriptstyle\det}\downarrow & \nwarrow \Delta & \uparrow & & \\
 & & S^1 & = & S^1 & \underset{\det}{\longleftarrow} & \mathrm{U}(2) & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \qquad (1)$$

where $\Delta : \mathrm{U}(2) \longrightarrow \mathrm{Spin}^c(4) \subset \mathrm{U}(2)\times\mathrm{U}(2)$ acts by $a \longrightarrow \left(\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \det a \end{pmatrix}, a\right)$, and

$$\begin{array}{ccccccccccc}
 & & & & & & & & 1 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & 1 & & 1 & & \mathbb{Z}_2 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \rightarrow & S^1 & \rightarrow & \mathrm{Spin}^c(4) & \rightarrow & \mathrm{SO}(4) & \rightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle (\lambda^+,\lambda^-)} & & \\
 & & 1 & \rightarrow & S^1\times S^1 & \rightarrow & \mathrm{U}(2)\times\mathrm{U}(2) & \stackrel{\mathrm{ad}}{\rightarrow} & \mathrm{SO}(3)\times\mathrm{SO}(3) & \rightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & S^1 & \stackrel{(\cdot)^2}{\rightarrow} & S^1 & \rightarrow & 1 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 1 & & 1 & & & &
\end{array}$$

where $\lambda^\pm : \mathrm{SO}(4) \longrightarrow \mathrm{SO}(3)$ are induced by the two projections of $\mathrm{Spin}(4) = \mathrm{SU}(2)^+ \times \mathrm{SU}(2)^-$ [HH]. $\lambda^\pm$ can be also be seen as the representations of $\mathrm{SO}(4)$ in $\Lambda^2_\pm(\mathbb{R}^4) \simeq \mathbb{R}^3$ induced by the canonical representation in $\mathbb{R}^4$.

Let X be a closed, oriented 4-manifold. Given any principal $\mathrm{SO}(4)$-bundle P over X, we denote by $P^\pm$ the induced principal $\mathrm{SO}(3)$-bundles. If $\hat{P}$ is a $\mathrm{Spin}^c(4)$-bundle, we let $\Sigma^\pm$ be the associated $\mathrm{U}(2)$-vector bundles, and we set (via the vertical determinant-map in (1)) $\det(\hat{P}) = L$, so that $\det(\Sigma^\pm) = L$.

Lemma 1.1 *Let P be a principal $SO(4)$-bundle over X with characteristic classes $w_2(P) \in H^2(X, \mathbb{Z}_2)$, and $p_1(P), e(P) \in H^4(X, \mathbb{Z})$. Then*
i) P lifts to a principal $\mathrm{Spin}^c(4)$*-bundle $\hat{P}$ iff $w_2(P)$ lifts to an integral cohomology class.*
ii) Given a class $L \in H^2(X, \mathbb{Z})$ with $w_2(P) \equiv \bar{L} (mod\ 2)$, the $\mathrm{Spin}^c(4)$*-lifts $\hat{P}$ of P with* $\det \hat{P} = L$ *are in 1-1 correspondence with the 2-torsion elements in $H^2(X, \mathbb{Z})$.*
iii) Let $\hat{P}$ be a $\mathrm{Spin}^c(4)$*-principal bundle with $P \simeq \hat{P}/S^1$, and let $L = \det \hat{P}$. Then the Chern classes of $\Sigma^\pm$ are:*

$$\begin{array}{ll} c_1(\Sigma^\pm) & = L \\ c_2(\Sigma^\pm) & = \frac{1}{4}\left(L^2 - p_1(P) \mp 2e(P)\right) \end{array}$$

Proof: [HH] and the diagrams above. ■

Consider now a Riemannian metric g on X, and let P be the associated principal SO(4)-bundle. In this case the real vector bundles associated to $P^{\pm}$ via the standard representations are the bundles $\Lambda^2_{\pm}$ of (anti-) self-dual 2-forms on X.

The integral characteristic classes of P are given by $p_1(P) = 3\sigma$ and $e(P) = e$, where σ and e denote the signature and the Euler number of the oriented manifold X. Furthermore, $w_2(P)$ always lifts to an integral class, the lifts are precisely the characteristic elements in $H^2(X, \mathbb{Z})$, i.e. the classes L with $x^2 \equiv x \cdot L$ for every $x \in H^2(X, \mathbb{Z})$ [HH].

Let T_X be the tangent bundle of X, and denote by Λ^p the bundle of p-forms on X. The choice of a $\mathrm{Spin}^c(4)$-lift $\hat{P}$ of P with associated U(2)-vector bundles $\Sigma^{\pm}$ defines a vector bundle isomorphism $\gamma : \Lambda^1 \otimes \mathbb{C} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\Sigma^+, \Sigma^-)$. There is also a $\mathbb{C}$-linear isomorphism $(\cdot)^{\#} : \mathrm{Hom}_{\mathbb{C}}(\Sigma^+, \Sigma^-) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\Sigma^-, \Sigma^+)$ which satisfies the identity:

$$\gamma(u)^{\#}\gamma(v) + \gamma(v)^{\#}\gamma(u) = 2g^{\mathbb{C}}(u, v)\mathrm{id}_{\Sigma^+},$$

and $\gamma(u)^{\#} = \gamma(u)^* = g(u, u)\gamma(u)^{-1}$ for real non-vanishing cotangent vectors u.

It is convenient to extend the homomorphisms $\gamma(u)$ to endomorphisms of the direct sum $\Sigma := \Sigma^+ \oplus \Sigma^-$. Putting $\gamma(u)|_{\Sigma^-} := -\gamma(u)^{\#}$, we obtain a vector-bundle homomorphism $\gamma : \Lambda^1 \otimes \mathbb{C} \longrightarrow \mathrm{End}_0(\Sigma)$, which maps the bundle Λ^1 of real 1-forms into the bundle of trace-free skew-Hermitian endomorphisms of Σ. With this convention, we get:

$$\gamma(u) \circ \gamma(v) + \gamma(v) \circ \gamma(u) = -2g^{\mathbb{C}}(u, v)\mathrm{id}_{\Sigma}.$$

Consider the induced homomorphism

$$\Gamma : \Lambda^2 \otimes \mathbb{C} \longrightarrow \mathrm{End}_0(\Sigma)$$

defined on decomposable elements by

$$\Gamma(u \wedge v) := \frac{1}{2}[\gamma(u), \gamma(v)].$$

The restriction $\Gamma|_{\Lambda^2}$ identifies the bundle Λ^2 with the bundle $\mathrm{ad}_0(\hat{P}) \simeq \mathrm{ad}(P)$ of skew-symmetric endomorphisms of the tangent bundle of X.

Λ^2 splits as an orthogonal sum $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ and Γ maps the bundle $\Lambda^2_\pm \otimes \mathbb{C}$ (respectively $\Lambda^2_\pm$) isomorphically onto the bundle $\mathrm{End}_0(\Sigma^\pm) \subset \mathrm{End}(\Sigma)$ ($su(\Sigma^\pm) \subset su(\Sigma)$) of trace-free (trace free skew-Hermitian) endomorphisms of $\Sigma^\pm$.

We give now an explicit description of the two spinor bundles $\Sigma^\pm$ and of the map Γ in the case of a $\mathrm{Spin}^c(4)$-structure coming from an almost Hermitian structure.

Definition 1.2 *A characteristic element $K \in H^2(X, \mathbb{Z})$ is an almost canonical class if $K^2 = 3\sigma + 2e$.*

Such classes exist on X if and only if X admits an almost complex structure. More precisely:

Proposition 1.3 *(*Wu*) $K \in H^2(X, \mathbb{Z})$ is an almost canonical class if and only if there exists an almost complex structure J on X which is compatible with the orientation, such that $K = c_1(\Lambda^{10}_J)$.*

Proof: [HH] ■

Here we denote, as usual, by Λ^{pq}_J the bundle of (p, q)-forms defined by the almost complex structure J.

Notice that any almost complex structure J on X can be deformed into a g-orthogonal one, and that J is g-orthogonal iff g is J-Hermitian. The choice of a g-orthogonal almost complex structure J on X corresponds to to a reduction of the SO(4)-bundle P of X to a $U(2)$-bundle via the inclusion $\mathrm{U}(2) \subset \mathrm{SO}(4)$; since the inclusion factors through the embedding $\Delta : \mathrm{U}(2) \longrightarrow \mathrm{Spin}^c(4)$ (see diagram (1)), this reduction defines a unique $\mathrm{Spin}^c(4)$-bundle $\hat{P}_J$ over X. By construction we have $\hat{P}_J/S^1 \simeq P$, and $\det \hat{P}_J = -K$.

Proposition 1.4 *Let J be a g-orthogonal almost complex structure on X, compatible with the orientation.*
i) The spinor bundles $\Sigma^\pm_J$ of $\hat{P}_J$ are:

$$\Sigma^+_J \simeq \Lambda^{00} \oplus \Lambda^{02}_J, \quad \Sigma^-_J \simeq \Lambda^{01}_J.$$

ii) The map $\Gamma : \Lambda^2_+ \otimes \mathbb{C} \longrightarrow \mathrm{End}_0(\Sigma^+_J)$ is given by

$$\Lambda^{20}_J \oplus \Lambda^{02}_J \oplus \Lambda^{00}\omega_g \ni (\lambda^{20}, \lambda^{02}, \omega_g) \stackrel{\Gamma}{\longmapsto} 2 \begin{bmatrix} -i & -*(\lambda^{20} \wedge \cdot) \\ \lambda^{02} \wedge \cdot & i \end{bmatrix} \in \mathrm{End}_0(\Lambda^{00} \oplus \Lambda^{02}).$$

Proof: i) $c_1(\Sigma_J^+) = c_1(\Sigma_J^-) = -K$, $c_2(\Sigma_J^+) = \frac{1}{4}[K^2 - 3\sigma - 2e]$, $c_2(\Sigma_J^-) = \frac{1}{4}[K^2 - 3\sigma + 2e] = c_2(\Sigma^+) + e$, and U(2)-bundles on a 4-manifold are classified by their Chern classes.
ii) With respect to a suitable choice of the isomorphisms i), the Clifford map γ acts by

$$\gamma(u)(\varphi + \alpha) = \sqrt{2}\left(\varphi u^{01} - i\Lambda_g u^{10} \wedge \alpha\right),$$
$$\gamma(u)^{\#}(\theta) = \sqrt{2}\left(i\Lambda_g(u^{10} \wedge \theta) - u^{01} \wedge \theta\right), \tag{3}$$

where $\Lambda_g : \Lambda_J^{pq} \longrightarrow \Lambda_J^{p-1,q-1}$ is the adjoint of the map $\cdot \wedge \omega_g$ [H1]. ■

2 The coupled Seiberg-Witten equations

Let P be the principal SO(4)-bundle associated with the tangent bundle of the oriented, closed Riemannian 4-manifold (X, g), and fix a $\mathrm{Spin}^c(4)$ structure $\hat{P}$ over P with $L = \det(\hat{P})$. The choice of a $\mathrm{Spin}^c(4)$-connection in $\hat{P}$ projecting onto the Levi-Civita connection in P is equivalent to the choice of a connection b in the unitary line bundle L [H1]. We denote by $B(b)$ the $\mathrm{Spin}^c(4)$-connection in $\hat{P}$ corresponding to b, and also the induced connection in the vector bundle $\Sigma = \Sigma^+ \oplus \Sigma^-$. The curvature $F_{B(b)}$ of the connection $B(b)$ in Σ has the form

$$F_{B(b)} = \frac{1}{2}F_b \mathrm{id}_\Sigma + F_g = \begin{bmatrix} \frac{1}{2}F_b \mathrm{id}_{\Sigma^+} + F_g^+ & 0 \\ 0 & \frac{1}{2}F_b \mathrm{id}_{\Sigma^-} + F_g^- \end{bmatrix},$$

where F_g, and $F_g^{\pm}$ denote the Riemannian curvature operator, and its components with respect to the splitting $\mathrm{ad}(P) = \Lambda^2_+ \oplus \Lambda^2_-$.

Let now E be an arbitrary Hermitian bundle of rank r over X, and A a connection in E. We denote by A_b the induced connection in the tensor product $\Sigma \otimes E$, and by $\not{D}_{A,b} : A^0(\Sigma \otimes E) \longrightarrow A^0(\Sigma \otimes E)$ the associated Dirac operator. $\not{D}_{A,b}$ is defined as the composition:

$$A^0(\Sigma \otimes E) \xrightarrow{\nabla_{A_b}} A^1(\Sigma \otimes E) \xrightarrow{m} A^0(\Sigma \otimes E)$$

where m is the Clifford multiplication $m(u, \sigma \otimes e) := \gamma(u)(\sigma) \otimes e$. $\not{D}_{A,b}$ is an elliptic, self-adjoint operator and its square $\not{D}^2_{A,b}$ is related to the usual Laplacian $\nabla^*_{A_b}\nabla_{A_b}$ by the Weitzenböck formula

$$\not{D}^2_{A,b} = \nabla^*_{A_b}\nabla_{A_b} + \Gamma(F_{A_b}).$$

Here $\Gamma(F_{A_b}) \in A^0(\mathrm{End}(\Sigma \otimes E))$ is the Hermitian endomorphism defined as the composition

$$A^0(\Sigma \otimes E) \xrightarrow{F_{A_b}} A^0(\Lambda^2 \otimes \Sigma \otimes E) \xrightarrow{\Gamma} A^0(\mathrm{End}_0(\Sigma) \otimes \Sigma \otimes E) \xrightarrow{ev} A^0(\Sigma \otimes E).$$

We set $F_{A,b} := F_A + \frac{1}{2}F_b \mathrm{id}_E \in A^0(\Lambda^2 \otimes \mathrm{End}(E))$.

Proposition 2.1 *Let s be the scalar curvature of the Riemannian 4-manifold (X, g). Fix a $\mathrm{Spin}^c(4)$-structure on X and choose connections b and A in L and E respectively. Then*

$$\not{D}^2_{A,b} = \nabla^*_{A_b}\nabla_{A_b} + \Gamma(F_{A,b}) + \frac{s}{4}\mathrm{id}_{\Sigma\otimes E}.$$

Proof: Since $\Gamma(F_g) = \frac{s}{4}\mathrm{id}_\Sigma$ [H1], and $F_{A_b} = F_{B(b)} \otimes \mathrm{id}_E + \mathrm{id}_\Sigma \otimes F_A = \frac{1}{2}F_b \mathrm{id}_\Sigma \otimes \mathrm{id}_E + F_g \otimes \mathrm{id}_E + \mathrm{id}_\Sigma \otimes F_A = \mathrm{id}_\Sigma \otimes (F_A + \frac{1}{2}F_b\mathrm{id}_E) + F_g \mathrm{id}_E$, we find $\Gamma(F_{A_b}) = \Gamma(F_{A,b}) + \frac{s}{4}\mathrm{id}_{\Sigma\otimes E}$. ■

Remark 2.2 *One has a Bochner-type result for spinors Ψ on which $\Gamma(F_{A,b}) + \frac{s}{4}\mathrm{id}_{\Sigma\otimes E}$ is positive: Such a spinor is harmonic if and only if it is parallel [H1].*

Let $(\ , \ )$ be the pointwise inner product on $\Sigma \otimes E$, $| \ |$ the associated pointwise norm, and $\| \ \|$ the corresponding L^2-norm. For a spinor $\Psi \in A^0(\Sigma^\pm \otimes E)$ we define $(\Psi\bar{\Psi})_0 \in A^0(\mathrm{End}_0(\Sigma^\pm \otimes E))$ as the image of the Hermitian endomorphism $\Psi \otimes \bar{\Psi} \in A^0(\mathrm{End}(\Sigma^\pm \otimes E))$ under the projection $\mathrm{End}(\Sigma^\pm \otimes E) \longrightarrow \mathrm{End}_0(\Sigma^\pm) \otimes \mathrm{End}(E)$.

Corollary 2.3 *With the notations above, we have*

$$(\not{D}^2_{A,b}\Psi, \Psi) = (\nabla^*_{A_b}\nabla_{A_b}\Psi, \Psi) + (\Gamma(F^+_{A,b}), (\Psi_+\bar{\Psi}_+)_0) + (\Gamma(F^-_{A,b}), (\Psi_-\bar{\Psi}_-)_0) + \frac{s}{4}|\Psi|^2,$$

where $(F^-_{A,b})$ $F^+_{A,b}$ is the (anti-)self-dual component of $F_{A,b}$.

Proof: Indeed, since $\Gamma(F^\pm_{A,b})$ vanishes on $\Sigma^\mp$ and is trace free with respect to $\Sigma^\pm$, the inner product $(\Gamma(F_{A,b}), (\Psi\bar{\Psi}))$ in the Weitzenböck formula simplifies for a spinor $\Psi \in A^0(\Sigma^\pm \otimes E)$:

$$(\Gamma(F_{A,b}), (\Psi\bar{\Psi})) = (\Gamma(F^\pm_{A,b}), (\Psi\bar{\Psi})_0)$$

■

For a positive spinor $\Psi \in A^0(E\otimes\Sigma^+)$, the following important identity follows immediately:

$$(\not{D}^2_{A,b}\Psi,\Psi)+\frac{1}{2}|\Gamma(F^+_{A,b})-(\Psi\bar{\Psi})_0|^2 = (\nabla^*_{A_b}\nabla_{A_b}\Psi,\Psi)+\frac{1}{2}|F^+_{A,b}|^2+\frac{1}{2}|(\Psi\bar{\Psi})_0|^2+\frac{s}{4}|\Psi|^2 \tag{4}$$

If we integrate both sides of (4) over X, we get:

Proposition 2.4 *Let (X,g) be an oriented, closed Riemannian 4-manifold with scalar curvature s, E a Hermitian bundle over X. Choose a* $\mathrm{Spin}^c(4)$*-structure on X and a connection b in the determinant line bundle $L=$* $\det(\Sigma^+)=\det(\Sigma^-)$*. Let A be a connection in E. For any $\Psi\in A^0(\Sigma^+\otimes E)$ we have:*

$$\| \not{D}_{A,b}\Psi \|^2 +\frac{1}{2} \| \Gamma(F^+_{A,b})-(\Psi\bar{\Psi})_0 \|^2=$$
$$=\| \nabla_{A_b}\Psi \|^2 +\frac{1}{2} \| F^+_{A,b} \|^2 +\frac{1}{2} \| (\Psi\bar{\Psi})_0 \|^2 +\frac{1}{4}\int_X s|\Psi|^2 .$$

We introduce now our coupled variant of the Seiberg-Witten equations. The unknown is a pair (A,Ψ) consisting of a connection in the Hermitian bundle E and a section $\Psi\in A^0(\Sigma^+\otimes E)$. The equations ask for the vanishing of the left-hand side in the above formula.

$$\left\{\begin{array}{lcl} \not{D}_{A,b}\Psi & = & 0 \\ \Gamma(F^+_{A,b}) & = & (\Psi\bar{\Psi})_0 \end{array}\right. \qquad (SW)$$

Proposition 2.4 and the inequality $|(\Psi\bar{\Psi})_0|^2 \geq \frac{1}{2}|\Psi|^4$ give immediately:

Remark 2.5 *If the scalar curvature s is nonnegative on X, then the only solutions of the equations are the pairs $(A,0)$, with $F^+_{A,b}=0$.*

If L is the square of a line bundle $L^{\frac{1}{2}}$, and if we choose a connection $b^{\frac{1}{2}}$ in $L^{\frac{1}{2}}$ with square b, then $F_{A,b}$ is simply the curvature of the connection $A_{b^{\frac{1}{2}}}$ in $E\otimes L^{\frac{1}{2}}$. The solution of the coupled Seiberg-Witten equations on a manifold with $s\geq 0$ are in this case just $\mathrm{U}(r)$-instantons on $E\otimes L^{\frac{1}{2}}$.

In the case of a Kähler surface (X,g), the coupled Seiberg-Witten equation can be reformulated in terms of complex geometry. The point is that if we consider the canonical $\mathrm{Spin}^c(4)$-structure associated to the Kähler structure, the Dirac operator has a very simple form [H1]. The determinant of this $\mathrm{Spin}^c(4)$-structure is the anti-canonical bundle $K_X^\vee$ of the surface, which comes with a holomorphic structure and a natural metric inherited from the holomorphic tangent bundle.

Let c be the Chern connection in $K_X^\vee$. With this choice, the induced connection $B(c)$ in $\Sigma = \Lambda^{00} \oplus \Lambda^{02} \oplus \Lambda^{01}$ coincides with the connection defined by the Levi-Civita connection. Recall that on a Kähler manifold, the almost complex structure is parallel with respect to the Levi-Civita connection, so that the splitting of the exterior algebra $\bigoplus_p \Lambda^p \otimes \mathbb{C}$ becomes parallel, too.

Proposition 2.6 *Let (X, g) be a Kähler surface with Chern connection c in $K_X^\vee$. Choose a connection A in a Hermitian vector bundle E over X and a section $\Psi = \varphi + \alpha \in A^0(E) + A^0(\Lambda^{02} \otimes E)$.*

The pair (A, Ψ) satisfies the Seiberg-Witten equations iff the following identities hold:

$$\begin{array}{lcl} F_{A,c}^{20} & = & -\frac{1}{2}\varphi \otimes \bar{\alpha} \\ F_{A,c}^{02} & = & \frac{1}{2}\alpha \otimes \bar{\varphi} \\ i\Lambda_g F_{A,c} & = & -\frac{1}{2}\left(\varphi \otimes \bar{\varphi} - *(\alpha \otimes \bar{\alpha})\right) \\ \bar{\partial}_A \varphi & = & i\Lambda_g \partial_A \alpha \end{array}$$

Proof: The Dirac operator is in this case $\not{D}_{A,c} = \sqrt{2}(\bar{\partial}_A - i\Lambda_g \partial_A)$, and the endomorphism $(\Psi\bar{\Psi})_0$ has the form:

$$\left[\begin{array}{cc} \frac{1}{2}(\varphi \otimes \bar{\varphi} - *(\alpha \otimes \bar{\alpha})) & *(\varphi \otimes \bar{\alpha} \wedge \cdot) \\ \alpha \otimes \bar{\varphi} & -\frac{1}{2}(\varphi \otimes \bar{\varphi} - *(\alpha \otimes \bar{\alpha})) \end{array}\right].$$

Since $\Gamma(F_{A,c}^+) = \Gamma(F_{A,c}^{20} + F_{A,c}^{02} + \frac{1}{2}\Lambda_g F_{A,c} \cdot \omega_g)$ equals

$$2\left[\begin{array}{cc} -\frac{i}{2}\Lambda_g F_{A,c} & -*\left(F_{A,c}^{20} \wedge \cdot\right) \\ F_{A,c}^{20} \wedge \cdot & \frac{i}{2}\Lambda_g F_{A,c} \end{array}\right],$$

the equivalence of the two systems of equations follows. ■

3 Monopoles on Kähler surfaces and the generalized vortex equation

Let (X, g) be a Kähler surface with canonical $\mathrm{Spin}^c(4)$-structure, and Chern connection c in the anti-canonical bundle $K_X^\vee$.

We fix a unitary vector bundle E of rank r over X, and define $J(E) := \deg_g(\Sigma^+ \otimes E)$, i.e. $J(E) = 2r(\mu_g(E) - \frac{1}{2}\mu_g(K_X))$, where μ_g denotes the slope with respect to ω_g.

Every spinor $\Psi \in A^0(\Sigma^+ \otimes E)$ has the form $\Psi = \varphi + \alpha$ with $\varphi \in A^0(E)$ and $\alpha \in A^0(\Lambda^{02} \otimes E)$.

We have seen that the coupled Seiberg-Witten equations are equivalent to the system:

$$\left\{\begin{array}{lcl} F_{A,c}^{20} & = & -\frac{1}{2}\varphi \otimes \bar{\alpha} \\ F_{A,c}^{02} & = & \frac{1}{2}\alpha \otimes \bar{\varphi} \\ i\Lambda_g F_{A,c} & = & -\frac{1}{2}\left(\varphi \otimes \bar{\varphi} - *(\alpha \otimes \bar{\alpha})\right) \\ \bar{\partial}_A \varphi & = & i\Lambda_g \partial_A \alpha \end{array}\right. \qquad (SW^*)$$

Lemma 3.1
A. Suppose $J(E) < 0$*:*
A pair $(A, \varphi + \alpha)$ *is a solution of the system* (SW^*) *if and only if*
i) $F_A^{20} = F_A^{02} = 0$
ii) $\alpha = 0$, $\bar{\partial}_A \varphi = 0$
iii) $i\Lambda_g F_A + \frac{1}{2}\varphi \otimes \bar{\varphi} + \frac{1}{2} s \mathrm{id}_E = 0$.
B. Suppose $J(E) > 0$*, and put* $a := \bar{\alpha} \in A^{20}(\bar{E}) = A^0(E^\vee \otimes K_X)$*:*
A pair $(A, \varphi + \bar{a})$ *is a solution of the system* (SW^*) *if and only if*
i) $F_A^{20} = F_A^{02} = 0$
ii) $\varphi = 0$, $\bar{\partial}_A a = 0$
iii) $i\Lambda_g F_A - \frac{1}{2} * (a \otimes \bar{a}) + \frac{1}{2} s \mathrm{id}_E = 0$.

Proof: (cf. [W]) The splitting $\Sigma^+ \otimes E = \Lambda^{00} \otimes E \oplus \Lambda^{02} \otimes E$ is parallel with respect to $\nabla_{A,c}$, so that, by Proposition 2.4

$$\| \not{D}_{A,c}\Psi \|^2 + \frac{1}{2} \| \Gamma(F_{A,c}^+) - (\Psi\bar{\Psi})_0 \|^2 =$$

$$= \| \nabla_{A_c}\varphi \|^2 + \| \nabla_{A_c}\alpha \|^2 + \frac{1}{2} \| F_{A,c}^+ \|^2 + \frac{1}{2} \| (\Psi\bar{\Psi})_0 \|^2 + \frac{1}{4}\int_X s(|\varphi|^2 + |\alpha|^2).$$

The right-hand side is invariant under the transformation $(A,\varphi,\alpha) \longmapsto (A,\varphi,-\alpha)$, hence any solution $(A,\varphi+\alpha)$ must have $F_A^{20} = F_A^{02} = 0$ and $\varphi\otimes\bar\alpha = \alpha\otimes\bar\varphi = 0$; the latter implies obviously $\alpha = 0$ or $\varphi = 0$. Integrating the trace of the equation $i\Lambda F_{A,c} = -\frac{1}{2}\left(\varphi\otimes\bar\varphi - *(\alpha\otimes\bar\alpha)\right)$, we find:

$$J(E) = c_1(\Sigma^+\otimes E)\cup[\omega_g] = (2c_1(E) - rc_1(K_X))\cup[\omega_g] =$$

$$= 2\int\limits_X \frac{i}{2\pi}\mathrm{Tr}(F_{A,c})\wedge\omega_g = \frac{1}{4\pi}\int\limits_X \mathrm{Tr}(i\Lambda F_{A,c})\omega_g^2 = \frac{1}{8\pi}\int\limits_X \mathrm{Tr}(-\varphi\otimes\bar\varphi)+*(\alpha\otimes\bar\alpha))\omega_g^2$$

This equation shows that we must have $\alpha = 0$, if $J(E) < 0$, and $\varphi = 0$, if $J(E) > 0$. Notice that, replacing E by $E^\vee\otimes K_X$, the second case reduces to the first one.

The assertion follows now immediately from the identity $i\Lambda_g F_c = s$.

■

Notice that the last equation

$$i\Lambda_g F_A + \frac{1}{2}\varphi\otimes\bar\varphi + \frac{1}{2}s\mathrm{id}_E = 0$$

has the form of a generalized vortex equation as studied by Bradlow [B1], [B2] and by Garcia-Prada [G2]; it is precisely the vortex equation with constant $\tau = -s$, if (X,g) has constant scalar curvature.

Let s_m be the mean scalar curvature defined by $\int\limits_X s\omega_g^2 = s_m\int\limits_X \omega^2 = 2s_m\mathrm{Vol}_g(X)$.

We are going to prove that the system

$$\begin{cases} \bar\partial_A^2 & = 0 \\ \bar\partial_A\varphi & = 0 \\ i\Lambda_g F_A + \frac{1}{2}\varphi\otimes\bar\varphi + \frac{1}{2}s\mathrm{id}_E & = 0 \end{cases}$$

for the pair (A,φ) consisting of a unitary connection in E, and a section in E, is always equivalent to the vortex system with parameter $\tau = -s_m$, i.e. to the system obtained by replacing the third equation with

$$i\Lambda_g F_A + \frac{1}{2}\varphi\otimes\bar\varphi + \frac{1}{2}s_m\mathrm{id}_E = 0.$$

"Equivalent" means here that the corresponding moduli spaces of solutions are naturally isomorphic.

Let generally t be a smooth real function on X with mean value t_m, and consider the following system of equations:

$$\left\{\begin{array}{ll} \bar\partial_A^2 & =0 \\ \bar\partial_A\varphi & =0 \\ i\Lambda_g F_A+\frac{1}{2}\varphi\otimes\bar\varphi-\frac{1}{2}t\mathrm{id}_E & =0 \end{array}\right. \qquad (V_t)$$

(V_t) is defined on the space $\mathcal{A}(E)\times A^0(E)$, where $\mathcal{A}(E)$ is the space of unitary connections in E. The product $\mathcal{A}(E)\times A^0(E)$ (completed with respect to sufficiently large Sobolev indices) carries a natural L^2 Kähler metric $\tilde g$ and a natural right action of the gauge group $U(E)$: $(A,\varphi)^f := (A^f, f^{-1}\varphi)$, where $d_{A^f} := f^{-1}\circ d_A\circ f$.

For every real function t let

$$m_t : \mathcal{A}(E)\times A^0(E) \longrightarrow A^0(\mathrm{ad}(E))$$

be the map given by $m_t := \Lambda_g F_A - \frac{i}{2}\varphi\otimes\bar\varphi + \frac{i}{2}t\mathrm{id}_E$.

Proposition 3.2 *m_t is a moment map for the action of $U(E)$ on $\mathcal{A}(E)\times A^0(E)$.*

Proof: Let $a^{\#}$ be the vector field on $\mathcal{A}(E)\times A^0(E)$ associated with the infinitesimal transformation $a\in A^0(\mathrm{ad}(E)) = \mathrm{Lie}(U(E))$, and define the real function $m_t^a : \mathcal{A}(E)\times A^0(E)\longrightarrow \mathbb{R}$ by:

$$m_t^a(x) = \langle m_t(x), a\rangle_{L^2}.$$

We have to show that m_t satisfies the identities:

$$\iota_{a^{\#}}\omega_{\tilde g} = dm_t^a\ , \quad m_t^a\circ f = m^{\mathrm{ad}_f(a)} \quad \text{for all} \quad a\in A^0(\mathrm{ad}(E)),\quad f\in U(E).$$

It is well known that, in general, a moment map for a group action in a symplectic manifold is well defined up to a constant central element in the Lie algebra of the group. In our case, the center of the Lie algebra $A^0(\mathrm{ad}(E))$ of the gauge group is just $iA^0\mathrm{id}_E$, hence it suffices to show that m_0 is a moment map. This has already been noticed by Garcia-Prada [G1], [G2]. ■

Note also that in our case every moment map has the form m_t for some function t, which shows that from the point of view of symplectic geometry, the natural equations are the generalized vortex equations (V_t).

In order to show that Bradlow's main result [B2] also holds for the generalized system (V_t), we have to recall some definitions.

Let $\mathcal{E}$ be a holomorphic vector bundle of topological type E, and let $\varphi \in H^0(\mathcal{E})$ be a holomorphic section. The pair $(\mathcal{E}, \varphi)$ is λ-stable with respect to a constant $\lambda \in \mathbb{R}$ iff the following conditions hold:
(1) $\mu_g(\mathcal{E}) < \lambda$ and $\mu_g(\mathcal{F}) < \lambda$ for all reflexive subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < r$.
(2) $\mu_g(\mathcal{E}/\mathcal{F}) > \lambda$ for all reflexive subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < r$ and $\varphi \in H^0(\mathcal{F})$.

Theorem 3.3 *Let (X, g) be a closed Kähler manifold, $t \in A^0$ a real function, and $(\mathcal{E}, \varphi)$ a holomorphic pair over X. Set $\lambda := \frac{1}{4\pi} t_m \mathrm{Vol}_g(X)$. $\mathcal{E}$ admits a Hermitian metric h such that the associated Chern connection A_h satisfies the vortex equation*

$$i\Lambda_g F_A + \frac{1}{2}\varphi \otimes \bar{\varphi} - \frac{1}{2} t \mathrm{id}_E = 0$$

iff one of the following conditions holds:
(i) $(\mathcal{E}, \varphi)$ is λ-stable
(ii) $\mathcal{E}$ admits a splitting $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$ with $\varphi \in H^0(\mathcal{E}')$ such that $(\mathcal{E}', \varphi)$ is λ-stable, and $\mathcal{E}''$ admits a weak Hermitian-Einstein metric with factor $\frac{t}{2}$. In particular $\mathcal{E}''$ is polystable of slope λ.

Proof: In the case of a constant function $t = \tau$, the theorem was proved by Bradlow [B2], and his arguments work in the general context, too: The fact that the existence of a solution of the vortex equation implies (i) or (ii) follows by replacing the constant τ in [B2] everywhere with the function t. The difficult part consists in showing that every λ-stable pair $(\mathcal{E}, \varphi)$ admits a metric h such that (A_h, φ) satisfies the vortex equation (V_t). To this end let $Met(E)$ be the space of Hermitian metrics in E, and fix a background metric $k \in Met(E)$. Bradlow constructs a functional $M_{\varphi,\tau}(\cdot,\cdot) : Met(E) \times Met(E) \longrightarrow \mathbb{R}$, which is convex in the second argument, such that any critical point of $M_{\varphi,\tau}(k,\cdot)$ is a solution of the vortex equation; the point is then to find an absolute minimum of $M_{\varphi,\tau}(k,\cdot)$. The existence of an absolute minimum follows from the following basic $\mathcal{C}^0$ estimate:

Lemma 3.4 (Bradlow) *Let $Met_2^p(E, B) := \{h = ke^a | a \in L_p^2(\mathrm{End}(E)), a^{*k} = a, \| \mu_t(A_h, \varphi) \|_{L^p} \leq B\}$. If $(\mathcal{E}, \varphi)$ is $\frac{\tau}{4\pi}\mathrm{Vol}_g(X)$-stable, then there exist positive* *constants C_1, C_2 such that*

$$\sup |a| \leq C_1 M_{\varphi,\tau}(k, ke^a) + C_2,$$

for all k-Hermitian endomorphisms $a \in L^2_p(\mathrm{End}(E))$. Moreover, any absolute minimum of $M_{\varphi,t}(k,\cdot)$ on $Met^p_2(E,B)$ is a critical point of $M_{\varphi,t}(k,\cdot)$, and gives a solution of the vortex equation V_τ.

Let now t be a real function on X, and choose a solution v of the Laplace equation $i\Lambda_g\bar\partial\partial v = \frac{1}{2}(t - t_m)$. If we make the substituion $h = h'e^v$, then h solves the vortex equation (V_t) iff h' is a solution of

$$i\Lambda_g F_{h'} + \frac{1}{2}e^v\varphi\otimes\bar\varphi^{h'} - \frac{1}{2}t_m \mathrm{id}_E = 0.$$

Define $\mu_{t_m,v}(h') := i\Lambda_g F_{h'} + \frac{1}{2}e^v\varphi\otimes\bar\varphi^{h'} - \frac{1}{2}t_m\mathrm{id}_E = 0$, and

$$M_{\varphi,t_m,v}(k,h) := M_D(k,h) + \parallel e^{\frac{v}{2}}\varphi \parallel^2_h - \parallel e^{\frac{v}{2}}\varphi \parallel^2_k - t_m \int\limits_X \mathrm{Tr}(\log(k^{-1}h)),$$

where M_D is the Donaldson functional [D]. Then it is not difficult to show that all arguments of Bradlow remain correct after replacing μ_{t_m} and M_{φ,t_m} with $\mu_{t_m,v}$ and $M_{\varphi,t_m,v}$ respectively. Indeed, let l be a positive bound from below for the map e^v. Then

$$\begin{aligned} M_{\varphi,t_m}(k, ke^{a+\log l}) &\leq M_D(k,ke^a) + M_D(ke^a, lke^a) + \parallel l\varphi \parallel^2_h - t_m \int\limits_X \mathrm{Tr}\log(lk^{-1}h) \\ &\leq M_{\varphi,t_m,v}(k,ke^a) + \parallel e^{\frac{v}{2}}\varphi \parallel^2_k + 2\log l \deg_g(E) - rt_m \log l \mathrm{Vol}_g(X) \\ &\leq M_{\varphi,t_m,v}(k,ke^a) + C'(k,\varphi,v,l). \end{aligned}$$

Similarly, we get constants $n > 0$, C'' and an inequality

$$M_{\varphi,t_m,v}(k, ke^{a+\log n}) \leq M_{\varphi,t_m}(k,ke^a) + C'',$$

which shows that the basic $\mathcal{C}^0$ estimate in the Lemma above holds for $M_{\varphi,t_m,v}$ iff it holds for Bradlow's functional M_{φ,t_m}. ■

Remark 3.5 *In the special case of a rank-1 bundle E, a much more elementary proof based on [B1] is possible.*

4 Moduli spaces of monopoles, vortices, and stable pairs

Let (X, g) be a closed Kähler manifold of arbitrary dimension, and fix a unitary vector bundle E of rank r over X. We denote by $\bar{\mathcal{A}}(E)$ the affine space of semiconnection of type $(0,1)$ in E. the complex gauge group $GL(E)$ acts on $\bar{\mathcal{A}}(E) \times A^0(E)$ from the right by $(\bar{\partial}_A, \varphi)^g := (g^{-1} \circ \bar{\partial}_A \circ g, g^{-1}\varphi)$; this action becomes complex analytic after suitable Sobolev completions. We denote by $\bar{\mathcal{S}}(E)$ the set of pairs $(\bar{\partial}_A, \varphi)$ with trivial isotropy group. Notice that $\varphi \neq 0$ when $(\bar{\partial}_A, \varphi) \in \bar{\mathcal{S}}(E)$, and that $\bar{\mathcal{S}}(E)$ is an open subset of $\bar{\mathcal{A}}(E) \times A^0(E)$, by elliptic semi-continuity [K].

The action of $GL(E)$ on $\bar{\mathcal{S}}(E)$ is free, by definition, and we denote the Hilbert manifold ${}^{\bar{\mathcal{S}}(E)}/_{GL(E)}$ by $\bar{\mathcal{B}}^s(E)$. The map $p : \bar{\mathcal{A}}(E) \times A^0(E) \longrightarrow A^{02}(\mathrm{End}(E) \oplus A^{01}(E)$ defined by $p(\bar{\partial}_A, \varphi) = (F_A^{02}, \bar{\partial}_A\varphi)$ is equivariant with respect to the natural actions of $GL(E)$, hence it gives rise to a section $\hat{p}$ in the associated vector bundle $\bar{\mathcal{S}}(E) \times_{GL(E)} (A^{02}(\mathrm{End}(E) \oplus A^{01}(E))$ over $\bar{\mathcal{B}}^s(E)$. We define the moduli space of <u>simple pairs</u> of type E to be the zero-locus $Z(\hat{p})$ of this section. $Z(\hat{p})$ can be identified with the set of isomorphism classes consisting of a holomorphic bundle $\mathcal{E}$ of differentiable type E, and a holomorphic section $\varphi \neq 0$, such that the kernel of the evaluation map $ev(\varphi) : H^0(\mathrm{End}(\mathcal{E})) \longrightarrow H^0(\mathcal{E})$ is trivial.

In a similar way we define the moduli space $\mathcal{V}_t^g$ of gauge-equivalence classes of irreducible solutions of the generalized vortex equation V_t:

Let B^+ denote as usual the subbundle $((\Lambda^{02} + \Lambda^{20}) \cap \Lambda^2) \oplus \Lambda^0\omega$ of the bundle Λ^2 of real 2-forms on X. We denote by $\mathcal{D}^*$ the open subset of the product $\mathcal{D} := \mathcal{A}(E) \times A^0(E) \simeq \bar{\mathcal{A}}(E) \times A^0(E)$ consisting of pairs with trivial isotropy group with respect to the action of the gauge group $U(E)$. The quotient $\mathcal{B}^*(E) := {}^{\mathcal{D}^*(E)}/_{U(E)}$ comes with the structure of a real-analytic manifold.

Let $v : \mathcal{D}(E) \longrightarrow A^0(B^+ \otimes \mathrm{ad}(E)) \oplus A^{01}(E)$ be the map given by:

$$v(A, \varphi) = (F^{20} + F^{02}, m_t(A, \varphi)\omega_g \mathrm{id}_E, \bar{\partial}_A\varphi).$$

Again v is $U(E)$-equivariant, and the moduli space $\mathcal{V}_t^g$ of t-vortices is defined to be the zero-locus $Z(\hat{v})$ of the induced section $\hat{v}$ of $\mathcal{D}^*(E) \times_{U(E)} A^0(B^+ \otimes \mathrm{ad}(E)) \oplus A^{01}(E)$ over $\mathcal{B}^*(E)$.

Notice now that by Proposition 3.2, the second component v^2 of v is a moment map for the $U(E)$ action. It is easy to see that (at least in a neighbourhood of $Z(v) \cap \mathcal{D}^*$) it has the general property of a moment map in the finite dimensional Kähler geometry: Its zero locus $Z(v^2)$ is smooth and intersects every $GL(E)$ orbit along a $U(E)$ orbit, and the intersection is transversal. This means that the natural map $A \longrightarrow \bar{\partial}_A$ defines in a neighbourhood of $Z(\hat{v}) \cap \mathcal{B}^*(E)$ an open embedding $i : Z(\hat{v}^2) \longrightarrow \bar{\mathcal{B}}^s$ of smooth Hilbert manifolds.

Regard now $\mathcal{V}^g_t$ as the subspace of $Z(\hat{v}^2) \subset \mathcal{B}^*(E)$ defined by the equation $(\hat{v}^1, \hat{v}^3) = 0$. On the other hand, the pullback of the equation $\hat{p} = 0$, cutting out the moduli space $Z(\hat{p})$ of simple holomorphic pairs, via the open embedding i is precisely the equation $(\hat{v}^1, \hat{v}^3) = 0$, cutting out $\mathcal{V}^g_t$. We get therefore an open embedding $i_0 : \mathcal{V}^g_t \longrightarrow Z(\hat{p})$ of real analytic spaces induced by i, and by Theorem 3.3 the image of i_0 consists of the set of λ-stable pairs, with $\lambda := \frac{1}{4\pi} t_m \mathrm{Vol}_g(X)$.

Finally we denote by $\mathcal{M}^g_X(E, \lambda) \subset Z(\hat{p})$ the open subspace of λ-stable pairs, with the induced complex space-structure. Putting everything together, we have:

Theorem 4.1 *Let (X, g) be a closed Kähler manifold, $t \in A^0$ a real function, and $\lambda := \frac{1}{4\pi} t_m \mathrm{Vol}_g(X)$. Fix a unitary vector bundle E of rank r over X. There are natural real-analytic isomorphisms of moduli spaces*

$$\mathcal{V}^g_t(E) \simeq \mathcal{V}^g_{t_m}(E) \simeq \mathcal{M}^g_X(E, \lambda).$$

Let us come back now to the monopole equation (SW^*) on a Kähler surface. In this case the function t is the negative of the scalar curvature s, so that the corresponding constant λ becomes:

$$\lambda = \frac{-s_m}{4\pi} \mathrm{Vol}_g(X) = -\frac{1}{8\pi} \int_X s\omega^2 = -\frac{1}{8\pi} \int_X (i\Lambda F_c)\omega^2 = -\frac{1}{4\pi} \int_X iF_c \wedge \omega = \frac{1}{2}\mu_g(K).$$

This yields our main result:

Theorem 4.2 *Let (X, g) be a Kähler surface with canonical* $\mathrm{Spin}^c(4)$*-structure, and Chern connection c in $K^\vee_X$. Fix a unitary vector bundle E of rank r over X, and suppose $J(E) = \deg_g(\Sigma^+ \otimes E) < 0$. The moduli space of solutions of the coupled Seiberg-Witten equations is isomorphic to the moduli space $\mathcal{M}^g_X(E, \frac{1}{2}\mu_g(K))$ of $\frac{1}{2}\mu_g(K)$-stable pairs of topological type E.*

At this point it is natural to study the properties of the moduli spaces $\mathcal{M}^g_X(E,\lambda)$. We do not want to go into details here, and we content ourselves by describing some of the basic steps.

The infinitesimal structure of the moduli space around a point $[(A,\varphi)]$ is given by a deformation complex $(C^*_{\bar\partial_A,\varphi}, d^*_{A,\varphi})$ which is the cone over the evaluation map ev^*, $ev^q(\varphi) : A^{0q}(\mathrm{End}(E)) \longrightarrow A^{0q}(E)$. More precisely $C^q_{\bar\partial_A,\varphi} = A^{0q}(\mathrm{End}(E)) \oplus A^{0,q-1}(E)$ and the differential $d^q_{A,\varphi}$ is given by the matrix

$$d^q_{A,\varphi} = \begin{bmatrix} -\bar D_A & 0 \\ ev(\varphi) & \bar\partial_A \end{bmatrix},$$

where $\bar\partial_A$ and $\bar D_A$ are the operators of the Dolbeault complexes $(A^{0*}(E), \bar\partial_A)$ and $(A^{0*}\mathrm{End}(E), \bar D_A)$ respectively.

Associated to the morphism $ev^*(\varphi)$ is an exact sequence

$$\ldots \longrightarrow H^q(\mathrm{End}(\mathcal{E}_A)) \xrightarrow{ev^q(\varphi)} H^q(\mathcal{E}_A) \longrightarrow H^{q+1}_{\bar\partial_A,\varphi} \longrightarrow H^{q+1}(\mathrm{End}(\mathcal{E}_A)) \longrightarrow \ldots$$

with finite dimensional vector spaces

$$H^q_{\bar\partial_A,\varphi} = \ker(ev^q(\varphi)) \oplus \mathrm{coker}(ev^{q-1}(\varphi)).$$

$H^0_{\bar\partial_A,\varphi}$ vanishes for a simple pair $(\bar\partial_A,\varphi)$, and $H^1_{\bar\partial_A,\varphi}$ is the Zariski tangent space of $\mathcal{M}^g_X(E,\lambda)$ at $[\bar\partial_A,\varphi]$.

A Kuranishi type argument yields local models of the moduli space, which can be locally described as the zero loci of holomorphic map germs

$$K_{[\bar\partial_A,\varphi]} : H^1_{\bar\partial_A,\varphi} \longrightarrow H^2_{\bar\partial_A,\varphi}$$

at the origin.

One finds that $H^2_{\bar\partial_A,\varphi} = 0$ is a sufficient smoothness criterion in the point $[\bar\partial_A,\varphi]$ of the moduli space, and that the expected dimension is $\chi(E) - \chi(\mathrm{End}(E))$. The necessary arguments are very similar to the ones in [BD1], [BD2].

The moduli spaces $\mathcal{M}^g(E,\lambda)$ will be quasi-projective varieties if the underlying manifold (X,g) is Hodge, i.e. if X admits a projective embedding such that a multiple of the Kähler class is a polarisation [G1].

A GIT construction for projective varieties of any dimension has been given in [HL2]. The spaces $\mathcal{M}^g_X(E,\lambda)$ vary with the parameter λ, and flip-phenomena occur just like in the case of curves [T].

5 Applications

The equations considered by Seiberg and Witten are associated to a $\mathrm{Spin}^c(4)$-structure, and correspond to the case when (in our notations) the unitary bundle E is the trivial line bundle. Alternatively, we can fix a $\mathrm{Spin}^c(4)$ structure $\mathfrak{s}_0$ on X , and regard the Seiberg-Witten equations corresponding to the other $\mathrm{Spin}^c(4)$-structures as *coupled* Seiberg-Witten equations associated to $\mathfrak{s}_0$ and to a unitary line bundle E. The $\mathrm{Spin}^c(4)$-structure we fix will always be the canonical structure defined by a Kähler metric. In the most interesting case of rank-1 bundles E over Kähler surfaces the central result is:

Proposition 5.1 *Let (X, g) be a Kähler surface with canonical class K, and let L be a complex line bundle over X with $L \equiv K$ (mod 2). Denote by $\mathcal{W}^g_X(L)$ the moduli space of solutions of the Seiberg-Witten equation for all* $\mathrm{Spin}^c(4)$*-structures with determinant L. Then*
i) If $\mu(L) < 0$, $\mathcal{W}^g_X(L)$ is isomorphic to the space of all linear systems $|D|$, where D is a divisor with $c_1(\mathcal{O}_X(2D - K)) = L$.
ii) If $\mu(L) > 0$, $\mathcal{W}^g_X(L)$ is isomorphic to the space of all linear systems $|D|$, where D is a divisor with $c_1(\mathcal{O}_X(2D - K)) = -L$.

Proof: Use Theorem 4.2 and Bradlow's description of the moduli spaces of stable pairs in the case of line bundles [B1]. ■

We have already noticed (Remark 2.5) that in the case of a Riemannian 4-manifold with nonnegative scalar curvature s_g, the Seiberg-Witten equations have only reducible solutions. In the Kähler case, the same result can be obtained under the weaker assumption $\sigma_g \geq 0$ on the total scalar curvature.

Corollary 5.2 *Let (X, g) be a Kähler surface with nonnegative total scalar curvature σ_g. Then all solutions of the Seiberg-Witten equations in rank 1 are reducible. If moreover the surface has $K^2 > 0$, then for every almost canonical class L, the corresponding Seiberg-Witten equations are incompatible.*

Proof: The first assertion follows directly from the theorem, since the condition $\sigma_g \geq 0$ is equivalent to $K \cup [\omega_g] \leq 0$. For the second assertion, note that if L is an almost canonical class, then $L^2 = K^2 > 0$, hence (regarded as line bundle) it cannot admit anti-selfdual connections. ■

Remark 5.3 *The Seiberg-Witten invariants associated to almost canonical classes are well-defined for oriented, closed 4-manifolds X satisfying $3\sigma + 2e > 0$.*

Proof: Recall that if L is an almost canonical class, then the expected dimension of the moduli space of solutions of the perturbed Seiberg-Witten equations [W, KM]corresponding to a $\mathrm{Spin}^c(4)$-structure of determinant L is 0. Seiberg and Witten associate to every such class L the number n_L of points (counted with the correct signs [W]) of such a moduli space chosen to be smooth and of the expected dimension. In the case $b_+ \geq 2$, using the same cobordism argument as in Donaldson theory, it follows that that these numbers are well-defined, i.e. independent of the metric, provided the moduli space has the expected dimension [KM]. The point is that the space of L-good metrics [KM] (i.e. metrics with the property that the space of harmonic anti-selfdual forms does not contain the harmonic representative of $c_1^{\mathbb{R}}(L)$) is in this case path-connected. On the other hand, under the assumption $3\sigma + 2e > 0$, it follows that $L^2 > 0$ for any almost canonical class L, hence all metrics are L-good. ■

Proposition 5.4 *Let (X, H_0) be a polarised surface with K nef and big, and choose a Kähler metric g with Kähler class $[\omega_g] = H_0 + nK =: H$ for some $n \geq KH_0$. Then $\mathcal{W}_X^g(L)$ is empty for all almost canonical classes, except for $L = \pm K$, when it consists of a simple point.*

Proof: Let L be an almost canonical class with $LH < 0$. Suppose D is an effective divisor with $c_1(\mathcal{O}_X(2D - K)) = L$, so that $D(D - K) = 0$. Then $D^2 = DK \geq 0$ since K is nef. If D^2 were strictly positive, the Hodge index theorem would give $(D - K)^2 \leq 0$, i.e. $K^2 \leq D^2$. But from $LH < 0$ we get $0 > (2D - K)(H_0 + nK) = (2D - K)H_0 + n(2D^2 - K^2) \geq (2D - K)H_0 + n$, which leads to the contradiction $n < (K - 2D)H_0 \leq KH_0$. Therefore $D^2 = DK = 0$, so that, again by the Hodge index theorem, D must be numerically zero. Since D is effective, it must be empty, and $L = -K$.

Replacing L by $-L$ if L is an almost canonical class with $LH > 0$, we find $L = K$ in this case. The corresponding Seiberg-Witten moduli spaces are simple points in both cases, since $H^2_{\bar\partial_{A,\varphi}} = H^1(\mathcal{O}(D)|_D) = 0$. ■

Corollary 5.5 *There exists no orientation-preserving diffeomorphism between a rational surface and a minimal surface of general type.*

Proof: Indeed, any rational surface X admits a Hodge metric with positive total scalar curvature [H2]. If X was orientation-preservingly diffeomorphic to a minimal surface of general type, then $K^2 > 0$, hence the Seiberg-Witten invariants are well defined (Remark 5.3), and vanish by Corollary 5.2. Proposition 5.4 shows, however, that the Seiberg-Witten invariants of a minimal surface of general type are non-trivial for two almost canonical classes. ■

Witten has already proved [W] that for a minimal surface of general type with $p_g > 0$ ($b_+ \geq 2$), the only almost canonical classes which give non-trivial invariants are K and $-K$. Their proof uses the moduli space of solutions of the perturbation of the Seiberg-Witten equation with a holomorphic form. Proposition 5.4 shows that a stronger result can be obtained with the non-perturbed equations by choosing the Hodge metric $H = H_0 + nK,\ n \gg 0$.

For the proof of Corollary 5.5, we need in fact only the mod. 2 version of the Seiberg-Witten invariants [KM2].

Bibliography

[AHS] Atiyah M., Hitchin N. J., Singer I. M.: *Selfduality in four-dimensional Riemannian geometry*, Proc. R. Lond. A. 362, 425-461 (1978)
[BPV] Barth, W., Peters, C., Van de Ven, A.: *Compact complex surfaces*, Springer Verlag (1984)
[B] Bertram, A.: *Stable pairs and stable parabolic pairs*, J. Alg. Geometry 3, 703-724 (1994)
[B1] Bradlow, S. B.: *Vortices in holomorphic line bundles over closed Kähler manifolds*, Comm. Math. Phys. 135, 1-17 (1990)
[B2] Bradlow, S. B.: *Special metrics and stability for holomorphic bundles with global sections*, J. Diff. Geom. 33, 169-214 (1991)
[BD1] Bradlow, S. B.; Daskalopoulos, G.: *Moduli of stable pairs for holomorphic bundles over Riemann surfaces I*, Intern. J. Math. 2, 477-513 (1991)
[BD2] Bradlow, S. B.; Daskalopoulos, G.: *Moduli of stable pairs for holomorphic bundles over Riemann surfaces II*, Intern. J. Math. 4, 903-925 (1993)
[D] Donaldson, S.: *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. 3, 1-26 (1985)
[DK] Donaldson, S.; Kronheimer, P.B.: *The Geometry of four-manifolds*, Oxford Science Publications (1990)
[FM] Friedman, R.; Morgan, J.W.: *Smooth 4-manifolds and Complex Surfaces*, Springer Verlag 3. Folge, Band 27 (1994)
[FQ] Friedman, R.; Qin, Z.: *On complex surfaces diffeomorphic to rational surfaces*, Preprint (1994)
[G1] Garcia-Prada, O.: *Dimensional reduction of stable bundles, vortices and stable pairs*, Int. J. of Math. Vol. 5, No 1, 1-52 (1994)
[G2] Garcia-Prada, O.: *A direct existence proof for the vortex equation over a compact Riemann surface*, Bull. London Math. Soc., 26, 88-96 (1994)
[HH] Hirzebruch, F., Hopf H.: *Felder von Flächenelementen in 4-dimensionalen 4-Mannigfaltigkeiten*, Math. Ann. 136 (1958)
[H1] Hitchin, N.: *Harmonic spinors*, Adv. in Math. 14, 1-55 (1974)
[H2] Hitchin, N.: *On the curvature of rational surfaces*, Proc. of Symp. in Pure Math., Stanford, Vol. 27 (1975)
[HL1] Huybrechts, D.; Lehn, M.: *Stable pairs on curves and surfaces*, J. Alg. Geometry 4, 67-104 (1995)

[HL2] Huybrechts, D.; Lehn, M.: *Framed modules and their moduli.* Int. J. Math. 6, 297-324 (1995)
[JT] Jaffe, A., Taubes, C.: *Vortices and monopoles*, Boston, Birkhäuser (1980)
[K] Kobayashi, S.: *Differential geometry of complex vector bundles*, Princeton University Press (1987)
[KM1] Kronheimer, P.; Mrowka, T.: *Recurrence relations and asymptotics for four-manifold invariants*, Bull. Amer. Math. Soc. 30, 215 (1994)
[KM2] Kronheimer, P.; Mrowka, T.: *The genus of embedded surfaces in the projective plane*, Preprint (1994)
[OSS] Okonek, Ch.; Schneider, M.; Spindler, H: *Vector bundles on complex projective spaces*, Progress in Math. 3, Birkhäuser, Boston (1980)
[Q] Qin, Z.: *Equivalence classes of polarizations and moduli spaces of stable locally free rank-2 sheaves*, J. Diff. Geom. 37, No 2 397-416 (1994)
[S] Simpson, C. T.: *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. 1 867-918 (1989)
[T] Thaddeus, M.: *Stable pairs, linear systems and the Verlinde formula*, Invent. math. 117, 181-205 (1994)
[UY] Uhlenbeck, K. K.; Yau, S. T.: *On the existence of Hermitian Yang-Mills connections in stable vector bundles*, Comm. Pure App. Math. 3, 257-293 (1986)
[W] Witten, E.: *Monopoles and four-manifolds*, Mathematical Research Letters 1, 769-796 (1994)

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