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The coupling and attenuation of nearly resonant multiplets in the Earth's free oscillation spectrum

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Summary. The effect of attenuation on the coupling of nearly resonant multiplets in the Earth's free oscillation spectrum is investigated and numerical results are presented for several of the most strongly coupled low frequency multiplet pairs. The coupling influences considered are those of the Coriolis forces due to rotation and of the Earth's hydrostatic ellipticity of figure. It is found that the effects of attenuation (in particular the difference in Q^{-1} for the two multiplets) can significantly change the splitting diagrams and the degree to which coupling takes place. The Q values for the coupled singlets are, in general, all different and lie between the two Q values of the uncoupled multiplet pair. In addition it is shown that the diagonal sum rule may be readily extended to sets of coupled multiplets.

Introduction

Dahlen (1969) and Luh (1973, 1974) have drawn attention to the strong coupling which can exist between modes of free oscillation of the Earth whose eigenfrequencies are very close to one another. Coupling between such nearly resonant modes is induced by a slight asphericity of the model, such as rotation, ellipticity or lateral heterogeneity in mechanical properties. Strong coupling may exist between multiplets when the difference between their unperturbed, degenerate eigenfrequencies is comparable to the frequency splitting predicted for the modes by ordinary degenerate perturbation theory for the given kind of asphericity (Dahlen 1968; Woodhouse & Dahlen 1978). As a result degenerate perturbation theory is inapplicable to these multiplets and quasi-degenerate perturbation theory must be employed (Dahlen 1969; Luh 1973, 1974). The eigenfunctions of the perturbed, slightly aspherical earth model are not even approximately eigenfunctions of the unperturbed model, but will be, to zero order in small quantities, linear combinations of the unperturbed eigenfunctions of all the singlets within the multiplets which are coupled.

While it is a necessary condition for coupling that the modes have nearly equal eigenfrequencies this may not be sufficient; the Coriolis forces due to rotation, for instance, couple either a pair of multiplets of different type (one spheroidal, the other toroidal) whose angular orders (l) differ by unity, or a pair of spheroidal multiplets belonging to the same angular order. Ellipticity also couples multiplets of the above kinds and, in addition, couples modes of the same type whose angular orders differ by two.

If attenuation is introduced the eigenfrequencies may be thought of as possessing an imaginary part $i\omega/2Q$ where ω is the unperturbed eigenfrequency and Q is the quality factor of the mode. If the quality factors, Q_1, Q_2, say, of two interacting multiplets were very different one would expect the modes to be less strongly coupled than they would be for a perfectly elastic earth model since, viewed in the complex plane, their frequencies would not be so nearly equal. It is the purpose of the present paper to investigate this effect. It will be shown that the introduction of attenuation can produce large changes in the splitting diagram of the coupled multiplets and that, even though we shall consider only spherically symmetric distributions of the material attenuation parameters Q_μ, Q_κ, the quality factors for the singlets in the perturbed spectrum will, in general, all be different. This is in contrast to the result in degenerate perturbation theory, where spherically symmetric distributions of Q_μ, Q_κ lead to the same value of Q for all members of a given multiplet.

For the low frequency multiplets considered by Luh (1974) the dominant coupling influences were shown to be rotation and ellipticity, and in this study of the effect of attenuation on quasi-degenerate multiplets we shall confine our attention to these asphericities alone. In the Appendix, however, we give the completely general matrix element, correcting the part pertaining to ellipticity (Dziewonski & Sailor 1976; Dahlen 1976; Woodhouse & Dahlen 1978) and also correcting some minor misprints in Luh's (1974) result. This matrix element, given here for reference, has rather wider applications than those considered in the present paper; it is needed, for example, in determining the first order corrections to the eigenfunctions in degenerate perturbation theory and also in calculating the second and higher order corrections to the eigenfrequencies (Dahlen 1968; Dahlen & Sailor 1979).

In Section 1 of the present paper notation is introduced and the necessary perturbation theory reviewed. It is also shown that the diagonal sum rule (Gilbert 1971) can be generalized to include the case of quasi-degenerate multiplets. In Section 2 the necessary formulae are derived for the particular application which is the subject of this paper and the results for some strongly coupled multiplets are discussed.

1 Perturbation theory

1.1 NOTATION

The normal mode eigenvalue problem for a spherically symmetric, non-rotating, elastic isotropic (SNREI) earth model may be written symbolically

$$(H_0 - \rho_0 \omega^2)\mathbf{s} = 0 \tag{1}$$

where H_0 is a self-adjoint integro-differential operator, also thought of as incorporating the appropriate boundary conditions; ρ_0 is the spherically symmetric density distribution of the model and the eigenvalue ω^2 is the squared angular frequency of a mode of free oscillation. The vector field s is the elastic displacement eigenfunction, which takes the form:

$$\mathbf{s} = |n\, q\, l\, m\rangle \equiv {}_nU_l^q(r)\,Y_l^m(\theta,\phi)\,\hat{\mathbf{r}} + {}_nV_l^q(r)\,\boldsymbol{\nabla}_1 Y_l^m(\theta,\phi) - {}_nW_l^q(r)\,\hat{\mathbf{r}}\times\boldsymbol{\nabla}_1 Y_l^m(\theta,\phi). \tag{2}$$

In equation (2) (r, θ, ϕ) are spherical polar coordinates and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ will denote unit vectors in the coordinate directions. The operator $\boldsymbol{\nabla}_1$ is $\hat{\boldsymbol{\theta}}\partial_\theta + \operatorname{cosec}\theta\,\hat{\boldsymbol{\phi}}\partial_\phi$ and $Y_l^m(\theta, \phi)$ are scalar spherical harmonics, as defined by Edmonds (1960),* which are normalized so that

$$\int [Y_{l'}^{m'}(\theta,\phi)]^* Y_l^m(\theta,\phi)\, d\Omega = \delta_{mm'}\delta_{ll'} \tag{3}$$

* Note that some geophysical applications (e.g. Gilbert & Dziewonski 1975) use a definition which differs from this by a factor $(-1)^m$ for $m < 0$ (*cf.* Schiff 1968).

where $\int d\Omega$ denotes integration over the unit sphere and * denotes complex conjugation. The parameters n, l, m are respectively, the radial order, angular order and azimuthal order of the eigenfunction and q takes the values T or S, signifying an eigenfunction of toroidal or spheroidal type; thus if q = T then ${}_nU_l^q = {}_nV_l^q = 0$ and if q = S, ${}_nW_l^q = 0$. For brevity we shall use the symbol k to denote the triplet n, q, l which defines a multiplet and write the unperturbed eigenvalues and eigenfunctions simply as ω_k, $|k\,m)$. Thus equation (1) becomes:

$$H_0|k\,m) = \omega_k^2 \rho_0 |k\,m). \tag{4}$$

For any tensor operator $\mathcal{O}$ which associates a vector field $\mathcal{O}\mathbf{u}$ with a given vector field $\mathbf{u}$, and for any two eigenfunctions $\mathbf{s} = |k\,m)$, $\mathbf{s}' = |k'\,m')$ we shall write for the 'matrix element':

$$(k\,m|\mathcal{O}|k'\,m') \equiv \int_V \mathbf{s}^* \cdot \mathcal{O}\mathbf{s}'\,dv, \tag{5}$$

where V is the volume of the SNREI earth model. The eigenfunctions will be normalized in such a way that

$$(k\,m|\rho_0|k'\,m') = \delta_{kk'}\delta_{mm'} \tag{6}$$

where, of course, $\delta_{kk'} = 1$ if $n = n'$, $q = q'$, $l = l'$ and is zero otherwise. In view of equations (2) and (3) this requires, using an obvious notation:

$$\delta_{ll'} \int_0^a \rho_0 [UU' + l(l+1)(VV' + WW')] r^2\,dr = \delta_{nn'}\delta_{qq'}\delta_{ll'} \tag{7}$$

where a is the radius of the earth model. If all eigenfunctions are included (not omitting the toroidal modes of the inner core or the internal wave modes of the fluid outer core) the vector fields $|k\,m)$ form a complete set, thus for any vector field $\mathbf{u}$ we may write

$$\mathbf{u} = \sum_{k\,m} |k\,m)(k\,m|\rho_0|u) \tag{8}$$

so that symbolically the identity tensor operator $\mathbf{I}$ may be written

$$\mathbf{I} = \sum_{k\,m} |k\,m)(k\,m|\rho_0. \tag{9}$$

1.2 THE PERTURBED EIGENVALUE PROBLEM

When the SNREI earth model is perturbed by the influences of rotation, ellipticity and asphericity we obtain the perturbed eigenvalue equation

$$[H_0 + \epsilon H_1 - (\rho_0 + \epsilon\rho_1)\sigma^2]\mathbf{u} = 0 \tag{10}$$

where σ^2 is an eigenvalue of the perturbed system and $\mathbf{u}$ the corresponding eigenfunction. The perturbation in density is $\epsilon\rho_1$ and ϵH_1 is a perturbing operator. The small parameter ϵ has been introduced to identify small quantities in the equations; ϵ will later be set to unity, and the small factor incorporated into ρ_1, H_1. Expressing $\mathbf{u}$ in the form (8) and making use of the unperturbed eigenvalue equation (4) we find

$$\sum_{k'\,m'} [\epsilon(H_1 - \rho_1\sigma^2) - (\sigma^2 - \omega_{k'}^2)\rho_0]\,|k'\,m')(k'\,m'|\rho_0|u) = 0. \tag{11}$$

Taking the scalar product with $|k''\,m'')^*$ and integrating over V we may also write

$$\sum_{k'\,m'} (k''\,m''|[\epsilon(H_1 - \rho_1\sigma^2) - (\sigma^2 - \omega_{k'}^2)\rho_0]|k'\,m')(k'\,m'|\rho_0|u) = 0 \tag{12}$$

which may be regarded as an (infinite dimensional) algebraic eigenvalue problem for σ^2.

Now let us seek an eigenvalue σ in the neighbourhood of some chosen frequency ω_0 and let K denote a set of multiplets with unperturbed frequencies close to ω_0, i.e.

$$\omega_k^2 - \omega_0^2 = O(\epsilon) \qquad \text{for } k \in K \tag{13}$$

Each multiplet $k = (n\ q\ l)$ contains $2l + 1$ singlets so that the number of singlets in K is

$$N \equiv \sum_{k \in K} (2l + 1). \tag{14}$$

Writing $\sigma^2 = \omega_0^2 + \epsilon\eta$ we find that the zeroth order terms of equation (12) give, using equation (6)

$$\sum_{\substack{k' \notin K \\ m'}} (\omega_0^2 - \omega_{k'}^2)\delta_{k'k''}\delta_{m'm''}(k'\ m' \,|\, \rho_0 \,|\, u) = 0$$

and consequently, for k' not in K

$$(k'\ m' \,|\, \rho_0 \,|\, u) = 0 \tag{15}$$

to zero order in ϵ. The first order terms of equation (12) give

$$\sum_{\substack{k' \in K \\ m'}} [(k''\ m'' \,|\, \epsilon Z \,|\, k'\ m') - (\omega_0^2 - \omega_{k'}^2 + \epsilon\eta)\delta_{k'k''}\delta_{m'm''}]\,(k'\ m' \,|\, \rho_0 \,|\, u) = 0 \tag{16}$$

where $Z = H_1 - \omega_0^2 \rho_1$.

Thus $\epsilon\eta$ is an eigenvalue of the square matrix of dimension N (equation (14)) whose elements are:*

$$A_{k''m''k'm'} = (k''\ m'' \,|\, \epsilon Z \,|\, k'\ m') - (\omega_0^2 - \omega_{k'}^2)\delta_{k'k''}\delta_{m'm''}, \qquad k'', k' \in K. \tag{17}$$

The eigenvalues η of equation (16) will be denoted by η_i, $i = 1, 2, \ldots, N$ and the components of the corresponding eigenvector by $(k'\ m' \,|\, \rho_0 \,|\, u_i)$. Using equations (8) and (15), the corresponding displacement eigenfunction of the perturbed system is given by

$$u_i = \sum_{k \in K} |k\ m)\,(k\ m \,|\, \rho_0 \,|\, u_i) \tag{18}$$

to zero order in ϵ. The parameter ϵ will now be set to unity, incorporating the small factor into H_1, ρ_1, Z, η.

1.3 THE DIAGONAL SUM RULE

For a purely aspherical perturbation and to first order in ordinary degenerate perturbation theory, Gilbert (1971) has shown that the average squared eigenfrequency of a split multiplet is the corresponding squared eigenfrequency of the SNREI starting model. That is to say that to this order of approximation the average squared eigenfrequency of a split multiplet is an eigenfrequency of the 'terrestrial monopole' – the spherically averaged earth. This result fails for quasi-degenerate multiplets since, as is clear from the above analysis, it is no longer possible to associate a singlet of the aspherical model with a single multiplet of the SNREI starting model, but only with the sets of coupled multiplets K. A simple generalization of Gilbert's result is possible, however, as we now show.

First we note that, since Z is a tensor operator, the Wigner–Eckart theorem (Edmonds 1960) tells us that the matrix element $(k'\ m' \,|\, Z \,|\, k\ m)$ may be written in the form

$$(k'\ m' \,|\, Z \,|\, k\ m) = \sum_{l''m''} (-1)^{-m'} \begin{pmatrix} l' & l'' & l \\ -m' & m'' & m \end{pmatrix} (k' \,\|\, Z^{(l'', m'')} \,\|\, k) \tag{19}$$

*The rows of A are labelled by the pair $(k''\ m'')$, the columns by $(k'\ m')$; $k'', k' \in K$; $-l'' \leqslant m'' \leqslant l''$, $-l' \leqslant m' \leqslant l'$.

where the reduced matrix element $(k'\|Z^{(l'',m'')}\|k)$, itself defined by equation (19), is independent of m, m' and where the 3-j symbol is that defined by Edmonds (1960). The term on the right hand side with $l''=0$ characterizes the spherical part of the perturbation and thus is identically zero if the SNREI starting model is chosen to represent the terrestrial monopole. From the property of the 3-j symbol that

$$\sum_m (-1)^m \begin{pmatrix} l & l'' & l \\ -m & m'' & m \end{pmatrix} = 0 \qquad (l'' \neq 0) \tag{20}$$

it follows that

$$\sum_m (k\,m|Z|k\,m) = 0. \tag{21}$$

Now the sum of the eigenvalues η_i of equation (16) is equal to the trace of the matrix **A**, i.e.

$$\sum_{i=1}^{N} \eta_i = \sum_{\substack{k \in K \\ m}} A_{k\,m\,k\,m} \tag{22}$$

so that making use of equations (17) and (21) we obtain simply

$$\sum_{i=1}^{N} \eta_i = \sum_{\substack{k \in K \\ m}} (\omega_k^2 - \omega_0^2) \tag{23}$$

so

$$\frac{1}{N}\sum_{i=1}^{N} (\omega_0^2 + \eta_i) = \frac{1}{N}\sum_{\substack{k \in K \\ m}} \omega_k^2 = \frac{1}{N}\sum_{k \in K} (2l+1)\,\omega_k^2. \tag{24}$$

Thus, since $\omega_0^2 + \eta_i$, $i = 1, 2, \ldots, N$ are squared eigenfrequencies of the aspherical model, we have shown that the average squared eigenfrequency for all the coupled singlets in K, calculated for the aspherical model, is equal to the same average calculated for the terrestrial monopole.

When K contains only a single multiplet the above results reduce to those of ordinary degenerate perturbation theory and equation (24) represents the diagonal sum rule of Gilbert (1971); equation (24) gives the extension of this rule to a set of nearly resonant multiplets. Just as Gilbert's result can be used to obtain data about the terrestrial monopole from the average eigenfrequency of a split multiplet, the generalized result equation (24) shows that if, for instance, two multiplets are strongly coupled, then the average squared eigenfrequency of all their singlets is equal to the average (weighted according to the number of singlets within each multiplet) of the corresponding squared eigenfrequencies of the terrestrial monopole. Thus, while it is not possible in this case to identify individually the two unperturbed eigenfrequencies, it is possible to determine the average, which may then be used as a datum in an inversion for the spherically averaged earth.

2 Coupling by rotation and ellipticity

2.1 DERIVATION

We shall restrict our attention to the coupling influences of rotation and ellipticity in an earth model possessing spherically symmetric distributions of the intrinsic attenuation parameters Q_μ, Q_κ. If a coordinate system is chosen in which the axis, $\theta = 0$, coincides with the axis of steady rotation the matrix elements $(k'\, m' | Z | k\, m)$ are different from zero only if $m' = m$ (equation (A17)), so that only singlets of the same azimuthal order are coupled. An examination of equation (A17) shows that only matrix elements of the following six kinds are different from zero:

$$(n'\, S\, l \pm 2m | Z | n\, S\, l\, m) = \begin{cases} \tfrac{3}{2}\, S_{l+2\,m} S_{l+1\,m} \displaystyle\int_0^a E^{(+)} r^2\, dr \\[2ex] \tfrac{3}{2}\, S_{l\,m} S_{l-1\,m} \displaystyle\int_0^a E^{(+)} r^2\, dr \end{cases} \tag{25}$$

$$(n'\, T\, l \pm 2m | Z | n\, T\, l\, m) = \begin{cases} \tfrac{3}{2}\, S_{l+2\,m} S_{l+1\,m} \displaystyle\int_0^a E^{(+)} r^2\, dr \\[2ex] \tfrac{3}{2}\, S_{l\,m} S_{l-1\,m} \displaystyle\int_0^a E^{(+)} r^2\, dr \end{cases} \tag{26}$$

$$(n'\, S\, l \pm 1m | Z | n\, T\, l\, m) = \begin{cases} iS_{l+1\,m} \left[2\omega_0 \Omega \displaystyle\int_0^a \rho_0 C^{(-)} r^2\, dr + 3m \int_0^a E^{(-)} r^2\, dr \right] \\[2ex] iS_{l\,m} \left[2\omega_0 \Omega \displaystyle\int_0^a \rho_0 C^{(-)} r^2\, dr + 3m \int_0^a E^{(-)} r^2\, dr \right] \end{cases} \tag{27}$$

$$(n'\, T\, l \pm 1m | Z | n\, S\, l\, m) = \begin{cases} iS_{l+1\,m} \left[2\omega_0 \Omega \displaystyle\int_0^a \rho_0 C^{(-)} r^2\, dr + 3m \int_0^a E^{(-)} r^2\, dr \right] \\[2ex] iS_{l\,m} \left[2\omega_0 \Omega \displaystyle\int_0^a \rho_0 C^{(-)} r^2\, dr + 3m \int_0^a E^{(-)} r^2\, dr \right] \end{cases} \tag{28}$$

$$(n'\, T\, l\, m | Z | n\, T\, l\, m) = \frac{2\omega_0}{l(l+1)}\, \Omega m \delta_{nn'} + T_{l\,m} \int_0^a E^{(+)} r^2\, dr + \omega_0 \delta_{nn'} (iQ^{-1} - d) \tag{29}$$

$$(n'\, S\, l\, m | Z | n\, s\, l\, m) = \tfrac{2}{3}\, \Omega^2 \left[\delta_{nn'} - l(l+1) \int_0^a \rho_0 C^{(+)} r^2\, dr \right] + 2\omega_0 \Omega m \int_0^a \rho_0 C^{(+)} r^2\, dr$$
$$+ T_{l\,m} \int_0^a E^{(+)} r^2\, dr + \omega_0^2 \delta_{nn'} (iQ^{-1} - d) \tag{30}$$

where the notation is that of the Appendix. The kernels $E^{(\pm)}$, $C^{(\pm)}$ are those given by equations (A18) to (A33) and it is understood that they should be evaluated using the scalar eigenfunctions (U, V, W, ϕ_1), (U', V', W', ϕ_1') of the multiplets appearing in the right and left brackets, respectively, of the matrix elements on the left hand sides of the equations. The parameters Q^{-1} in equations (29) and (30) are the quality factors of the multiplets in the absence of asphericity, defined by

$$\omega_0^2 Q^{-1} = \int_0^a [\kappa Q_\kappa^{-1}(\omega_0) K' + \mu Q_\mu^{-1}(\omega_0) M'] r^2\, dr \tag{31}$$

where

$$K' = (\dot{U} + F)^2 \tag{32}$$

$$M' = \tfrac{1}{3}(2\dot{U} - F)^2 + r^{-2}l(l+1)\,[(r\dot{V} - V + U)^2 + (r\dot{W} - W)^2] + r^{-2}(l-1)l(l+1)(l+2)\,[V^2 + W^2] \tag{33}$$

$$F = r^{-1}\,[2U - l(l+1)V]. \tag{34}$$

Similarly the parameters d in equations (29) and (30) characterize the frequency shifts of the multiplets arising from the frequency dependent real parts of the elastic moduli (Liu, Anderson & Kanamori 1976), defined to be $-\kappa d_\kappa(\omega)$, $-\mu d_\mu(\omega)$.* Thus we have

$$\omega_0^2 d = \int_0^a [\kappa d_\kappa(\omega_0)K' + \mu d_\mu(\omega_0)M']\,r^2\,dr. \tag{35}$$

In the calculations which follow, however, these dispersive terms have been omitted. The reason is that the calculations were performed for model 1066A (Gilbert & Dziewonski 1975) which was obtained without taking into account the dispersive consequences of attenuation. The eigenfrequencies of 1066A, therefore, will be close to the eigenfrequencies of the real terrestrial monopole, already modified by dispersion and it would be erroneous to incorporate these dispersive effects a second time. We therefore set d equal to zero in equations (29) and (30), although it would clearly be preferable to use a model which correctly accounted for dispersion and to make use of equation (35).

We shall consider coupling between a pair of multiplets, so that K contains two multiplets $k = (n, q, l)$, $k' = (n', q', l')$ say. Without loss of generality we assume $l' \leqslant l$. Since coupling occurs only between multiplets of the same azimuthal order, those singlets (if any) of multiplet k for which $l' < |m| \leqslant l$ remain uncoupled and equation (16) gives simply

$$(k\,m|Z|k\,m) - (\omega_0^2 + \eta - \omega_k^2) = 0 \qquad (l' < |m| \leqslant l) \tag{36}$$

so that the perturbed squared eigenfrequency is

$$\omega_0^2 + \eta = \omega_k^2 + (k\,m|Z|k\,m) \qquad (l' < |m| \leqslant l) \tag{37}$$

and the corresponding eigenfunction is simply $|k\,m)$, to zeroth order. This is, of course, the same result as would be obtained from ordinary degenerate perturbation theory. For singlets belonging to multiplets k, k' and having $|m| \leqslant l'$ equation (16) reduces to the 2×2 eigenvalue equation

$$\begin{pmatrix} (k\,m|Z|k\,m) + \omega_k^2 & (k\,m|Z|k'\,m) \\ (k'\,m|Z|k\,m) & (k'\,m|Z|k'\,m) + \omega_{k'}^2 \end{pmatrix} \begin{pmatrix} (k\,m|\rho_0|u) \\ (k'\,m|\rho_0|u) \end{pmatrix} = (\omega_0^2 + \eta) \begin{pmatrix} (k\,m|\rho_0|u) \\ (k'\,m|\rho_0|u) \end{pmatrix}. \tag{38}$$

Writing

$$\gamma = \tfrac{1}{2}(\omega_k^2 + \omega_{k'}^2) + \tfrac{1}{2}(k\,m|Z|k\,m) + \tfrac{1}{2}(k'\,m|Z|k'\,m) \tag{39}$$

$$\nu = \tfrac{1}{2}(\omega_k^2 - \omega_{k'}^2) + \tfrac{1}{2}(k\,m|Z|k\,m) - \tfrac{1}{2}(k'\,m|Z|k'\,m) \tag{40}$$

$$\mu = \{\nu^2 + (k\,m|Z|k'\,m)\,(k'\,m|Z|k\,m)\}^{1/2} \tag{41}$$

equation (38) gives the perturbed eigenfrequencies

$$\omega_\pm^2 \equiv \omega_0^2 + \eta_\pm = \gamma \pm \mu \tag{42}$$

*The complex elastic moduli of the spherical part of the perturbed model are thus $\kappa\,[1 - d_\kappa(\omega) + iQ_\kappa^{-1}(\omega)]$, $\mu\,[1 - d_\mu(\omega) + iQ_\mu^{-1}(\omega)]$, where κ, μ are the elastic moduli of the SNREI starting model.

and the corresponding zero order eigenfunctions

$$u_\pm = a_\pm |k\, m) + b_\pm |k'\, m) \tag{43}$$

where

$$\begin{aligned} a_\pm &= (r_+ + r_-)^{1/2} (r_\pm)^{1/2} \exp\,[\tfrac{1}{2} i(\theta_\pm - \theta_c)] \\ b_\pm &= (r_+ + r_-)^{1/2} (r_\mp)^{1/2} \exp\,[\tfrac{1}{2} i(-\theta_\pm + \theta_c)] \end{aligned} \tag{44}$$

and $r_\pm, \theta_\pm, \theta_c$ are defined through

$$r_\pm \exp(i\theta_\pm) = \nu \pm \mu \qquad (r_\pm > 0, -\pi < \theta_\pm \leqslant \pi) \tag{45}$$

$$r_c \exp(i\theta_c) = (k'\, m|Z|k\, m) \qquad (r_c > 0, -\pi < \theta_c \leqslant \pi).$$

The eigenfunctions (43) have been normalized so that

$$(u_\pm|\rho_0|u_\pm) = 1$$

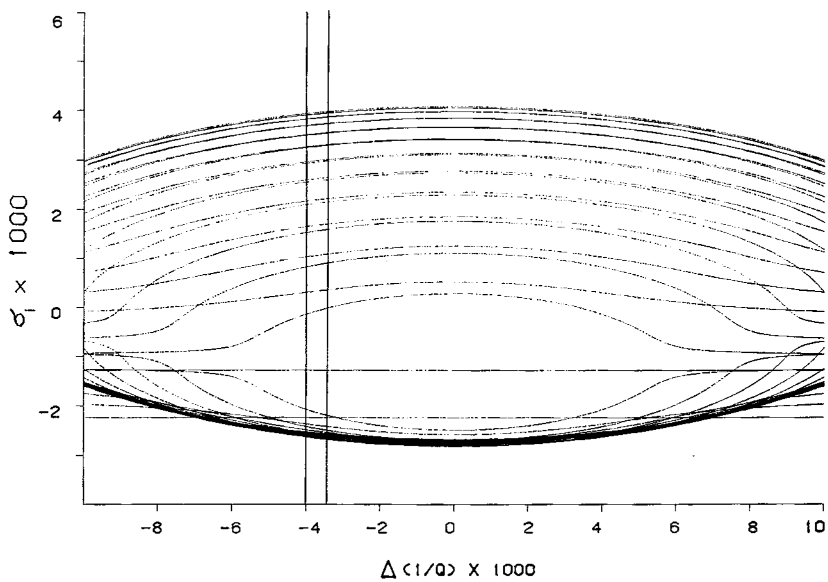
but it should be noted that they are not necessarily orthogonal. In fact we have

$$(u_-|\rho_0|u_+) = 2(r_+ r_-)^{1/2}(r_+ + r_-)^{-1} \cos \tfrac{1}{2}(\theta_+ - \theta_-) \tag{46}$$

and it is readily shown that this vanishes only if $(k\, m|Z|k\, m) - (k'\, m|Z|k'\, m)$ is real, which occurs only when the Q values for the multiplets are identical.

2.2 RESULTS

For each of the $N[= 2(l + l' + 1)]$ singlets belonging to the coupled multiplets the above analysis enables complex eigenfrequencies ω_i $(i = 1, 2, \ldots, N)$ to be determined, whence we may define real eigenfrequencies $(1 + \sigma_i)\omega_0$ $(i = 1, 2, \ldots, N)$ and Q^{-1} values $\tfrac{1}{2}(Q_k^{-1} +$



(a)

Figure 1. (a) The splitting diagram (σ_i equation (47)) of ${}_0S_{11}-{}_0T_{12}$. (b) The parameters q_i (equation (47)) of ${}_0S_{11}-{}_0T_{12}$. (c) The parameters E_i of ${}_0S_{11}-{}_0T_{12}$.

$Q_{k'}^{-1}) + q_i$ ($i = 1, 2, \ldots, N$) where Q_k, $Q_{k'}$ are the Q values belonging to the two multiplets k, k' in the absence of asphericities. Thus σ_i, q_i are defined by

$$\sigma_i = \frac{1}{\omega_0} \operatorname{Re}(\omega_i) - 1$$
$$q_i = \frac{2}{\omega_0} \operatorname{Im}(\omega_i) - \tfrac{1}{2}(Q_k^{-1} + Q_{k'}^{-1}). \tag{47}$$

We also define $E_i = |(k\, m\, |\rho_0| u_i)|^2$ where u_i is the normalized eigenfunction corresponding to the complex eigenfrequency ω_i and thus E_i is simply $|a_\pm| = r_\pm/(r_+ + r_-)$ (equations (43),

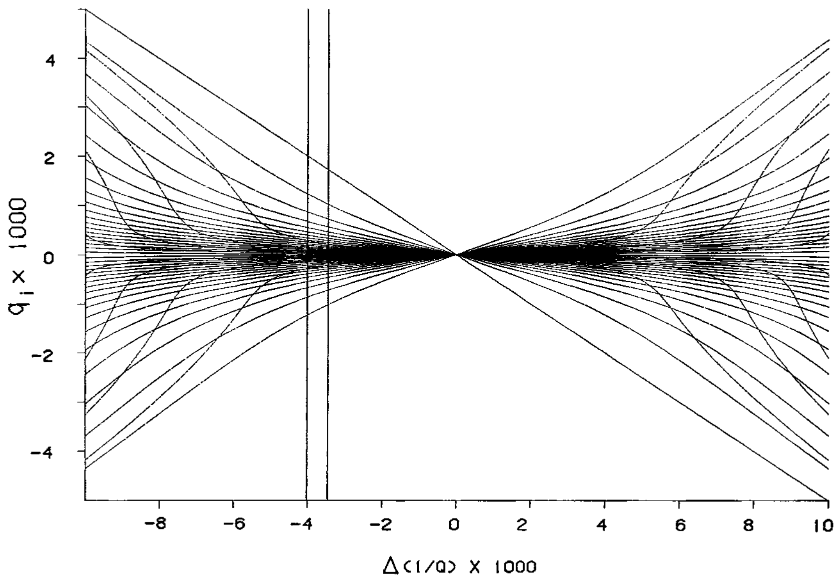


Fig. 1 (b)

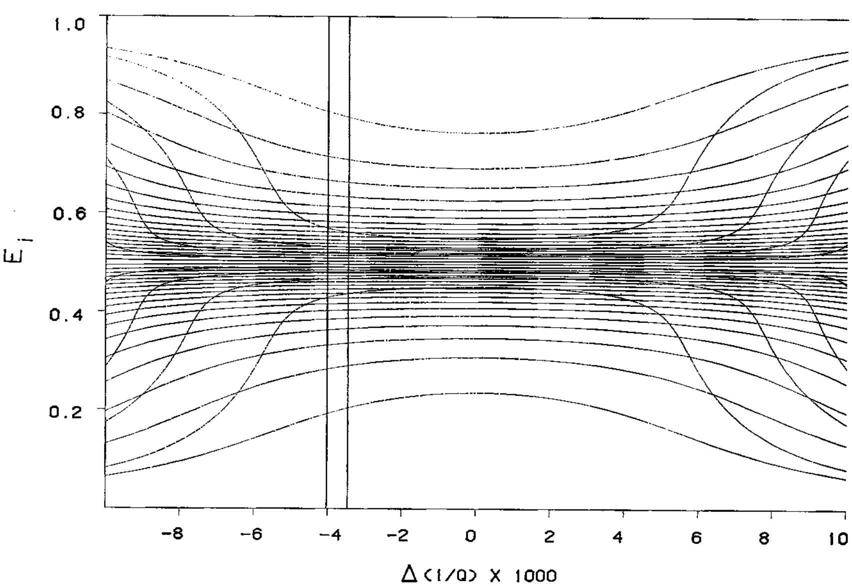


Fig. 1 (c)

(44)) where the upper or lower sign is taken according to whether ω_i is ω_+ or ω_- given by equation (42) for some value of m. Since the results are sensitive to the difference in Q^{-1} between multiplets the results for σ_i, q_i, E_i $(i = 1, 2, \ldots, N)$ have been calculated as a function of

$$\Delta Q^{-1} \equiv Q_k^{-1} - Q_{k'}^{-1} \tag{48}$$

which is the abscissa in Figs 1–5.

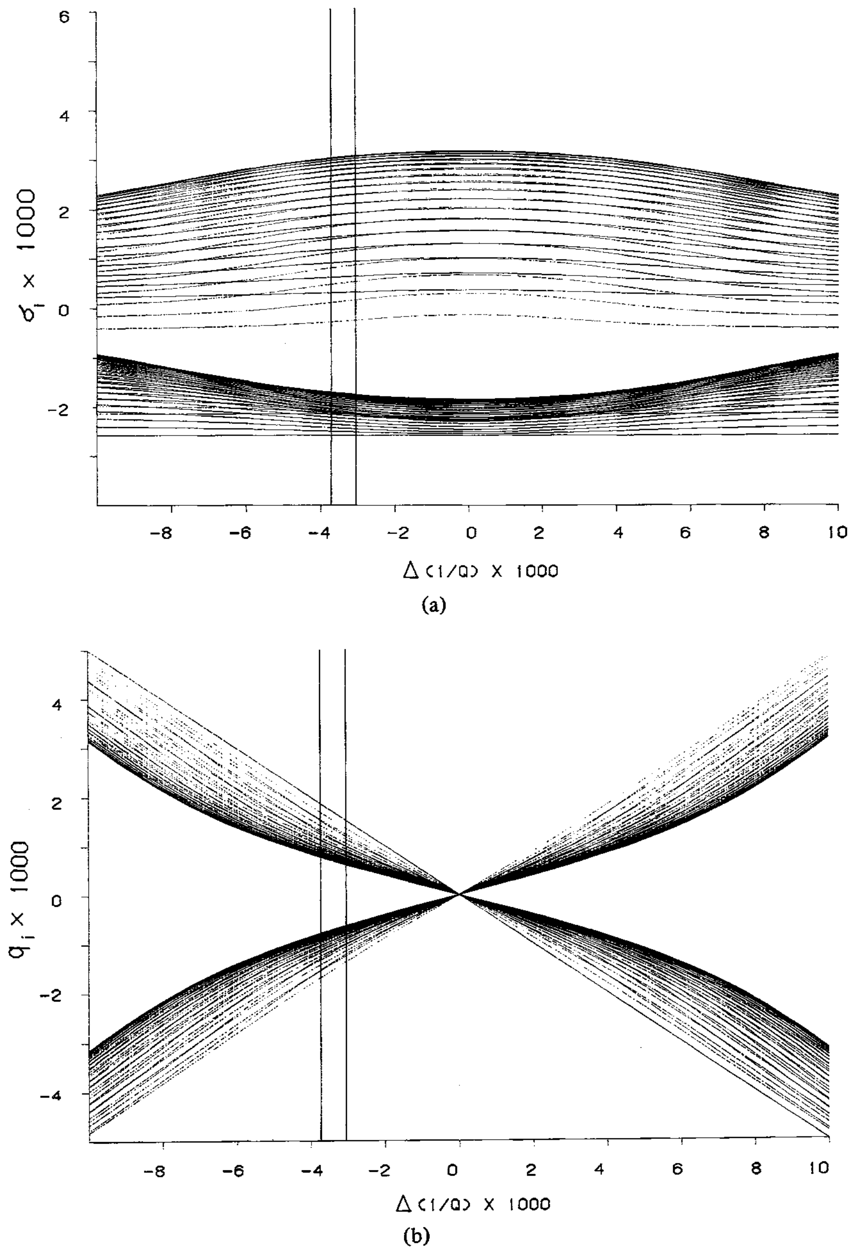


Figure 2. (a) The splitting diagrams of ${}_0S_{18}$–${}_0T_{19}$. (b) The parameters q_i (equation (47)) of ${}_0S_{18}$–${}_0T_{19}$. (c) The parameters E_i of ${}_0S_{18}$–${}_0T_{19}$.

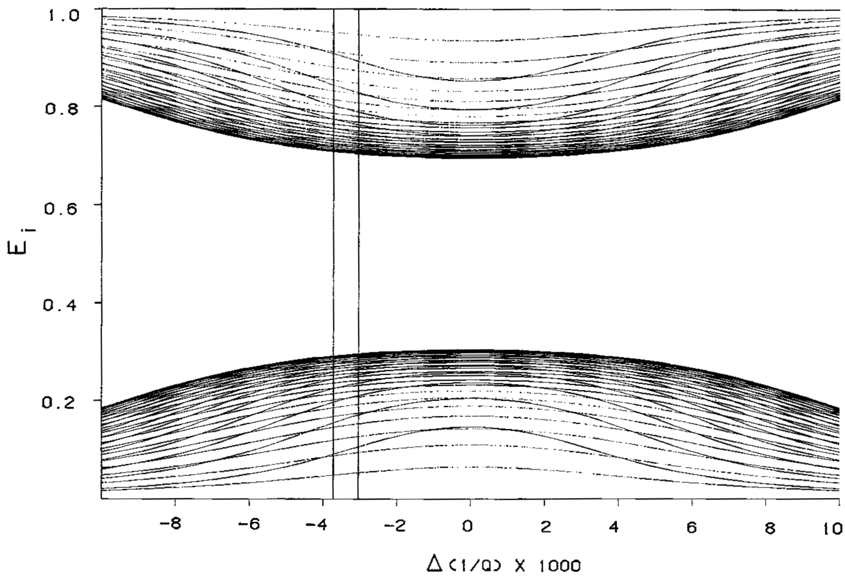
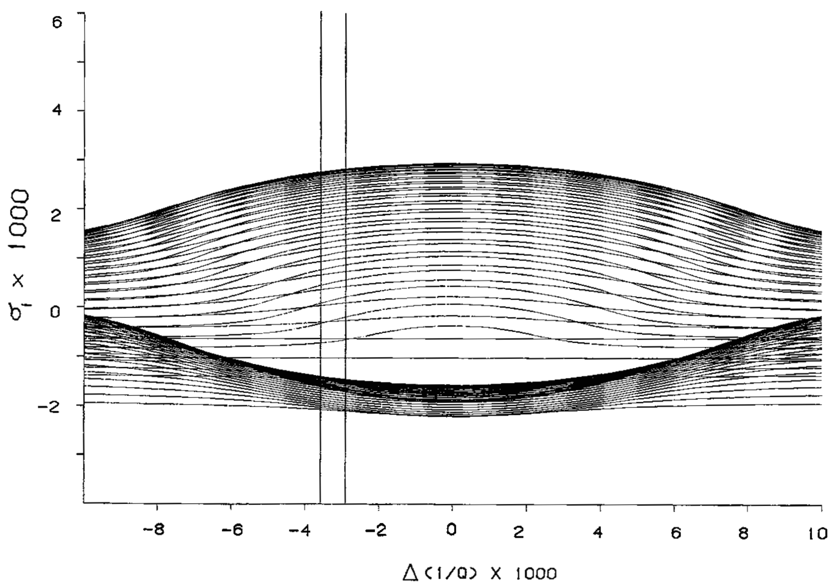
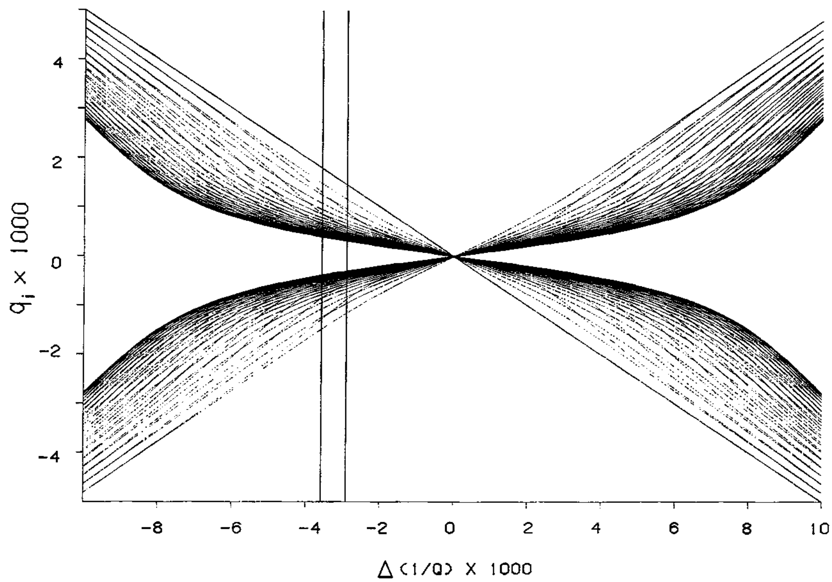


Fig. 2 (c)



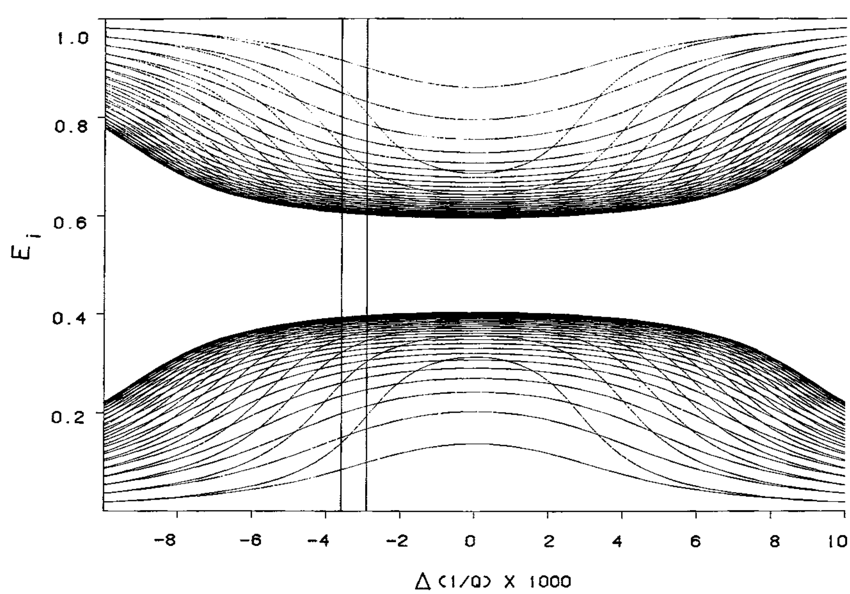
(a)

Figure 3. (a) The splitting diagrams of ${}_0S_{19}-{}_0T_{20}$; (b) The parameters q_i (equation (47)) of ${}_0S_{19}-{}_0T_{20}$. (c) The parameters E_i of ${}_0S_{19}-{}_0T_{20}$.



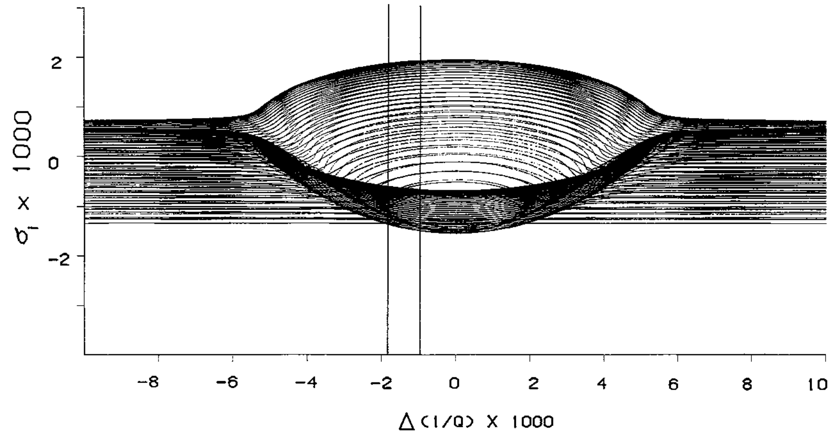
4
2
0
-2
-4
$q_i \times 1000$
-8
-6
-4
-2
0
2
4
6
8
10
Δ (1/Q) X 1000

Fig. 3 (b)

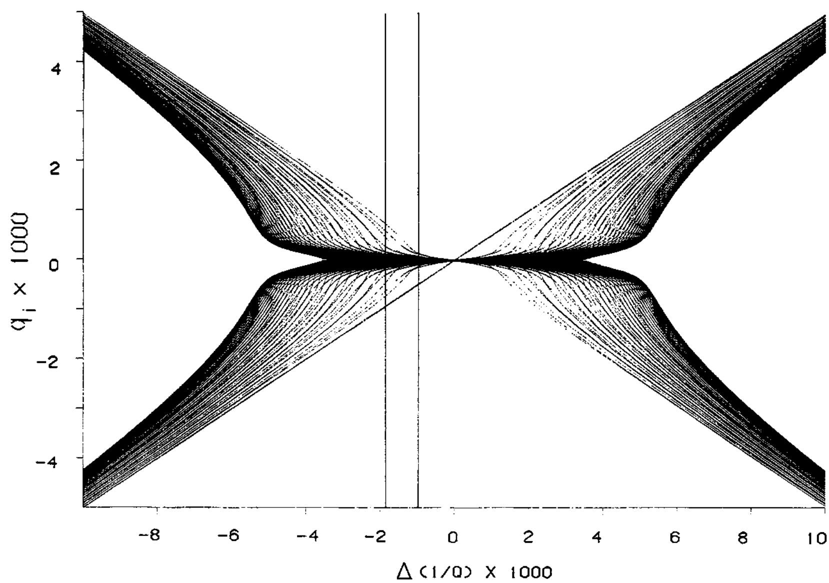


1.0
0.8
0.6
0.4
0.2
E_i
-8
-6
-4
-2
0
2
4
6
8
10
Δ (1/Q) X 1000

Fig. 3 (c)



(a)



(b)

Figure 4. (a) The splitting diagrams of ${}_0S_{32}-{}_0T_{31}$. (b) The parameters q_i (equation (47)) of ${}_0S_{32}-{}_0T_{31}$. (c) The parameters E_i of ${}_0S_{32}-{}_0T_{31}$.

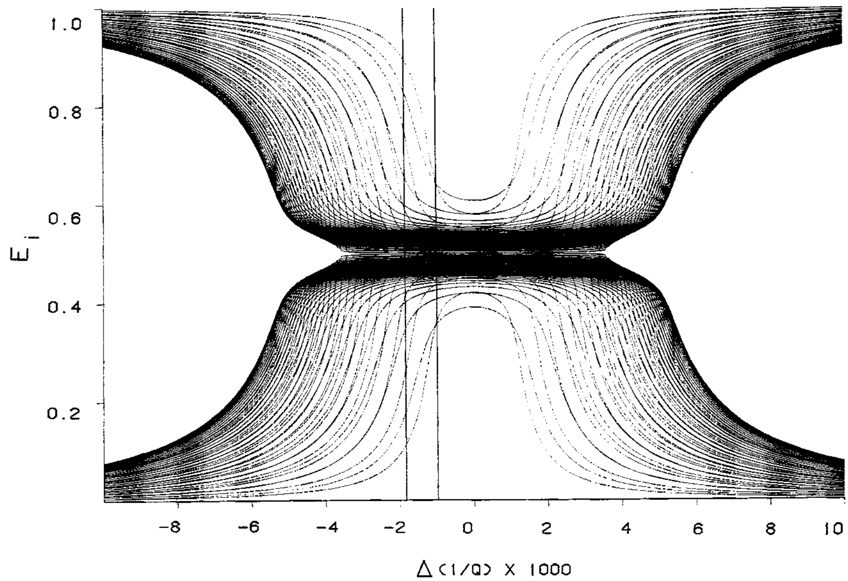
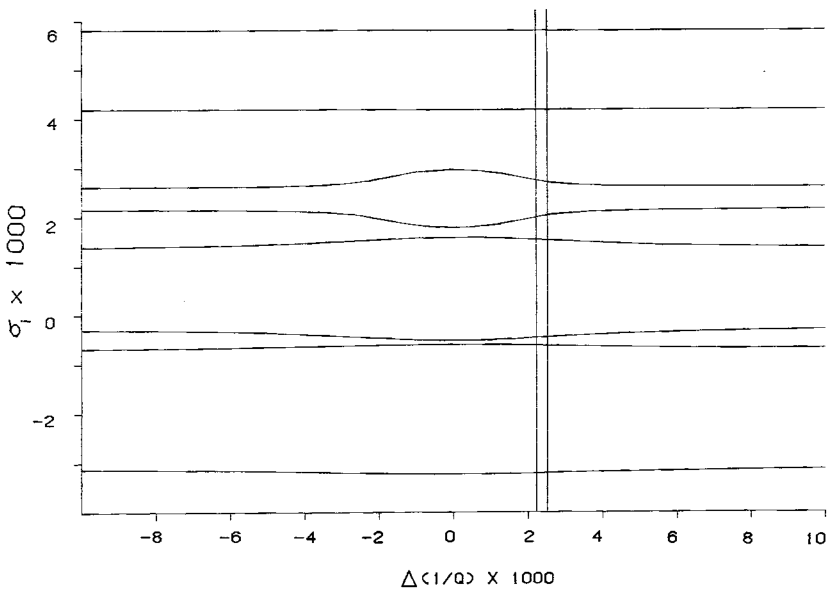
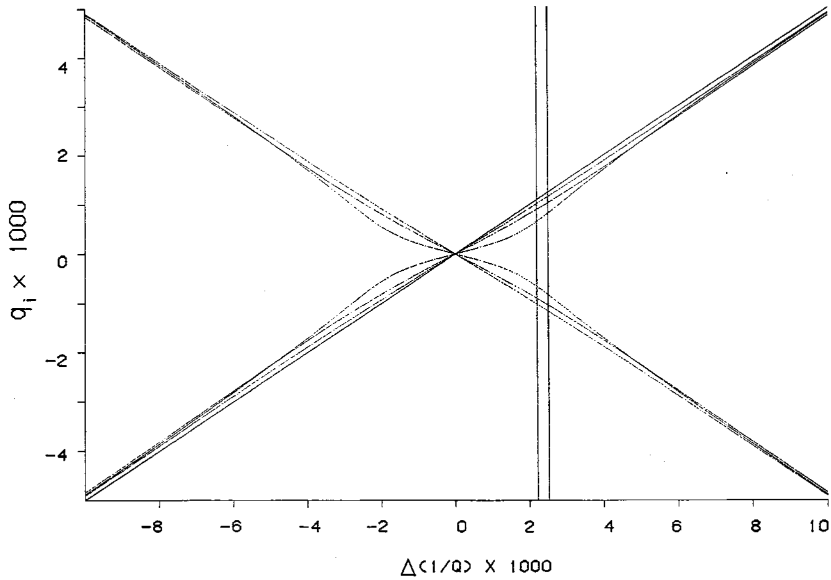


Fig. 4(c)



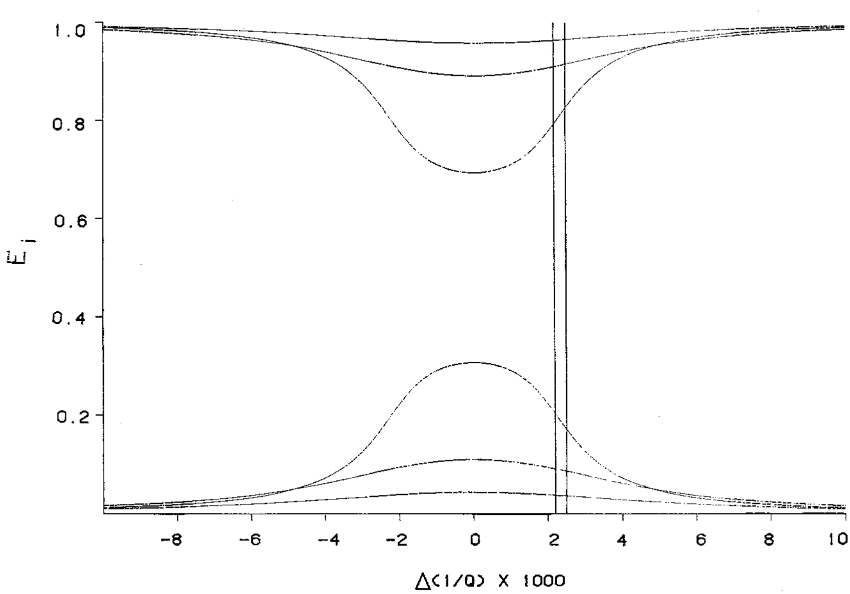
(a)

Figure 5. (a) The splitting diagrams of ${}_1S_3 - {}_3S_1$. The two uncoupled singlets ${}_1S_3^{-2}$, ${}_1S_3^{-3}$ are beyond the range of the diagram. (b) The parameters q_i of ${}_1S_3 - {}_3S_1$. (c) The parameters E_i of ${}_1S_3 - {}_3S_1$.



4
2
0
-2
-4
$q_i \times 1000$
-8
-6
-4
-2
0
2
4
6
8
10
Δ(1/Q) X 1000

Fig. 5 (b)



1.0
0.8
0.6
0.4
0.2
E_i
-8
-6
-4
-2
0
2
4
6
8
10
Δ(1/Q) X 1000

Fig. 5 (c)

The calculations were performed for model 1066A of Gilbert & Dziewonski (1975) using the catalogue of eigenfunctions calculated by Buland & Gilbert (1975). The parameters σ_i, q_i, E_i ($i = 1, 2, \ldots, N$) are plotted in Figs 1–5 for the following five pairs of multiplets:

$k = {}_0S_{11}, \quad k' = {}_0T_{12},$

$k = {}_0S_{18}, \quad k' = {}_0T_{19},$

$k = {}_0S_{19}, \quad k' = {}_0T_{20},$

$k = {}_0S_{32}, \quad k' = {}_0T_{31},$

$k = {}_1S_3, \quad k' = {}_3S_1.$

Sailor & Dziewonski (1978) have given several models for the distributions of $Q_\kappa(r), Q_\mu(r)$ consistent with normal mode observations and in each of Figs 1–5 is shown the band of ΔQ^{-1} predicted by their five Q models QMU, QDQ, QBS, QKB, QML.

Fig. 1(a) shows the splitting diagrams $\sigma_i = \sigma_i(\Delta Q^{-1})$ for the strongly coupled multiplets ${}_0S_{11}$, ${}_0T_{12}$. Notable here is the clustering in frequency of many singlets. Twenty-one of the 48 singlets, including each azimuthal order $|m| \leqslant 10$ have $-0.00282 < \sigma_i < -0.00267$ at $\Delta Q^{-1} = 0$, with some separation as $|\Delta Q^{-1}|$ increases. The remaining singlets form (at ΔQ^{-1} = 0) pairs with azimuthal orders $+m, -m$, $1 \leqslant |m| \leqslant 12$, and these pairs tend to separate as $|\Delta Q^{-1}|$ increases. It can be seen in Fig. 1(c) that there is strong clustering about $E_i = ½$, corresponding to perturbed eigenfunctions in which the toroidal and spheroidal components are of roughly equal magnitude. Correspondingly in Fig. 1(b) it can be seen that the values of Q^{-1} of the coupled singlets cluster about the mean Q^{-1} of the initial uncoupled multiplets.

The diagrams for the pairs ${}_0S_{19}$–${}_0T_{20}$, ${}_0S_{18}$–${}_0T_{19}$, shown in Figs 2 and 3, are basically similar, though in Figs 2(c) and 3(c) it can be seen that a distinct separation is maintained *between singlets predominantly of the toroidal and spheroidal types.*

The diagrams for ${}_0S_{32}$–${}_0T_{31}$ (Fig. 4) show a very complicated pattern. Clearly seen here is the decoupling effect for large values of ΔQ^{-1}; the splitting diagram (Fig. 4(a)) tends to the combined ordinary degenerate splitting diagram for the uncoupled multiplets and the parameters E_i (Fig. 4(c)) tend asymptotically to 1 or 0, corresponding to purely toroidal or purely spheroidal eigenfunctions. Correspondingly the Q^{-1} values for the singlets (Fig. 4(b)) tend asymptotically to the Q^{-1} values of the initial multiplets. It will be seen, however, that the differential Q^{-1} predicted for these multiplets by the Q models of Sailor & Dziewonski (1978) are not large enough to effect this decoupling.

For the pair ${}_1S_3$–${}_3S_1$ only the six singlets $|m| \leqslant 1$ are coupled. The only contributing coupling influence here is ellipticity and it can be seen that a significant degree of decoupling is achieved for the most strongly coupled singlets ($m = 1$) by introducing the differential Q^{-1} from the models of Sailor & Dziewonski (1978).

Acknowledgments

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Appendix: the general matrix element $(k'\ m'|Z|k\ m)$

The analysis given in Section 1 of this paper was purely formal and no attention was paid to the form of the perturbing operator H_1 or to the boundary conditions. These aspects of the problem have received detailed attention in a recent paper by Woodhouse & Dahlen (1978) and we may take our initial expressions for the general matrix element (equation (19)) directly from that paper (equations (68)–(71)) which will be referred to hereinafter as I. These expressions differ from the corresponding results used by Luh (1974) only in those parts relating to perturbations in the locations of discontinuities; similarly our result for the part relating to ellipticity will differ from that of Dahlen (1969). We shall omit here the effects of anistropy and non-hydrostatic initial stress, so that our perturbation is specified by the angular velocity $\mathbf{\Omega}$ and the consequent centrifugal potential ψ, the perturbation in density $\delta\rho(r, \theta, \phi)$ and the consequent perturbation in gravitational potential $\delta\phi_0(r, \theta, \phi)$, the perturbations in bulk and shear moduli $\delta\kappa(r, \theta, \phi)$, $\delta\mu(r, \theta, \phi)$ and the normal displacement $h(\theta, \phi)$ characterizing the perturbation in the location of each of the discontinuity surfaces of the model. To incorporate the effects of attenuation $\delta\kappa$, $\delta\mu$ may possess imaginary parts κ/Q_κ, μ/Q_μ and to include the effects of dispersion they may be frequency dependent. When applying the results to multiplets with eigenfrequencies close to ω_0,

$\delta\kappa$, $\delta\mu$ will be taken to be those evaluated at $\omega = \omega_0$ since variations within the band of splitting lead to only second-order effects.

Writing $\mathbf{s} = |k\,m) = |n\,q\,l\,m)$, $\mathbf{s}' = |k'\,m') = |n'\,q'\,l'\,m')$ and using the results of I, we find

$$
\begin{aligned}
(k'\,m'|Z|k\,m) = \tfrac{1}{2} \int_V \rho_0 [&4\omega\mathbf{s}\cdot(i\boldsymbol{\Omega}\times\mathbf{s}')^* + \mathbf{s}\cdot\boldsymbol{\nabla}(\mathbf{s}'^*\cdot\boldsymbol{\nabla}\psi) + \mathbf{s}'^*\cdot\boldsymbol{\nabla}(\mathbf{s}\cdot\boldsymbol{\nabla}\psi) \\
&- (\mathbf{s}\cdot\boldsymbol{\nabla}\psi)(\boldsymbol{\nabla}\cdot\mathbf{s}'^*) - (\mathbf{s}'^*\cdot\boldsymbol{\nabla}\psi)(\boldsymbol{\nabla}\cdot\mathbf{s})]\,dV \\
+ \int_V [&\delta\kappa(\boldsymbol{\nabla}\cdot\mathbf{s})(\boldsymbol{\nabla}\cdot\mathbf{s}'^*) + 2\delta\mu\boldsymbol{\Gamma}:\boldsymbol{\Gamma}'^* + \delta\rho_0(\mathbf{s}\cdot\boldsymbol{\nabla}\phi_1'^* + \mathbf{s}'^*\cdot\boldsymbol{\nabla}\phi_1 \\
&+ 4\pi G\rho_0 s_r s_r'^* + g_0\Lambda - \omega_0^2\mathbf{s}\cdot\mathbf{s}'^*]\,dV \\
+ \tfrac{1}{2} \int_V \rho_0 [&\mathbf{s}\cdot\boldsymbol{\nabla}(\mathbf{s}'^*\cdot\boldsymbol{\nabla}\delta\phi_0) + \mathbf{s}'^*\cdot\boldsymbol{\nabla}(\mathbf{s}\cdot\boldsymbol{\nabla}\delta\phi_0) - (\mathbf{s}\cdot\boldsymbol{\nabla}\delta\phi_0)(\boldsymbol{\nabla}\cdot\mathbf{s}'^*) \\
&- (\mathbf{s}'^*\cdot\boldsymbol{\nabla}\delta\phi_0)(\boldsymbol{\nabla}\cdot\mathbf{s})]\,dV \\
- \int_\Sigma h [&\tfrac{1}{2}\kappa(\boldsymbol{\nabla}\cdot\mathbf{s})(\boldsymbol{\nabla}\cdot\mathbf{s}'^* - 2\partial_r s_r'^*) + \tfrac{1}{2}\kappa(\boldsymbol{\nabla}\cdot\mathbf{s}'^*)(\boldsymbol{\nabla}\cdot\mathbf{s} - 2\partial_r s_r) \\
&+ \mu\boldsymbol{\Gamma}:(\boldsymbol{\Gamma}'^* - 2\hat{\mathbf{r}}\partial_r\mathbf{s}'^*) + \mu\boldsymbol{\Gamma}'^*:(\boldsymbol{\Gamma} - 2\hat{\mathbf{r}}\partial_r\mathbf{s}) + \rho_0(\mathbf{s}\cdot\boldsymbol{\nabla}\phi_1'^* \\
&+ \mathbf{s}'^*\cdot\boldsymbol{\nabla}\phi_1 + 8\pi G\rho_0 s_r s_r'^* + g_0\Lambda - \omega^2\mathbf{s}\cdot\mathbf{s}'^*]_-^+\,d\Sigma \\
- \int_\Sigma \boldsymbol{\nabla}_\Sigma h \cdot [&\kappa(\boldsymbol{\nabla}\cdot\mathbf{s})\mathbf{s}'^* + \kappa(\boldsymbol{\nabla}\cdot\mathbf{s}'^*)\mathbf{s} + 2\mu(\hat{\mathbf{r}}\cdot\boldsymbol{\Gamma}\cdot\hat{\mathbf{r}})\mathbf{s}'^* \\
&+ 2\mu(\hat{\mathbf{r}}\cdot\boldsymbol{\Gamma}'^*\cdot\hat{\mathbf{r}})\mathbf{s}]_-^+\,d\Sigma
\end{aligned}
\tag{A1}
$$

where

$$\Lambda = \tfrac{1}{2}\left(\mathbf{s}\cdot\boldsymbol{\nabla}s_r'^* + \mathbf{s}'^*\cdot\boldsymbol{\nabla}s_r - s_r\boldsymbol{\nabla}\cdot\mathbf{s}'^* - s_r'^*\boldsymbol{\nabla}\cdot\mathbf{s} - \frac{4}{r}s_r s_r'^*\right) \tag{A2}$$

and s_r, s_r' denote the radial components of $\mathbf{s}$, $\mathbf{s}'$.

As in I we shall express the total perturbations $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$, $\delta\mu$, h as a sum of the perturbations $\delta\rho_0^e$, $\delta\phi_0^e$, $\delta\kappa^e$, $\delta\mu^e$, h^e relating to the hydrostatic, isotropic, rotating, elliptical model which has the same spherical average as the SNREI starting model, together with an expansion of the remaining asphericities in terms of spherical harmonics, i.e.

$$\delta\rho_0 = \delta\rho_0^e + \sum_{l''m''} \delta\rho_{l''}^{m''} Y_{l''}^{m''} \tag{A3}$$

$$\delta\phi_0 = \delta\phi_0^e + \sum_{l''m''} \delta\phi_{l''}^{m''} Y_{l''}^{m''} \tag{A4}$$

$$\delta\kappa = \delta\kappa^e + \sum_{l''m''} \delta\kappa_{l''}^{m''} Y_{l''}^{m''} \tag{A5}$$

$$\delta\mu = \delta\mu^e + \sum_{l''m''} \delta\kappa_{l''}^{m''} Y_{l''}^{m''} \tag{A6}$$

$$h = h^e + \sum_{l''m''} h_{l''}^{m''} Y_{l''}^{m''} \tag{A7}$$

where (Dahlen 1968)

$$\delta\rho_0^e = (4\pi/5)^{1/2}\cdot\tfrac{2}{3}r\epsilon(r)\partial_r\rho_0 Y_2^0 \quad \text{(A8)}$$

$$\delta\mu^e = (4\pi/5)^{1/2}\cdot\tfrac{2}{3}r\epsilon(r)\partial_r\mu Y_2^0 \quad \text{(A9)}$$

$$\delta\kappa^e = (4\pi/5)^{1/2}\cdot\tfrac{2}{3}r\epsilon(r)\partial_r\kappa Y_2^0 \quad \text{(A10)}$$

$$\delta\phi^e = (4\pi/5)^{1/2}[\tfrac{2}{3}r\epsilon(r)g_0 - \tfrac{1}{3}\Omega^2 r^2]Y_2^0 \quad \text{(A11)}$$

$$h^e = -(4\pi/5)^{1/2}\cdot\tfrac{2}{3}r\epsilon(r)Y_2^0 \quad \text{(A12)}$$

and $\epsilon(r)$ is the hydrostatic ellipticity, the solution of Clairaut's equation (Jeffreys 1970).

The form used in the present paper for the eigenfunctions (equation (2)) and the definition used for the scalar spherical harmonics were chosen to be concordant with I and with many other papers on the Earth's free oscillations; for the calculation of matrix elements, however, the formalism developed by Burridge (1969) and Phinney & Burridge (1973) is very advantageous so we point out here the connection between their notation and our own. Following Phinney & Burridge (1973) we may define complex, spherical, contravariant components of s (and similarly s′)

$$s^{\mp} = 2^{-1/2}(\mp s_\theta + is_\phi), \qquad s^0 = s_r \quad \text{(A13)}$$

giving $s^{\pm} = \gamma_l\Omega_0^l(V \pm iW)Y_l^{\pm 1m}$, $s^0 = \gamma_l UY_l^{0m}$ where $\gamma_l = (4\pi)^{-1/2}(2l+1)^{1/2}$, $\Omega_N^l = [\tfrac{1}{2}(l+N)(l-N+1)]^{1/2}$ and $Y_l^{Nm}(\theta,\phi)$ are the generalized spherical harmonics defined by Phinney & Burridge (1973).

[Note $Y_l^m = \gamma_l Y_l^{0m}$, $\;2^{-1/2}(\mp\partial_\theta + i\operatorname{cosec}\theta\,\partial_\phi)Y_l^{0m} = \Omega_0^l Y_l^{\pm 1m}$]

The integrands in equation (A1) may then be readily expressed in terms of the contravariant components of the various vectors and tensors which appear, and these in turn may be written in terms of U, V, W, ϕ_1. The contravariant components of strain $e = \tfrac{1}{2}(\nabla \mathbf{s} + \mathbf{s}\nabla)$, for example, are:

$$\begin{aligned} e^{\mp\mp} &= \gamma_l\Omega_0^l\Omega_2^l r^{-1}(V \pm iW)Y_l^{\pm 2m}\\ e^{00} &= \gamma_l \dot{U} Y_l^{0m}\\ e^{\mp\pm} &= -\tfrac{1}{2}\gamma_l F Y_l^{0m}\\ e^{0\pm} &= e^{\pm 0} = \tfrac{1}{2}\gamma_l\Omega_0^l(X \pm iZ)Y_l^{\pm 1m} \end{aligned} \quad \text{(A14)}$$

where

$$F = r^{-1}(2U - l(l+1)V), \qquad X = \dot{V} + r^{-1}(U - V), \qquad Z = \dot{W} - r^{-1}W \quad \text{(A15)}$$

and '˙' denotes differentiation with respect to r. It is then a simple matter to evaluate the angular integrals in equation (A1) by making use of the formula

$$\frac{1}{4\pi}\int (Y_{l'}^{N'm'})^* Y_{l''}^{N''m''} Y_l^{Nm}\, d\Omega = (-1)^{N'-m'}\begin{pmatrix} l' & l'' & l\\ -N' & N'' & N\end{pmatrix}\begin{pmatrix} l' & l'' & l\\ -m' & m'' & m\end{pmatrix} \quad \text{(A16)}$$

which is obtainable from equation (4.6.2) of Edmonds (1960) when it is noted that

$$Y_l^{Nm}(\theta,\phi) = \mathscr{D}_{Nm}^{(l)}(\phi,\theta,0) = d_{Nm}^{(l)}(\theta)\exp(im\phi)$$

(for notations $\mathscr{D}_{Nm}^{(l)}$, $d_{Nm}^{(l)}$ see Edmonds 1960).

If the axis $\theta = 0$ of our spherical coordinate system is chosen to coincide with the angular velocity vector $\boldsymbol{\Omega}$ the final result for the matrix element may be written:

$$
\begin{aligned}
(k'\,m'|Z|k\,m) = \delta_{mm'} \Big\{ & 2\omega_0 \Omega m \delta_{ll'} \int_0^a \rho_0 C^{(+)} r^2\, dr + 2i\omega_0 \Omega (S_{l'm}\delta_{l'\,l+1} \\
& + S_{lm}\delta_{l\,l'+1}) \int_0^a \rho_0 C^{(-)} r^2\, dr + \tfrac{2}{3}\Omega^2 \delta_{ll'} \left[\delta_{qq'}\delta_{nn'} - L \int_0^a \rho_0 C^{(+)} r^2\, dr \right] \\
& + (T_{lm}\delta_{ll'} + \tfrac{3}{2} S_{lm} S_{l'+1\,m} \delta_{ll'+2} + \tfrac{3}{2} S_{l'm} S_{l+1\,m} \delta_{l'\,l+2}) \int_0^a E^{(+)} r^2\, dr \\
& + 3im (S_{l'm}\delta_{l'\,l+1} + S_{lm}\delta_{l\,l'+1}) \int_0^a E^{(-)} r^2\, dr \Big\} \\
& + \sum_{l''m''} \Bigg(\int_0^a [\delta\kappa_{l''}^{m''} K_{l''} + \delta\mu_{l''}^{m''} M_{l''} + \delta\rho_{l''}^{m''} R_{l''}^{(1)} + \delta\phi_{l''}^{m''} G_{l''}^{(1)} \\
& + \delta\dot{\phi}_{l''}^{m''} G_{l''}^{(2)}] r^2\, dr - \sum_d r^2 h_{l''}^{m''} [\kappa \tilde{K}_{l''} + \mu \tilde{M}_{l''} + \rho_0 R_{l''}^{(1)}]_-^+ \Bigg) \\
& \times (2l+1)^{1/2} (2l'+1)^{1/2} (2l''+1)^{1/2} (4\pi)^{-1/2} (-1)^{m'} \begin{pmatrix} l' & l'' & l \\ -m' & m'' & m \end{pmatrix}. \qquad \text{(A17)}
\end{aligned}
$$

The notations used above are as follows:

$$
S_{lm} = \left[\frac{(l+m)(l-m)}{(2l+1)(2l-1)} \right]^{1/2}, \quad T_{lm} = \frac{l(l+1) - 3m^2}{(2l-1)(2l+3)} \qquad \text{(Luh 1974)} \qquad \text{(A18)}
$$

$$
C^{(+)} = VV' + WW' + UV' + U'V
$$

$$
C^{(-)} = \tfrac{1}{2}[(L' - L + 2)U - (L' + L - 2)V]W' - \tfrac{1}{2}[(L - L' + 2)U' - (L + L' - 2)V']W \qquad \text{(A19)}
$$

with

$$
L = l(l+1), \quad L' = l'(l'+1), \qquad \text{(A20)}
$$

$$
E^{(+)} = \tfrac{2}{3}\epsilon(r)[\kappa(\bar{K}^{(+)} - (\eta+1)\tilde{K}^{(+)}) + \mu(\bar{M}^{(+)} - (\eta+1)\tilde{M}^{(+)}) + \rho_0(\bar{R}^{(+)} - (\eta+3)\tilde{R}^{(+)})] \qquad \text{(A21)}
$$

$$
E^{(-)} = \tfrac{2}{3}\epsilon(r)[-\kappa(\eta+2)\tilde{K}^{(-)} + \mu(\bar{M}^{(-)} - (\eta+1)\tilde{M}^{(-)}) + \rho_0(\bar{R}^{(-)} - (\eta+3)\tilde{R}^{(-)})],
$$

$$
\eta = \eta(r) = r\dot{\epsilon}(r)/\epsilon(r) \qquad \text{(A22)}
$$

and

$$
\bar{K}^{(+)} = -(\dot{U} + F)(\dot{U}' + \tfrac{1}{2}(L' - L + 2) r^{-1} V') - (\dot{U}' + F')(\dot{U} + \tfrac{1}{2}(L - L' + 2) r^{-1} V), \qquad \text{(A23)}
$$

$$
\begin{aligned}
\bar{M}^{(+)} = & -\tfrac{1}{3}(2\dot{U} - F)[2\dot{U}' + \tfrac{1}{2}(L' - L + 6)(3\dot{V}' - 4r^{-1}V')] + X[\tfrac{1}{2}(L - L' + 6)\dot{U}' \\
& - \tfrac{1}{2}(L + L' - 6)\dot{V}' - \tfrac{1}{2}(L' - L + 6) r^{-1} L V'] + r^{-1}[\tfrac{1}{2}L'(L' - L + 6) \\
& + 3(L + L' - 6)](W'\dot{W} + V'\dot{V}) - \tfrac{1}{2}(L' + L - 6)\dot{W}Z' - \tfrac{1}{3}(2\dot{U}' - F')[2\dot{U} \\
& + \tfrac{1}{2}(L - L' + 6)(3\dot{V} - 4r^{-1}V)] + X'[\tfrac{1}{2}(L' - L + 6)\dot{U} - \tfrac{1}{2}(L' + L - 6)\dot{V} \\
& - \tfrac{1}{2}(L - L' + 6) r^{-1} L' V] + r^{-1}[\tfrac{1}{2}L(L - L' + 6) + 3(L' + L - 6)](W\dot{W}' \\
& + V\dot{V}') - \tfrac{1}{2}(L + L' - 6)\dot{W}' Z, \qquad \text{(A24)}
\end{aligned}
$$

$$\bar{R}^{(+)} = F(r\dot{\phi}_1' + 4\pi G\rho_0 rU' + g_0 U') + \tfrac{1}{2}(L' - L + 6)UV'(\omega_0^2 - r^{-1}g_0) + 3r^{-1}g_0 UU'$$
$$+ r^{-1}\phi_1' [\tfrac{1}{2}(L' + L - 6)V - L'U] + F'(r\dot{\phi}_1 + 4\pi G\rho_0 rU + g_0 U)$$
$$+ \tfrac{1}{2}(L - L' + 6)U'V(\omega_0^2 - r^{-1}g_0) + 3r^{-1}g_0 U'U + r^{-1}\phi_1 [\tfrac{1}{2}(L + L' - 6)V' - LU'] \tag{A25}$$

where

$$\left.\begin{aligned} F &= r^{-1}(2U - LV), & F' &= r^{-1}(2U' - L'V'),\\ X &= \dot{V} + r^{-1}(U - V), & X' &= \dot{V}' + r^{-1}(U' - V'),\\ Z &= \dot{W} - r^{-1}W, & Z' &= \dot{W}' - r^{-1}W'. \end{aligned}\right\} \tag{A26}$$

$g_0 = g_0(r)$ is the gravitational acceleration in the SNREI starting model and G is the gravitational constant.

$$\tilde{K}^{(+)} = \tfrac{1}{2}(\dot{U} + F)(-\dot{U}' + F' + (L' - L + 6)r^{-1}V') + \tfrac{1}{2}(\dot{U}' + F')(-\dot{U} + F$$
$$+ (L - L' + 6)r^{-1}V), \tag{A27}$$

$$\tilde{M}^{(+)} = \tfrac{1}{2}r^{-2}(VV' + WW')((L' + L - 8)(L' + L - 6) - 2LL') + \tfrac{1}{2}(L' + L - 6)$$
$$\times (XX' + ZZ' - \dot{V}'X - \dot{W}'Z - \dot{V}X' - \dot{W}Z') - \tfrac{1}{3}(2\dot{U} - F)$$
$$\times (\dot{U}' + \tfrac{1}{2}F' - (L' - L + 6)r^{-1}V') - \tfrac{1}{3}(2\dot{U}' - F')(\dot{U} + \tfrac{1}{2}F - (L - L' + 6)r^{-1}V) \tag{A28}$$

$$\tilde{R}^{(+)} = \tfrac{1}{4}(L + L' - 6)(2r^{-1}V'\phi_1 - \omega_0^2 VV' - \omega_0^2 WW') + \tfrac{1}{2}U'[2\dot{\phi}_1 + 8\pi G\rho_0 U - \omega_0^2 U$$
$$- (L - L' + 6)g_0 r^{-1}V] + \tfrac{1}{4}(L' + L - 6)(2r^{-1}V\phi_1' - \omega_0^2 V'V - \omega_0^2 W'W)$$
$$+ \tfrac{1}{2}U[2\dot{\phi}_1' + 8\pi G\rho_0 U' - \omega_0^2 U' - (L' - L + 6)g_0 r^{-1}V'] \tag{A29}$$

$$\bar{M}^{(-)} = \dot{W}'[2\dot{V} - \dot{U} + 3r^{-1}U - (L - L' + 7)r^{-1}V] + r^{-1}W'[7\dot{U} - 7\dot{V} + 5Lr^{-1}V$$
$$- (L + 8)r^{-1}U] - \dot{W}[2\dot{V}' - \dot{U}' + 3r^{-1}U' - (L' - L + 7)r^{-1}V']$$
$$- r^{-1}W[7\dot{U}' - 7\dot{V}' + 5L'r^{-1}V' - (L' + 8)r^{-1}U'] \tag{A30}$$

$$\bar{R}^{(-)} = g_0(2r^{-1}W'U - \dot{W}'U - W'\dot{U}) + W'(\omega_0^2 U - r^{-1}\phi_1 - 4\pi G\rho_0 U)$$
$$- g_0(2r^{-1}WU' - \dot{W}U' - W\dot{U}') - W(\omega_0^2 U' - r^{-1}\phi_1' - 4\pi G\rho_0 U') \tag{A31}$$

$$\tilde{K}^{(-)} = r^{-1}W'(\dot{U} + F) - r^{-1}W(\dot{U}' + F') \tag{A32}$$

$$\tilde{M}^{(-)} = r^{-2}W'(U - V) + \tfrac{2}{3}r^{-1}W'(2\dot{U} - F) + \dot{V}\dot{W}' - r^{-2}VW'(L + L' - 8)$$
$$- r^{-2}W(U' - V') - \tfrac{2}{3}r^{-1}W(2\dot{U}' - F') - \dot{V}'\dot{W} + r^{-2}V'W(L' + L - 8). \tag{A33}$$

$$\tilde{R}^{(-)} = W'(\omega_0^2 V - r^{-1}\phi_1) - W(\omega_0^2 V' - r^{-1}\phi_1') \tag{A34}$$

The above definitions (A18) to (A34) specify completely the first five terms on the right hand side of equation (A17) which represent the contributions to the matrix element arising from the effects of the Coriolis forces, centrifugal forces and ellipticity. The first two terms are the contribution from Coriolis forces and the fourth and fifth terms represent the contribution from ellipticity, together with the aspherical part of the centrifugal forces (the aspherical part of ψ). These two terms result from a transformation similar to that outlined in the Appendix to I, which enables the ellipticity contribution to be written in a form which does not involve the radial derivatives μ, κ, ρ_0 of the model parameters. The third term in equation (A17) is the contribution from the spherically averaged centrifugal forces. It may be remarked that this term does not satisfy the diagonal sum rule as formulated in Section 1.3 of the present paper but the reason for this is clear; if the 'terrestrial monopole' is defined as the spherically averaged Earth then it should include a spherically symmetric body force distribution, everywhere directed outwards along the radius vector, representing the spherical average of the centrifugal force. Such a force has not been included, however, in our SNREI starting model, and hence a term appears in equation (A17) which does not satisfy the diagonal sum rule.

It is only these first five terms which are needed for the particular application which is the subject of the present paper. For completeness, however, the contribution to the matrix element from additional arbitrary asphericities has been included as the final summation in equation (A17). To complete the specification of this contribution we now list the kernels which appear. First we define the following coefficients:

$$B^{(N)\pm}_{l'l''l} = \tfrac{1}{2}(1 \pm (-1)^{l'+l''+l})\left[\frac{(l'+N)!(l+N)!}{(l'-N)!(l-N)!}\right]^{1/2}(-1)^N\begin{pmatrix} l' & l'' & l \\ -N & 0 & N\end{pmatrix}. \tag{A35}$$

The kernels may then be written as follows:

$$K_{l''} = (\dot{U}'+F')(\dot{U}+F)B^{(0)+}_{l'l''l} \tag{A36}$$

$$\begin{aligned} M_{l''} &= r^{-2}(V'V+W'W)B^{(2)+}_{l'l''l} + r^{-2}(V'W-W'V)iB^{(2)-}_{l'l''l} + (XX'+ZZ')B^{(1)+}_{l'l''l} \\ &\quad + (X'Z-XZ')iB^{(1)-}_{l'l''l} + \tfrac{1}{3}(2\dot{U}'-F')(2\dot{U}-F)B^{(0)+}_{l'l''l} \end{aligned} \tag{A37}$$

$$\begin{aligned} R^{(1)}_{l''} &= [-\omega_0^2(VV'+WW') + r^{-1}(\phi_1'V+\phi_1V') + \tfrac{1}{2}g_0r^{-1}(U'V+V'U)]B^{(1)+}_{l'l''l} \\ &\quad + [-\omega_0^2(V'W-W'V) + r^{-1}(\phi_1'W-\phi_1W') + \tfrac{1}{2}g_0r^{-1}(U'W-W'U)]\,iB^{(1)-}_{l'l''l} \\ &\quad + [8\pi G\rho_0UU' + \dot{\phi}_1'U + \dot{\phi}_1U' - \omega_0^2UU' - \tfrac{1}{2}g_0(4r^{-1}UU' + U'F + UF')]B^{(0)+}_{l'l''l} \end{aligned} \tag{A38}$$

$$\begin{aligned} G^{(1)}_{l''} &= \tfrac{1}{2}\rho_0r^{-1}(U\dot{V}' + r^{-1}UV' - \dot{U}V' - 2FV')B^{(1)+}_{l'll''} \\ &\quad + \tfrac{1}{2}\rho_0r^{-1}(U'\dot{V} + r^{-1}U'V - \dot{U}'V - 2F'V)B^{(1)+}_{ll'l''} \\ &\quad + \tfrac{1}{2}\rho_0r^{-1}(U\dot{W}' + r^{-1}UW' - \dot{U}W' - 2FW')iB^{(1)-}_{l'll''} \\ &\quad - \tfrac{1}{2}\rho_0r^{-1}(U'\dot{W} + r^{-1}U'W - \dot{U}'W - 2F'W)iB^{(1)-}_{ll'l''} \\ &\quad + \rho_0r^{-2}UU'l''(l''+1)B^{(0)+}_{l'l''l} \end{aligned} \tag{A39}$$

$$\begin{aligned} G^{(2)}_{l''} &= \tfrac{1}{2}\rho_0r^{-1}UV'B^{(1)+}_{l'll''} + \tfrac{1}{2}\rho_0r^{-1}U'VB^{(1)+}_{ll'l''} \\ &\quad + \tfrac{1}{2}\rho_0r^{-1}UW'iB^{(1)-}_{l'll''} - \tfrac{1}{2}\rho_0r^{-1}U'WiB^{(1)-}_{ll'l''} \\ &\quad - \rho_0(F'U+U'F)B^{(0)+}_{l'l''l} \end{aligned} \tag{A40}$$

$$\begin{aligned} \tilde{K}_{l''} &= K_{l''} - (2\dot{U}\dot{U}' + \dot{U}'F + \dot{U}F')B^{(0)+}_{l'l''l} \\ &\quad + r^{-1}(\dot{U}+F)(V'B^{(1)+}_{l'll''} + iW'B^{(1)-}_{l'll''}) \\ &\quad + r^{-1}(\dot{U}'+F')(VB^{(1)+}_{ll'l''} - iWB^{(1)-}_{ll'l''}) \end{aligned} \tag{A41}$$

$$\begin{aligned} \tilde{M}_{l''} &= M_{l''} - (\dot{V}'X + \dot{V}X' + \dot{W}'Z + \dot{W}Z')B^{(1)+}_{l'l''l} \\ &\quad - (\dot{V}'Z - \dot{V}Z' + \dot{W}X' - X\dot{W}')iB^{(1)-}_{l'l''l} \\ &\quad + \tfrac{2}{3}(2\dot{U}-F)(-\dot{U}'B^{(0)+}_{l'l''l} + r^{-1}V'B^{(1)+}_{l'll''} + r^{-1}W'iB^{(1)-}_{l'll''}) \\ &\quad + \tfrac{2}{3}(2\dot{U}'-F')(-\dot{U}B^{(0)+}_{ll''l'} + r^{-1}VB^{(1)+}_{ll'l''} - r^{-1}WiB^{(1)-}_{ll'l''}). \end{aligned} \tag{A42}$$

This completes the specification of the quantities appearing in equation (A17). If desired, the kernels (A36)–(A42) may be reduced to expressions involving only 3-j symbols of the form:

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}$$

by making use of the identities:

$$B^{(1)+}_{l'l''l} = \tfrac{1}{2}(L' + L - L'')B^{(0)+}_{l'l''l} \tag{A43}$$

$$B^{(2)-}_{l'l''l} = (L' + L - L'' - 2)B^{(1)-}_{l'l''l} \tag{A44}$$

$$B^{(2)+}_{l'l''l} = \tfrac{1}{2}[(L' + L - L'' - 2)(L' + L - L'') - 2L'L]B^{(0)+}_{l'l''l} \tag{A45}$$

$$B^{(1)-}_{l'l''l} = \tfrac{1}{2}\left\{\frac{(\Sigma + 2)(\Sigma + 4)}{\Sigma + 3}\cdot(\Sigma + 1 - 2l)(\Sigma + 1 - 2l')(\Sigma + 1 - 2l'')\right\}^{1/2} B^{(0)+}_{l'+1\,l''+1\,l+1} \tag{A46}$$

where

$$L = l(l + 1), \quad L' = l'(l' + 1), \quad L'' = l''(l'' + 1), \quad \Sigma = l' + l'' + l. \tag{A47}$$

Finally we note that, as in I, the perturbation in gravitational potential $\delta\phi^{m''}_{l''}$ may be eliminated from the result (A17) by making use of the explicit representation of $\delta\phi^{m''}_{l''}$ in terms of $\delta\rho^{m''}_{l''}$ (Dahlen 1974, equation 11). The general matrix element is then obtained purely in terms of the specified perturbations $\delta\kappa^{m''}_{l''}$, $\delta\mu^{m''}_{l''}$, $\delta\rho^{m''}_{l''}$, $h^{m''}_{l''}$. This form may be obtained from equation (A17) by performing the following substitutions, whose combined effect is to leave the result unchanged:

$$R^{(1)}_{l''} \to R^{(2)}_{l''} = R^{(1)}_{l''} + \frac{4\pi G}{2l'' + 1}\left\{r^{l''}\int_r^a r^{-l''}[(l'' + 1)G^{(2)}_{l''} - rG^{(1)}_{l''}]\,dr - r^{-l''-1}\int_0^r r^{l''+1}[l''G^{(2)}_{l''} + rG^{(1)}_{l''}]\,dr\right\}, \tag{A48}$$

$$G^{(1)}_{l''} \to 0, \tag{A49}$$

$$G^{(2)}_{l''} \to 0. \tag{A50}$$