The cover pebbling theorem

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Abstract

For any configuration of pebbles on the nodes of a graph, a pebbling move replaces two pebbles on one node by one pebble on an adjacent node. A cover pebbling is a move sequence ending with no empty nodes. The number of pebbles needed for a cover pebbling starting with all pebbles on one node is trivial to compute and it was conjectured that the maximum of these simple cover pebbling numbers is indeed the general cover pebbling number of the graph. That is, for any configuration of this size, there exists a cover pebbling. In this note, we prove a generalization of the conjecture. All previously published results about cover pebbling numbers for special graphs (trees, hypercubes et cetera) are direct consequences of this theorem. We also prove that the cover pebbling number of a product of two graphs equals the product of the cover pebbling numbers of the graphs.

1 Introduction

Pebbling, peg solitaire, chip firing and checker jumping are some kindred combinatorial games on graphs. Put some tokens on the nodes, define local moves and you can start asking questions about convergence, reachability and enumeration! But this is not a true description of how these games came into existence. Each one has its own roots in areas such as number theory, statistical mechanics, economics and of course recreational mathematics.

The pebbling game appeared in the 1980s and comes in two flavours. In the first version, played on a directed graph, a move consists in replacing a pebble on one node by new pebbles on the adjacent nodes, moving along directed edges. The 1995 paper by Eriksson [3] seems to have solved this game completely.

The second pebbling version, which is still very hot, was introduced in 1989 by Chung [1] and is played on a connected graph, directed or undirected. A move replaces two pebbles on one node by one pebble on an adjacent node, and this is the pebbling rule for the remainder of this paper.

The most important reachability questions concern the *pebbling number* and the *cover pebbling number* of a graph, that is the smallest n such that from any initial distribution of n pebbles, it is possible to pebble any desired node, respectively pebble all nodes. In a series of recent papers by Crull et al. [2], Watson and Yerger [8], Hurlbert and Munyan [5], and Tomova and Wyels [6], the cover pebbling number has been derived for several classes of graphs. These results are all special cases of our main theorem, conjectured by Crull et al. in [2].

The results in this paper were found independently by Vuong and Wyckoff [7]. However, our proofs and presentations are different and in a way complementary.

2 General covers and simple distributions.

Following Crull et al. we generalize the situation like this: Instead of trying to place at least one pebble on each node, we define a goal distribution w of pebbles. A *w-cover* is a distribution of pebbles such that every node has at least as many pebbles as in w. We write $w(v)$ for the number of pebbles on the node v in w. In this terminology, the usual cover is the special case where w is the *1-distribution*, i.e. $w(v) = 1$ for all nodes v. The *w-cover pebbling number* is the smallest n such that, from any initial distribution of n pebbles, it is possible to obtain a w-cover. We will require w to be positive, i.e. there should be at least one pebble on each node.

A node v is *fat*, *thin* respectively *perfect* if the number of pebbles on it is greater than, less than, respectively equal to $w(v)$.

The initial distribution is said to be *simple* if all pebbles are on one single node. For two nodes v and u, the *distance* $d(v,u)$ *from* v *to* u is the length of the minimal path from v to u. (For a directed graph, $d(v,u) \neq d(u,v)$ in general.) The *cost* from a node v of a pebble on a node u is $2^{d(v,u)}$, and the sum of the costs from v of all pebbles in w is the cost of cover pebbling from v.

In the graph below, $8+8+4+2+1$ pebbles on v are necessary and sufficient for a cover pebbling if w is the 1-distribution.

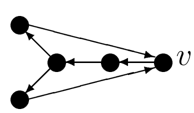


Figure 1: A cover pebbling from v needs 23 pebbles.

In each pebbling move, the total number of pebbles decreases, but the total *value* is invariant, if the value of a pebble is defined to be the number of pebbles that have gone into it. Recursively speaking, the value of a newborn pebble is the sum of the values of its demised parents, the original pebbles being unit valued.

3 The cover pebbling theorem

For nonsimple initial distributions, costs are ill-defined and there is no easy way to see if a cover pebbling exists. The following theorem tells us not to worry about that when it comes to computing the cover pebbling number of a graph, for this number is always determined by a simple distribution.

Theorem 1 *Let w be a positive goal distribution. To determine the w-cover pebbling number of a (directed or undirected) connected graph, it is sufficient to consider simple initial distributions. In fact, for any initial distribution that admits no cover pebbling, all pebbles may be concentrated to one of the fat nodes*[1] *with cover pebbling still not possible.*

Proof. Start with a distribution that admits no cover pebbling.

If there are no fat nodes, we can concentrate all the pebbles to any of the nodes. The cost of cover pebbling from this node is of course no less than the number of pebbles in w, so cover pebbling is still not possible.

If some node is fat, we will have to do some pebbling. During the pebbling we will always maintain the following efficiency condition: *Every pebble has a value no greater than the cost from its nearest fat node (the fat node that minimizes this cost).* At the beginning all pebbles have the value one, so the efficiency condition is trivially satisfied.

Now pebble like this: Among all pairs (f, t) of a fat and a thin node, take one that minimizes the distance $d(f, t)$. Let $fp_1p_2 \cdots p_{d-1}t$ be a minimal path from f to t. Every inner node p_i of this path must be perfect, since if it were thin, then (f, p_i) would be a (fat,thin)-pair with $d(f, p_i) < d(f, t)$, and if it were fat, then (p_i, t) would be a (fat,thin)-pair with $d(p_i, t) < d(f, t)$. Furthermore, f must be a nearest fat node to t and to every p_i. Now we play two pebbles on f to p_1, then we play the new pebble on p_1 together with any old pebble from p_1 to p_2, then the new pebble and an old one on p_2 to p_3, and so on, until we reach t.

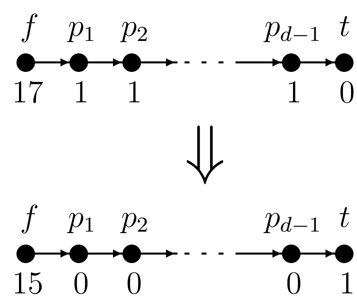


Figure 2: Playing two pebbles from f and continuing all the way to t in the case where w is the 1-distribution.

The value of the new pebble on t is 2 plus the sum of the values of the old pebbles on $p_1, \ldots, p_{d-1}$ that were consumed. By the efficiency condition this is no greater than $2 + 2^1 + 2^2 + \cdots + 2^{d-1}$ which equals 2^d. Thus the condition is satisfied even after this operation. It is possible that f is no longer fat, but this only makes it easier to fulfil the condition.

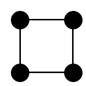
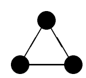
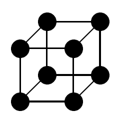
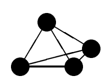
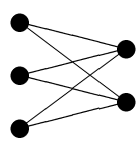
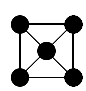
[1] Of course, this is not true if there are no fat nodes, but then any node will do.

We iterate the above procedure (choosing a new pair (f, t), and so on) until no node is fat. During each iteration the total number of pebbles on fat nodes decreases, so we cannot continue forever.

Let f be the fat node that survived the longest. Then each pebble value is at most equal to its cost from f. But there are still thin nodes, so the cost of cover pebbling from f exceeds the total value of the pebbles. Therefore, cover pebbling is not possible with all pebbles initially on f. $\square$

4 The cover number for some classes of graphs

The cover number is now easy to compute for any graph. Here is a table of some classes of undirected graphs, for the case that w is the 1-distribution.

class	**example**	**cover number**
n-path	$n = 4$: 	$2^n - 1$
$2n$-cycle	$n = 2$: 	$3 \cdot (2^n - 1)$
$(2n - 1)$-cycle	$n = 2$: 	$2^{n+1} - 3$
n-dimensional hypercube	$n = 3$: 	3^n
complete graph K_n	$n = 4$: 	$2n - 1$
complete multipartite graph $K_{n_1,\ldots,n_k}$ where $n_1 \geq \cdots \geq n_k$	$n_1 = 3$ $n_2 = 2$ 	$4n_1 + 2n_2 + \cdots + 2n_k - 3$
n-wheel	$n = 4$: 	$4n - 5$

All the results in the table were previously known, but now, in the light of our theorem, they are reduced to simple exercises.

5 Product graphs

One of the conjectures in [5] is also an easy consequence of our theorem. We will prove it in a much more general form. Introduce the notation $G_1 \square G_2$ for the product[2] of two

[2] $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ and there is an edge from (u_1, u_2) to (v_1, v_2) if $u_1 = v_1$ and there is an edge from u_2 to v_2 in G_2, or if $u_2 = v_2$ and there is an edge from u_1 to v_1 in G_1.

(directed or undirected) graphs.

Let w_1 and w_2 be goal distributions on G_1 respectively G_2. Define a goal distribution $w_1 \Box w_2$ on $G_1 \Box G_2$ by $(w_1 \Box w_2)(v_1, v_2) = w_1(v_1)w_2(v_2)$. Finally, let γ_w denote the w-cover pebbling number. Then we have the following theorem.

Theorem 2 $\gamma_{w_1 \Box w_2}(G_1 \Box G_2) = \gamma_{w_1}(G_1)\gamma_{w_2}(G_2)$.

Proof. The distance from (v_1, v_2) to (u_1, u_2) in $G_1 \Box G_2$ is equal to the sum of the distance from v_1 to u_1 in G_1 and the distance from v_2 to u_2 in G_2, so, for any v_1 and v_2,

$$\gamma_{w_1 \Box w_2}(G_1 \Box G_2) \geq \sum_{(u_1,u_2) \in V(G_1 \Box G_2)} (w_1 \Box w_2)(u_1, u_2) 2^{d((v_1,v_2),(u_1,u_2))} =$$

$$= \sum_{u_1 \in V(G_1)} w_1(u_1) 2^{d(v_1,u_1)} \sum_{u_2 \in V(G_2)} w_2(u_2) 2^{d(v_2,u_2)} \leq \gamma_{w_1}(G_1)\gamma_{w_2}(G_2).$$

By the cover pebbling theorem, we can choose (v_1, v_2) to make the first inequality an equality. On the other hand, there are v_1 and v_2 that make the second inequality an equality. $\Box$

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