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THE CRITICAL DIMENSIONS OF HAMACHI SHIFTS

ANTHONY H. DOOLEY AND GENEVIEVE MORTISS

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Abstract. In 1981, Hamachi introduced an interesting class of type III shifts. Since these are not measure-preserving, one cannot use metric entropy to study them. As a possible substitute, we give estimates for the upper and lower critical dimensions of Hamachi shifts. These invariants have previously been used by the authors to study odometer actions.

1. Introduction. A measurable dynamical system is a standard Lebesgue space $(X, \mathcal{B}, \mu)$ and an automorphism $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ which preserves the measure class of μ. In the measure-preserving case, that is when $\mu \circ T = \mu$, the entropy $H(T)$ has been an important tool for studying the system. If $\mathcal{P}$ is a partition, we define the *entropy* $H(\mathcal{P})$ of $\mathcal{P}$ as $H(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P)$. A partition is called generating if $\sigma(T, \mathcal{P})$, the smallest complete σ-algebra containing all the partitions $T^i\mathcal{P}$, $i \in \boldsymbol{Z}$, is all of $\mathcal{B}$. The *metric entropy* of T is defined by taking any generating partition $\mathcal{P}$ and defining: $H(T) = H(\mathcal{P}, T) = \lim_{n \to \infty}(1/n)H(\bigvee_0^{n-1} T^i\mathcal{P})$. To show that this limit exists, one uses in an essential way the fact that T preserves μ.

A theorem of Sinai shows that the limit is the same for all generating partitions. Rokhlin, Kolmogorov and Sinai [15, 16] used entropy to distinguish between shifts. Also, Ornstein's celebrated result tells us that Bernoulli systems of the same entropy are isomorphic. Much further work has been done, and is too extensive to mention here.

In the general non-singular case where $\mu \circ T \sim \mu$, the limit which defines $H(T)$ may no longer exist. As a consequence, whereas the theory of measure-preserving shifts is well-developed, relatively little is known about non-singular shifts. In [8], Hamachi introduced a notion of non-singular ergodic conservative shift. More details are given in Section 3 below.

It is natural to enquire whether there is some quantity which can be used to replace entropy in the study of Hamachi shifts, and which allows one to distinguish between them. For non-singular systems which are not measure-preserving, Silva and Thielluen [17] gave a definition of entropy, but it takes values only 0, or ∞, and therefore is not as close a discriminator between non-isomorphic systems as one might wish.

In her 1997 thesis [10], Mortiss tried some new approaches to the entropy of the odometer action on an infinite product space. This led to the development of the *upper critical dimension* $\alpha(T)$ and the *lower critical dimension* $\beta(T)$ ([12, 6, 7]), which are invariants for metric isomorphism of non-singular system $(X, \mathcal{B}, T, \mu)$. These are defined using the growth rate of sums of Radon derivatives, and one has $0 \le \alpha(T) \le \beta(T) \le 1$. In the case of

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odometer actions, these have been shown to have entropy-like properties—if the two quantities coincide, then they are equal to the average coordinate entropy introduced in [10], there is a Shannon-McMillan-Breiman theorem, and one can prove a version of Katok's lemma for this quantity.

It is therefore natural to try to use the critical dimension in the study of Hamachi shifts. The aim of this paper is to estimate the critical dimension of Hamachi shifts. Specifically, we show the following.

THEOREM 1. *Let $\varepsilon > 0$. There exists a Hamachi shift S with $\alpha(S) < \varepsilon$ and $\beta(S) > 1 - \varepsilon$.*

In fact, our estimates hold for Hamachi shifts with reasonably mild hypotheses on the rates of growth of their defining parameters. We conjecture that, in some senses, Hamachi shifts generically have $\alpha(S) = 0$ and $\beta(S) = 1$.

The fact that the Hamachi shifts have small lower critical dimension and high upper critical dimension is not so surprising: see Proposition 2 of [8], where it is shown that the Radon-Nikodym derivatives are either very large or very small. What is interesting is that they can be estimated at all. It would be interesting to have a better way of calculating these invariants.

From this result, it follows that the Hamachi shifts which we construct are not isomorphic to certain odometers (for example, they cannot be isomorphic to standard type III_λ odometers, $\bigotimes\{(1/1+\lambda), (\lambda/1+\lambda)\}$, on the infinite product of two-point spaces).

The plan of the paper is as follows. In Section 2, we introduce the main ideas of critical dimension. Section 3 then introduces Hamachi shifts. In Sections 4 and 5 we prove Theorem 1(i) and (ii), respectively.

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2. Critical dimension. Given $(X, \mathcal{B}, \mu, T)$ a non-singular measurable dynamical system, that is, when $\mu \circ T$ is equivalent to μ, we let $\omega_i(x) = d\mu \circ T^i(x)/d\mu$.

DEFINITION 1. Let $(X, \mathcal{B}, \mu, T)$ be a non-singular conservative ergodic dynamical system with $\mu(X) = 1$.

(i) Let

$$X_{\alpha'} = \left\{ x \in X \,;\, \liminf_{n\to\infty} \left(\sum_{i=0}^{n-1} \omega_i(x) \right) \Big/ n^{\alpha'} > 0 \right\},$$

and notice that $X_{\alpha'}$ is an invariant set. The supremum over the set of $\alpha' \geq 0$ for which $\mu(X_{\alpha'}) = 1$ is called the *lower critical dimension* $\alpha = \alpha(T)$ of $(X, \mathcal{B}, \mu, T)$.

(ii) Let

$$X_{\beta'} = \left\{ x \in X \,;\, \limsup_{n\to\infty} \left(\sum_{i=1}^{n} \omega_i(x) \right) \Big/ n^{\beta'} = 0 \right\}.$$

Let β be the infimum of the set $\{\beta' \geq 0\,;\, \mu(X_{\beta'}) = 1\}$. We call $\beta = \beta(T)$ the *upper critical dimension.*

Notice that we have $\alpha = \liminf_{n\to\infty}(\log \sum_{i=1}^{n} \omega_i(x))/\log n$ and $\beta = \limsup_{n\to\infty}(\log \sum_{i=1}^{n} \omega_i(x))/\log n$. In the case where $\alpha = \beta$, we say that the system has *critical dimension* α. The following theorem was proved in [7].

THEOREM 2. *If* $(X, \mathcal{B}, \mu, T)$ *and* $(X', \mathcal{B}', \mu', T')$ *are metrically isomorphic, then* $\alpha(T) = \alpha(T')$ *and* $\beta(T) = \beta(T')$.

It is not too hard to see that $0 \leq \alpha \leq \beta \leq 1$. Indeed, if β were strictly greater than one, then for any $1 < \beta' < \beta$, letting $\phi_n = (1/n) \sum^{n} \omega_i$, we would have $\limsup_n (n^{1-\beta'})\phi_n > 0$ a.e. At the same time, since μ is a probability measure, $\int \phi_n d\mu = 1$ for all n. An elementary argument in measure theory shows that this is impossible.

Notice that if $\mu \circ T = \mu$, then $\alpha = \beta = 1$. By a theorem of Maharam [9], we know that the following are equivalent:

(i) There is a finite T-invariant measure.

(ii) $\lim_{n\to\infty}(1/n) \sum_{i=1}^{n-1} \omega_i(x)$ exists as a positive number a.e.

It follows that $0 \leq \alpha \leq \beta \leq 1$ for all systems of type II_∞ or type III, $\alpha > 0$ for systems of type II_∞ and that $\alpha = \beta = 1$ for systems of type II_1. (We recall the von Neumannn classification of dynamical systems, see [18]. A system is of type II if it has a preserved measure equivalent to μ: it is of type II_1 if the measure is probability and of type II_∞ if the space has infinite measure. An ergodic system is of type III if there is no measure equivalent to μ which is T-invariant.)

We summarize briefly the main results from [6, 7], on odometer actions, although in this article, we will consider the shift S. Let $l(i) \geq 2$ be a bounded sequence of integers and consider the infinite product space $X = \prod_{i=1}^{\infty} \boldsymbol{Z}_{l(i)}$, where we write $\boldsymbol{Z}_{l(i)} = \{0, \ldots, l(i) - 1\}$. The odometer T acts on X by the standard method: $Tx = y$ if y is the smallest element greater than x in the lexicographic order, and if $\ell = (l(1) - 1, l(2) - 1, \ldots, l(n) - 1, \ldots)$, then $T\ell = 0 = (0, 0, 0, \ldots)$. We denote $s(n) = l(0) \cdots l(n)$.

We shall assume that X is equipped with the usual product σ-algebra, and an infinite product measure $\mu = \bigotimes_{i=1}^{\infty} \mu_i$, where μ_i is a fully supported probability measure on the finite space $\boldsymbol{Z}_{l(i)}$. The entropy $H(\mathcal{P}_n)$ of the partition of the first n coordinates with respect to μ is given by

$$H(\mathcal{P}_n) = -\sum_{i=0}^{n} (\mu_i(0) \log \mu_i(0) + \cdots + \mu_i(l_i - 1) \log \mu_i(l_i - 1))\,.$$

Let α and β denote the upper and lower critical dimensions for $(X, \mathcal{B}, \mu, T)$.

THEOREM 3. *Let T be the odometer action on X. Then the following hold.*

(i)

$$\alpha = \liminf_{n\to\infty} -\frac{\sum_{i=1}^{n} \log \mu_i(x_i)}{\log(s(n))} = \liminf_{n\to\infty} \frac{H(\mathcal{P}_n)}{\log(s(n))}$$

for μ almost all $x \in X$.

(ii)

$$\beta = \limsup_{n\to\infty} -\frac{\sum_{i=1}^{n} \log \mu_i(x_i)}{\log(s(n))} = \limsup_{n\to\infty} \frac{H(\mathcal{P}_n)}{\log(s(n))}$$

for μ almost all $x \in X$.

For each of the two statements above, the left hand equality is like a version of the theorem of Shannon-MacMillan-Breiman, with α (resp. β) playing the part of the entropy.

In the case where $\alpha = \beta$, the two expressions on the right hand sides coincide with $\lim_{n\to\infty} H(\mathcal{P}_n)/\log(s(n))$. In [10], this expression (when it exists) was called the *average coordinate entropy* and denoted as $h_{\text{AC}}(\mu)$. For an odometer on the infinite product of 2-point spaces, taking logarithms to base 2, $h_{\text{AC}}(\mu) = \lim_{n\to\infty}(\sum_{i=1}^{n} H(\mu_i))/n$ and therefore it actually is the limiting average of the entropy of the individual coordinate measures.

Note that if we were to use the usual definition [13, 14] of metric entropy, we would evaluate $\lim_{n\to\infty} H(\mathcal{P}_n)/s(n)$, as the odometer takes $s(n)$ steps to produce $\mathcal{P}_n$. Thus, the usual entropy limit of these odometer actions is zero. By contrast, the critical dimension can have any value between 0 and 1.

Similar results were proved in [7] for Markov odometers, where we replace infinite product measures by Markov measures in the sense of [5].

3. Description of Hamachi shifts. Let $X = \prod_{i=-\infty}^{\infty} \boldsymbol{Z}_2$ and S be the shift action on X, so that

$$(Sx)_i = x_{i+1}\,.$$

Now we will equip X with a product measure $\mu = \bigotimes_{i=-\infty}^{\infty} \mu_i$, where

$$\mu_i(1) = \mu_i(0) = 1/2 \quad \text{for any } i \geq 0\,.$$

For $i < 0$, either $\mu_i(0) = 1/2$ or $\mu_i(0) = 1/(1+\lambda_t)$ according to the following rules:

$$\mu_k(0) = 1/(1+\lambda_t) \quad \text{if } -N_t < k \leq -M_{t-1}\,, \tag{1}$$

$$\mu_k(0) = 1/2 \quad \text{if } -M_t < k \leq -N_t\,, \tag{2}$$

where $\{\lambda_t\}_{t=1}^{\infty}$ is a decreasing sequence with all $\lambda_t > 1$ and $\sum_{t=1}^{\infty}(\log(\lambda_t))^2 < \infty$. The sequences M_t and N_t are defined by the following equations

$$N_t = M_{t-1} + n_t; \quad M_t = N_t + m_t; \quad M_0 = 1\,.$$

Here n_t and m_t are two series of positive integers determined by an inductive process in [8]. We omit the details of the construction here. The reader is referred to [8] for a full description of the definition.

The inductive steps involved in the construction require that for each $t \in \boldsymbol{N}$, after λ_{t-1}, n_{t-1} and m_{t-1} have been chosen, the order of parameter selection is λ_t, n_t, m_t. Prior to beginning the induction, one chooses two sequences $\{\varepsilon_t\}_{t=1}^{\infty}$ and $\{p_t\}_{t=1}^{\infty}$ such that

$$\sum_{t=1}^{\infty} \varepsilon_t < \infty \quad \text{and} \quad \sum_{t=1}^{\infty} p_t = \infty\,.$$

The sequence $\{\eta_t\}_{t=1}^{\infty}$ is then defined as

$$\eta_t = \sum_{u=t}^{\infty} \varepsilon_u \,.$$

Note that λ_t is chosen so that

$$\lambda_t^{M_{t-1}} < \exp(\varepsilon_t)\,. \tag{3}$$

Furthermore, extra conditions may be added by which each λ_t can be made smaller, and each n_t and m_t larger. Hamachi showed the following.

PROPOSITION 4 ([8]). *For a suitable choice of the parameters above,* $(X, \mathcal{B}, \mu, S)$ *is a nonsingular ergodic system which is type* III *and conservative*.

4. The lower critical dimension.

LEMMA 5 ([8]). *With notation as in Section* 2, *let*

$$K_{t,i}(x) = \prod_{u=t+1}^{\infty} \lambda_u^{-\{x_{-N_u+1}+x_{-N_u+2}+\cdots+x_{-N_u+i}\}+\{x_{-M_{u-1}+1}+x_{-M_{u-1}+2}+\cdots+x_{-M_{u-1}+i}\}} \,.$$

If $0 \le i < N_t$, *then*

$$\frac{dS^{-i}\mu}{d\mu}(x) = K_{t,i}(x) \times \prod_{k=-N_t+1}^{i-1} \frac{\mu_{k-i}(x_k)}{\mu_k(x_k)} \,.$$

Now $K_{t,i}(x)$ is bounded from above and below, in terms of η:

$$\exp -\eta_{t+1} < K_{t,i} < \exp \eta_{t+1} \,.$$

Hence, in order to determine the lower critical dimension, we need to estimate

$$\prod_{k=-N_t+1}^{i-1} \frac{\mu_{k-i}(x_k)}{\mu_k(x_k)} = \prod_{k=-N_t+1}^{-1} \frac{\mu_k(x_{k+i})}{\mu_k(x_k)} \quad \text{for } 0 \le i < N_t \,.$$

Now the above product is equal to

$$\prod_{s=1}^{t} \lambda_s^{\rho_s(i,x)} \,,$$

where the integer $\rho_s(i, x)$ lies between $-n_s$ and n_s for each $s \in \{1, 2, \ldots, t\}$. Note that $\rho_s(i) = \sum_{j=0}^{i} d_s(j)$, where $d_s(j) \in \{-1, 0, 1\}$ for each $j \ge 1$. Now $d_s(j)$ depends on only two coordinates of x, namely, x_{-N_s+1+j} and $x_{-M_{s-1}+1+j}$. Thus, for fixed s, and $i, j < n_s$, $d_s(i)$ and $d_s(j)$ are independent.

LEMMA 6. *For* $n < N_t$, *we have*

$$\sum_{i=0}^{n-1} \frac{dS^{-i}\mu}{d\mu}(x) \le \prod_{s=1}^{t-1} \lambda_s^{n_s} \sum_{i=0}^{n-1} \lambda_t^{\rho_t(i,x)} \,.$$

We now focus on the term $\lambda_t^{\rho_t(i,x)}$. Now $n_t >> M_{t-1}$. For $M_{t-1} < i < n_t$, $d_t(i) = -1$ with probability $\lambda/(2(1+\lambda))$, $d_t(i) = 0$ with probability $1/2$ and $d_t(i) = 1$ with probability $1/(2(1+\lambda))$. Again, for $i < n_t$ all the $d_t(i)$ are independent.

THEOREM 7. *Let $0 < \alpha < 1 < \lambda$ and $0 < p < 1$. Let Ω be a probability space, and choose independent random variables $\{d_i(\omega); i \in \mathbf{N}\}$ on Ω taking values $\{-1, 0, 1\}$ with*

$$d_i(\omega) = \begin{cases} +1 & \text{with probability } 1/(2(1+\lambda))\,, \\ 0 & \text{with probability } 1/2\,, \\ -1 & \text{with probability } \lambda/(2(1+\lambda))\,. \end{cases}$$

Let $\rho_k(\omega) = \sum_{i=1}^k d_i(\omega)$. Then

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \lambda^{\rho_i}}{n^\alpha} = 0$$

on a set of measure at least p.

PROOF. Choose $\varepsilon > 0$ and let $\eta = (1-\alpha+\varepsilon)/\log\lambda$, so that $\eta > 0$. Now define

$$E_k = \left\{\omega \in \Omega\,;\, \sum_{j=1}^k d_j(\omega) = \rho_k(\omega) \le -\eta \log k = C_k\right\}$$

and set $E^k = \bigcap_{\ell=k}^\infty E_\ell$.

LEMMA 8. *If $\omega \in E^k$ for some k, then*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{j=1}^n \lambda^{\rho_j(\omega)} = 0\,.$$

PROOF. From the definition, we have

$$\sum_{k=1}^n \lambda^{\rho_k(\omega)} \le \sum_{k=1}^n \lambda^{-\eta\log k} = \sum_{k=1}^n e^{-\eta\log\lambda\log k} = \sum_{k=1}^n e^{-(1-\alpha+\varepsilon)\log k}$$
$$= \sum_{k=1}^n k^{\alpha-1-\varepsilon} \le O(n^{\alpha-\varepsilon})\,.$$

Thus

$$\frac{1}{n^\alpha}\sum_{k=1}^n \lambda^{\rho_k(\omega)} = O(n^{-\varepsilon}) \to 0$$

as $n \to \infty$. □

PROPOSITION 9. *Let $0 < p < 1$. Then there exists k such that $P(E^k) > p$.*

PROOF. We first calculate $P(E_l)$. For simplicity, we assume that l is even. Entirely similar estimates apply for the case where l is odd.

Notice that $\rho_l(\omega) = a - b$, where a is the number of times that $d_i(\omega) = +1$ and b is the number of times that $d_i(\omega) = -1$ for $i = 1, \ldots, l$. Thus, $d_i(\omega) = 0$, $l - (a + b)$ times. For fixed a, b, there are

$$\binom{l}{a}\binom{l-a}{b} = \frac{l!}{a!\,b!\,(l-a-b)!}$$

possible places to put the ± 1's and the probability of a given disposition is

$$\left(\frac{1}{2(1+\lambda)}\right)^a \left(\frac{\lambda}{2(1+\lambda)}\right)^b \frac{1}{2^{l-a-b}} = \frac{1}{2^l}\frac{\lambda^b}{(1+\lambda)^{a+b}}\,.$$

Thus, the probability of a $+1$'s and b -1's is

$$\frac{1}{2^l}\frac{l!}{a!\,b!\,(l-a-b)!}\frac{\lambda^b}{(1+\lambda)^{a+b}}\,.$$

Putting $u = a + b$ and summing, we have

$$1 = \frac{1}{2^l}\sum_{u=0}^{l}\binom{l}{u}\frac{1}{(1+\lambda)^u}\sum_{b=0}^{u}\binom{u}{b}\lambda^b\,.$$

To estimate the probability that $\omega \in E_k$, we need to have $a - b \leq C_l$, and also $a + b \leq u$, so that $b \geq \left[(u + C_l)/2\right]$.

Thus, the probability of E_l is

$$P(E_l) = \frac{1}{2^l}\sum_{u=0}^{l}\binom{l}{u}\frac{1}{(1+\lambda)^u}\sum_{b\geq\left[(u+C_l)/2\right]}^{u}\binom{u}{b}\lambda^b\,. \tag{4}$$

We now use the following standard estimates:

LEMMA 10. *Suppose that $p + q = 1$ with $p > q$. Then for n even,*
(i)

$$\sum_{k=n/2}^{n}\binom{n}{k}p^k q^{n-k} = \frac{1}{1+(q/p)^{n/2}}\,.$$

(ii) *If $n/2 > r > 0$ then*

$$\sum_{k=n/2+r}^{n}\binom{n}{k}p^k q^{n-k} = \frac{1}{1+(q/p)^{n/2}} - r\binom{n}{n/2}(p/q)^r\,.$$

We apply Lemma 10 with $p = \lambda/1+\lambda$, $q = 1/1+\lambda$, $n = l$ and $r = [C_l/2]$. Hence we obtain

$$P(E_l) \geq \frac{1}{1+\lambda^{-l/2}} - \frac{C_l}{2}\binom{l}{l/2}$$

Now, by Stirling's approximation, $\binom{l}{l/2} \simeq 2^{l+1}/\sqrt{2\pi l}$, and so we get

$$P(E_l) \geq \frac{1}{1+\lambda^{-l/2}} - \frac{\eta}{\sqrt{2\pi}}(4\lambda/(1+\lambda)^2)^{l/2} l^{\eta\log\lambda - 1/2}\log l\,.$$

Notice that $4\lambda/(1+\lambda)^2 < 1$.

Entirely similar calculations give similar estimates for l odd. It now follows that

$$P((E^k)^c) \leq \sum_{l=k}^{\infty} P(E_l^c) \leq \sum_{l=k}^{\infty} \frac{\lambda^{-l/2}}{1+\lambda^{-l/2}} + \frac{\eta}{\sqrt{2\pi}} \sum_{l=k}^{\infty} \left(\frac{2\sqrt{\lambda}}{1+\lambda}\right)^l l^{\eta \log\lambda - 1/2} \log l\,.$$

As $k \to \infty$, the left hand side tends to zero. It follows that for k sufficiently large, we have $P(E^k) > p$. □

We now prove the first part of Theorem 1.

LEMMA 11. *For any $\alpha > 0$, a Hamachi shift can be constructed with lower critical dimension less than or equal to α.*

PROOF. Let $\alpha > 0$. We will define the measure $\mu = \bigotimes_{i=-\infty}^{0} \mu_\lambda \bigotimes_{i=1}^{\infty} \mu_{1/2}$ using, as above, an inductive choice of the parameters. Assume that λ_{t-1}, n_{t-1} and m_{t-1} have all been chosen. Choose λ_t as usual. Then select n_t large enough to satisfy the conditions described in [8] and in addition ensure that

$$\frac{2M_{t-1}\lambda_t^{2M_{t-1}} \prod_{s=1}^{t-1} \lambda_s^{n_s}}{{n_t}^{\alpha/2}} < \varepsilon_t$$

and

$$\frac{\sum_{i=2M_{t-1}+1}^{n_t-1} \lambda_t^{\rho_t(i,x)}}{n_t^{\alpha/2}} < \varepsilon_t$$

on a set B_t of measure at least p_t. We choose M_{t-1} sufficiently large to ensure that the two events are independent.

Note that B_t is $\bigvee_{-N_t+2M_{t-1}+1}^{-M_{t-1}} \mathcal{F}_j \bigvee_{M_{t-1}+1}^{n_t-M_{t-1}} \mathcal{F}_j$-measurable.

Combining these two conditions, we obtain for all $x \in B_t$,

$$\begin{aligned}\frac{1}{n_t^\alpha} \sum_{i=0}^{n_t-1} \frac{dS^{-i}\mu}{d\mu}(x) &\leq \frac{1}{n_t^\alpha} \sum_{i=0}^{2M_{t-1}} \lambda_t^{2M_{t-1}} \prod_{s=1}^{t-1} \lambda_s^{n_s} + \frac{1}{n_t^\alpha} \prod_{s=1}^{t-1} \lambda_s^{n_s} \sum_{i=2M_{t-1}+1}^{n_t-1} \lambda_t^{\rho_t(i,x)} \\ &\leq \varepsilon_t + \varepsilon_t = 2\varepsilon_t.\end{aligned}$$

As $\sum p_t = \infty$, and the B_t are independent, the Borel-Cantelli lemma implies that almost every x is an element of infinitely many B_t. Clearly, α must be greater than or equal to the lower critical dimension. As α can be chosen arbitrarily small, we have proved the first inequality of Theorem 1. □

5. The upper critical dimension. In this section, we complete the proof of Theorem 1, showing that the sequence m_t can be chosen so that the upper critical dimension is arbitrarily close to 1. This choice is independent of the choice of n_t made in Section 4.

We begin by recalling two results from [8].

LEMMA 12 ([8]). *With the notation of Section* 2, *and with* $K_{t,i}$ *as in Lemma* 5, *let further*

$$F_t(x) = \prod_{u=1}^{t} \left(\frac{2}{1+\lambda_u}\right)_u^n \lambda_u^{x_{N_t-N_u+1}+x_{N_t-N_u+2}+\cdots+x_{N_t-M_{u-1}}} .$$

Then, if $N_t \le i < m_t$, *we have*

$$\frac{dS^{-i}\mu}{d\mu}(x) = K_{t,i}(x) \times \prod_{u=1}^{t} \left(\frac{1+\lambda_u}{2}\right)^{n_u} \lambda_u^{-(x_{-N_u+1}+x_{-N_u+1}+\cdots+x_{-M_{u-1}})} \times F_t(S^{i-N_t}(x)) .$$

LEMMA 13 ([8]). *For every* $x \in X$ *and for* $N_t \le i < m_t$,

$$\frac{dS^{-i}\mu}{d\mu}(x) \ge \frac{\exp(-\eta_{t+1})}{\lambda_1^{N_t}} .$$

PROOF. As $F_t(x) > \prod_{u=1}^{t} (2/(1+\lambda_u))^{n_u}$, we have by the previous lemma,

$$\begin{aligned}\frac{dS^{-i}\mu}{d\mu}(x) &\ge \prod_{u=t+1}^{\infty} \lambda_u^{-(i+1)} \times \prod_{u=1}^{t} \lambda_u^{-n_u} \ge \prod_{u=t+1}^{\infty} \lambda_u^{-M_{u-1}} \times \prod_{u=1}^{t} \lambda_1^{-n_u} \\ &\ge \prod_{u=t+1}^{\infty} \exp(-\varepsilon_u) \times \lambda_1^{-N_t} \quad \text{(by (1))} \\ &= \exp(-\eta_{t+1}) \times \lambda_1^{-N_t} .\end{aligned}$$

□

This proof shows how using λ_t close to 1 ensures conservativity of the shift. If λ_t were approaching some other number (say 2), then a different technique would be required to guarantee that the sum of the Radon derivatives was infinite.

LEMMA 14. *For any* $\varepsilon > 0$ *we can construct a Hamachi shift so that the upper critical dimension is at least* $1 - \varepsilon$.

PROOF. Assume that λ_t and n_t (and hence N_t) have been chosen. Choose m_t large enough that it satisfies the original construction and in addition

$$(m_t - N_t)\exp(-\eta_{t+1})/m_t^{(1-\varepsilon)}\lambda_1^{N_t} \ge t .$$

Then for all $x \in X$,

$$\frac{1}{m_t^{(1-\varepsilon)}} \sum_{i=0}^{m_t-1} \frac{dS^{-i}\mu}{d\mu}(x) \ge (m_t - N_t)\frac{\exp(-\eta_{t+1})}{m_t^{(1-\varepsilon)}\lambda^{N_t}} \ge t .$$

Continuing this construction, we build a Hamachi shift with upper critical dimension no less than $(1 - \varepsilon)$. □

Since the choice of n_t made in Lemma 11 and the choice of m_t in Lemma 13 can be made independently of each other, this completes the proof of Theorem 1.

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SCHOOL OF MATHEMATICS
UNIVERSITY OF NEW SOUTH WALES
SYDNEY 2052
AUSTRALIA