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The Critical Probability of Bond Percolation on the Square Lattice Equals 1/2*

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Abstract. We prove the statement in the title of the paper.

1. Introduction

Broadbent and Hammersley [2], introduced the following percolation problem. Let $\mathscr{L}$ be the graph in the plane whose vertices are the integral vectors (i.e., elements of $\mathbb{Z}^2$) and whose edges or bonds are the segments connecting two adjacent vertices (we call two vertices v' and v'' of $\mathscr{L}$ adjacent if the distance between them equals 1). Let each bond of $\mathscr{L}$ be open or passable with probability p, and closed or blocked with probability $q=1-p$, and assume that open- or closedness for all different bonds is chosen independently. The percolation probability is defined as

$$\theta(p)=P\{\text{the origin is part of an infinite connected open set in } \mathscr{L}\}, \tag{1.1}$$

and the critical probability p_H as

$$p_H=\inf\{p:\theta(p)>0\}. \tag{1.2}$$

Hammersley [5], [6] proved

$$\frac{1}{\lambda} \leqq p_H \leqq 1-\frac{1}{\lambda}. \tag{1.3}$$

where λ is the socalled connectivity constant of $\mathscr{L}(\lambda \approx 2.639$, see [9]). Harris [7] improved the lower bound to

$$p_H \geqq \tfrac{1}{2}. \tag{1.4}$$

Various results and numerical evidence (see [17], or [15] Chap. III, for a brief summary) indicated that $p_H=\frac{1}{2}$, and most people seem to have accepted the truth

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of this conjecture. So far no rigorous proof seems to have been given. Our principal aim is to give such a proof here.

Theorem 1.

$$p_H = \tfrac{1}{2}.$$

We can combine this with previous results to obtain somewhat sharper information. Define an open (closed) *cluster* to be a maximal connected subgraph of $\mathscr{L}$ all of whose bonds are open (closed). Denote by W the open cluster which contains the origin. (W consists of the origin only if all four edges connected to 0 are closed.) When $p \leqq p_H = \frac{1}{2}$ there is with probability one no infinite open cluster [see (1.5) below], but when $p > p_H$ there exists with probability one exactly one infinite open cluster (see [7] or [15], Theorem 3.14). With this information we can formulate the following theorem.

Theorem 2.

$$p > \tfrac{1}{2} \textit{ implies } \theta(p) > 0, \tag{1.5}$$

$$p \leqq \tfrac{1}{2} \textit{ implies } \theta(p) = 0. \tag{1.6}$$

For any $p < \frac{1}{2}$ *there exists a constant* $C_1(p) > 0$ *such that for all* n [1]

$$P_p\,\{W \textit{ contains vertices at distance } \geqq n \textit{ from the origin}\} \leqq 2e^{-C_1(p)n}. \tag{1.7}$$

For $p = \frac{1}{2}$ *and all* $n \geqq 1$

$$P_{1/2}\,\{W \textit{ contains vertices at distance } \geqq n \textit{ from the origin}\} \geqq \frac{1}{8n}. \tag{1.8}$$

Finally, for $p > \frac{1}{2}$

$$P_p\,\{\textit{the infinite open cluster contains no vertices within distance } n \textit{ of the origin}\} \leqq 2\{1 - e^{-C_1(q)}\}^{-1} e^{-C_1(q)n} \tag{1.9}$$

[$q = 1 - p$, $C_1(\cdot)$ *is as in* (1.7)].

The principal tools for our proof are the recent results and estimates of Seymour and Welsh [14] and Russo [12] (see also [15], Chap. III for an exposition). They introduced the additional critical probabilities

$$p_T = \inf\{p : E_p\{\# W\} = \infty\}, \tag{1.10}$$

1 The subscript p of course indicates that for each edge e, $P\{e \text{ is open}\} = p$

where $\# W$ denotes the number of vertices in W, and

$$p_S = \inf\left\{p : \limsup_{n\to\infty} S_p(n,n) > 0\right\}, \tag{1.11}$$

where $S_p(n,n)$ is the probability that there exists an open path in the square $[1,n]\times[1,n]$ connecting the left and right edge [see (2.2) below for a precise definition]. [12] and [14] prove

$$0 < p_T = p_S \leqq \tfrac{1}{2}, \quad p_T + p_H = 1. \tag{1.12}$$

We shall prove in the next section that $p_S \geqq \frac{1}{2}$, which together with (1.12) and (1.4) implies Theorem 1, and of course $p_T = p_S = p_H = \frac{1}{2}$.

Remark. Sykes and Essam [16] have introduced a critical probability in terms of a singularity of the mean number of clusters per bond (see also [3] and [4]). In a future publication we shall discuss analyticity properties of $\theta(p)$, and show that for the bond percolation on $\mathbb{Z}^2$ considered here, the Sykes and Essam critical probability also equals $\frac{1}{2}$. This result was already obtained in [16] under some unverified assumptions.

Sykes and Essam also argued that the critical probability for the symmetric triangular lattice equals the root in (0,1) of $1-3p+p^3$. We hope that the present method can be carried over to other lattices. In particular we hope that it can be used to complete the argument of [16] for the triangular lattice.[2]

2. Proofs

We first introduce some notation and collect some useful results. The recent monograph [15] by Smythe and Wierman contains all the results which we need, with their proofs. As much as possible we adopt the notation of [15].

A *path* on $\mathscr{L}$ is a sequence $(v_0, e_1, v_1, \ldots, e_\nu, v_\nu)$ with each v_i a vertex in $\mathscr{L}$, v_{i+1} adjacent to v_i, $0 \leqq i < \nu$, and e_i the edge connecting v_{i-1} and v_i. Such a path is called *open* (*closed*) if all its edges are open (respectively closed). *Throughout this paper a path will always be understood to be self-avoiding.* If t is a path or subgraph of $\mathscr{L}$ we denote by $|t|$ its carrier, i.e.,

$$|t| = \{z \in \mathbb{R}^2 : z \text{ is a vertex of } t \text{ or belongs to an edge of } t\}. \tag{2.1}$$

The analogous convention will be used for subgraphs of the dual lattice $\mathscr{L}^*$. $\mathscr{L}^*$ has a vertices the vectors $v^* = (x^*, y^*)$ with $x^* = n + \frac{1}{2}$, $y^* = m + \frac{1}{2}$, n, m integers, and as edges the segments connecting two adjacent vertices.

$T(m,n)$ denotes the "sponge" consisting of all vertices and bonds in the rectangle

$$[1,n]\times[1,m] = \{(x,y) : 1 \leqq x \leqq n, 1 \leqq y \leqq m\}.$$

2 Since this paper was completed John Wierman has carried out this argument for bond percolation on the triangular and honeycomb lattice, and L. Russo proved an analogous result for site percolation on $\mathbb{Z}^2$.

$S_p(m,n)$ denotes the probability that there is an open connection between the left and right edge of $T(m,n)$:

$$S_p(m,n)=P_p\{\exists \text{ open path } r=(v_0, e_1, \dots, e_\nu, v_\nu)\subset T(m,n) \\ \text{with } v_i=(x_i, y_i), x_0=1, x_\nu=n\}. \tag{2.2}$$

Of course the subscript p in (2.1) refers to the probability measure, according to which each edge is open with probability p. These crossing probabilities satisfy the trivial inequalities

$$S_p(m,n)\geqq S_p(m,n+1), \tag{2.3}$$

$$S_p(m+1,n)\geqq S_p(m,n) \tag{2.4}$$

(see [15], Eqs. (2.1) and (2.2)). It is also not too difficult to show that

$$S_p(m,n)+S_q(n-1,m+1)=1 \qquad (q=1-p) \tag{2.5}$$

and

$$S_{1/2}(n,n)\geqq S_{1/2}(n,n+1)=\tfrac{1}{2} \tag{2.6}$$

(see [15], Theorem 2.2, and last line of p. 42 or [14], Theorem 4.1). Far more difficult is the following inequality of Seymour and Welsh [14], Lemmas 5.2 and 5.3, [15], Lemmas 3.3 and 3.4,

$$S_p(2n,4n)\geqq S_p(2n,6n)\geqq \Gamma(S_p(2n,2n)), \tag{2.7}$$

where the function $\Gamma(\cdot)$ is defined by

$$\Gamma(\sigma)=\sigma^3\{1-(1-\sigma)^{1/2}\}^{16}, \qquad 0\leqq\sigma\leqq 1. \tag{2.8}$$

Let p_H, p_T and p_S be as in (1.2), (1.10), and (1.11). Seymour and Welsh [14], Theorem 2.1, and independently Russo [12], Theorem 2 and Sect. 5, proved

$$p_T+p_H=1. \tag{2.9}$$

Also, [14], p. 244

$$p_T=p_S. \tag{2.10}$$

([15], Chap. 3 gives a good exposition of these results). In view of these results Theorem 1 will follow once we show that $p<\frac{1}{2}$ implies

$$\lim_{k\to\infty} S_p(2^k,2^{k+1})=0. \tag{2.11}$$

Indeed, if $2^{k-1}\leqq n<2^k$, then by (2.3)

$$S_p(n,n)\leqq S_p(2^k,2^{k-1}),$$

whereas by (2.11) and (2.7)

$$\Gamma(S_p(2^k,2^k))\to 0 \quad \text{and hence} \quad S_p(2^k,2^k)\to 0.$$

By (2.3)–(2.5) this implies

$$S_q(2^k,2^k)\geqq S_q(2^k-1,2^k+1)=1-S_p(2^k,2^k)\to 1.$$

In turn, by (2.7) we obtain $S_q(2^k, 6.2^k) \to 1$, and once more by (2.3)–(2.5)

$$S_p(2^k, 2^{k-1}) = 1 - S_q(2^{k-1} - 1, 2^k + 1) \leqq 1 - S_q(2^{k-2}, 6.2^{k-2}) \to 0.$$

Thus (2.11) implies

$$\lim_{n\to\infty} S_p(n,n) = 0 \quad \text{and} \quad p \leqq p_S = 1 - p_H.$$

If this holds for all $p < \frac{1}{2}$, then $p_H \leqq \frac{1}{2}$ which together with Harris' bound (1.4) yields Theorem 1. We shall prove Theorem 1 by proving (2.11) for $p < \frac{1}{2}$.

Consider a path $r = (v_0, e_1, \ldots, e_v, v_v)$ in $T(m,n)$ which is a minimal connection between the left and right edge of $T(m,n)$, i.e., if $v_i = (x_i, y_i)$ then

$$x_0 = 1, x_v = n, 1 < x_i < n \quad \text{for} \quad 1 \leqq i \leqq v - 1, \quad \text{and} \quad 1 \leqq y_i \leqq n \quad \text{for} \quad 0 \leqq i \leqq v. \tag{2.12}$$

Such a path will be called a *left-right crossing* of $T(m,n)$. Any such crossing divides the open rectangle

$$C = C(m,n) \equiv (1,n) \times (\tfrac{1}{2}, m + \tfrac{1}{2}) \tag{2.13}$$

into two components ([11], Theorems V.11.7, 8). Let $C^+(r) = C^+(r; m, n)$ be the part of $C(m,n)$ "above r", i.e., the interior domain of the simple closed curve consisting of r, followed by the segment $\{n\} \times [y_v, m + \frac{1}{2}]$ of the right edge of C, followed by the upper edge of C, $[1,n] \times \{m + \frac{1}{2}\}$ (reversed), followed by the segment $\{1\} \times [x_0, m + \frac{1}{2}]$ (also reversed) of the left edge of C. $C^-(r)$, the part "below r" is defined similarly. If v^+ and v^- lie in $C^+(r)$, respectively $C^-(r)$, then any continuous curve in $C(m,n)$ from v^+ to v^- must intersect r. We shall write $\bar{C}^+(r)$ and $\bar{C}^-(r)$ for the closure of $C^+(r)$, respectively $C^-(r)$.[3]

We shall need the fact that if there exists any open left-right crossing, then there exists a lowest open left-right crossing. Russo [13], Aizenman [1], and Higuchi [8] recently exploited with great ingenuity the existence of similar extremal open paths in the proof of the non-existence of non-translation invariant Gibbs states in the two-dimensional Ising model.

Lemma 1. *If there exist an open left-right crossing of $T(m,n)$, then there exists an open left-right crossing $R = R(m,n)$ such that no left-right crossing*[4] *$s \subset \bar{C}^-(R)$ with $s \neq R$ is open. Moreover R is unique and any open left-right crossing s must satisfy*

$$s \subset \bar{C}^+(R) \quad \text{and} \quad R \subset \bar{C}^-(s). \tag{2.14}$$

This lemma is fairly intuitive and was in fact used without proof in [12], Lemma 3, and [14], Lemma 5.2. We give a formal proof in the Appendix. In the sequel we shall use the notation $\{R(m,n) = r\}$ to indicate that there exists an open left-right crossing of $T(m,n)$, and that the lowest such crossing equals r. It is crucial for our argument that the event $\{R = r\}$ depends only on the open- or closedness of

3 In general we use $\bar{A}$ to denote the closure of a set A in the plane

4 Strictly speaking we should write $|s| \subset \bar{C}^-(R)$ and similarly in (2.14). We shall often abuse notation and write $s \subset A$ for $|s| \subset A$

the edges in $\bar{C}^-(r)$. Indeed for a fixed left-right crossing r, $\{R=r\}$ occurs if and only if r is open, but any left-right crossing $s\in\bar{C}^-(r)$, $s\neq r$, contains a closed edge.

Clearly

$$\begin{aligned} S_p(2^k,2^{k+1}) &= P\{\exists \text{ open left-right crossing of } T(2^k,2^{k+1})\} \\ &= P\{\exists \text{ open left-right crossing of } T(2^{k+1},2^{k+1}) \\ &\quad \text{which lies in } T(2^k,2^{k+1})\} \\ &= P\{R(2^{k+1},2^{k+1}) \text{ exists and lies in } T(2^k,2^{k+1})\}. \end{aligned} \tag{2.15}$$

To prove (2.11) we now introduce an artifice. Let $p<\frac{1}{2}$ be fixed. Choose an integer κ and $0<p_1,p_2,\ldots,p_\kappa<1$ such that

$$p=\tfrac{1}{2}\prod_{i=1}^{\kappa} p_i. \tag{2.16}$$

We shall determine whether an edge e is open in $(\kappa+1)$ stages, rather than as usual in one stage. We take binomial variables $J_i(e)$, $0\leqq i\leqq\kappa$, such that all random variables

$$\{J_i(e): 0\leqq i\leqq\kappa, e\in\mathscr{L}\}$$

are independent, and such that

$$P\{J_0(e)=0\}=P\{J_0(e)=1\}=\tfrac{1}{2},$$

$$P\{J_i(e)=0\}=1-P\{J_i(e)=1\}=p_i, \qquad 1\leqq i\leqq\kappa. \tag{2.17}$$

We call e *l-open* if $J_0(e)\cdot J_1(e)\ldots J_l(e)=1$, or equivalently if $J_0(e)=J_1(e)=\ldots=J_l(e)=1$. Clearly

$$P\{e \text{ is } l\text{-open}\}=\tfrac{1}{2}p_1\ldots p_l, \tag{2.18}$$

and in particular, by (2.16),

$$P\{e \text{ is } \kappa\text{-open}\}=p. \tag{2.19}$$

Thus, for our choice of parameters, the distribution of the configuration of κ-open edges is the same as of the configuration of open edges in the original problem when $P\{e \text{ is open}\}=p$. We may therefore replace "open" by "κ-open" in (2.15) without changing the problem. We shall write $R_l(m,n)$ for the lowest l-open left-right crossing of $T(m,n)$, if it exists. Thus R_l is defined in the same way as R in Lemma 1, but with "open" replaced by "l-open". We shall now see that Theorem 1 reduces to the following proposition.

Proposition 1. *There exists a constant $\gamma_0>0$, independent of κ, $p_1,\ldots,p_\kappa$ and $l\leqq\kappa$, and a k_0 such that for $k\geqq k_0$, $0\leqq l<\kappa$, and for each left-right crossing r of $T(2^{k+1},2^{k+1})$ which is contained in $T(2^k,2^{k+1})$ one has*

$$P\{\exists(l+1)\text{-open left-right crossing of } T(2^{k+1},2^{k+1})\mid R_l(2^{k+1},2^{k+1})=r\}\leqq 1-\gamma_0. \tag{2.20}$$

Before starting on the proof of this proposition we show how it implies (2.11). We note that any $(l+1)$-open path is also l-open. Therefore, if $R_\kappa(2^{k+1},2^{k+1})$ exists, so do $R_0(2^{k+1},2^{k+1}, \ldots, R_{\kappa-1}(2^{k+1},2^{k+1})$. Moreover, by Lemma 1 (with "open" replaced by "l-open"), the l-open crossing R_{l+1} must lie in

$$\bar{C}^+(R_l)=\bar{C}^+(R_l;2^{k+1},2^{k+1}) \quad \text{and} \quad R_l\subset\bar{C}^-(R_{l+1})=\bar{C}^-(R_{l+1};2^{k+1},2^{k+1}).$$

Thus if $R_l(2^{k+1},2^{k+1})$ exists, and lies in $T(2^k,2^{k+1})$, then also

$$R_0(2^{k+1},2^{k+1}), \ldots, R_{l-1}(2^{k+1},2^{k+1})$$

must lie in $T(2^k,2^{k+1})$. Therefore,

$$\begin{aligned}
S_p(2^k,2^{k+1}) &= P\{R_\kappa(2^{k+1},2^{k+1}) \text{ exists and lies in } T(2^k,2^{k+1})\} \\
&= P\{R_0(2^k,2^{k+1}) \text{ exists and lies in } T(2^k,2^{k+1})\} \\
&\quad \cdot \prod_{l=0}^{\kappa-1} P\{R_{l+1}(2^{k+1},2^{k+1}) \text{ exists and lies in} \\
&\quad T(2^k,2^{k+1})\,|\,R_0(2^{k+1},2^{k+1}), \ldots, R_l(2^{k+1},2^{k+1}) \\
&\quad \text{exist and lie in } T(2^k,2^{k+1})\} \\
&\leqq \prod_{l=0}^{\kappa-1} P\{R_{l+1}(2^{k+1},2^{k+1}) \text{ exists}\,|\,R_0(2^{k+1},2^{k+1}), \ldots, R_l(2^{k+1},2^{k+1}) \\
&\quad \text{exist and lie in } T(2^k,2^{k+1})\} \\
&= \prod_{l=0}^{\kappa-1} P\{R_{l+1}(2^{k+1},2^{k+1}) \text{ exists}\,|\,R_l(2^{k+1},2^{k+1}) \\
&\quad \text{exists and lies in } T(2^k,2^{k+1})\} \\
&\leqq (1-\gamma_0)^\kappa. \qquad (2.21)
\end{aligned}$$

Since γ_0 is independent of κ and $p_1, \ldots, p_\kappa$, we can take κ arbitrarily large, so that

$$\limsup_{k\to\infty} S_p(2^k,2^{k+1}) \leqq \lim_{\kappa\to\infty}(1-\gamma_0)^\kappa = 0$$

will follow.

Before we can turn to the proof of Proposition 1 proper, we need one extra bit of preparation. We already introduced $\mathscr{L}^*$, the dual lattice of $\mathscr{L}$. We shall use e^* or f^* to denote generic edges of $\mathscr{L}^*$, u^*, v^* or w^* to denote vertices of $\mathscr{L}^*$. One trivially sees that each edge e^* of $\mathscr{L}^*$ crosses exactly one edge e of $\mathscr{L}$. We shall say that e^* and this e correspond to each other. We shall call e^* l-open (l-closed) if the corresponding $e\in\mathscr{L}$ is l-open (l-closed). We write $T^*(m,n)$ for the collection of all vertices and edges of $\mathscr{L}^*$ in the rectangle

$$\{(x,y): \tfrac{3}{2}\leqq x\leqq n-\tfrac{1}{2}, \tfrac{1}{2}\leqq y\leqq m+\tfrac{1}{2}\}.$$

As a graph $T^*(m,n)$ is isomorphic to $T(n-1,m+1)$. Now consider a fixed left-right crossing r of $T(2^{k+1},2^{k+1})$ and let $s^*=(w_0^*,e_1^*,\ldots,e_\varrho^*,w_\varrho^*)$ be a path in $T^*(2^{k+1},2^{k+1})$ which starts at the upper edge

$$E^*=\{(x,y): \tfrac{3}{2}\leqq x\leqq 2^{k+1}-\tfrac{1}{2}, y=2^{k+1}+\tfrac{1}{2}\},$$

and "goes down" until it first enters $C^-(r)=C^-(r;2^{k+1},2^{k+1})$. To be precise, if $w_i^*=(x_i^*,y_i^*)$, then we assume

$$\begin{aligned} &y_0^*=2^{k+1}+\tfrac{1}{2}, \quad \tfrac{1}{2}\leqq y_i^*\leqq 2^{k+1}-\tfrac{1}{2}, \quad 1\leqq i\leqq \varrho, \\ &\tfrac{3}{2}\leqq x_j^*\leqq 2^{k+1}-\tfrac{1}{2}, \qquad\qquad\qquad 0\leqq j\leqq \varrho, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} &w_i^*\in C^+(r), \quad 0<i<\varrho, \\ &w_\varrho^*\in C^-(r) \quad \text{or} \quad y_\varrho^*=\tfrac{1}{2}. \end{aligned} \tag{2.23}$$

Note that the last condition of (2.23) means $w_\varrho^*\in \bar{C}^-(r)$, and since $w_\varrho^*\in \mathscr{L}^*$ it cannot lie on r, i.e., $w_\varrho^*\in \bar{C}^-(r)\backslash r$. Since e_ϱ^* goes from $C^+(r)=C^+(r;2^{k+1},2^{k+1})$ to $\bar{C}^-(r)\backslash r$ it must cross r. Thus there is a unique edge e in r which is crossed by e_ϱ^*; we shall denote this edge by $e(s^*)$. A path s^* satisfying (2.22) and (2.23) is a cross-cut of $C^+(r)$ and divides $C^+(r)$ into two components, a "left" and a "right" one, which we shall denote by $V^L(r,s^*)$ and $V^R(r,s^*)$ (see [11], Theorems V.11.7 and 11.8). If $v_0=(1,y_0)$ denotes the left-endpoint of r and a is the intersection of e_ϱ^* and $e(s^*)$ [i.e., the midpoint of e_ϱ^* as well as of $e(s^*)$], then the boundary of V^L consists of the simple curve made up of the segment from $(1,2^{k+1}+\frac{1}{2})$ to $(1,y_0)=v_0$, followed by the piece of r from v_0 to a, followed by s^* from a to w_0^*, and then the segment from $w_0^*=(x_0^*,2^{k+1}+\frac{1}{2})$ to the starting point $(1,2^{k+1}+\frac{1}{2})$; see Fig. 1. The boundary of V^R can be described similarly. Since V^L and V^R are the two components of $C^+(r)\backslash s^*$ it is impossible to connect any point b^L in V^L to a point b^R in V^R by a continous curve $\psi:[0,1]\to C^+(r)$, without intersecting s^*. The same remains true if we allow b^L to be in $\bar{V}^L$, b^R in $\bar{V}^R$ and ψ in $\bar{C}^+(r)$. This is easily seen by observing that $\psi(\bar{t})$ must lie in $\bar{V}^L\cap \bar{V}^R$ where

$$\bar{t}=\max\{t:\psi(t)\in \bar{V}^L\}.$$

We call a *weak cut-set (with respect to r)* any path s^* in $T^*(2^{k+1},2^{k+1})$ which satisfies (2.22) and (2.23) and for which

$$e_1^*,e_2^*,\ldots,e_{\varrho-1}^* \text{ are 0-closed}. \tag{2.24}$$

We call s^* an *$(l+1)$-strong cut set (with respect to r)* if it is a weak cut set with respect to r, and in addition

$$e_\varrho^* \text{ is } (l+1)\text{-closed, or equivalently } J_{l+1}(e(s^*))=0. \tag{2.25}$$

In step (i) below we shall use the following simple observation: Call a path $t^*=(w_0^*,e_1^*,\ldots,e_\lambda^*,w_\lambda^*)$ a top-bottom crossing of $T^*(2^{k+1},2^{k+1})$ if it is a minimal connecting path on $\mathscr{L}^*$ of the upper and lower edge of $T^*(2^{k+1},2^{k+1})$, i.e., again with $w_i^*=(x_i^*,y_i^*)$, (2.22) holds with ϱ replaced by λ, as well as

$$y_\lambda^*=\tfrac{1}{2}. \tag{2.26}$$

Such a path connects[5] $w_0^*\in\partial C^+(r)\backslash\partial C^-(r)$ to $w_\lambda^*\in\partial C^-(r)\backslash\partial C^+(r)$, and except for its endpoints lies in $C(2^{k+1},2^{k+1})$. It therefore must intersect r, and if ϱ is the smallest

5 ∂A denotes the boundary of A for a set A in the plane

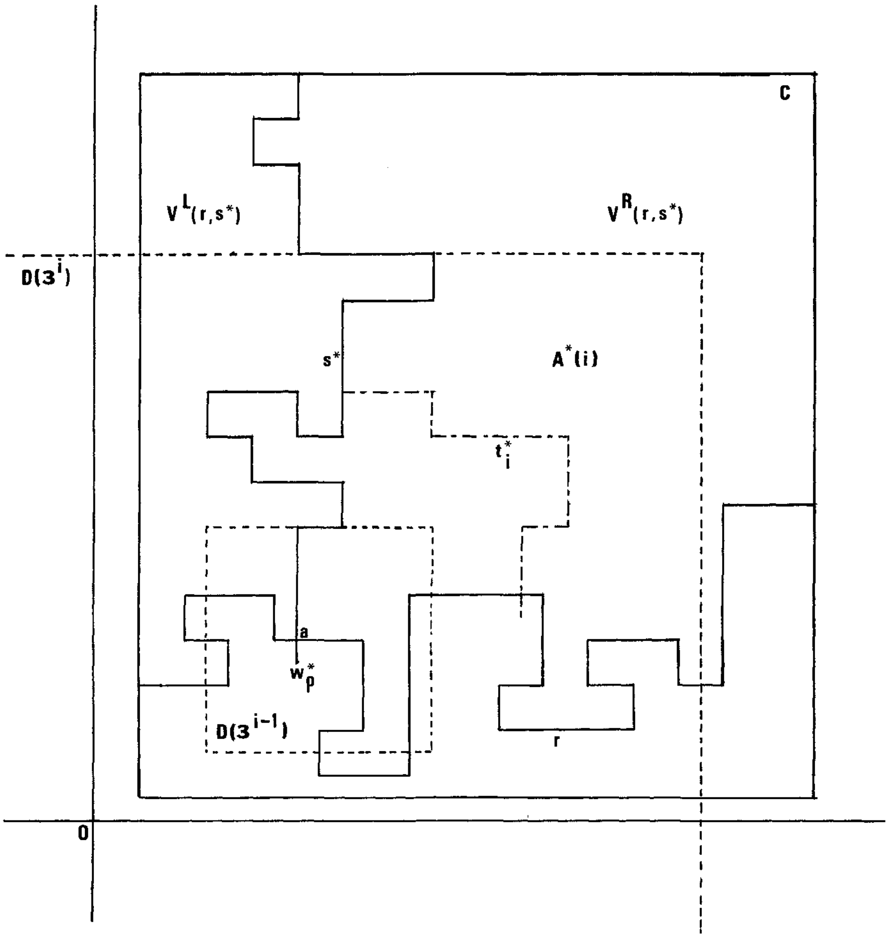

Fig. 1. The left-right crossing r and the cut set s^* are drawn in solid lines. The --- lines indicate the boundaries of $D(3^{i-1})$, respectively, $D(3^i)$. The path t_i^* is drawn as ---

index for which e_ϱ^* intersects r, then the initial piece $s^* = (w_0^*, e_1^*, \ldots, e_\varrho^*, w_\varrho^*)$ of t^* satisfies (2.22) and (2.23). In particular, if $e_1^*, \ldots, e_{\varrho-1}^*$ are 0-closed, then s^* is a weak cut-set with respect to r.

Just as we could find a lowest open left-right crossing in Lemma 1, we can find a left-most weak cut set. We formulate this as Lemma 2, whose proof is almost identical to that of Lemma 1 (see Appendix).

Lemma 2. *Let r be a fixed left-right crossing of $T(2^{k+1}, 2^{k+1})$. If there exists any weak cut set with respect to r in $T^*(2^{k+1}, 2^k)$, then there exists a weak cut set with respect to r, $S^* = S^*(r, k)$ say, in $T^*(2^{k+1}, 2^k)$ such that there exists no weak cut set with respect to r in $\bar{V}^L(S^*, r) \cap T^*(2^{k+1}, 2^k)$, other than S^*. Moreover, S^* is unique, and if t^* is any weak cut set with respect to r in $T^*(2^{k+1}, 2^k)$, then t^* (minus the last half of its last edge) belongs to $\bar{V}^R(S^*, r)$.*

We stress that we restrict ourselves to cut sets in $T^*(2^{k+1}, 2^k)$, the left half of $T^*(2^{k+1}, 2^{k+1})$, just as we have concentrated on left-right crossings in $T(2^k, 2^{k+1})$, the lower half of $T(2^{k+1}, 2^{k+1})$. The reasons for this will become apparent later [see (2.34) and (2.35)]. Use of the notation $\{S^*(r,k)=s^*\}$ means that there exists a weak cut set with respect to r in $T^*(2^{k+1}, 2^k)$ and S^*, the left most such cut set, equals s^*. Also, the event $\{S^*(r,k)=s^*\}$ depends only on the 0-open- or closedness of the edges e^* in $\bar{V}^L(r, s^*)$, since $\{S^*(r,k)=s^*\}$ occurs if and only if s^* is a weak-cut with respect to r, and any path t^* in $T^*(2^{k+1}, 2^k) \cap \bar{V}^L(s^*, r)$ from E^* to $C^-(r)$, other than s^*, has a 0-closed edge. Translated in terms of the $J_i(e)$, this means that for fixed r, the event $\{S^*(r,k)=s^*\}$ depends only on the collection

$$\{J_0(e) : |e| \cap V^L(r, s^*) \neq \emptyset\}. \tag{2.27}$$

Proof of Proposition 1. We first state the principal steps and show how these imply Proposition 1. After that the individual steps will be proven.

Step (i): The following inequality holds: for every left-right crossing r of $T(2^{k+1}, 2^{k+1})$, $k \geqq 1$, and $l \leqq \kappa$ one has

$$P\{\exists \text{ weak cut set with respect to } r \text{ which lies in } T^*(2^{k+1}, 2^k) \mid R_l(2^{k+1}, 2^{k+1}) = r\} \geqq \gamma_1 \equiv \Gamma(\tfrac{1}{2}) > 0. \tag{2.28}$$

Step (ii): For each integer τ there exists a $k_1 = k_1(\tau)$ such that for $k \geqq k_1$, $l \leqq \kappa$ and for each left-right crossing r of $T(2^{k+1}, 2^{k+1})$ which lies in $T(2^k, 2^{k+1})$, and each path s^* in $T^*(2^{k+1}, 2^k)$ which satisfies (2.22) and (2.23) one has

$$P\{\exists \text{ at least } \tau \text{ weak cut sets with respect to } r \text{ with distinct final edges} \mid R_l(2^{k+1}, 2^{k+1}) = r, S^*(r,k) = s^*\} \geqq \tfrac{1}{2}. \tag{2.29}$$

Step (iii): For each left-right crossing r of $T(2^{k+1}, 2^{k+1})$, each path s^* in $T^*(2^{k+1}, 2^k)$ satisfying (2.22) and (2.23), $\tau \geqq 1$, $l \leqq \kappa - 1$ one has

$$P\{\exists\ (l+1)\text{-strong cut set with respect to } r \mid R_l(2^{k+1}, 2^{k+1}) = r\ S^*(r,k) = s^*, \exists \text{ at least } \tau \text{ weak cut sets with respect to } r \text{ with distinct final edges}\} \geqq 1 - p_{l+1}^{\tau}. \tag{2.30}$$

We show now that (2.28)–(2.30) imply (2.20) with $\gamma_0 = \gamma_1/4$. As we observed, R_{l+1} if it exists, must lie in $\bar{C}^+(R_l)$ and since R_{l+1} connects the left and right edges of $C(2^{k+1}, 2^{k+1})$ it must intersect any cut set with respect to R_l, in particular, any $(l+1)$-strong cut set, assuming such a cut set exists. This is impossible, since the edge of R_{l+1} which intersects an $(l+1)$-strong cut set has to be $(l+1)$-open [because it belongs to R_{l+1}] as well as $(l+1)$-closed [because it crosses an $(l+1)$-strong cut set]. Thus

$$P\{\exists\ (l+1)\text{-open left-right crossing of } T(2^{k+1}, 2^{k+1}) \mid R_l(2^{k+1}, 2^{k+1}) = r\} \leqq 1 - P\{\exists\ (l+1)\text{-strong cut set with respect to } r \mid R_l(2^{k+1}, 2^{k+1}) = r\}. \tag{2.31}$$

On the other hand, for any left-right crossing r of $T(2^{k+1}, 2^{k+1})$,

$$P\{\exists\ (l+1)\text{-strong cut set with respect to } r \mid R_l(2^{k+1}, 2^{k+1}) = r\} \tag{2.32}$$

is at least as large as the inf over paths s^* in $T^*(2^{k+1}, 2^k)$ which satisfy (2.22) and (2.23), of the product of the left hand sides of (2.28)–(2.30). Thus, for $k \geqq k(\tau)$ (2.32) is at least $\frac{1}{2}\gamma_1(1-p^\tau_{l+1})$ and by (2.31), the left hand side of (2.20) is at most

$$1-\tfrac{1}{2}\gamma_1(1-p^\tau_{l+1}).$$

It remains to take $\tau(l)$ such that

$$p^{\tau(l)}_{l+1} \leqq \tfrac{1}{2} \text{ and then } k_0 = \max\{k(\tau(l)) : l \leqq \kappa - 1\}.$$

We turn to the proofs of (2.28)–(2.30). For brevity we write R_l instead of $R_l(2^{k+1}, 2^{k+1})$.

Proof of Step (i). As in Russo [13], Aizenman [1], and Higuchi [8] we use a sort of generalized form of the strong Markov property with respect to R_l (a set valued random variable, rather than the usual kind of stopping time). As we observed just after (2.25), the left hand side of (2.28) is bounded below by

$$\begin{aligned} P\{&\exists \text{ top-bottom crossing } t^* \text{ of } T^*(2^{k+1}, 2^{k+1}) \text{ which lies} \\ &\text{in } T^*(2^{k+1}, 2^k) \text{ and has } J_0(e_i^*) = 0 \text{ for all edges} \\ &e_i^* \text{ of } t^* \text{ with interior } (e_i^*) \subset C^+(r) \mid R_l = r\}. \end{aligned} \tag{2.33}$$

However, the edges e^* of $\mathcal{L}^*$ with interior $(e_i^*) \subset C^+(r)$ only cross edges e of $\mathcal{L}$ with interior $(e) \subset C^+(r)$. All these edges are independent of the edges in $\bar{C}^-(r)$, and as we pointed out after Lemma 1, the event $\{R_l = r\}$ depends only on edges e in $\bar{C}^-(r)$. Consequently, the conditional probability in (2.33) is the same as the unconditional probability

$$\begin{aligned} P\{&\exists \text{ top-bottom crossing } t^* \text{ of } T^*(2^{k+1}, 2^{k+1}) \text{ which lies in} \\ &T^*(2^{k+1}, 2^k) \text{ and has } J_0(e_i^*) = 0 \text{ for all edges } e_i^* \text{ of } t^* \\ &\text{with interior } (e_i) \subset C^+(r)\} \\ \geqq P\{&\exists \text{ top-bottom crossing } t^* \text{ of } T^*(2^{k+1}, 2^{k+1}) \text{ which lies in} \\ &T^*(2^{k+1}, 2^k) \text{ and has all its edges 0-closed}\} \\ = S_{1/2}&(2^k - 1, 2^{k+1} + 1). \end{aligned}$$

The last equality holds, because

$$P\{e^* \text{ is 0-closed}\} = P\{e^* \text{ is 0-open}\} = \tfrac{1}{2},$$

and $T^*(2^{k+1}, 2^k)$ is isomorphic to $T(2^k - 1, 2^{k+1} + 1)$. Finally, by (2.3), (2.4), (2.7), and (2.6)

$$\begin{aligned} S_{1/2}(2^k - 1, 2^{k+1} + 1) &\geqq S_{1/2}(2^{k-1}, 6.2^{k-1}) \\ &\geqq \Gamma(S_{1/2}(2^{k-1}, 2^{k-1})) \\ &\geqq \Gamma(\tfrac{1}{2}) = \gamma_1. \end{aligned}$$

(2.28) follows by combining these inequalities.

Proof of Step (ii). This step needs a further kind of separating set, or cut set. Let r be a fixed left-right crossing of $T(2^{k+1}, 2^{k+1})$ which lies in $T(2^k, 2^{k+1})$, and $s^*=(w_0^*, e_1^*, \ldots, e_\varrho^*, w_\varrho^*)$ a fixed path in $T^*(2^{k+1}, 2^k)$ satisfying (2.22) and (2.23). We consider the annuli $A^*(i)=A^*(i, w_\varrho^*)$ in $\mathscr{L}^*$ centered at $w_\varrho^*+(\frac{1}{2}, \frac{1}{2})$, consisting of all vertices and edges of $\mathscr{L}^*$ in $D(3^i)\backslash(3^{i-1})$, where $D(3^i)$ is the closed square bounded by portions of the lines

$$x=x_\varrho^*-3^i+1, \quad x=x_\varrho^*+3^i,$$
$$y=y_\varrho^*-3^i+1, \quad y=y_\varrho^*+3^i$$

[recall $w_\varrho^*=(x_\varrho^*, y_\varrho^*)$; see Fig. 1].

We take

$$2 \leqq i < k\frac{\log 2}{\log 3}. \tag{2.34}$$

Since we assumed $r \subset T(2^k, 2^{k+1})$ and $s^* \in T^*(2^{k+1}, 2^k)$, we have $x_\varrho^* \leqq 2^k$, $y_\varrho^* \leqq 2^k+\frac{1}{2}$, so that $D(3^i)$ and $A^*(i)$ do not intersect the top or right edge of $C(2^{k+1}, 2^{k+1})$, i.e.,

$$x_\varrho^*+3^i<2^{k+1}, \quad y_\varrho^*+3^i<2^{k+1}. \tag{2.35}$$

Now let $G^*=G^*(i; r, s^*)$ be a connected subgraph of $\mathscr{L}^*$ which is contained in $A^*(i)$ and which separates w_ϱ^* from ∞, i.e., has the following property:

$$\begin{aligned}&G^* \text{ is connected and any continuous curve from } w_\varrho^*\\ &\quad\text{to the exterior of } D(3^i) \text{ must intersect } G^*.\end{aligned} \tag{2.36}$$

It is again quite intuitive that in this case G^* contains a path on $\mathscr{L}^*$ which connects s^* to r through $V^R(r, s^*)$. Again we only formulate the lemma here, and leave its proof for the Appendix.

Lemma 3. *Let r be a left-right crossing of $T(2^{k+1}, 2^{k+1})$ which is contained in $T(2^k, 2^{k+1})$, s^* a path in $T^*(2^{k+1}, 2^k)$ satisfying* (2.22) *and* (2.23). *Let i satisfy* (2.34) *and let $G^*(i)$ be a subgraph of $\mathscr{L}^*$ in $A^*(i, w_\varrho^*)$ which has property* (2.36). *Then there exists a path $t^*=(u_0^*, f_1^*, \ldots, f_\lambda^*, u_\lambda^*)$ in $G^* \cap T^*(2^{k+1}, 2^{k+1})$ such that*

$$u_0^* \text{ is a vertex of } s^*, \tag{2.37}$$

$$f_1^*, u_1^*, \ldots, f_{\lambda-1}^*, u_{\lambda-1}^* \subset V^R(r, s^*), \tag{2.38}$$

$$f_\lambda^* \text{ crosses an edge of } r. \tag{2.39}$$

With this lemma it is not hard to complete step (ii), by means of another application of the analogue of the strong Markov property, quite similar to step (i). Fix r, a left-right crossing of $T(2^{k+1}, 2^{k+1})$ which lies in $T(2^k, 2^{k+1})$, fix a path s^* in $T^*(2^{k+1}, 2^k)$ which satisfies (2.22) and (2.23) and assume $R_l=R_l(2^{k+1}, 2^{k+1})=r$, $S^*=S^*(r, k)=s^*$, and let $G^*(i)$ be a subgraph of $A^*(i)=A^*(i, w_\varrho^*)$ which has property (2.36) as well as

$$\text{all edges of } G^*(i) \text{ whose interior lies in } V^R(r, s^*) \text{ are 0-closed}. \tag{2.40}$$

Then, by Lemma 3 there exists a path $t_i^* = (u_{0,i}^*, f_{1,i}^*, \ldots, f_{\lambda,i}^*, u_{\lambda,i}^*)$ in $A^*(i)$ with the properties (2.37)–(2.39). Since we assumed (2.40) it also satisfies

$$f_{j,i}^* \text{ is 0-closed}, \quad 1 \leqq j \leqq \lambda - 1. \tag{2.41}$$

Moreover, $u_{0,i}^*$ is some vertex of s^*, say $u_{0,i}^* = w_{\beta(i)}^*$. Then the path

$$(w_0^*, e_1^*, \ldots, e_{\beta(i)-1}^*, w_{\beta(i)}^* = u_{0,i}^*, f_{1,i}^*, \ldots, f_{\lambda,i}^*, u_{\lambda,i}^*) \tag{2.42}$$

consists of an initial piece of s^* followed by t_i^*. It begins on E^* [by (2.22)], its last edge crosses r [by (2.39)] and all its other edges are in $\bar{C}^+(r)$ [by (2.23) and (2.38); note $\beta(i) \neq \varrho$, since $w_\varrho^* \notin A^*(i)$]. All edges, but the last one of the path in (2.42) are 0-closed [since we assumed $s^* = S^*$ to be a weak cut set, and on account of (2.41)], so that this path is a weak cut set with respect to r. Also its last edge is in $A^*(i)$, and different $A^*(i)$ are disjoint. It follows from this that the left hand side of (2.29) is bounded below by

$$P\left\{\text{at least } \tau \text{ of the annuli } A^*(i, w_\varrho^*) \text{ with } 2 \leqq i < k\frac{\log 2}{\log 3} \text{ contain} \right.$$
$$\left. \text{a subgraph } G^*(i) \text{ which satisfies (2.36) and (2.40)} \,|\, R_l = r,\ S^* = s\right\}. \tag{2.43}$$

The event in (2.43) only involves the 0-open- or closedness of edges e^* in $V^R(r, s^*)$. In other words, it is determined by the collection

$$\{J_0(e) : \text{interior } (e) \subset V^R(r, s^*)\}.$$

On the other hand the event $\{R_l = r,\ S^* = s\}$ only involves the random variables

$$\{J_i(e) : 0 \leqq i \leqq l,\ e \in \bar{C}^-(r)\} \tag{2.44}$$

and those of (2.27), by our comments after Lemmas 1 and 2. Thus, as in step (i), the conditional probability in (2.43) is the same as the unconditional probability.

$$P\left\{\text{at least } \tau \text{ of the annuli } A^*(i) \text{ with } 2 \leqq i < k\frac{\log 2}{\log 3} \text{ contain} \right.$$
$$\left. \text{a subgraph } G^*(i) \text{ which satisfies (2.36) and (2.40)}\right\}$$
$$\geqq P\left\{\text{at least } \tau \text{ of the annuli } A^*(i) \text{ with } 2 \leqq i < k\frac{\log 2}{\log 3} \text{ contain} \right.$$
$$\left. \text{a subgraph } G^*(i) \text{ with property (2.36) and all its edges 0-closed}\right\}. \tag{2.45}$$

It was proved by Seymour and Welsh [14], Lemma 5.4 (see also [15], Lemma 3.5) that

$$P\{A^*(i) \text{ contains a subgraph } G^*(i) \text{ with property (2.36)}$$
$$\text{and all its edges 0-closed}\}$$
$$\geqq \gamma^{12}(1-(1-\gamma)^{1/2})^{64}, \tag{2.46}$$

where

$$\gamma = S_{1/2}(2.3^{i-1}, 2.3^{i-1}) \geqq \tfrac{1}{2} \quad \text{[cf. (2.6)]}.$$

Thus, the right hand side of (2.46) is at least

$$\gamma_2 \equiv 2^{-12}(1-2^{-1/2})^{64}.$$

Since the $A^*(i)$ are disjoint, the right hand side of (2.45) is bounded below by a right tail of the binomial distribution with success probability γ_2, to wit by

$$\sum_{j=\tau}^{N} \binom{N}{j} \gamma_2^j (1-\gamma_2)^{N-j}, \tag{2.47}$$

where

$$N = \left[k \frac{\log 2}{\log 3}\right] - 2.$$

As $k \to \infty$ and hence $N \to \infty$, (2.47) tends to 1, and (2.29) follows.

Proof of Step (iii). This is very easy indeed. The event $\{R_l = r\}$ is defined in terms of the random variables in (2.44), whereas any condition about weak cut sets with respect to r involves only the $\{J_0(e): e \in T(2^{k+1}, 2^{k+1})\}$. Thus neither of these involve the $J_{l+1}(e)$. However, if $s_1^*, \ldots, s_\tau^*$ are τ paths satisfying (2.22) and (2.23) which are also weak cut sets, and with the distinct last edges $e^*(\varrho(1)), \ldots, e^*(\varrho(\tau))$, then at least one of these paths will be a strong cut set with respect to r, as soon as one of the edges $e^*(\varrho(1)), \ldots, e^*(\varrho(\tau))$ is $(l+1)$-closed. Conditionally on any information on the $\{J_0(e), \ldots, J_l(e): e \in \mathscr{L}\}$, the probability of at least one of $e^*(\varrho(1)), \ldots, e^*(\varrho(\tau))$ being $(l+1)$-closed is $1 - p_{l+1}^\tau$. This proves (2.30). □

The proof of Proposition 1 and Theorem 1 is complete.

Proof of Theorem 2. (1.5) and (1.6) are immediate from the definition of p_H and Theorem 1, except when $p = \frac{1}{2}$. But $\theta(\frac{1}{2}) = 0$ was already proved by Harris [7]. (1.7) is a special case of Theorem 1 of [10], whereas (1.8) is in Remark 4 of [10], now that we know $p_T = p_H = \frac{1}{2}$ (see also [15], p. 61).

As for (1.9), if the infinite open cluster does not contain any vertices v of $\mathscr{L}$ with $|v| \leq n$, then by Whitney's theorem (see [15], Theorem 2.1) there exists a closed cut set on $\mathscr{L}^*$ which separates the infinite open component on $\mathscr{L}$ from the subgraph of $\mathscr{L}$ containing all edges and vertices within distance n from the origin. As in Lemma 3 this implies that there exists a path on $\mathscr{L}^*$ connecting some point $(0, l+\frac{1}{2})$ on the y-axis with some point $(m+\frac{1}{2}, 0)$ on the x-axis, $l, m \geq n$, and with all its edges closed. Such a path contains at least l edges, and the probability of an edge e^* of $\mathscr{L}^*$ being closed is $q = 1 - p$. Thus, the left hand side of (1.8) is at most

$$\begin{aligned} &\sum_{l \geq n} P_p\{\exists \text{ closed path on } \mathscr{L}^* \text{ starting at } (0, l+\tfrac{1}{2}) \text{ and} \\ &\qquad \text{containing vertices at distance} \geq l \text{ from } (0, l+\tfrac{1}{2})\} \\ &\quad \leq \sum_{l \geq n} P_q\{\exists \text{ open path on } \mathscr{L} \text{ starting at 0 and containing vertices} \\ &\qquad \text{at distance} \geq l \text{ from } 0\} \\ &\quad \leq 2\{1 - e^{-C_1(q)}\}^{-1} e^{-C_1(q)n} \end{aligned}$$

[by (1.7)].

Appendix

We give here the purely topological proofs of Lemmas 1–3. They involve only standard (but tedious) arguments.

Proof of Lemma 1. Let r_1 and r_2 be two left-right crossings of $T(m,n)$. We first show that one can find a left-right crossing r such that

$$r \subset \bar{C}^-(r_1;m,n) \cap \bar{C}^-(r_2;m,n) \tag{A.1}$$

and

$$|r| \subset |r_1| \cup |r_2| . \tag{A.2}$$

This part of the proof has nothing to do with the open-ness of any of these crossings. However, if r_1 and r_2 are open, then r will be open as well, since (A.2) implies that each edge of r is an edge of r_1 or of r_2. For the remainder of this proof we suppress the m,n from the notation.

To construct the desired r, assume that $|r_2|$ contains some point z from $C^-(r_1)$. [If no such z exists, then $r_2 \subset \bar{C}^+(r_1)$, and we can take $r=r_1$, as will be apparent from the proof below.] We follow r_2 from z backwards to the left edge E_1 of $C=C(m,n)$;

$$E_1 = \{(x,y): x=1, \tfrac{1}{2} \leqq y \leqq m+\tfrac{1}{2}\} .$$

Then there exists a first point, b say, at which we hit r_1 or E_1. b is necessarily a vertex of $\mathscr{L}$ on r_1 or E_1. Similarly, going forward from z along r_2 we will hit r_1 or

$$E_2 \equiv \{(x,y): x=n, \tfrac{1}{2} \leqq y \leqq m+\tfrac{1}{2}\} , \tag{A.3}$$

the right edge of C, at some vertex c. Denote the polygonal curve consisting of the piece of r_2 between b and c, excluding the endpoints b and c themselves, by ϱ. By construction ϱ is disjoint from $|r_1|$ and ∂C and contains the point $z \in C^-(r_1)$. Therefore

$$\varrho \subset C^-(r_1) .$$

For the sake of argument assume b and c belong to $|r_1| \backslash E_1 \cup E_2$. A simple modification of the argument suffices if $b \in E_1$ and/or $c \in E_2$. Denote by $\tilde{r}_1$ the path which starts at the initial point of r_1 on E_1, follows r_1 till it reaches b or c (c may be reached before b on r_1), then follows ϱ till c (respectively b) and then continues along r_1 to its endpoint on E_2. $\tilde{r}_1$ is a path on $\mathscr{L}$ with initial (end) point on E_1 (respectively E_2) and no other point on ∂C, and since $|\varrho| \cap |r_1| = \emptyset$, $\tilde{r}_1$ is also selfavoiding. Thus $\tilde{r}_1$ is a left-right crossing of C with

$$|\tilde{r}_1| \subset |r_1| \cup |r_2| .$$

In fact, $\tilde{r}_1 \subset |r_1| \cup |\varrho|$, so that even

$$\tilde{r}_1 \subset \bar{C}^-(r_1) . \tag{A.5}$$

We claim that (A.5) implies

$$r_1 \subset \bar{C}^+(r_1) \subset \bar{C}^+(\tilde{r}_1) \tag{A.6}$$

and

$$\bar{C}^-(\tilde{r}_1) \subset \bar{C}^-(r_1). \tag{A.7}$$

To see this, let $z_1 = (\frac{5}{4}, m + \frac{1}{4})$. z_1 is a point of C, near the upper left hand corner of C, but above the line $y = m$. Thus, z_1 can be connected to

$$E_3 \equiv \{(x, y) : 1 \leqq x \leqq n, y = m + \tfrac{1}{2}\}, \tag{A.8}$$

the upper edge of C, by the vertical segment $\{\frac{5}{4}\} \times [m + \frac{1}{4}, m + \frac{1}{2}]$, which does not intersect r_1. Since E_3 is part of $\partial C^+(r_1)$ but disjoint from $\partial C^-(r_1)$, we have $z_1 \in C^+(r_1)$. Similarly $z_1 \in C^+(\tilde{r}_1)$. Now let $z_2 \in \bar{C}^+(r_1)$. Then there exists a continuous curve φ from z_2 to z_1, such that φ belongs to $C^+(r_1)$, except possibly for its initial point z_2 [in case $z_2 \in \partial C^+(r_1)$]. In particular, φ has at most the point z_2 in common with $\bar{C}^-(r_1)$. By virtue of (A.5) φ also has at most the point z_2 in common with $\tilde{r}_1$. Thus, φ intersects $\partial C^+(\tilde{r}_1)$ at most in z_2, while its endpoint z_1 lies in $C^+(\tilde{r}_1)$. Thus $z_2 \in \bar{C}^+(\tilde{r}_1)$. This proves the second inclusion of (A.6) since z_2 was arbitrary in $\bar{C}^+(r_1)$. The first inclusion in (A.6) is true by definition, and (A.7) is immediate from (A.6) and

$$C^-(\tilde{r}_1) = C \backslash \bar{C}^+(\tilde{r}_1) \subset C \backslash \bar{C}^+(r_1) = C^-(r_1).$$

From (A.6) we see that any edge of r_2 which belongs to $\bar{C}^+(r_1)$ still belongs to $\bar{C}^+(\tilde{r}_1)$. However, some edge e of r_2 which contains z now belongs to $\tilde{r}_1 \subset \bar{C}_1^+(\tilde{r}_1)$, whereas the interior of e belonged to $C^-(r_1)$. Thus, there are strictly fewer edges of r_2 with their interior in $C^-(\tilde{r}_1)$ than in $C^-(r_1)$. We can now iterate this procedure. If $|r_2|$ still has a point $z' \in C^-(\tilde{r}_1)$, then we can modify $\tilde{r}_1$ to an $\tilde{r}_2$ such that

$$\tilde{r}_2 \subset |\tilde{r}_1| \cup |r_2| \subset |r_1| \cup |r_2|,$$

$$\tilde{r}_2 \subset \bar{C}^-(\tilde{r}_2) \subset \bar{C}^-(\tilde{r}_1) \subset \bar{C}^-(r_1),$$

and so on. Since each time r_2 has fewer edges below the new path, we obtain after a finite number of steps a left-right crossing $\tilde{r}_\lambda$ such that

$$|\tilde{r}_\lambda| \subset |\tilde{r}_{\lambda-1}| \cup |r_1| \subset \ldots \subset |r_1| \cup |r_2|, \tag{A.9}$$

$$\tilde{r}_\lambda \subset \bar{C}^-(\tilde{r}_{\lambda-1}) \subset \ldots \subset \bar{C}^-(\tilde{r}_1) \subset \bar{C}^-(r_1), \tag{A.10}$$

and r_2 has no more points in $C^-(\tilde{r}_\lambda)$, i.e.,

$$r_2 \subset \bar{C}^+(\tilde{r}_\lambda). \tag{A.11}$$

We now take $r = \tilde{r}_\lambda$. This r satisfies (A.1) and (A.2) by (A.9)–(A.11). Indeed (A.11) implies

$$r = \tilde{r}_\lambda \subset \bar{C}^-(\tilde{r}_\lambda) \subset \bar{C}^-(r_2), \tag{A.12}$$

by the proof which led from (A.5) to (A.6) (with the roles of $+$ and $-$ interchanged). Now that we found the crossing r which "lies below r_1 and r_2" we proceed with the proof of Lemma 1 by induction. Assume that $T(m, n)$ has at least one open left-right crossing, and let $r_1, r_2, \ldots, r_\alpha$ be the collection of all open left-right crossings of $T(m, n)$; α is necessarily finite. If $\alpha = 1$, take $R = r_1$. If $\alpha \geqq 2$ first construct r as above. If $\alpha=2$, take $R=r$. If $\alpha\geqq 3$ construct from r and r_3 a crossing, $\bar{r}$ say, such that

$$\bar{r}\subset\bar{C}^-(r)\cap\bar{C}^-(r_3)\subset\bar{C}^-(r_1)\cap\bar{C}^-(r_2)\cap\bar{C}^-(r_3).$$

and

$$|\bar{r}|\subset|r|\cup|r_3|\subset|r_1|\cup|r_2|\cup|r_3|.$$

Continuing in this way we finally obtain a left right crossing R with

$$R\subset\bigcap_{i=1}^{\alpha}\bar{C}^-(r_i), \tag{A.13}$$

$$|R|\subset\bigcup_{i=1}^{\alpha}|r_i|. \tag{A.14}$$

R is open by virtue of (A.14), and satisfies (2.14) by virtue of (A.13) [and the proof which led from (A.5) to (A.6)].

We have therefore proved the existence of an R with the properties stated in the lemma. It remains to prove the uniqueness. This, however, is immediate. If $R'\neq R$ is an open left-right crossing with no other open left-right crossing in $\bar{C}^-(R')$, then R' must be one of the r_i, so that $R\subset\bar{C}^-(R')$ on account of (A.13). This, however, contradicts our assumption on R'. □

Proof of Lemma 2. Except for a change in notation this proof is virtually identical to that of Lemma 1. The principal change is that the collection $r_1,\ldots,r_\alpha$ now has to be replaced by $s_1^*,\ldots,s_\beta^*$, the collection of all weak cut sets with respect to r which lie in $T^*(2^{k+1},2^k)$. Anologously to (A.14) we will now have

$$|S^*|\subset\bigcup_{i=1}^{\beta}|s_i^*|,$$

so that automatically $|S^*|\subset T^*(2^{k+1},2^k)$. □

Proof of Lemma 3. As before we write $r=(v_0,e_1,\ldots,e_v,v_v)$ and $s^*=(w_0^*,e_1^*,\ldots,e_\varrho^*,w_\varrho^*)$. a is the intersection of e_ϱ^*, the last edge in s^*, and $e(s^*)$, its corresponding edge in $\mathscr{L}$ [$e(s^*)$ belongs to r]. We shall write V^R for $V^R(r,s^*)$ in this proof. By definition $a\in\partial V^R$, so that we can pick a point $z_0\in V^R$ with $|a-z_0|<\frac{1}{2}$. Since $|a-w_\varrho^*|=\frac{1}{2}$, we have $|z_0-w_\varrho^*|<1$ and

$$z_0\in V^R\cap\text{interior }(D(3^{i-1})),\qquad i\geqq 2. \tag{A.15}$$

We take $z_1=(2^{k+1},2^{k+1}+\frac{1}{2})$, the upper right hand corner of $C=C(2^{k+1},2^{k+1})$. Note that

$$z_1\in\partial C\cap\partial V^R. \tag{A.16}$$

Let $G^*=G^*(i)$ be as in the lemma and define

$$F_1=|s^*|\cup\{z^*\in|G^*|:\exists\text{ continuous curve in }|G^*|\cap\bar{V}^R$$
$$\text{connecting a vertex of }s^*\text{ with }z^*\}.$$

Clearly F_1 is closed; in fact it is a finite union of closed edges of $\mathscr{L}^*$. We define a further closed set of the same nature. Let $e(s^*)$ be the edge e_γ of r, and let $\bar{r}$ be the piece of r from the center a of e_γ on, i.e.,

$$\bar{r}=(\text{segment from } a \text{ to } v_{\gamma+1})\cup|(v_{\gamma+1}, e_{\gamma+2}, \ldots, e_\nu, v_\nu)|.$$

Then we define

$$F_2=|\bar{r}|\cup\{z^*\in|G^*|:\exists \text{ continuous curve in } |G^*|\cap\bar{V}^R \text{ connecting a point of } \bar{r} \text{ with } z^*\}.$$

Now assume first that there exists a $z^*\in|G^*|$ which can be connected by continuous curves in $|G^*|\cap\bar{V}^R$ to $|s^*|$ as well as to $|\bar{r}|$. Then we can combine the curves from z^* to $|s^*|$ and from z^* to $|r|$ to obtain a continuous curve in $|G^*|\cap\bar{V}^R$ connecting some point of s^* with some point of $\bar{r}$. The successive edges of G^* traversed by this curve yield a path t^* connecting $|s^*|$ with $|\bar{r}|$ and

$$|t^*\backslash \text{ last half edge of } t^*|\subset|G^*|\cap\bar{V}^R\subset|\mathscr{L}^*|\cap\bar{V}^R\subset|T^*(2^{k+1},2^{k+1})|.$$

Thus, $t^*\subset T^*(2^{k+1},2^{k+1})$, and without loss of generality t^* can be taken self-avoiding and such that (2.37)–(2.39) hold. In this situation the lemma holds. We may therefore restrict ourselves to the situation where there is no $z^*\in|G^*|$ which can be connected by continuous curves in $|G^*|\cap\bar{V}^R$ to $|s^*|$ as well as to $|\bar{r}|$. This situation, however, cannot arise as we shall now demonstrate. Indeed, in this situation

$$F_1\cap F_2=|s^*|\cap|\bar{r}|=\{a\}. \tag{A.17}$$

Moreover, we can then connect z_0 to z_1 by a continuous curve which does not intersect F_1, by connecting z_0 to some point of $z_2\in\bar{r}\backslash|s^*|$ without hitting ∂V^R before z_2 (this can be done since $\bar{r}\subset\partial V^R$, and V^R has a simple structure), and then continuing along $\bar{r}$ to the endpoint v_ν of r and proceeding along the right edge E_2 of $C(2^{k+1}, 2^{k+1})$ [see (A.3) with $n=m=2^{k+1}$]. Similarly we can connect z_0 and z_1 by a continuous curve which does not intersect F_2 by first going to some point $z_3\in s^*\backslash|\bar{r}|$ and then following s^* to the upper edge E_3 of $C(2^{k+1}, 2^{k+1})$ [see (A.8) with $n=m=2^{k+1}$] and E_3 itself. By virtue of these facts and the connectedness of $F_1\cap F_2$ [see (A.17)] and Alexander's separation lemma. [11], Theorem V.9.2, there exists a continuous curve $\varphi:[0,1]\to\mathbb{R}^2\backslash F_1\cup F_2$ connecting z_0 with z_1 without hitting $F_1\cup F_2$. φ begins at $\varphi(0)=z_0\in V^R$ and ends at $\varphi(1)=z_1\in\partial V^R$. Let

$$\bar{t}=\min\{t:\varphi(t)\in\partial V^R\}.$$

Then, $\varphi([0,\bar{t}))\subset V^R$. We claim that also

$$\varphi([0,\bar{t}))\cap|G^*|=\emptyset. \tag{A.18}$$

This is so, because (as we shall show)

$$|G^*|\cap V^R\subset F_1\cup F_2, \tag{A.19}$$

and thus any point in the left hand side of (A.18) would have to lie in $F_1\cup F_2$, which in fact is disjoint from φ. To prove (A.19) we observe that s^* connects w_ϱ^* to E_3 which lies in the unbounded component of $D(3^i)$, by (2.35). Thus, by (2.36), s^* intersects G^* at some point, say z_2^*. Now if $z^*\in|G^*|\cap V^R$, then, by the connectedness of G^*, there exists a continuous curve ψ in G^* from $z^* \in V^R$ to $z_2^* \in s^* \subset \partial V^R$. Let z_3^* be the first point on ψ which lies in ∂V^R. By definition of V^R,

$$\partial V^R \subset |s^*| \cup \bar{r} \cup E_2 \cup E_3 , \tag{A.20}$$

whereas $z_3^* \in G^* \subset A^*(i)$ which is disjoint from $E_2 \cup E_3$ by (2.35). Thus $z_3^* \in s^*$ or $z_3^* \in \bar{r}$, so that the piece of ψ from z^* to z_3^* connects z^* with s^* or $\bar{r}$ in $|G^*| \cap \bar{V}^R$. Thus, $z^* \in F_1 \cup F_2$, which proves (A.19), and hence (A.18).

Now, $\varphi(\bar{t}) \in \partial V^R$, and since φ does not intersect $F_1 \cup F_2$ (by construction), $\varphi(\bar{t}) \notin |s^*| \cup \bar{r}$. By (A.20) this means $\varphi(\bar{t}) \in E_2 \cup E_3$. But then the restriction of φ to $[0, \bar{t}]$ connects z_0 with $E_2 \cup E_3$, which lies in the exterior of $D(3^i)$, while φ restricted to $[0, \bar{t}]$ does not intersect $|G^*|$ [by (A.18) and $\varphi(\bar{t}) \in E_2 \cup E_3$]. Since we can extend φ in the beginning by the segment from w_ϱ^* to z_0 without hitting G^* [this segment lies in the interior of $D(3^{i-1})$] we found a curve which connects w_ϱ^* to the exterior of $D(3^i)$ without hitting G^*. Since this violates (2.36) the proof is complete.

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