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THE CUBIC DIRAC EQUATION: SMALL INITIAL DATA IN $H^1(\mathbb{R}^3)$

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ABSTRACT. We establish global well-posedness and scattering for the cubic Dirac equation for small data in the critical space $H^1(\mathbb{R}^3)$. The main ingredient is obtaining a sharp end-point Strichartz estimate for the Klein-Gordon equation. In a classical sense this fails and it is related to the failure of the endpoint Strichartz estimate for the wave equation in space dimension three. In this paper, systems of coordinate frames are constructed in which end-point Strichartz estimates are recovered and energy estimates are established.

1. INTRODUCTION AND MAIN RESULTS

For $m > 0$, consider the scalar homogeneous Klein-Gordon equation

$$\Box u(t,x) + m^2 u(t,x) = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{1.1}$$

A fundamental problem is the validity of Strichartz estimates for solutions of this equation. In the low frequency regime, the dispersive properties of the Klein-Gordon equation are similar to the Schrödinger equation, while in the high frequency regime they are similar to the wave equation (this will be detailed later in the paper). This hints at the range of available Strichartz estimates for (1.1).

In dimensions $n \geq 4$, it is known that all the Strichartz estimates including the end-point hold true both for the Schrödinger and the wave equation [14]. Therefore all the Strichartz estimates including the end-point hold true for the Klein-Gordon equation as well.

A major problem arises in dimension $n = 3$ since the endpoint Strichartz estimate $L_t^2 L_x^\infty$ fails for the wave equation due to the slow dispersion of type t^{-1}. On the positive side, the end-point Strichartz

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estimate $L_t^2 L_x^6$ holds true for the Schrödinger equation. Therefore, the problem one encounters for the Klein-Gordon equation is in the high frequency regime only.

Strichartz estimates lead to well-posedness results for various non-linear equations. The endpoint Strichartz estimate plays a crucial role in certain critical problems. The application discussed in this paper, the cubic Dirac equation, is such an example. In fact this equation motivated our research in the direction of obtaining a replacement for the false endpoint Strichartz estimate for (1.1).

In a future work we will address the same problem in two dimensions where the $L_t^2 L_x^\infty$ estimate fails for the Schrödinger equation and it is not even the correct end-point for the wave equation.

Throughout the rest of this paper the physical dimension is set to $n = 3$ and the mass is fixed to $m = 1$ in (1.1). By rescaling, estimates for any other $m \neq 0$ can be obtained. It is well-known that in the case of the wave equation,

$$\Box u = 0, u(0, x) = f_0(x), u_t(0, x) = f_1(x),$$

the end-point Strichartz estimate

$$\|u\|_{L_t^2 L_x^\infty} \lesssim \|\nabla f_0\|_{L^2} + \|f_1\|_{L^2} \tag{1.2}$$

does not hold true, see [26]. In fact it fails for any $P(D)u$ where $P(D)$ is a Fourier multiplier whose symbol lies in C_0^∞, vanishes near the origin and it is not identically zero, see [40]. In particular it fails for $P_k u$, where P_k is the standard Fourier multiplier localizing at frequency $|\xi| \approx 2^k$, see Subsection 1.1. As a consequence the estimate (1.2) cannot hold true for (1.1) either. To be more precise, the estimate (1.2) for $P_k u$ with a bound independent of k cannot be true. This obstruction comes as $k \to \infty$ where the symbol of the Klein-Gordon equation is essentially the same as the one for the wave equation.

An important observation needs to be made here. While for the wave equation (1.2) is false regardless on how much regularity is added to the right hand side, that is to f_0, f_1, some extra regularity fixes the estimate for the Klein-Gordon equation. To be more precise, if

$$(\Box + 1)u = 0, u(0, x) = f_0(x), u_t(0, x) = f_1(x),$$

the end-point Strichartz estimate

$$\|P_k u\|_{L_t^2 L_x^\infty} \lesssim_\epsilon 2^{(1+\epsilon)k} \|P_k f_0\|_{L^2} + 2^{\epsilon k} \|P_k f_1\|_{L^2}, \qquad k \geq 0, \tag{1.3}$$

holds true for any $\epsilon > 0$, see [22]. But this fails to be true for $\epsilon = 0$!

Our goal in this paper is to provide a lucrative replacement for (1.3) in the case $\epsilon = 0$ and for its inhomogeneous counterpart. This will done in adapted frames in Section 2.1, see Theorem 2.1. In applications to nonlinear problems, the end-point Strichartz estimate is used in conjunction with the energy estimate $L^\infty_t L^2_x$ to generate the bilinear $L^2_{t,x}$ estimate

$$\|u \cdot v\|_{L^2_{t,x}} \leq \|u\|_{L^2_t L^\infty_x} \|u\|_{L^\infty_t L^2_x}.$$

Since the $L^2 L^\infty$ estimate is generated in adapted frames, one has to derive energy estimates in similar frames in order to recoup the above $L^2_{t,x}$ bilinear estimate. We will provide this type of energy estimates in Subsection 2.2. In fact, the combination of the energy and the Strichartz estimate to a uniform L^2 estimate is only possible by using a null structure, see Subsection 3.2.

The use of adapted frames to generate a replacement for the missing $L^2_t L^\infty_x$ end-point Strichartz estimate was initiated by Tataru [41] in the context of the Wave Map problem. Another context in which such estimates were derived was the Schrödinger Map problem, see [1]. Our work is closer in spirit to the work of Tataru [41], although the geometry of the characteristic surface for the Klein-Gordon equation requires a more involved construction.

As an application, we study the cubic Dirac equation which we describe below. For $M > 0$, the cubic Dirac equation for the spinor field $\psi : \mathbb{R}^4 \to \mathbb{C}^4$ is given by

$$(-i\gamma^\mu \partial_\mu + M)\psi = \langle \gamma^0 \psi, \psi \rangle \psi, \tag{1.4}$$

where we use the summation convention. Here, $\gamma^\mu \in \mathbb{C}^{4\times 4}$ are the Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. The $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{C}^4$, hence $\langle \gamma^0 \psi, \psi \rangle = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2 \in \mathbb{R}$. It then follows that $\langle \gamma^0 \psi, \psi \rangle$ equals its conjugate which is written as $\bar{\psi}\psi = \psi^\dagger \gamma^0 \psi$, where $\bar{\psi} = \psi^\dagger \gamma^0$ and $\psi^\dagger$ is the conjugate transpose of ψ. The conclusion is that $\langle \gamma^0 \psi, \psi \rangle = \psi^\dagger \gamma^0 \psi$ and we made this point so as to avoid confusion between the two apparently different ways the nonlinear term appears in literature.

The matrices γ^μ satisfy the following properties

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2 g^{\alpha\beta} I_4, \qquad (g^{\alpha\beta}) = \mathrm{diag}(1, -1, -1, -1).$$

The physical background for this equation is provided in [9, 33]. Existence and stability of bound state solutions of (1.4) has been investigated in [36, 4, 25].

Using scaling arguments, it turns out that the problem becomes critical in $H^1(\mathbb{R}^3)$. Local well-posedness was obtained in $H^s(\mathbb{R}^3)$, $s > 1$ (subcritical range) in [8]. Global well-posedness and scattering was proved in [22] for small initial data in $H^s(\mathbb{R}^3)$, $s > 1$ as well as for small initial data in $H^1(\mathbb{R}^3)$ with some regularity in the angular variable in [21].

The main idea in the above mentioned papers is as follows. The linear part of the Dirac equation is closely related to a half-Klein-Gordon equation. In the subcritical case one can make use of the (1.3) with $\epsilon > 0$, while in the critical case certain spherically averaged versions (1.3) with $\epsilon = 0$ hold true, see [21, 13], which is similar to the Schrödinger case [38] in dimension $n = 2$.

Both of the above strategies reach their limitations when one considers the (1.4) with small but general data in $H^1(\mathbb{R}^3)$, cp. [22, p. 181, l. 1-5]. Using our strategy to fix (1.3) in the case $\epsilon = 0$ and the null structure exhibited by the nonlinearity we are able to prove the following result in the critical space:

Theorem 1.1. *The initial value problem associated to the cubic Dirac equation* (1.4) *is globally well-posed for small initial data in* $H^1(\mathbb{R}^3)$*. Moreover, small solutions scatter to free solutions for* $t \to \pm\infty$*.*

In addition, the result includes persistence of initial regularity, i.e. if $\psi(0) \in H^\sigma(\mathbb{R}^3)$ for some $\sigma \geq 1$, the solution $t \mapsto \psi(t)$ is a continuous curve in $H^\sigma(\mathbb{R}^3)$, which in the case $\sigma > 1$ is already known from the previous work [22].

In a future work we intend to address the initial value problem for the cubic Dirac equation in the critical space in space dimension $n = 2$.

For a subcritical result for the cubic Dirac equation in space dimension $n = 2$, see [29], for results in space dimension $n = 1$, see [23, 3]. Concerning nonlinear Klein-Gordon equations we refer the reader to [6, 17, 15, 31].

The plan for the paper is as follows. In the following subsection we introduce the main notation which will be used throughout the rest of the paper. In Section 2 we derive the major linear estimates of the paper: the end-point L^2L^∞ in frames in subsection 2.1 and the energy estimates in similar frames in subsection 2.2. The proofs of some of the decay estimates are postponed to Appendix A. In Section 3 we prepare the setup for the Dirac equation and unveil the null condition present in the nonlinearity. In Section 4 we introduce our function spaces, in Section 5 we prove useful bilinear estimates, which are applied in Section 6 to prove the main result concerning the cubic Dirac equation.

1.1. **Notation.** We define $A \prec B$ by $A \leq B - c$ for some absolute constant $c > 0$. Also, we define $A \ll B$ to be $A \leq dB$ for some absolute small constant $0 < d < 1$. Similarly, we define $A \lesssim B$ to be $A \leq eB$ for some absolute constant $e > 0$, and $A \approx B$ iff $A \lesssim B \lesssim A$.

Similar to [21], we set $\langle \xi \rangle_k := (2^{-2k} + |\xi|^2)^{\frac{1}{2}}$ for $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$, and we also write $\langle \xi \rangle := \langle \xi \rangle_0$. We note that $\langle \xi \rangle_k$ coincides with the euclidean norm of the vector $(\xi, 2^{-k}) \in \mathbb{R}^{n+1}$. Since the euclidean norm is a smooth function, homogeneous of degree 1, on $\mathbb{R}^{n+1} \setminus \{0\}$, we conclude that for all multi-indices $\beta \in \mathbb{N}_0^n$ there are $c_{\beta,n} > 0$, such that

$$\forall k \in \mathbb{Z},\ \xi \in \mathbb{R}^n: \quad |\partial_\xi^\beta \langle \xi \rangle_k| \leq c_{\beta,n} \langle \xi \rangle_k^{1-|\beta|}. \tag{1.5}$$

Thoughout the paper, let $\rho \in C_c^\infty(-2,2)$ be a fixed smooth, even, cutoff satisfying $\rho(s) = 1$ for $|s| \leq 1$ and $0 \leq \rho \leq 1$. For $k \in \mathbb{Z}$ we define $\chi_k : \mathbb{R}^n \to \mathbb{R}$, $\chi_k(y) := \rho(2^{-k}|y|) - \rho(2^{-k+1}|y|)$, such that $A_k := \operatorname{supp}(\chi_k) \subset \{y \in \mathbb{R}^n \colon 2^{k-1} \leq |y| \leq 2^{k+1}\}$. Let $\tilde{\chi}_k = \chi_{k-1} + \chi_k + \chi_{k+1}$ and $\tilde{A}_k := \operatorname{supp}(\tilde{\chi}_k)$.

We denote by $P_k = \chi_k(D)$ and $\tilde{P}_k = \tilde{\chi}_k(D)$. Note that $P_k \tilde{P}_k = \tilde{P}_k P_k = P_k$. Further, we define $\chi_{\leq k} = \sum_{l=-\infty}^{k} \chi_l$, $\chi_{>k} = 1 - \chi_{\leq k}$ as well as the corresponding operators $P_{\leq k} = \chi_{\leq k}(D)$ and $P_{>k} = \chi_{>k}(D)$.

We denote by $\mathcal{K}_l$ a collection of spherical caps of diameter 2^{-l} which provide a symmetric and finitely overlapping cover of the unit sphere $\mathbb{S}^2$. Let $\omega(\kappa)$ to be the "center" of κ and let Γ_κ be the cone generated by κ and the origin, in particular $\Gamma_\kappa \cap \mathbb{S}^2 = \kappa$.

For $M_1, M_2 \subset \mathbb{R}^n$ we set

$$\mathrm{d}(M_1, M_2) = \inf\{|x - y| \colon x \in M_1, y \in M_2\}.$$

Further, let η_κ be smooth partition of unity subordinate to the covering of $\mathbb{R}^3 \setminus \{0\}$ with the cones Γ_κ, such that each η_κ is supported in $2\Gamma_\kappa$ and is homogeneous of degree zero and satisfies

$$|\partial_\xi^\beta \eta_\kappa(\xi)| \leq C_\beta 2^{l|\beta|} |\xi|^{-\beta}, \quad |(\omega(\kappa) \cdot \nabla)^N \eta_\kappa(\xi)| \leq C_N |\xi|^{-N},$$

as described in detail in [34, Chapt. IX, §4.4 and formula (66)]. Let $\tilde{\eta}_\kappa$ with similar properties but slightly bigger support, such that $\tilde{\eta}_\kappa \eta_\kappa = 1$. We define $P_\kappa = \eta_\kappa(D)$, $\tilde{P}_\kappa = \tilde{\eta}_\kappa(D)$. With $P_{k,\kappa} := \eta_\kappa(D)\chi_k(D)$ and $\tilde{P}_{k,\kappa} := \tilde{\eta}_\kappa(D)\tilde{\chi}_k(D)$, we obtain the angular decomposition

$$P_k = \sum_{\kappa \in \mathcal{K}_l} P_{k,\kappa}$$

and $P_{k,\kappa}\tilde{P}_{k,\kappa} = \tilde{P}_{k,\kappa}P_{k,\kappa} = P_{k,\kappa}$. We further define $A_{k,\kappa} = \text{supp}(\eta_\kappa\chi_k)$ and $\tilde{A}_{k,\kappa} = \text{supp}(\tilde{\eta}_\kappa\tilde{\chi}_k)$.

We define $\widehat{Q_m^\pm u}(\tau,\xi) = \chi_m(\tau \mp \langle\xi\rangle)\widehat{u}(\tau,\xi)$, and $\widehat{Q_{\le m}^\pm u}(\tau,\xi) = \chi_{\le m}(\tau \mp \langle\xi\rangle)\widehat{u}(\tau,\xi)$. We also define $\tilde{Q}_m^\pm = Q_{m-1}^\pm + Q_m^\pm + Q_{m+1}^\pm$. We set $B_{k,m}^\pm$ to be the Fourier support of $Q_m^\pm$, and $\tilde{B}_{k,m}^\pm$ to be the Fourier support of $\tilde{Q}_m^\pm$. Additionally, we define $Q_{\prec m}^\pm = \sum_{l=-\infty}^{m-c} Q_l^\pm$ for a large integer $c > 0$, and $Q_{\succeq m}^\pm = I - Q_{\prec m}^\pm$. Given $k \in \mathbb{Z}$, and $\kappa \in \mathcal{K}_l$ for some $l \in \mathbb{N}$ we set $B_{k,\kappa}^\pm$ to be the Fourier-support of $Q_{\prec k-2l}^\pm P_{k,\kappa}$. Similarly we define $\tilde{B}_{k,\kappa}^\pm$.

Given an angle ω and a parameter λ we define the directions $\Theta_{\lambda,\omega} = \frac{1}{\sqrt{1+\lambda^2}}(\lambda,\omega)$, $\Theta_{\lambda,\omega}^\perp = \frac{1}{\sqrt{1+\lambda^2}}(-1,\lambda\omega)$ and the associated orthogonal co-ordinates $(t_\Theta, x_\Theta^1, x_\Theta')$

$$t_{\lambda,\omega} = (t,x)\cdot\Theta_{\lambda,\omega}, \quad x_{\lambda,\omega}^1 = (t,x)\cdot\Theta_{\lambda,\omega}^\perp.$$

If $\lambda = 1$ we obtain the characteristic directions (null co-ordinates) as in [41, p. 42] and [39, p. 476]. However, our analysis requires more flexibility in the choice of the frames with respect to which the estimates are available. With $\omega(\kappa)$ defined above and $\lambda(k) = (1+2^{-2k})^{-\frac{1}{2}}$ let $(t_{k,\kappa}^\pm, x_{k,\kappa}^\pm) = (t_{\pm\lambda(k),\omega(\kappa)}, x_{\pm\lambda(k),\omega(\kappa)})$.

For $1 \le p,q \le \infty$ we use the spaces $L_t^pL_x^q$ of all equivalence classes of measurable (weak-$*$-measurable if $q=\infty$) functions $f:\mathbb{R}\to L^q(\mathbb{R}^3)$ such that the norm

$$\|f\|_{L^pL^q} = \|t\mapsto \|f(t)\|_{L^q(\mathbb{R}^3)}\|_{L^p(\mathbb{R})}$$

is finite.

2. Linear estimates

The decay rates of solutions to the linear wave equation and Klein-Gordon equation have been analyzed e.g. in [42, 30, 37, 27, 32, 16, 10, 2, 24], see also the references therein. From the harmonic analysis point of view, the decay is determined by the curvature properties of the characteristic sets. In particular, we refer the reader to [28, Section 2.5] for a detailed discussion of decay and Strichartz estimates in the context of the Klein-Gordon equation.

For convenience, we set $m = 1$ in the Klein-Gordon equation (1.1). By rescaling our analysis extends to (1.1) with any $m \neq 0$. With $m = 1$, the solution is given by

$$u(t) = \frac{1}{2}(e^{it\langle D\rangle} + e^{-it\langle D\rangle})u_0 + \frac{1}{2i}(e^{it\langle D\rangle} - e^{-it\langle D\rangle})\frac{u_1}{\langle D\rangle}. \tag{2.1}$$

where $\langle D\rangle$ is the Fourier multiplier with symbol $\langle\xi\rangle$. It then becomes clear that the key operator to study is $e^{\pm it\langle D\rangle}$. To keep things simple, we work all estimates for the $+$ sign choice, that is for $e^{it\langle D\rangle}$. The estimates for $e^{-it\langle D\rangle}$ are obtained in a similar way by simply reversing time in the estimates for $e^{it\langle D\rangle}$.

2.1. **End-point L^2L^∞ type Strichartz estimate.** Our main result in this section provides the end-point Strichartz estimates available for functions localized in frequency.

Theorem 2.1. i) *For all $k\lesssim 1$ and $f\in L^2(\mathbb{R}^3)$ satisfying* $\operatorname{supp}(\widehat{f})\subset \tilde{A}_k$,

$$\|e^{it\langle D\rangle}f\|_{L^2_tL^\infty_x}\lesssim 2^{\frac{k}{2}}\|f\|_{L^2} \tag{2.2}$$

ii) *For all $k\gtrsim 1$, $\kappa\in\mathcal{K}_k$ and $f\in L^2(\mathbb{R}^3)$ satisfying* $\operatorname{supp}(\widehat{f})\subset\tilde{A}_{k,\kappa}$,

$$2^{-k}\|e^{it\langle D\rangle}f\|_{L^2_tL^\infty_x}+\|e^{it\langle D\rangle}f\|_{L^2_{t_{k,\kappa}}L^\infty_{x_{k,\kappa}}}\lesssim\|f\|_{L^2}. \tag{2.3}$$

iii) *For all $k\gtrsim 1$, $1\le l\le k$, $\kappa_1\in\mathcal{K}_l$ and $f\in L^2(\mathbb{R}^3)$ satisfying* $\operatorname{supp}(\widehat{f})\subset\tilde{A}_{k,\kappa_1}$,

$$\sum_{\kappa\in\mathcal{K}_k}\|e^{it\langle D\rangle}\tilde{P}_\kappa f\|_{L^2_{t_{k,\kappa}}L^\infty_{x_{k,\kappa}}}\lesssim 2^{k-l}\|f\|_{L^2}. \tag{2.4}$$

Part i) claims that for the low frequencies the end-point Strichartz estimates holds in a standard fashion. Given that in that regime the evolution is Schrödinger-like, the correct end-point would be $L^2_tL^6_x$ from which the estimate (2.2) can be obtained using the Sobolev embedding theorem.

In (2.3) we reveal the main Strichartz estimates in high frequencies. If we localize $\hat{f}$ in the angular variable at scale 2^{-k} we obtain two Strichartz estimates. The standard one $L^2_tL^\infty_x$ is obtained without any logarithmic loss, which would be the case in the absence of angular localization. The Strichartz estimate in characteristic coordinates is better adapted to the direction in which the waves propagate and hence it comes with a much better prefactor. The other key advantage that the Strichartz estimate in characteristic coordinates has is revealed in (2.4) where at each scale (larger than 2^{-k}) of angular localization we obtain the l^1 structure on pieces measured in L^2L^∞ in characteristic coordinates. In particular when no angular localization is present ($l=0$) one obtains a replacement of the missing end-point $L^2_tL^\infty_x$ with the correct factor of 2^k. The use of so many frames to capture the L^2L^∞ estimate will require more flexibility in the corresponding energy estimates.

The rest of this subsection is devoted to the proof of Theorem 2.1. Define the kernel

$$K_k(t,x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{it\langle\xi\rangle} \tilde{\chi}_k^2(|\xi|)\, d\xi. \tag{2.5}$$

We identify $L_t^2 L_x^\infty$ as the dual of $L_t^2 L_x^1$, see [7, Theorem 8.20.3] and the definitions in Subsection 1.1. Through the usual TT^* argument, see e.g. [11, Lemma 2.1], the low frequency case (2.2) follows from

$$\|K_k * g\|_{L_t^2 L_x^\infty} \lesssim 2^k \|g\|_{L_t^2 L_x^1}. \tag{2.6}$$

The following result can be found in [28, Corollaries 2.36 and 2.38], it can be traced back to [10, 2, 24].

Lemma 2.2. i) *For all $k \in \mathbb{Z}$, $k \lesssim 1$, we have*

$$|K_k(t,x)| \lesssim 2^{3k}(1 + 2^{2k}|(t,x)|)^{-\frac{3}{2}}. \tag{2.7}$$

ii) *For all $k \in \mathbb{Z}, k \gtrsim 1$ we have*

$$|K_k(t,x)| \lesssim 2^{3k}(1 + 2^k|(t,x)|)^{-1} \min(1, (1 + 2^k|(t,x)|)^{-\frac{1}{2}} 2^k)) \tag{2.8}$$

Estimate (2.7) easily follows from the classical result on Fourier transforms of surface carried measures [34, p. 348, Theorem 1]. The idea behind estimate (2.8) is the following: After rescaling to unit frequency size, K_k essentially is the (inverse) Fourier transform of an approximately cone-like surface with 2 principal curvatures which are uniformly bounded from below, cp. [20] or [34, p. 361], which implies (2.8) for $|(t,x)| \le 2^k$. By taking into account that the surface actually has n non-vanishing principal curvatures, one of which is of size 2^{-2k}, cp. [34, p. 360] or [12, Section 7] one obtains (2.8) for $|(t,x)| \ge 2^k$. For convenience of the reader, we provide a proof in Appendix A.

Using the above Lemma, we obtain $\|K_k\|_{L_t^1 L_x^\infty} \lesssim 2^k$ from which (2.6) and therefore (2.2) follows. We are now left with completing the most interesting part of the argument, namely the proof of (2.3). Through the TT^* argument, the estimate (2.3) is reduced to the following

$$\|K_{k,\kappa} * g\|_{L_t^2 L_x^\infty} \lesssim 2^{2k} \|g\|_{L_t^2 L_x^1}, \quad \|K_{k,\kappa} * g\|_{L^2_{t_{k,\kappa}} L^\infty_{x_{k,\kappa}}} \lesssim \|g\|_{L^2_{t_{k,\kappa}} L^1_{x_{k,\kappa}}}$$

for $\kappa \in \mathcal{K}_k$, where

$$K_{k,\kappa}(t,x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{it\langle\xi\rangle} \tilde{\chi}_k^2(|\xi|) \tilde{\eta}_\kappa(\xi)\, d\xi. \tag{2.9}$$

Again by Young's inequality, this reduces to showing that

$$2^{-2k}\|K_{k,\kappa}\|_{L_t^1 L_x^\infty} + \|K_{k,\kappa}\|_{L^1_{t_{k,\kappa}} L^\infty_{x_{k,\kappa}}} \lesssim 1. \tag{2.10}$$

This estimate follows from the Proposition below.

Proposition 2.3. *For all* $k \in \mathbb{Z}, k \gtrsim 1$, $\kappa \in \mathcal{K}_k$ *and all* (t,x)

$$|K_{k,\kappa}(t,x)| \lesssim 2^k(1+2^{-k}|(t,x)|)^{-\frac{3}{2}}. \tag{2.11}$$

In addition, for $N = 1, 2$, *we have the following:*

$$|K_{k,\kappa}(t,x)| \lesssim_N 2^k(1+2^k|t_{k,\kappa}|)^{-N}, \quad \text{if } |t_{k,\kappa}| \gg 2^{-2k}|(t,x)|. \tag{2.12}$$

We remark that (2.12) holds with any $N \in \mathbb{N}$, but as stated it suffices for our purposes.

Before turning to the proof of this Proposition, we show how (2.10) follows from the statements above. The first part of (2.10) is straightforward:

$$\begin{aligned}\|K_{k,\kappa}\|_{L^1_tL^\infty_x} &\lesssim \int_{-2^k}^{2^k} \|K_{k,\kappa}\|_{L^\infty_x} dt + \int_{-\infty}^{-2^k} \|K_{k,\kappa}\|_{L^\infty_x} dt + \int_{2^k}^{\infty} \|K_{k,\kappa}\|_{L^\infty_x} dt \\ &\lesssim 2^{2k} + \int_{2^k}^{\infty} 2^{\frac{5k}{2}} t^{-\frac{3}{2}} dt \lesssim 2^{2k}.\end{aligned}$$

For the second part of (2.10), we want to understand $\|K_{k,\kappa}(t_{k,\kappa},\cdot)\|_{L^\infty_{x_{k,\kappa}}}$ for some fixed $t_{k,\kappa}$ such that $|t_{k,\kappa}| \approx 2^j$ with $j \geq -k$. If the point $(t_{k,\kappa}, x_{k,\kappa})$ belongs to the region $|t_{k,\kappa}| \gg 2^{-2k}|(t,x)|$, then we have the bound $|K_k(t,x)| \lesssim 2^k(2^{k+j})^{-2}$, while if it belongs to the region $|t_{k,\kappa}| \lesssim 2^{-2k}|(t,x)|$ then we have the bound $|K_k(t,x)| \lesssim 2^k(2^{-k}|(t,x)|)^{-\frac{3}{2}} \lesssim 2^k(2^{k+j})^{-\frac{3}{2}}$. The conclusion is that if $|t_{k,\kappa}| \approx 2^j$ with $j \geq -k$ then

$$\|K_{k,\kappa}(t_{k,\kappa},\cdot)\|_{L^\infty_{x_{k,\kappa}}} \lesssim 2^k 2^{-\frac{3}{2}(k+j)}.$$

From this we estimate

$$\begin{aligned}\|K_{k,\kappa}\|_{L^1_{t_{k,\kappa}}L^\infty_{x_{k,\kappa}}} &\lesssim \int_0^{2^{-k}} 2^k dt_{k,\kappa} + \sum_{j=-k}^{\infty} \int_{2^j}^{2^{j+1}} \|K_{k,\kappa}(t_{k,\kappa},\cdot)\|_{L^\infty_{x_{k,\kappa}}} dt_{k,\kappa} \\ &\lesssim 1 + \sum_{j=-k}^{\infty} 2^{k+j} 2^{-\frac{3}{2}(k+j)} \lesssim 1\end{aligned}$$

and this finishes the argument for the second part of (2.10). With this, the proof of (2.3) is complete.

Proof of Proposition 2.3. We begin with the proof of (2.11). If $|(t,x)| \lesssim 2^k$ then the statement follows directly by using that size of the support of the integration has volume $\approx 2^k$. If $|(t,x)| \gtrsim 2^k$, then the estimate follows from (2.8) and Young's inequality.

It remains to provide a proof of (2.12). For compactness of notation, we write $\lambda = \lambda(k)$, $\omega = \omega(\kappa)$. By rescaling it suffices to consider

$$B_{k,\kappa}(s,y) := \int_{\mathbb{R}^3} e^{iy\cdot\xi + is\langle\xi\rangle_k} \tilde{\chi}_1^2(|\xi|)\tilde{\eta}_\kappa(\xi) d\xi$$

and establish, for $N=1,2$

$$|B_{k,\kappa}(s,y)| \lesssim_N 2^{-2k}(1+|s_{\lambda,\omega}|)^{-N}, \qquad |s_{\lambda,\omega}| \gg 2^{-2k}|(s,y)|. \tag{2.13}$$

If $|s_{\lambda,\omega}| \lesssim 1$, the estimate follows from the fact that the support of the integration has volume $\approx 2^{-2k}$. For the rest of the argument we work under the hypothesis $|s_{\lambda,\omega}| > 1$.

We write $(s,y) = \beta(r,z)$ with $\beta = |(s,y)|$ and the integral above becomes

$$C_{k,\kappa}(\beta,r,z) = \int_{\mathbb{R}^3} e^{i\beta\phi(r,z,\xi)} \tilde{\chi}_1^2(|\xi|)\tilde{\eta}_\kappa(\xi) d\xi$$

with $\phi(r,z,\xi) = z\cdot\xi + r\langle\xi\rangle_k$. The phase function satisfies $\partial_{\xi_j}\phi(r,z,\xi) = z_j + r\frac{\xi_j}{\langle\xi\rangle_k}$. Define $\partial_\omega = \omega\cdot\nabla_\xi$, $d_\phi := \frac{1}{\partial_\omega\phi}\partial_\omega$ and $d_\phi^* := -\partial_\omega\left(\frac{\cdot}{\partial_\omega\phi}\right)$. Integrating by parts, we compute

$$\begin{aligned}\int_{\mathbb{R}^3} e^{i\beta\phi(r,z,\xi)} \tilde{\chi}_1^2(|\xi|)\tilde{\eta}_\kappa(\xi) d\xi &= \int_{\mathbb{R}^3} \frac{1}{(i\beta)^N} d_\phi^N(e^{i\beta\phi(r,z,\xi)})\tilde{\chi}_1^2(|\xi|)\tilde{\eta}_\kappa(\xi) d\xi \\ &= (i\beta)^{-N}\int_{\mathbb{R}^3} e^{i\beta\phi(r,z,\xi)}(d_\phi^*)^N(\tilde{\chi}_1^2(|\xi|)\tilde{\eta}_\kappa(\xi)) d\xi\end{aligned} \tag{2.14}$$

For $\zeta(\xi) = \tilde{\chi}_1^2(|\xi|)\tilde{\eta}_\kappa(\xi)$ we claim the bounds

$$|(d_\phi^*)^N(\zeta)(\xi)| \lesssim_N \left(\frac{\beta}{|s_{\lambda,\omega}|}\right)^N, \qquad N=1,2. \tag{2.15}$$

Since the support of the integration above has volume $\approx 2^{-2k}$, (2.13) follows from (2.14) and (2.15). Hence all that is left is an argument for (2.15).

Let $N=1$. Let $(\omega,\omega_2,\omega_3)$ be an orthonormal basis of $\mathbb{R}^3$. For ξ in the support of the integration we have

$$\frac{\xi}{|\xi|} = \omega + \mathcal{O}(2^{-k})\omega_2 + \mathcal{O}(2^{-k})\omega_3 + \mathcal{O}(2^{-2k}), \qquad \frac{|\xi|}{\langle\xi\rangle_k} = \lambda + \mathcal{O}(2^{-2k}),$$

where we recall that $\lambda = \lambda(k) = \frac{1}{\sqrt{1+2^{-2k}}}$. Using these facts we obtain

$$\begin{aligned}\partial_\omega\phi &= \omega\cdot(z + r\frac{\xi}{\langle\xi\rangle_k}) = \omega\cdot z + r\frac{|\xi|}{\langle\xi\rangle_k} + \mathcal{O}(2^{-2k}) \\ &= \omega\cdot z + r\lambda + \mathcal{O}(2^{-2k}) = \frac{s_{\lambda,\omega}}{\beta\sqrt{1+\lambda^2}} + \mathcal{O}(2^{-2k})\end{aligned}$$

Therefore we obtain $|\partial_\omega\phi| \gtrsim \frac{|s_{\lambda,\omega}|}{\beta} \gg 2^{-2k}$. In particular it follows that

$$|\frac{\partial_\omega\zeta}{\partial_\omega\phi}| \lesssim \frac{\beta}{|s_{\lambda,\omega}|}. \tag{2.16}$$

where we used that $|\partial_\omega \zeta| \lesssim 1$. In addition, we have

$$\partial_\omega^2 \phi(\xi) = \partial_\omega\Big(r\frac{\omega\cdot\xi}{\langle\xi\rangle_k}\Big) = r\left(\frac{\omega\cdot\omega}{\langle\xi\rangle_k} - \frac{(\omega\cdot\xi)^2}{\langle\xi\rangle_k^3}\right) = \frac{r}{\langle\xi\rangle_k}\left(1-(\frac{\omega\cdot\xi}{\langle\xi\rangle_k})^2\right)$$

from which, using the above arguments, we conclude that in the domain of integration we have $|\partial_\omega^2\phi| \lesssim 2^{-2k}$. This allows us to estimate

$$|\partial_\omega\Big(\frac{1}{\partial_\omega\phi}\Big)| \lesssim \frac{2^{-2k}}{|\partial_\omega\phi|^2} \lesssim \frac{1}{|\partial_\omega\phi|} \lesssim \frac{\beta}{|s_{\lambda,\omega}|}.$$

From this and (2.16) we obtain (2.13) for $N=1$. Now let $N=2$ and compute

$$(d_\phi^*)^2\zeta = \partial_\omega\Big(\frac{1}{\partial_\omega\phi}\partial_\omega\frac{\zeta}{\partial_\omega\phi}\Big) = \frac{\partial_\omega^2\zeta}{(\partial_\omega\phi)^2} - 3\frac{\partial_\omega\zeta\partial_\omega^2\phi}{(\partial_\omega\phi)^3} - \frac{\zeta\partial_\omega^3\phi}{(\partial_\omega\phi)^3} + 3\frac{\zeta(\partial_\omega^2\phi)^2}{(\partial_\omega\phi)^4}$$

We compute

$$\partial_\omega^3\phi = \frac{3r}{\langle\xi\rangle_k^5}\Big((\omega\cdot\xi)^3 - (\omega\cdot\xi)\langle\xi\rangle_k^2\Big) = \mathcal{O}(2^{-2k}).$$

Recalling that $|\partial_\omega\phi| \gtrsim \frac{|s_{\lambda,\omega}|}{\beta} \gg 2^{-2k}$, $|\partial_\omega^2\phi| \lesssim 2^{-2k}$ and $|\partial_\omega^N\zeta| \lesssim_N 1$ we conclude that

$$|(d_\phi^*)^N| \lesssim \frac{\beta^2}{|s_{\lambda,\omega}|^2} + \frac{2^{-2k}\beta^3}{|s_{\lambda,\omega}|^3} + \frac{2^{-4k}\beta^4}{|s_{\lambda,\omega}|^4} \lesssim \frac{\beta^2}{|s_{\lambda,\omega}|^2}.$$

This finishes the proof of (2.15) and, in turn, the proof of (2.12). □

We end this section with the proof of (2.4). Since there are $\approx 2^{2(k-l)}$ caps $\kappa\in\mathcal{K}_k$ such that $P_\kappa f \neq 0$, we obtain from (2.3)

$$\sum_{\kappa\in\mathcal{K}_k} \|e^{it\langle D\rangle}\tilde P_\kappa f\|_{L^2_{t_{k,\kappa}}L^\infty_{x_{k,\kappa}}} \lesssim 2^{k-l}\left(\sum_{\kappa\in\mathcal{K}_k}\|e^{it\langle D\rangle}\tilde P_\kappa f\|^2_{L^2_{t_{k,\kappa}}L^\infty_{x_{k,\kappa}}}\right)^{\frac12}$$
$$\lesssim 2^{k-l}\left(\sum_{\kappa\in\mathcal{K}_k}\|\tilde P_\kappa f\|^2_{L^2_x}\right)^{\frac12} \lesssim 2^{k-l}\|f\|_{L^2_x}.$$

2.2. **Energy estimates in the** (λ,ω) **frames.** Given a pair (λ,ω) with $\lambda\in\mathbb{R}$ and $\omega\in\mathbb{S}^2$ we recall that we defined

$$\Theta_{\lambda,\omega} = \frac{1}{\sqrt{1+\lambda^2}}(\lambda,\omega), \quad \Theta^\perp_{\lambda,\omega} = \frac{1}{\sqrt{1+\lambda^2}}(-1,\lambda\omega)$$

to be two orthogonal vectors in $\mathbb{R}^4$. This can be completed to an orthonormal basis in $\mathbb{R}^4$ by considering any two vectors $\Theta_{2,\omega} = (0,\omega_2)$ and $\Theta_{3,\omega} = (0,\omega_3)$ such that $(\omega,\omega_2,\omega_3)$ form a positively oriented orthonormal basis in $\mathbb{R}^3$.

With respect to this basis, understanding the vectors $\Theta_{\lambda,\omega}$, $\Theta_{\lambda,\omega}^{\perp}$, $\Theta_{2,\omega}$, $\Theta_{3,\omega}$ as column vectors, we introduce the new coordinates $t_{\lambda,\omega}, x_{\lambda,\omega}$, with $x_{\lambda,\omega}=(x_{\lambda,\omega}^1, x_\omega^2, x_\omega^3)$, defined by

$$\begin{pmatrix} t_{\lambda,\omega} \\ x_{\lambda,\omega}^1 \\ x_\omega^2 \\ x_\omega^3 \end{pmatrix} = \begin{pmatrix} \Theta_{\lambda,\omega} & \Theta_{\lambda,\omega}^{\perp} & \Theta_{2,\omega} & \Theta_{3,\omega} \end{pmatrix}^t \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

In many of the computations we will write $x'_\omega=(x_\omega^2, x_\omega^3)$.

We denote by $(\tau_{\lambda,\omega}, \xi_{\lambda,\omega})$ the corresponding Fourier variables which are given by

$$\begin{pmatrix} \tau_{\lambda,\omega} \\ \xi_{\lambda,\omega}^1 \\ \xi_\omega^2 \\ \xi_\omega^3 \end{pmatrix} = \begin{pmatrix} \Theta_{\lambda,\omega} & \Theta_{\lambda,\omega}^{\perp} & \Theta_{2,\omega} & \Theta_{3,\omega} \end{pmatrix} \begin{pmatrix} \tau \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

where we also write $\xi'_\omega=(\xi_\omega^2,\xi_\omega^3)$. In the following theorem and its proof we set $B_{k,\kappa}=B_{k,\kappa}^+$ and $\tilde{B}_{k,\kappa}=\tilde{B}_{k,\kappa}^+$.

Theorem 2.4. *Let $k,j\geq 100$, $0\leq l\leq \min(j,k)-10$ and $\kappa\in\mathcal{K}_l$. Let $\Theta_{\lambda,\omega}$ be a direction with $\lambda=\lambda(j)=\frac{1}{\sqrt{1+2^{-2j}}}$, and we assume $\alpha=\mathrm{d}(\omega,\kappa)$ satisfies $2^{-3-l}\leq\alpha\leq 2^{3-l}$.*

i) *If $f\in L^2(\mathbb{R}^3)$ has the property that $\hat{f}$ is supported in $A_{k,\kappa}$, then for the free solution the following holds true*

$$\alpha\|e^{it\langle D\rangle}f\|_{L^\infty_{t_{\lambda,\omega}}L^2_{x_{\lambda,\omega}}}\lesssim\|f\|_{L^2}. \tag{2.17}$$

ii) *Let $\hat{g}$ be supported in the set $B_{k,\kappa}$ and $g\in L^1_{t_{\lambda,\omega}}L^2_{x_{\lambda,\omega}}$. Then, the solution u of the inhomogeneous equation*

$$(i\partial_t+\langle D\rangle)u=g,\quad u(0)=0, \tag{2.18}$$

satisfies the estimate

$$\alpha\|u\|_{L^\infty_{t_{\lambda,\omega}}L^2_{x_{\lambda,\omega}}}\lesssim\alpha^{-1}\|g\|_{L^1_{t_{\lambda,\omega}}L^2_{x_{\lambda,\omega}}}. \tag{2.19}$$

iii) *Under the hypothesis of Part ii) the solution u can be written as*

$$u(t)=e^{it\langle D\rangle}\tilde{v}_0+\int_{-\infty}^{\infty}u_s(t)\chi_{t_{\lambda,\omega}>s}ds \tag{2.20}$$

where $u_s(t)=e^{it\langle D\rangle}v_s$ (homogeneous solution in the original coordinates) and

$$\|\tilde{v}_0\|_{L^2_x}+\int_{-\infty}^{\infty}\|v_s\|_{L^2_x}ds\lesssim\alpha^{-1}\|g\|_{L^1_{t_{\lambda,\omega}}L^2_{x_{\lambda,\omega}}}. \tag{2.21}$$

In addition $\hat{v}_s$ *and* $\hat{\tilde{v}}_0$ *are supported in* $\tilde{A}_{k,\kappa}$.

Proof. i) The space-time Fourier of $w(t,x) = e^{it\langle D\rangle} f(x)$ is given by the distribution $\mathcal{F}w = \hat{f} d\sigma$ where $d\sigma(\tau,\xi) = \delta_{\tau=\sqrt{|\xi|^2+1}}$ is comparable with the standard measure on the surface $\tau = \sqrt{|\xi|^2+1}$. We change the variables: $(\tau,\xi) \to (\tau_{\lambda,\omega}, \xi_{\lambda,\omega})$ where $\xi_{\lambda,\omega} = (\xi^1_{\lambda,\omega}, \xi'_{\lambda,\omega})$. The goal is to write $\hat{f} d\sigma = F \delta_{\tau_{\lambda,\omega}=h(\xi_{\lambda,\omega})}$. We then would have

$$\|F\|_{L^2_{\xi_{\lambda,\omega}}} \lesssim (1 + \|\nabla h\|_{L^\infty})^{\frac{1}{2}} \|f\|_{L^2} \tag{2.22}$$

where the L^∞ norms is taken on the support of F.

The equation of the characteristic surface $\tau = \sqrt{|\xi|^2+1}$ can be rewritten as

$$\tau^2 - |\xi|^2 - 1 = 0.$$

In the new frame this takes the form

$$\frac{1}{\lambda^2+1}(\lambda\tau_{\lambda,\omega} - \xi^1_{\lambda,\omega})^2 - \frac{1}{\lambda^2+1}(\tau_{\lambda,\omega} + \lambda\xi^1_{\lambda,\omega})^2 - |\xi'_{\lambda,\omega}|^2 - 1 = 0.$$

We solve this equation for $\tau_{\lambda,\omega}$, hence we rewrite it as follows

$$\frac{\lambda^2-1}{\lambda^2+1}(\tau_{\lambda,\omega})^2 - \frac{4\lambda}{\lambda^2+1}\tau_{\lambda,\omega}\xi^1_{\lambda,\omega} + \frac{1-\lambda^2}{\lambda^2+1}(\xi^1_{\lambda,\omega})^2 - |\xi'_{\lambda,\omega}|^2 - 1 = 0. \tag{2.23}$$

The solutions of this quadratic equation are given by

$$\tau_{\lambda,\omega} = h^\pm(\xi_{\lambda,\omega}) = \frac{2\lambda\xi^1_{\lambda,\omega} \pm \sqrt{(\lambda^2+1)^2(\xi^1_{\lambda,\omega})^2 + (\lambda^4-1)(|\xi'_{\lambda,\omega}|^2+1)}}{\lambda^2-1}. \tag{2.24}$$

We will identify which one of the two solutions is the correct one. The positivity of the discriminant $\Delta_{\lambda,\omega} = (\lambda^2+1)^2(\xi^1_{\lambda,\omega})^2 + (\lambda^4-1)(|\xi'_{\lambda,\omega}|^2+1)$ is implicit, as we know a priori that (2.23) has at least one solution. We will come back shortly to these issues. We continue with the following computation:

$$\begin{aligned}
\frac{\partial h^\pm}{\partial \xi^1_{\lambda,\omega}} &= \frac{1}{\lambda^2-1}\Big(2\lambda + \frac{(\lambda^2+1)^2\xi^1_{\lambda,\omega}}{\pm\sqrt{(\lambda^2+1)^2(\xi^1_{\lambda,\omega})^2 + (\lambda^4-1)(|\xi'_{\lambda,\omega}|^2+1)}}\Big) \\
&= \frac{1}{\lambda^2-1}\Big(2\lambda + \frac{(\lambda^2+1)^2\xi^1_{\lambda,\omega}}{(\lambda^2-1)\tau_{\lambda,\omega} - 2\lambda\xi^1_{\lambda,\omega}}\Big) \\
&= \frac{2\lambda\tau_{\lambda,\omega} + (\lambda^2-1)\xi^1_{\lambda,\omega}}{(\lambda^2-1)\tau_{\lambda,\omega} - 2\lambda\xi^1_{\lambda,\omega}} \\
&= -\frac{\xi^1_{\lambda,-\omega}}{\tau_{\lambda,-\omega}}
\end{aligned}$$

In a similar manner we obtain $\nabla_{\xi'_\omega} h^\pm = (\lambda^2+1)\frac{\xi'_{\lambda,\omega}}{\tau_{\lambda,-\omega}}$, from which, using (2.22), it follows

$$\|e^{it\langle D\rangle} f\|_{L^\infty_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}} \lesssim \left(1+\sup_{\xi\in A_{k,\kappa}} \frac{2^k}{|\tau_{\lambda,-\omega}|}\right)^{\frac12} \|f\|_{L^2}. \tag{2.25}$$

To finish the argument we need a lower bound for $|\tau_{\lambda,-\omega}|$. We provide below lower bounds for $\Delta_{\lambda,\omega}$ and $\tau_{\lambda,-\omega}$ for $(\tau,\xi)\in B_{k,\kappa}$, as these more general bounds are needed in Part ii).

For $(\tau,\xi)\in B_{k,\kappa}$ it holds that $\tau-\sqrt{|\xi|^2+1}=\epsilon(\tau,\xi)$ with $|\epsilon(\tau,\xi)|\leq 2^{k-2l-10}$, hence

$$\begin{aligned}\tau_{\lambda,-\omega} &= \lambda\tau-\xi\cdot\omega = \lambda\sqrt{|\xi|^2+1}+\lambda\epsilon-\xi\cdot\omega\\ &= |\xi|\Big(\lambda\sqrt{1+|\xi|^{-2}}+\frac{\lambda\epsilon}{|\xi|}-\frac{\xi\cdot\omega}{|\xi|}\Big)\end{aligned}$$

Given the hypothesis of the Theorem, we obtain $1-2^{-2l-6}\leq\frac{\xi\cdot\omega}{|\xi|}\leq 1-2^{-2l+6}$, $\frac{|\lambda\epsilon|}{|\xi|}\leq 2^{-2l-8}$, and $|\lambda\sqrt{1+|\xi|^{-2}}-1|\leq 2^{-2\min(j,k)+2}$. Thus we conclude that $\tau_{\lambda,-\omega}\approx 2^k\alpha^2$ and $\tau_{\lambda,-\omega}\geq 2^{k-2}\alpha^2$.

In particular, using (2.25) we obtain (2.17). Since the solutions in (2.24) can be recast in the form $\tau_{\lambda,-\omega}=\pm\sqrt{\Delta_{\lambda,\omega}}$ and we just proved that $\tau_{\lambda,-\omega}>0$ in $B_{k,\kappa}$, it follows that the solutions h^+ in (2.24) correspond to the choice of the surface $\tau=\sqrt{|\xi|^2+1}$.

We now continue with the more general bounds for $\Delta_{\lambda,\omega}$ in the set $B_{k,\kappa}$. Since $|\tau-\langle\xi\rangle|\leq 2^{k-10}\alpha^2$ hence $|\tau^2-|\xi|^2-1|\lesssim 2^{2k-8}\alpha^2$ or equivalently, $\tau^2-|\xi|^2-1=\epsilon(\tau,\xi)$ with $|\epsilon(\tau,\xi)|\lesssim 2^{2k-8}\alpha^2$. We rewrite the equation in characteristic coordinates as above, to obtain

$$\tau^2_{\lambda,-\omega}=\Delta_{\lambda,\omega}+(1-\lambda^4)\epsilon$$

We have already shown that $\tau_{\lambda,-\omega}\geq 2^{k-2}\alpha^2$ and since $|(1-\lambda^4)\epsilon|\leq 2^{2k-6}\alpha^2|1-\lambda|\leq 2^{2k-6}\alpha^4$, it follows that $\Delta_{\lambda,\omega}\geq 2^{2k-4}\alpha^4$ in $B_{k,\kappa}$. A similar argument proves $\Delta_{\lambda,\omega}\approx 2^{2k}\alpha^4$ in $B_{k,\kappa}$.

ii) On the Fourier side the inhomogeneous problem (2.18) becomes

$$(-\tau+\langle\xi\rangle)\hat{u}=\hat{g}$$

which we rewrite as follows

$$(\tau^2-|\xi|^2-1)\hat{u}=(-\tau-\langle\xi\rangle)\hat{g}:=\hat{G}.$$

Due to the localization in $B_{k,\kappa}$ it follows that $\hat{G}=a\hat{g}$ where

$$a(\tau,\xi)=(-\tau-\langle\xi\rangle)\tilde{\chi}_k(\xi)\tilde{\eta}_\kappa\tilde{\chi}_{\leq k-2l}(\tau-\langle\xi\rangle)$$

has the property $\|\mathcal{F}_{t,x}^{-1}a\|_{L^1_{t,x}} \lesssim 2^k$. From this it follows that

$$\|G\|_{L^1_{t_{\lambda,\omega}}L^2_{x_{\lambda,\omega}}} \lesssim 2^k \|g\|_{L^1_{t_{\lambda,\omega}}L^2_{x_{\lambda,\omega}}} \tag{2.26}$$

In the new coordinates the equation above becomes

$$\frac{\lambda^2-1}{\lambda^2+1}(\tau_{\lambda,\omega} - h^+(\xi_{\lambda,\omega}))(\tau_{\lambda,\omega} - h^-(\xi_{\lambda,\omega}))\hat{u} = \hat{G}$$

where $h^\pm(\xi_{\lambda,\omega})$ are the two roots in (2.24) of the quadratic equation (2.23). We have

$$\begin{aligned}&|(\lambda^2-1)(\tau_{\lambda,\omega} - h^\pm(\xi_{\lambda,\omega}))| = |(\lambda^2-1)\tau_{\lambda,\omega} - 2\lambda\xi^1_{\lambda,\omega} \pm \sqrt{\Delta_{\lambda,\omega}}| \\ = &|(\lambda^2+1)\tau_{\lambda,-\omega} \pm \sqrt{\Delta_{\lambda,\omega}}|\end{aligned}$$

From part i) we have that $|(\lambda^2+1)\tau_{\lambda,\omega} + \sqrt{\Delta_{\lambda,\omega}}| \approx 2^k\alpha^2$ in $B_{k,\kappa}$. We then rewrite the equation above as follows

$$(\tau_{\lambda,\omega} - h^-(\xi_{\lambda,\omega}))\hat{u} = m^{-1}\tilde{\chi}_{B_{k,\kappa}}\hat{G}$$

where $m(\tau_{\lambda,\omega},\xi_{\lambda,\omega}) = \frac{1-\lambda^2}{1+\lambda^2}(\tau_{\lambda,\omega} - h^+(\xi_{\lambda,\omega}))$ and $\tilde{\chi}_{B_{k,\kappa}}$ is a smooth function which equals 1 in $B_{k,\kappa}$ and is supported in the double of the set $B_{k,\kappa}$. Taking the inverse Fourier transform with respect to $\tau_{\lambda,\omega}$ only gives

$$(-i\partial_{t_{\lambda,\omega}} - h^-(\xi_{\lambda,\omega}))\mathcal{F}_{x_{\lambda,\omega}}u = K *_{t_{\lambda,\omega}} \mathcal{F}_{x_{\lambda,\omega}}G$$

where $K(t_{\lambda,\omega},\xi_{\lambda,\omega}) = \mathcal{F}^{-1}_{\tau_{\lambda,\omega}}(m^{-1}\tilde{\chi}_{B_{k,\kappa}})$. A solution for the above problem is given by the Duhamel formula

$$\mathcal{F}_{x_{\lambda,\omega}}v(t_{\lambda,\omega},\xi_{\lambda,\omega}) = \int_{-\infty}^{t_{\lambda,\omega}} e^{i(t_{\lambda,\omega}-s)h^-(\xi_{\lambda,\omega})}(K *_{t_{\lambda,\omega}} G)(s,\xi_{\lambda,\omega})ds \tag{2.27}$$

In integral form the kernel K is given by

$$K(t_{\lambda,\omega},\xi_{\lambda,\omega}) = \frac{1+\lambda^2}{1-\lambda^2}\int \frac{e^{it_{\lambda,\omega}\tau_{\lambda,\omega}}}{\tau_{\lambda,\omega} - h^+(\xi_{\lambda,\omega})}\tilde{\chi}_{B_{k,\kappa}}(\tau_{\lambda,\omega},\xi_{\lambda,\omega})d\tau_{\lambda,\omega}$$

We fix $\xi_{\lambda,\omega}$ and by using stationary phase it follows that

$$|K_\alpha(t_{\lambda,\omega},\xi_{\lambda,\omega})| \lesssim_N \frac{1}{1-\lambda^2}\langle t_{\lambda,\omega}(1-\lambda^2)^{-1}2^k\alpha^2\rangle^{-N}$$

which has the advantage that it holds uniformly with respect to $\xi_{\lambda,\omega}$. From this we obtain

$$\|K\|_{L^1_{t_{\lambda,\omega}}L^\infty_{\xi_{\lambda,\omega}}} \lesssim (2^k\alpha^2)^{-1}.$$

This implies that

$$\|K *_{t_{\lambda,\omega}} G\|_{L^1_{t_{\lambda,\omega}}L^2_{x_{\lambda,\omega}}} \lesssim (2^k\alpha^2)^{-1}\|G\|_{L^1_{t_{\lambda,\omega}}L^2_{x_{\lambda,\omega}}}.$$

from which, when combined with (2.26), we obtain

$$\|v\|_{L^\infty_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}} \lesssim \alpha^{-2} \|g\|_{L^1_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}}.$$

Thus we have produced a solution v of the inhomogeneous equation

$$(i\partial_t + \langle D \rangle)v = g$$

satisfying the bounds in (2.19) but without satisfying the initial condition $v(0) = 0$. Therefore we have that

$$u(t) = v(t) - e^{it\langle D \rangle} v(0).$$

We rewrite (2.27) as follows

$$v = \int_{-\infty}^{\infty} v_s \chi_{t_{\lambda,\omega} \geq s} ds$$

where $\mathcal{F}_{\xi_{\lambda,\omega}} v_s = e^{i(t_{\lambda,\omega}-s)h^-(\xi_{\lambda,\omega})}(K *_{t_{\lambda,\omega}} G)(s, \xi_{\lambda,\omega})$. Thus v is a superposition of free waves truncated across the hyperplanes $t_{\lambda,\omega} = s$. In addition, by reversing the computations in part i) we obtain

$$\|v_s\|_{L^\infty_t L^2_x} \lesssim \alpha^{-1} \|(K *_{t_{\lambda,\omega}} G)(s)\|_{L^2_{x_{\lambda,\omega}}}$$

from which it follows

$$\int_{-\infty}^{\infty} \|v_s\|_{L^\infty_t L^2_x} ds \lesssim \alpha^{-1} \|g\|_{L^1_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}}.$$

In particular this implies that

$$\|v(0)\|_{L^2_x} \lesssim \alpha^{-1} \|g\|_{L^1_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}}$$

and by invoking part i) we obtain

$$\|e^{it\langle D \rangle} v(0)\|_{L^\infty_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}} \lesssim \alpha^{-2} \|g\|_{L^1_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}}$$

which finishes the argument for part ii). In fact this also proves part iii) of the Theorem. □

2.3. **Estimates for the Klein-Gordon equation.** Let us specifically describe how the above estimates read in the context of the Klein-Gordon equation

$$(\Box + m^2)u = g, u(0) = f_0, u_t(0) = f_1, \tag{2.28}$$

where $m \neq 0$ is fixed. The analogue of Theorem 2.1 is

Corollary 2.5. *Let $m \neq 0$. Suppose that u is the solution of* (2.28) *with $g = 0$ and the initial data $f_0, f_1 \in L^2(\mathbb{R}^3)$ satisfy*

$$\mathrm{supp}(\widehat{f_0}),\ \mathrm{supp}(\widehat{f_1}) \subset \tilde{A}_k, \quad k \in \mathbb{Z}.$$

i) *For all* $k \lesssim 1$,

$$\|u\|_{L_t^2 L_x^\infty} \lesssim 2^{\frac{k}{2}} \|f_0\|_{L^2} + \|f_1\|_{L^2} \tag{2.29}$$

ii) *For all* $k \gtrsim 1$, $\kappa \in \mathcal{K}_k$,

$$2^{-k}\|P_\kappa u\|_{L_t^2 L_x^\infty} + \|P_\kappa u\|_{L^2_{t_{k,\kappa}} L^\infty_{x_{k,\kappa}}} \lesssim \|P_\kappa f_0\|_{L^2} + 2^{-k}\|P_\kappa f_1\|_{L^2} \tag{2.30}$$

iii) *For all* $k \gtrsim 1$, $1 \le l \le k$, $\kappa_1 \in \mathcal{K}_l$,

$$\sum_{\kappa \in \mathcal{K}_k} \|P_\kappa P_{\kappa_1} u\|_{L^2_{t_{k,\kappa}} L^\infty_{x_{k,\kappa}}} \lesssim 2^{k-l}\|P_{\kappa_1} f_0\|_{L^2} + 2^{-l}\|P_{\kappa_1} f_1\|_{L^2}. \tag{2.31}$$

The proof is obvious, see (2.1). Of course, there is also an analogue of Theorem 2.4 for (2.28).

Corollary 2.6. *Let* $k, j \ge 100$, $0 \le l \le \min(j,k) - 10$ *and* $\kappa \in \mathcal{K}_l$. *Let* $\Theta_{\lambda,\omega}$ *be a direction with* $\lambda = \lambda(j) = \frac{1}{\sqrt{1+2^{-2j}}}$, *and we assume* $\alpha = \mathrm{d}(\omega,\kappa)$ *satisfies* $2^{-3-l} \le \alpha \le 2^{3-l}$.

i) *If* $f_0, f_1 \in L^2(\mathbb{R}^3)$ *have the property that* $\hat{f}_0, \hat{f}_1$ *are supported in* $A_{k,\kappa}$, *then the solution* u *to* (2.28) *with* $g = 0$ *satisfies*

$$\alpha \|u\|_{L^\infty_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}} \lesssim \|f_0\|_{L^2} + 2^{-k}\|f_1\|_{L^2}. \tag{2.32}$$

ii) *Assume that* $f_0 = f_1 = 0$ *and let* $\hat{g}$ *be supported in the set* $B^+_{k,\kappa} \cup B^-_{k,-\kappa}$ *and* $g \in L^1_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}$. *Then, the solution* u *of* (2.28) *satisfies*

$$\alpha \|u\|_{L^\infty_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}} \lesssim 2^{-k}\alpha^{-1}\|g\|_{L^1_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}} \tag{2.33}$$

iii) *Under the hypothesis of Part ii) the solution* u *can be written as*

$$u(t) = v(t) + \int_{-\infty}^{\infty} u_s(t) \chi_{t_{\lambda,\omega} > s} ds \tag{2.34}$$

where v *and* u_s *are homogeneous solutions of the Klein-Gordon equation (in the original coordinates) and*

$$\begin{aligned} &\int_{-\infty}^{\infty} (\|u_s(0)\|_{L^2_x} + 2^{-k}\|\partial_t u_s(0)\|_{L^2_x}) ds \\ &+ \|v(0)\|_{L^2_x} + 2^{-k}\|\partial_t v(0)\|_{L^2_x} \lesssim 2^{-k}\alpha^{-1}\|g\|_{L^1_{t_{\lambda,\omega}} L^2_{x_{\lambda,\omega}}}. \end{aligned} \tag{2.35}$$

In addition, $\hat{u}_s$ *and* $\hat{v}$ *are supported in* $\tilde{B}^+_{k,\kappa} \cup \tilde{B}^-_{k,-\kappa}$.

3. Setup of the cubic Dirac

As written in (1.4) the cubic Dirac equation has a linear part whose coefficients are matrices. We rewrite (1.4) as a new system whose linear parts are the two half Klein-Gordon equations, see (3.4) below.

In the new setup it is possible to identify a null-structure in the nonlinearity, which is very similar to the ideas for the Dirac-Klein-Gordon system presented in [5, Section 2 and 3]. This will play a key role in overcoming some logarithmic divergences in the bilinear estimates. The main difference is that we keep the mass term inside the operator.

3.1. **Reduction.** The cubic Dirac equation can be written as

$$-i(\partial_t + \alpha \cdot \nabla + i\beta)\psi = \langle \psi, \beta\psi \rangle \beta\psi. \tag{3.1}$$

where $\beta = \gamma^0$ and $\alpha^j = \gamma^0\gamma^j$ and $\alpha \cdot \nabla = \alpha^j \partial_j$. The new matrices satisfy

$$\alpha^j\alpha^k + \alpha^k\alpha^j = 2\delta^{jk} I_4, \qquad \alpha^j\beta + \beta\alpha^j = 0. \tag{3.2}$$

There is one more computation which we will use in this section, namely

$$\alpha^j\alpha^k = \delta^{jk} + i\epsilon^{jkl} S^l \tag{3.3}$$

where $\epsilon^{jkl} = 1$ if (j,k,l) is an even permutation of $(1,2,3)$, $\epsilon^{jkl} = -1$ if (j,k,l) is an odd permutation of $(1,2,3)$ and $\epsilon^{jkl} = 0$ otherwise (when it contains repeated indexes). The matrices S^l are defined by

$$S^l = \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix}.$$

Following [5, Section 2] we decompose the spinor field relative to a basis of the operator $\alpha \cdot \nabla + i\beta$ whose symbol is $\alpha \cdot \xi + \beta$. Since $(\alpha \cdot \xi + \beta)^2 = (|\xi|^2 + 1)I$, the eigenvalues are $\pm\langle \xi \rangle$. We introduce the projections $\Pi_\pm(D)$ with symbol

$$\Pi_\pm(\xi) = \frac{1}{2}[I \mp \frac{1}{\langle \xi \rangle}(\xi \cdot \alpha + \beta)].$$

In comparison to [5, formula (2.2)], note that in the definition of $\Pi_\pm$ we chose the opposite sign for internal consistency purposes. The key identity is

$$-i(\alpha \cdot \nabla + i\beta) = \langle D \rangle (\Pi_-(D) - \Pi_+(D))$$

where $\langle D \rangle$ has symbol $\sqrt{|\xi|^2 + 1}$. The following identity, which can be verified easily at the level of the symbols, will be important in our computations:

$$\Pi_\pm(D)\beta = \beta(\Pi_\mp(D) \mp \frac{\beta}{\langle D \rangle}).$$

We then define $\psi_\pm = \Pi_\pm(D)\psi$ and split $\psi = \psi_+ + \psi_-$. By applying the operators $\Pi_\pm(D)$ to the cubic Dirac equation we obtain the following system of equations

$$\begin{cases} (i\partial_t + \langle D\rangle)\psi_+ = -\Pi_+(D)(\langle\psi, \beta\psi\rangle\beta\psi) \\ (i\partial_t - \langle D\rangle)\psi_- = -\Pi_-(D)(\langle\psi, \beta\psi\rangle\beta\psi). \end{cases} \tag{3.4}$$

This system will replace (1.4) as the object of our research for the rest of the paper. It is obvious from the form of the operators $\Pi_\pm$ that $\|\psi\|_X \approx \|\psi_+\|_X + \|\psi_-\|_X$ for many reasonable function spaces X. In particular we use it for $X = H^1(\mathbb{R}^3)$ so that we conclude that the initial data for (3.4) satisfies $\psi_\pm(0) \in H^1(\mathbb{R}^3)$.

3.2. **Null Structure.** There is a subtle null structure hidden in the system (3.4), which we describe next. This is again inspired by the work on the Dirac-Klein-Gordon system in [5].

We start with $\langle\psi, \beta\psi\rangle$ which, in our decomposition, is rewritten as

$$\begin{aligned} \langle\psi, \beta\psi\rangle &= \langle(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-, \beta(\Pi_+(D)\psi_+ + \Pi_-(D))\psi_-\rangle \\ &= \langle\Pi_+(D)\psi_+, \beta\Pi_+(D)\psi_+\rangle + \langle\Pi_-(D)\psi_-, \beta\Pi_-(D))\psi_-\rangle \\ &\quad + \langle\Pi_+(D)\psi_+, \beta\Pi_-(D)\psi_-\rangle + \langle\Pi_-(D)\psi_-, \beta\Pi_+(D)\psi_+\rangle \end{aligned}$$

The following Lemma analyses the symbols of the bilinear operators above, which is very similar to [5, Lemma 2] and its proof.

Lemma 3.1. *The following holds true*

$$\begin{aligned} \Pi_\pm(\xi)\Pi_\mp(\eta) &= \mathcal{O}(\angle(\xi, \eta)) + \mathcal{O}(\langle\xi\rangle^{-1} + \langle\eta\rangle^{-1}) \\ \Pi_\pm(\xi)\Pi_\pm(\eta) &= \mathcal{O}(\angle(-\xi, \eta)) + \mathcal{O}(\langle\xi\rangle^{-1} + \langle\eta\rangle^{-1}) \end{aligned} \tag{3.5}$$

Proof. We use the notation $\hat{\xi} := \frac{\xi}{|\xi|}$. Since $\frac{\xi}{\langle\xi\rangle} = \frac{\xi}{|\xi|} + \mathcal{O}(\langle\xi\rangle^{-1})$, and similarly for η, it follows, cp. [5, p.886], that

$$\begin{aligned} 4\Pi_\pm(\xi)\Pi_\mp(\eta) &= [I \mp \frac{1}{\langle\xi\rangle}(\xi\cdot\alpha + \beta)][I \pm \frac{1}{\langle\eta\rangle}(\eta\cdot\alpha + \beta)] \\ &= I - \hat{\xi}_j\hat{\eta}_k\alpha^j\alpha^k \mp (\hat{\xi} - \hat{\eta})\cdot\alpha + \mathcal{O}(\langle\xi\rangle^{-1} + \langle\eta\rangle^{-1}) \\ &= (1 - \hat{\xi}\cdot\hat{\eta})I - i(\hat{\xi}\times\hat{\eta})\cdot S \mp (\hat{\xi} - \hat{\eta})\cdot\alpha + \mathcal{O}(\langle\xi\rangle^{-1} + \langle\eta\rangle^{-1}) \\ &= \mathcal{O}(\angle(\xi, \eta)) + \mathcal{O}(\langle\xi\rangle^{-1} + \langle\eta\rangle^{-1}) \end{aligned}$$

where in passing from the second to the third line we have used (3.2) and (3.3). The second estimate in (3.5) follows from the first and the fact that $\Pi_\pm(\xi) = \Pi_\mp(-\xi) + \mathcal{O}(\langle\xi\rangle^{-1})$. □

We now explain why the above result plays the role of a null structure. Taking the spatial Fourier transform yields

$$\mathcal{F}_x\langle \Pi_+(D)\psi_1, \beta\Pi_+(D)\psi_2\rangle(\nu) = \int\limits_{\nu=\xi+\eta} \langle \Pi_+(\xi)\widehat{\psi_1}(\xi), \beta\Pi_+(\eta)\widehat{\psi_2}(\eta)\rangle$$

where we suppose that $\widehat{\psi_1}, \widehat{\psi_2}$ are supported at high frequencies $|\xi|, |\eta| \gg 1$. In this regime the equation is of wave type and it is well-known that the strongest interactions are the parallel ones, i.e. when $\angle(\xi, \eta) = 0$. On the other hand we have

$$\begin{aligned}
&\langle \Pi_+(\xi)\widehat{\psi_1}(\xi), \beta\Pi_+(\eta)\widehat{\psi_2}(\eta)\rangle \\
=& \langle \Pi_+(\xi)\widehat{\psi_1}(\xi), \Pi_-(\eta)\beta\widehat{\psi_2}(\eta)\rangle - \langle \Pi_+(\xi)\widehat{\psi_1}(\xi), \frac{1}{\langle\eta\rangle}\widehat{\psi_2}(\eta)\rangle \\
=& \langle \Pi_-(\eta)\Pi_+(\xi)\widehat{\psi_1}(\xi), \beta\widehat{\psi_2}(\eta)\rangle - \langle \Pi_+(\xi)\widehat{\psi_1}(\xi), \frac{1}{\langle\eta\rangle}\widehat{\psi_2}(\eta)\rangle
\end{aligned}$$

From the above computation it follows that, when $\angle(\xi, \eta) = 0$,

$$\Pi_-(\eta)\Pi_+(\xi) = \mathcal{O}(\langle\xi\rangle^{-1} + \langle\eta\rangle^{-1}),$$

thus greatly improving the structure of the bilinear form.

4. Function Spaces

Based on the structures developed in Section 2 we are now ready to define the function spaces in which we will perform the Picard iteration for (3.4). Notice that there are similarities to the function spaces used in the wave map problem [18, 39, 41], which we highlight by using a similar notation.

For $1 \le p \le \infty$, $b \in \mathbb{R}$, we define

$$\|f\|_{\dot{X}^{\pm,b,p}} = \big\| \big(2^{bm}\|Q_m^\pm f\|_{L^2}\big)_{m\in\mathbb{Z}}\big\|_{\ell^p_m},$$

For the low frequency part we define

$$\|f\|_{S^\pm_{\le 99}} = \|f\|_{L^\infty_t L^2_x} + \|f\|_{L^2_t L^\infty_x} + \|f\|_{X^{\pm,\frac12,\infty}} + \sup_{m\in\mathbb{Z}} 2^m \|Q_m^\pm f\|_{L_t^{\frac43} L^2_x}.$$

For the large frequencies, that is $k \ge 100$, the norm has a multiscale structure. For $l \le k-10$ and $\kappa \in \mathcal{K}_l$ we define

$$\|f\|_{S^\pm[k,\kappa]} = \|f\|_{L^\infty_t L^2_x} + \sup_{j\ge l+10} \sup_{\substack{\kappa_1\in\mathcal{K}_l: \\ 2^{-l-3}\le \mathrm{d}(\kappa,\kappa_1)\le 2^{-l+3}}} 2^{-l}\|f\|_{L^\infty_{t^\pm_{j,\kappa_1}} L^2_{x^\pm_{j,\kappa_1}}}$$

and

$$
\begin{aligned}
\|f\|_{S_k^\pm} =& \|f\|_{L_t^\infty L_x^2} + \|f\|_{\dot X^{\pm,\frac12,\infty}} + 2^{-\frac{k}{4}} \sup_{m\in\mathbb{Z}} 2^m \|Q_m^\pm f\|_{L_t^{\frac43} L_x^2} \\
&+ \Big(\sum_{\kappa\in\mathcal{K}_k} 2^{-2k} \|P_\kappa f\|^2_{L_t^2 L_x^\infty} + \|P_\kappa f\|^2_{L^2_{t^\pm_{k,\kappa}} L^\infty_{x^\pm_{k,\kappa}}} \Big)^{\frac12} \\
&+ \sup_{1\le l\le k-10} \Big(\sum_{\kappa\in\mathcal{K}_l} \|Q^\pm_{\prec k-2l} P_\kappa f\|^2_{S^\pm[k;\kappa]} \Big)^{\frac12}
\end{aligned} \tag{4.1}
$$

The resolution space corresponding to regularity at the level of $H^\sigma(\mathbb{R}^3)$ is the closed subspace of $C(\mathbb{R}, H^\sigma(\mathbb{R}^3))$ defined by the norm

$$
\|f\|_{S^{\pm,\sigma}} = \|P_{\le 99} f\|_{S^\pm_{\le 99}} + \Big(\sum_{k\ge 100} 2^{2k\sigma} \|P_k f\|^2_{S_k^\pm} \Big)^{\frac12}.
$$

Now we turn our attention to the construction of the space for the nonlinearity. For the low frequency part we define

$$
\|f\|_{N^{\pm,at}_{\le 99}} = \inf_{f=f_1+f_2} \Big\{ \|f_1\|_{\dot X^{\pm,-\frac12,1}} + \|f_2\|_{L_t^1 L_x^2} \Big\}.
$$

and

$$
\|f\|_{N^\pm_{\le 99}} = \|f\|_{N^{\pm,at}_{\le 99}} + \|f\|_{L_t^{\frac43} L_x^2}.
$$

An important property of these spaces is

$$
S^\mp_{\le 99} \subset (N^{\pm,at}_{\le 99})^* \subset S^{\mp,w}_{\le 99}. \tag{4.2}
$$

where $(N^{\pm,at}_{\le 99})^*$ is the dual of $N^{\pm,at}_{\le 99}$ and $S^{\pm,w}_{\le 99}$ is endowed with the norm

$$
\|f\|_{S^{\pm,w}_{\le 99}} = \|f\|_{L_t^\infty L_x^2} + \|f\|_{X^{\pm,\frac12,\infty}}. \tag{4.3}
$$

Next let $k\ge 100$. For $l\le k-10$ we consider $\kappa\in\mathcal{K}_l$ and define

$$
\|f\|_{N^\pm[k,\kappa]} = \inf \Big\{ 2^l \sum_{(j,\kappa_1)} \|f_{j,\kappa_1}\|_{L^1_{t^\pm_{j,\kappa_1}} L^2_{x^\pm_{j,\kappa_1}}} : f = \sum_{(j,\kappa_1)} f_{j,\kappa_1} \Big\}
$$

where the infimum is taken over pairs (j,κ_1) with $l\le j-10$ and $\kappa_1\in\mathcal{K}_l$ with $2^{-3}\le 2^l\,\mathrm{d}(\kappa_1,\kappa)\le 2^3$. Then we define the space for the following atomic structure

$$
\begin{aligned}
\|f\|_{N_k^{\pm,at}} = & \inf_{f=f_1+f_2+\sum_{1\le l\le k-10} g_l} \Big\{ \|f_1\|_{\dot X^{\pm,-\frac12,1}} + \|f_2\|_{L_t^1 L_x^2} \\
&+ \sum_{1\le l\le k-10} \Big(\sum_{\kappa\in\mathcal{K}_l} \|P_\kappa g_l\|^2_{N^\pm[k,\kappa]} \Big)^{\frac12} \Big\}
\end{aligned} \tag{4.4}
$$

where the atoms g_l in the above decomposition are assumed to be localized at frequency 2^k and modulation $\ll 2^{k-2l}$, more precisely that $\tilde Q^\pm_{\prec k-2l} \tilde P_k g_l = g_l$.

One important remark should be made about the third component in $N_k^{\pm,at}$, i.e. the $\sum_{1\leq l\leq k-10} g_l$, which we will henceforth call the cap-localized structure. The atoms g_l are localized in frequency and modulation, while when they are measured in $N^{\pm}[k,\kappa]$ the components in the decomposition there $g_l = \sum_{(j,\kappa_1)} g_{l,j,\kappa_1}$ are not assumed to keep that localization. However, by applying the operator $\tilde{Q}^{\pm}_{\prec k-2l}\tilde{P}_{k,\kappa}$ to the decomposition and using part i) in Lemma 4.1 below one obtains a new decomposition with similar norm. From now on we assume that the decomposition above comes with the correct frequency and modulation localization.

An important property of this construction is that

$$S_k^{\mp} \subset (N_k^{\pm,at})^* \subset S_k^{\mp,w} \tag{4.5}$$

where $(N_k^{\pm,at})^*$ is the dual of $N_k^{\pm,at}$ and $S_k^{\pm,w}$ is endowed with the norm
(4.6)

$$\|f\|_{S_k^{\pm,w}} = \|f\|_{L_t^\infty L_x^2} + \|f\|_{X^{\pm,\frac{1}{2},\infty}} + \sup_{1\leq l\leq k}\Big(\sum_{\kappa\in\mathcal{K}_l}\|Q^{\pm}_{\prec k-2l}P_\kappa f\|^2_{S^{\pm}[k;\kappa]}\Big)^{\frac{1}{2}}$$

and the embeddings are continuous, i.e.

$$\|f\|_{S_k^{\mp,w}} \lesssim \|f\|_{(N_k^{\pm,at})^*} \lesssim \|f\|_{S_k^{\mp}}.$$

For high frequencies, the space for dyadic pieces of the nonlinearity is the following

$$\|f\|_{N_k^{\pm}} = \|f\|_{N_k^{\pm,at}} + 2^{-\frac{k}{4}}\|f\|_{L_t^{\frac{4}{3}}L_x^2}.$$

The space for the nonlinearity at regularity H^σ is the following

$$\|f\|_{N^{\pm,\sigma}} = \|P_{\leq 99}f\|_{N^{\pm}_{\leq 99}} + \Big(\sum_{k\geq 100} 2^{2k\sigma}\|P_k f\|^2_{N_k^{\pm}}\Big)^{\frac{1}{2}}.$$

We now turn our attention to the relevance of the above structures for the equations we study. Our first result is of technical nature and it says that certain frequency and modulation localization operators preserve the structures involved above.

Lemma 4.1. i) *For all $k\geq 100$, $1\leq l\leq k$, $\kappa\in\mathcal{K}_l$, the operators $\tilde{P}_{k,\kappa}$ and $\tilde{Q}^{\pm}_{\prec k-2l}\tilde{P}_{k,\kappa}$ have bounded kernel in L^1_x, respectively $L^1_{t,x}$. As a consequence, they are uniformly bounded on all L^pL^q in all frame choices.*

ii) *For all $k,j\geq 100$, $1\leq l\leq \min(j,k)-10$, $\kappa,\kappa_1\in\mathcal{K}_l$ such that $2^{-3-l}\leq \mathrm{d}(\kappa,\kappa_1)\leq 2^{3-l}$, the operators $\tilde{Q}^{\pm}_m\tilde{P}_{k,\kappa}$ for $m\leq k-2l$ are bounded on the spaces $L^1_{t^{\pm}_{j,\kappa_1}}L^2_{x^{\pm}_{j,\kappa_1}}$.*

iii) *For all* $k \geq 100, 1 \leq l \leq k, \kappa \in \mathcal{K}_l$, *and functions* u *localized at frequency* 2^k, *we have*

$$\| \left(\Pi_\pm(D) - \Pi_\pm(2^k\omega(\kappa))\right) P_\kappa u\|_S \lesssim 2^{-l}\|P_\kappa u\|_S \tag{4.7}$$

for $S \in \{S_k^\pm, S_k^{\pm,w}\}$.

Proof. i) The kernel of the operator $\tilde{P}_{k,\kappa}$ is given by $\mathcal{F}_x^{-1}(\tilde{\eta}_\kappa \tilde{\chi}_k)$ and it is a straightforward exercise to prove that it belongs to L^1_x. Since

$$\tilde{P}_{k,\kappa} u = \mathcal{F}_x^{-1}(\tilde{\eta}_\kappa \tilde{\chi}_k) *_x u$$

the boundedness of $\tilde{P}_{k,\kappa}$ on all $L^p L^q$ spaces follows from the boundedness of its kernel in L^1_x.

Next, we prove the statement for the operator $\tilde{Q}^+_{\prec k-2l}\tilde{P}_{k,\kappa}$. With $a_{l,k,\kappa}(\tau,\xi) = \tilde{\chi}_{\leq k-2l}(\tau - \langle\xi\rangle)\tilde{\eta}_\kappa\tilde{\chi}_k$ and $R = \mathcal{F}^{-1}(a_{l,k,\kappa})$ we have

$$Q^+_{\prec k-2l} P_\kappa u = R * Q^+_{\prec k-2l} P_k u.$$

Since a is a smooth approximation of the characteristic function of a rectangular parallelepiped (of sizes $2^k \times 2^{k-2l} \times 2^{k-l} \times 2^{k-l}$ in the direction of $(\tau_{k,\kappa}, \xi^1_{k,\kappa}, \xi^2_{k,\kappa}, \xi^3_{k,\kappa})$), it is a straightforward exercice to prove that $\|R\|_{L^1_{t,x}} \lesssim 1$. The boundedness statement follows from the above.

ii) We give the proof for the operator $\tilde{Q}^+_m \tilde{P}_{k,\kappa}$, which is a Fourier multiplier whose symbol $a_{m,k,\kappa}(\tau,\xi) = \tilde{\chi}_m(\tau - \langle\xi\rangle)\tilde{\chi}_k(\xi)\tilde{\eta}_\kappa(\xi)$ satisfies

$$|\partial^\beta_{\tau_{j,\kappa_1}} a_{m,k,\kappa}| \lesssim (2^{m+2l})^{-\beta}.$$

The inverse Fourier transform of $a_{m,k,\kappa}$ with respect to τ_{j,κ_1} satisfies

$$|K_{l,k,\kappa}(t_{j,\kappa_1}, \xi_{j,\kappa_1})| \lesssim_N 2^{m+2l}(1 + |t_{j,\kappa_1}|2^{m+2l})^{-N}, \text{ for any } N \in \mathbb{N}.$$

From this we obtain the uniform bound

$$\|K_{l,k,\kappa}\|_{L^1_{t_{j,\kappa_1}} L^\infty_{\xi_{j,\kappa_1}}} \lesssim 1.$$

On the other hand we have

$$\mathcal{F}_{\xi_{j,\kappa_1}}(\tilde{Q}^+_m \tilde{P}_{k,\kappa} f) = K_{l,k,\kappa} *_{t_{j,\kappa_1}} \mathcal{F}_{\xi_{j,\kappa_1}} f,$$

where one performs convolution with respect to t_{j,κ_1} variable only. From the last two statements, the conclusion follows.

iii) We prove the statement for the $+$ choice above and $S = S^+_k$, the proof for the other choices being similar. A similar argument to the one used in i) shows that the operators $\left(\Pi_+(D) - \Pi_+(2^k\omega(\kappa))\right) P_{k,\kappa}$ and $\left(\Pi_+(D) - \Pi_+(2^k\omega(\kappa))\right) Q^+_{\prec k-2l} P_{k,\kappa}$ are, up to picking a factor of 2^{-l}, uniformly bounded on each component. □

The main result of this section is the following Proposition.

Proposition 4.2. *For all $g \in N_k^{\pm}$ and initial data $u_0 \in L^2(\mathbb{R}^3)$, both localized at (spatial) frequency 2^k, $k \geq 100$, the solution u of*

$$(i\partial_t \pm \langle D \rangle)u = g, \quad u(0) = u_0, \tag{4.8}$$

belongs to $S_k^{\pm}$ and the following estimate holds true:

$$\|u\|_{S_k^{\pm}} \lesssim \|g\|_{N_k^{\pm}} + \|u_0\|_{L^2}. \tag{4.9}$$

Proof. To simplify the exposition we write the argument for the $+$ choice above. The argument is organized as follows. In Part 1 we consider $g \in N_k^{+,at}$ and we derive all the properties in S_k for u, except the $L_t^{\frac{4}{3}} L_x^2$ structure. Since the $N_k^{+,at}$ contains three type of atoms, we split the argument in three cases. In Part 2, we prove that if $g \in 2^{\frac{k}{4}} L_t^{\frac{4}{3}} L_x^2$, then we obtain the similar structure for $Q_m^+ u$.

Further, since all estimates in S_k^+ were provided for homogeneous solutions in Section 2, it suffices to provide the argument for $u_0 = 0$. We note that the homogenous solutions belong to the kernel of the operators Q_m^+, hence the $\dot{X}^{+,\frac{1}{2},\infty}$ and $L_t^{\frac{4}{3}} L_x^2$ components are vacuous for them.

Part 1) $g \in N_k^{+,at}$. *Case a)* $g \in L_t^1 L_x^2$. The solution is given by

$$u(t) = -e^{it\langle D \rangle} \int_{-\infty}^0 e^{-is\langle D \rangle} g(s) ds + \int e^{i(t-s)\langle D \rangle} g(s) \chi_{t>s} ds.$$

Hence u is a superposition of homogeneous solutions with L^2 data which are truncated across hyperplanes $t > s$. The $L_t^\infty L_x^2$ bound is obvious. Theorem 2.1 and Theorem 2.4 i) imply the end-point Strichartz and energy estimates. The estimate in $\dot{X}^{+,\frac{1}{2},\infty}$ is proved as follows. Inserting the modulation operator Q_m^+ into the equation we obtain

$$(i\partial_t + \langle D \rangle) Q_m^+ u = Q_m^+ g.$$

Let $D_t = i\partial_t$. Then,

$$Q_m^+ = e^{it\langle D \rangle} \chi_m(D_t) e^{-it\langle D \rangle}$$

which yields

$$D_t \chi_m(D_t) e^{-it\langle D \rangle} u = \chi_m(D_t) e^{-it\langle D \rangle} g. \tag{4.10}$$

Now, the kernel of $D_t^{-1} \chi_m(D_t)$ satisfies

$$\left\| \mathcal{F}_t^{-1} \left(\frac{\chi_m}{\tau} \right) \right\|_{L^q(\mathbb{R})} \lesssim 2^{-\frac{m}{q}} \tag{4.11}$$

for all $1 \leq q \leq \infty$, hence

$$\begin{aligned}\|Q_m^+ u\|_{L^2} &= \|D_t^{-1}\chi_m(D_t)e^{-it\langle D\rangle}g\|_{L^2}\\ &\lesssim \Big\|\mathcal{F}_t^{-1}(\frac{\chi_m}{\tau})\Big\|_{L_t^2}\|e^{-it\langle D\rangle}g\|_{L_t^1L_x^2} \lesssim 2^{-\frac{m}{2}}\|g\|_{L_t^1L_x^2}.\end{aligned}$$

Case b) $g \in \dot{X}^{+,-\frac{1}{2},1}$. Let v defined by $\hat{v} = \frac{\hat{g}}{\tau-\langle\xi\rangle}$. As defined now, v may not even be a distribution. Using $g \in 2^{\frac{k}{4}}L_t^{4/3}L_x^2$ and the frequency localization of g, it follows from a Sobolev embedding that $g \in L^2$. Thus $g = \sum_{m\in\mathbb{Z}} Q_m^+ g$, and it follows further that $\hat{v} = \sum_{m\in\mathbb{Z}} \frac{\chi_m(\tau-\langle\xi\rangle)\hat{g}}{\tau-\langle\xi\rangle}$.

Then, by (4.11) with $q = 1$,

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} 2^{\frac{m}{2}}\|Q_m^+ v\|_{L^2} = \sum_{m\in Z} 2^{\frac{m}{2}}\|D_t^{-1}\chi_m(D_t)e^{-it\langle D\rangle}g\|_{L^2}\\ \lesssim &\sum_{m\in\mathbb{Z}} 2^{\frac{m}{2}}\Big\|\mathcal{F}_t^{-1}(\frac{\chi_m}{\tau})\Big\|_{L_t^1}\|(\chi_{m-1}(D_t)+\chi_m(D_t)+\chi_{m+1}(D_t))e^{-it\langle D\rangle}g\|_{L^2}\\ \lesssim &\sum_{m\in\mathbb{Z}} 2^{-\frac{m}{2}}\|Q_m^+ g\|_{L^2},\end{aligned}$$

hence $v \in \dot{X}^{+,\frac{1}{2},1}$ and $\|v\|_{L_t^\infty L_x^2} \lesssim \|v\|_{\dot{X}^{+,\frac{1}{2},1}} \lesssim \|g\|_{\dot{X}^{+,-\frac{1}{2},1}}$; in particular we upgraded v to a tempered distribution. Further, v can be written as

$$v = \sum_{m\in\mathbb{Z}} \int e^{it\tau}e^{it\langle D\rangle}\tilde{v}_m(\tau)d\tau, \text{ where } \tilde{v}_m = \mathcal{F}_t(e^{-it\langle D\rangle}g)\frac{\chi_m}{\tau},$$

i.e. as a superposition of modulated homogeneous solutions. Due to the estimate

$$\sum_{m\in\mathbb{Z}} \int \|\tilde{v}_m(\tau)\|_{L_x^2}d\tau \lesssim \sum_{m\in\mathbb{Z}} 2^{-\frac{m}{2}}\|Q_m^+ g\|_{L^2} = \|g\|_{\dot{X}^{+,-\frac{1}{2},1}}$$

the end-point Strichartz and energy estimates for v follow from Theorem 2.1 and Theorem 2.4 i). The only problem is that while v satisfies the inhomogeneous equation (4.8), it does not have to satisfy the initial condition. On the other hand

$$u = v - e^{it\langle D\rangle}v(0)$$

becomes a solution to (4.8) (with $u_0 = 0$) and since $\|v(0)\|_{L_x^2} \lesssim \|g\|_{\dot{X}^{+,-\frac{1}{2},1}}$, (4.9) follows in this case.

Case c) g belongs to the cap-localized structure. Given the l^1 structure in the l parameter, it suffices to establish the estimates for fixed

l. For each $\kappa \in \mathcal{K}_l$ we have the decomposition

$$P_\kappa g_l = \sum_{(j,\kappa_1)} g_{j,\kappa_1} \tag{4.12}$$

where we recall that we can choose g_{j,κ_1} such that $\tilde{Q}^+_{\prec k-2l}\tilde{P}_{k,\kappa} g_{j,\kappa_1} = g_{j,\kappa_1}$. Using part iii) of Theorem 2.4 with g_{j,κ_1} as forcing, we obtain that the solution generated satisfies

$$\|u_{j,\kappa_1}\|_{S^+[k,\kappa]} \lesssim \|g_{j,\kappa_1}\|_{L^1_{t_{j,\kappa_1}} L^2_{x_{j,\kappa_1}}}$$

and has Fourier support in the set $\tilde{B}_{k,\kappa}$. If u_κ is the solution of the equation with forcing $P_\kappa g_l$, then by adding all the components in the decomposition of g_l gives the following estimate

$$\|u_\kappa\|_{S^+[k,\kappa]} \lesssim \sum_{(j,\kappa_1)} \|g_{j,\kappa_1}\|_{L^1_{t_{j,\kappa_1}} L^2_{x_{j,\kappa_1}}}$$

and that u_κ has Fourier support in the set $\tilde{B}_{k,\kappa}$. In the last step we need to perform the summation with respect to $\kappa \in \mathcal{K}_l$. Given that each u_κ is supported in $\tilde{B}_{k,\kappa}$, the $L^\infty_t L^2_x$ and the end-point Strichartz estimate follow. Concerning the cap-localized structure, it is easy to see that one obtains the $S^+[k,\kappa']$ structures with $\kappa' \in \mathcal{K}_{l'}$ with $l' \geq l$. For the case when $l' \leq l$, one splits

$$P_{\kappa'} u = \sum_{\kappa \in \mathcal{K}_l} \tilde{P}_\kappa P_{\kappa'} u$$

and uses the almost orthogonality of $P_{\kappa'} u_\kappa$, $\kappa \in \mathcal{K}_l$ with respect to ξ_{j,κ_1} to obtain

$$\|P_{\kappa'} u\|^2_{L^\infty_{t_{j,\kappa_1}} L^2_{x_{j,\kappa_1}}} \lesssim \sum_{\kappa \in \mathcal{K}_l} \|P_{\kappa'} u_\kappa\|^2_{L^\infty_{t_{j,\kappa_1}} L^2_{x_{j,\kappa_1}}}.$$

We now prove that $u \in \dot{X}^{+,\frac{1}{2},\infty}$. We start from the decomposition (4.12). From this we obtain

$$\begin{aligned}\|Q^+_m g_{j,\kappa_1}\|_{L^2_{t,x}} = \|\mathcal{F}(Q^+_m g_{j,\kappa_1})\|_{L^2_{\tau_{j,\kappa_1},\xi_{j,\kappa_1}}} &\lesssim 2^{\frac{m+2l}{2}} \|\mathcal{F}(Q^+_m g_{j,\kappa_1})\|_{L^2_{\xi_{j,\kappa_1}} L^\infty_{\tau_{j,\kappa_1}}} \\ &\lesssim 2^{\frac{m+2l}{2}} \|Q^+_m g_{j,\kappa_1}\|_{L^1_{t_{j,\kappa_1}} L^2_{x_{j,\kappa_1}}} \\ &\lesssim 2^{\frac{m+2l}{2}} \|g_{j,\kappa_1}\|_{L^1_{t_{j,\kappa_1}} L^2_{x_{j,\kappa_1}}}.\end{aligned}$$

In the above we have used that the size of the support of Fourier transform of $Q^+_m g_{j,\kappa_1}$ in the direction of τ_{j,κ_1} is $\approx 2^{m+2l}$ and part ii) of Lemma 4.1. We sum the above estimates with respect to (j,κ_1) to obtain

$$\|Q^+_m P_{k,\kappa} g_l\|_{L^2} \lesssim 2^{\frac{m}{2}} \|Q^+_{\prec k-2l} P_{k,\kappa} g_l\|_{N[k,\kappa]}.$$

Finally, we sum the above with respect to $\kappa \in \mathcal{K}_l$ to conclude with

$$2^{-\frac{m}{2}} \|Q_m^+ g_l\|_{L^2} \lesssim \left(\sum_{\kappa \in \mathcal{K}_l} \|Q_{\prec k-2l}^+ P_{k,\kappa} g_l\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Since this is uniform with respect to $m \leq 2k - l$ we obtain that $g \in \dot{X}^{+,-\frac{1}{2},\infty}$. Since $\mathcal{F}(Q_m^+ u) = \frac{1}{\tau - \langle \xi \rangle} \mathcal{F}(Q_m^+ g)$, the estimate for u in $\dot{X}^{+,\frac{1}{2},\infty}$ follows.

Part 2) g belongs to $2^{\frac{k}{4}} L_t^{\frac{4}{3}} L_x^2$. From (4.10) it follows that

$$\begin{aligned} \|Q_m^+ u\|_{L_t^{4/3} L_x^2} &= \|D_t^{-1} \chi_m(D_t) e^{-it\langle D \rangle} g\|_{L_t^{4/3} L_x^2} \\ \lesssim \left\| \mathcal{F}_t^{-1}\left(\frac{\chi_m}{\tau}\right) \right\|_{L_t^1} &\|e^{-it\langle D \rangle} g\|_{L_t^{4/3} L_x^2} \lesssim 2^{-m} \|g\|_{L_t^{4/3} L_x^2}, \end{aligned}$$

where we used (4.11) with $q = 1$, and this finishes our proof. □

Corollary 4.3. *For all $u_0 \in H^\sigma(\mathbb{R}^3)$ and $g \in N^{\pm,\sigma}$, there exists a unique solution $u \in S^{\pm,\sigma}$ of (4.8), and the following estimate holds true*

$$\|u\|_{S^{\pm,\sigma}} \lesssim \|g\|_{N^{\pm,\sigma}} + \|u_0\|_{H^\sigma}. \tag{4.13}$$

Proof. The claim follows from its dyadic versions for high frequencies ($k \geq 100$), which is precisely Proposition 4.2. The low frequency part is standard, except the $L_t^{\frac{4}{3}} L_x^2$ part which is established as in Part 2) above. Alternatively it is an easy exercise to work out the whole argument following the same steps as for the high frequency case. □

5. Bilinear estimates

In this section we derive the main bilinear $L_{t,x}^2$-type estimate for functions in our spaces. As a convention, throughout the rest of the paper u's will denote complex scalars, $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, while ψ's will denote complex vectors $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$. To make the exposition simpler we will abuse notation and set $S_{99}^{\pm} := S_{\leq 99}^{\pm}$.

The main result of this section is the following

Proposition 5.1. i) *For all $k_1, k_2 \geq 99$ and $\psi_1 \in S_{k_1}^{\pm}$, $\psi_2 \in S_{k_2}^{\pm,w}$, where ψ_j localized at frequency 2^{k_j} for $j = 1, 2$, the following holds true:*

$$\left\| \langle \Pi_\pm(D)\psi_1, \beta \Pi_\pm(D)\psi_2 \rangle \right\|_{L^2} \lesssim 2^{k_1} \|\psi_1\|_{S_{k_1}^{\pm}} \|\psi_2\|_{S_{k_2}^{\pm,w}}, \tag{5.1}$$

ii) *If in addition* $l \leq \min(k_1, k_2)$, *then*

$$\left\| \sum_{\substack{\kappa_1,\kappa_2 \in \mathcal{K}_l: \\ d(\pm\kappa_1,\pm\kappa_2) \lesssim 2^{-l}}} \langle \Pi_\pm(D) \tilde{P}_{\kappa_1}\psi_1, \beta \Pi_\pm(D) \tilde{P}_{\kappa_2}\psi_2 \rangle \right\|_{L^2} \tag{5.2}$$
$$\lesssim 2^{k_1 - l} \|\psi_1\|_{S^\pm_{k_1}} \|\psi_2\|_{S^{\pm,w}_{k_2}}.$$

In both of the above estimates the sign of each $\Pi_\pm$ *and* $\pm\kappa_j$ *is chosen to be consistent with the one of the corresponding* $S^\pm_{k_j}$.

iii) *Let* $2 < q \leq \infty$. *For all* $100 \leq k_1 \leq k_2$ *and* $u_1 \in S^+_{k_1}$, $u_2 \in S^{+,w}_{k_2}$, *each localized at frequency* 2^{k_1} *resp.* 2^{k_2}, *the following holds true:*

$$\|u_1 \cdot \bar{u}_2\|_{L^2_t L^q_x} \lesssim_q 2^{k_1} 2^{3(\frac{1}{2}-\frac{1}{q})k_2} \|u_1\|_{S^+_{k_1}} \|u_2\|_{S^{+,w}_{k_2}}. \tag{5.3}$$

The same result holds true for $u_1 \in S^-_{k_1}$, $u_2 \in S^{-,w}_{k_2}$.

As an immediate consequence (5.3) we note the following Strichartz type estimate.

Corollary 5.2. *Let* $4 < q \leq \infty$. *For all* $k \geq 100$,

$$\|\tilde{P}_k u\|_{L^4_t L^q_x} \lesssim_q 2^{\frac{k}{2}} 2^{\frac{3}{2}(\frac{1}{2}-\frac{2}{q})k} \|\tilde{P}_k u\|_{S^\pm_k}. \tag{5.4}$$

By interpolation one can easily obtain all the "off the line" Strichartz estimates $L^p_t L^q_x$ with $p \geq 4$, following closely the ideas of [18, 19, 39] in the context of wave maps. In the case of wave maps, it has been observed later in [35, Section 5.4] that the usual "on the line" Strichartz estimates such as $L^4_{t,x}$ hold true in these spaces as well, but this is a little more difficult to prove and we do not need it here.

The low frequency counterpart of (5.2) is, for all $4 < q \leq \infty$,

$$\|\tilde{P}_k u\|_{L^4_t L^q_x} \lesssim \|\tilde{P}_{\leq 99} u\|_{L^4_{t,x}} \lesssim \|\tilde{P}_{\leq 99} u\|_{S^\pm_{\leq 99}}. \tag{5.5}$$

which is easily obtained from the $L^4_{t,x}$ using Sobolev embedding. The latter is obtained using interpolation between the $L^2_t L^\infty_x$ and $L^\infty_t L^2_x$ components of $S^\pm_{\leq 99}$.

Proof of Proposition 5.1. To make the exposition easier, we choose to prove all the estimates for the + choice in all terms. A careful examination of the argument reveals that the other choices follow in a similar manner. The focus of the argument is on the high frequency interactions, that is $\min(k_1, k_2) \geq 100$. It will be obvious that when $\min(k_1, k_2) = 99$, the argument carries on and in fact it becomes simpler. Note that (5.2) does not say anything new in the case $\min(k_1, k_2) = 99$, while (5.3) is not even stated in this case.

We will reduce (5.1),(5.2) and (5.3) to the following claim: For all u_1, u_2 be localized at frequencies 2^{k_1}, respectively 2^{k_2}, and $|l_1 - l_2| \leq 2$ with $l_1 \leq \min(k_1, k_2)$ the following estimate holds true:

$$\sum_{\kappa_1 \in \mathcal{K}_{l_1}, \kappa_2 \in \mathcal{K}_{l_2}} \|\tilde{P}_{\kappa_1} u_1 \tilde{P}_{\kappa_2} u_2\|_{L^2} \lesssim 2^{k_1} \|u_1\|_{S^+_{k_1}} \|u_2\|_{S^{+,w}_{k_2}}. \tag{5.6}$$

where the above sum is restricted to the range $\mathrm{d}(\kappa_1, \kappa_2) \approx 2^{-l_1}$ or $\mathrm{d}(\kappa_1, \kappa_2) \lesssim 2^{-l_1}$ in the case $|l_1 - \min(k_1, k_2)| \leq 2$.

First case: $k_1 \leq k_2$. If $l_1 \leq k_1 - 10$, then

$$\sum_{\kappa_1 \in \mathcal{K}_{l_1}, \kappa_2 \in \mathcal{K}_{l_2}} \|\tilde{P}_{\kappa_1} u_1 \cdot \tilde{P}_{\kappa_2} u_2\|_{L^2} \leq A_0 + A_1 + A_2 + A_3.$$

We will provide estimates for each contribution.

$$\begin{aligned} A_0 :=& \sum_{\kappa_1 \in \mathcal{K}_{l_1}, \kappa_2 \in \mathcal{K}_{l_2}} \|Q_{\succeq k_1 - 2l_1} \tilde{P}_{\kappa_1} u_1\|_{L^4_t L^\infty_x} \|\tilde{P}_{\kappa_2} Q_{\succeq k_1 - 2l_2} u_2\|_{L^4_t L^2_x} \\ \lesssim& 2^{\frac{3k_1 - 2l_1}{2}} \Big(\sum_{\kappa_1 \in \mathcal{K}_{l_1}} \|Q_{\succeq k_1 - 2l_1} \tilde{P}_{\kappa_1} u_1\|^2_{L^4_t L^2_x} \Big)^{\frac{1}{2}} \Big(\sum_{\kappa_2 \in \mathcal{K}_{l_2}} \|\tilde{P}_{\kappa_2} Q_{\succeq k_1 - 2l_2} u_2\|^2_{L^4_t L^2_x} \Big)^{\frac{1}{2}}. \end{aligned}$$

Now, we use

$$\begin{aligned} &\Big(\sum_{\kappa_1 \in \mathcal{K}_{l_1}} \|Q_{\succeq k_1 - 2l_1} \tilde{P}_{\kappa_1} u_1\|^2_{L^4_t L^2_x} \Big)^{\frac{1}{2}} \lesssim \sum_{m \succeq k_1 - 2l_1} \Big(\sum_{\kappa_1 \in \mathcal{K}_{l_1}} \|Q_m \tilde{P}_{\kappa_1} u_1\|^2_{L^4_t L^2_x} \Big)^{\frac{1}{2}} \\ \lesssim& \sum_{m \succeq k_1 - 2l_1} 2^{\frac{m}{4}} \Big(\sum_{\kappa_1 \in \mathcal{K}_{l_1}} \|Q_m \tilde{P}_{\kappa_1} u_1\|^2_{L^2_t L^2_x} \Big)^{\frac{1}{2}} \\ \lesssim& \sum_{m \succeq k_1 - 2l_1} 2^{\frac{m}{4}} \|Q_m u_1\|_{L^2_t L^2_x} \lesssim 2^{-\frac{k_1 - 2l_1}{4}} \|Q_{\succeq k_1 - 2l_1} u_1\|_{\dot{X}^{+,\frac{1}{2},\infty}} \end{aligned}$$

to complete the argument as follows:

$$\begin{aligned} A_0 \lesssim& 2^{\frac{3k_1 - 2l_1}{2}} 2^{-\frac{k_1 - 2l_1}{2}} \|Q_{\succeq k_1 - 2l_1} u_1\|_{\dot{X}^{+,\frac{1}{2},\infty}} \|Q_{\succeq k_1 - 2l_2} u_2\|_{\dot{X}^{+,\frac{1}{2},\infty}} \\ \lesssim& 2^{k_1} \|\tilde{P}_{\kappa_1} u_1\|_{S^+_{k_1}} \|\tilde{P}_{\kappa_2} u_2\|_{S^{+,w}_{k_2}}, \end{aligned}$$

and

$$
\begin{aligned}
A_1 :=& \sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}} \|Q_{\prec k_1-2l_1}\tilde{P}_{\kappa_1}u_1\|_{L^\infty}\|\tilde{P}_{\kappa_2}Q_{\succeq k_1-2l_2}u_2\|_{L^2}\\
\lesssim& \Big(\sum_{\kappa_1\in\mathcal{K}_{l_1}}\|Q_{\prec k_1-2l_1}\tilde{P}_{\kappa_1}u_1\|_{L^\infty}^2\Big)^{\frac12}\Big(\sum_{\kappa_2\in\mathcal{K}_{l_2}}\|\tilde{P}_{\kappa_2}Q_{\succeq k_1-2l_2}u_2\|_{L^2}^2\Big)^{\frac12}\\
\lesssim& 2^{\frac{3k_1-2l_1}{2}}\Big(\sum_{\kappa_1\in\mathcal{K}_{l_1}}\|Q_{\prec k_1-2l_1}\tilde{P}_{\kappa_1}u_1\|_{L_t^\infty L_x^2}^2\Big)^{\frac12}2^{-\frac{k_1-2l_2}{2}}\|Q_{\succeq k_1-2l_2}u_2\|_{\dot{X}^{+,\frac12,\infty}}\\
\lesssim& 2^{k_1}\|u_1\|_{S^+_{k_1}}\|u_2\|_{S^{+,w}_{k_2}},
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 :=& \sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}} \|Q_{\succeq k_1-2l_1}\tilde{P}_{\kappa_1}u_1\|_{L_t^2L_x^\infty}\|Q_{\prec k_1-2l_2}\tilde{P}_{\kappa_2}u_2\|_{L_t^\infty L_x^2}\\
\lesssim& \Big(\sum_{\kappa_1\in\mathcal{K}_{l_1}}\|Q_{\succeq k_1-2l_1}\tilde{P}_{\kappa_1}u_1\|_{L_t^2L_x^\infty}^2\Big)^{\frac12}\Big(\sum_{\kappa_2\in\mathcal{K}_{l_2}}\|Q_{\prec k_1-2l_2}\tilde{P}_{\kappa_2}u_2\|_{L_t^\infty L_x^2}^2\Big)^{\frac12}\\
\lesssim& 2^{\frac{3k_1-2l_1}{2}}\Big(\sum_{\kappa_1\in\mathcal{K}_{l_1}}\|Q_{\succeq k_1-2l_1}\tilde{P}_{\kappa_1}u_1\|_{L_{t,x}^2}^2\Big)^{\frac12}\|u_2\|_{S^{+,w}_{k_2}}\\
\lesssim& 2^{k_1}\|u_1\|_{S^+_{k_1}}\|u_2\|_{S^{+,w}_{k_2}},
\end{aligned}
$$

as well as

$$
\begin{aligned}
A_3 :=& \sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}}\sum_{\kappa\in\mathcal{K}_{k_1}} \|P_\kappa Q_{\prec k_1-2l_1}\tilde{P}_{\kappa_1}u_1\|_{L^2_{t_{k_1,\kappa}}L^\infty_{x_{k_1,\kappa}}}\|Q_{\prec k_1-2l_2}\tilde{P}_{\kappa_2}u_2\|_{L^\infty_{t_{k_1,\kappa}}L^2_{x_{k_1,\kappa}}}\\
\lesssim& 2^{l_1}\sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}}\|Q_{\prec k_1-2l_2}\tilde{P}_{\kappa_2}u_2\|_{S[k_2,\kappa_2]}\sum_{\kappa\in\mathcal{K}_{k_1}}\|P_\kappa\tilde{P}_{\kappa_1}u_1\|_{L^2_{t_{k_1,\kappa}}L^\infty_{x_{k_1,\kappa}}}\\
\lesssim& 2^{k_1}\sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}}\|Q_{\prec k_1-2l_2}\tilde{P}_{\kappa_2}u_2\|_{S[k_2,\kappa_2]}\Big(\sum_{\kappa\in\mathcal{K}_{k_1}}\|P_\kappa\tilde{P}_{\kappa_1}u_1\|^2_{L^2_{t_{k_1,\kappa}}L^\infty_{x_{k_1,\kappa}}}\Big)^{\frac12}\\
\lesssim& 2^{k_1}\Big(\sum_{\kappa_2\in\mathcal{K}_{l_2}}\|Q_{\prec k_1-2l_2}\tilde{P}_{\kappa_2}u_2\|^2_{S[k_2,\kappa_2]}\Big)^{\frac12}\Big(\sum_{\kappa\in\mathcal{K}_{k_1}}\|P_\kappa u_1\|^2_{L^2_{t_{k_1,\kappa}}L^\infty_{x_{k_1,\kappa}}}\Big)^{\frac12}\\
\lesssim& 2^{k_1}\|u_1\|_{S^+_{k_1}}\|u_2\|_{S^{+,w}_{k_2}}.
\end{aligned}
$$

If $k_1-10\leq l_1\leq k_1$, then the argument is entirely similar, but for the A_3 contribution we use $L_t^2L_x^\infty$ and $L_t^\infty L_x^2$.

Second case: $k_1\geq k_2$. The argument above works the same way for $l_1\leq k_2-10$. Consider now the case $k_2-10\leq l_1\leq k_2$. Again, the contributions analogous to A_0, A_1 and A_2 can be treated in the same way (now, the modulation threshold is k_2-2l_j). In the case of A_3 (low modulation), we face the problem that $\|P_{\kappa_1}u_1\|_{L^2_tL^\infty_x}$ gives suboptimal bounds, because $\kappa_1\in\mathcal{K}_{l_1}$ with $l_1\approx k_2$ instead of k_1. Therefore, we decompose

$$\tilde{P}_{\kappa_1}u_1=\sum_{\kappa\in\mathcal{K}_{k_1}}P_\kappa\tilde{P}_{\kappa_1}u_1$$

and note that the interactions $P_\kappa\tilde{P}_{\kappa_1}u_1\tilde{P}_{\kappa_2}u_2$ are almost orthogonal with respect to $\kappa\in\mathcal{K}_{k_1}$. Indeed this follows from the fact that both $P_\kappa\tilde{P}_{\kappa_1}u_1$ and $\tilde{P}_{\kappa_2}u_2$ have Fourier-support of size ≈ 1 in the orthogonal directions to $\omega(\kappa_2)$. Thus

$$\begin{aligned}&\|\tilde{P}_{\kappa_1}Q_{\prec k_2-2l_1}u_1\cdot\tilde{P}_{\kappa_2}Q_{\prec k_2-2l_2}u_2\|_{L^2}^2\\ \lesssim&\sum_{\kappa\in\mathcal{K}_{k_1}}\|P_\kappa\tilde{P}_{\kappa_1}Q_{\prec k_2-2l_1}u_1\cdot\tilde{P}_{\kappa_2}Q_{\prec k_2-2l_2}u_2\|_{L^2}^2\\ \lesssim&\sum_{\kappa\in\mathcal{K}_{k_1}}\|P_\kappa\tilde{P}_{\kappa_1}Q_{\prec k_2-2l_1}u_1\|_{L^2_tL^\infty_x}^2\cdot\|\tilde{P}_{\kappa_2}Q_{\prec k_2-2l_2}u_2\|_{L^\infty_tL^2_x}^2.\end{aligned}$$

For the contribution A_3, we obtain the bound

$$\begin{aligned}&\sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}}\Big(\sum_{\kappa\in\mathcal{K}_{k_1}}\|P_\kappa\tilde{P}_{\kappa_1}Q_{\prec k_2-2l_1}u_1\|_{L^2_tL^\infty_x}^2\cdot\|\tilde{P}_{\kappa_2}Q_{\prec k_2-2l_2}u_2\|_{L^\infty_tL^2_x}^2\Big)^{\frac12}\\ \lesssim&\sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}}\Big(\sum_{\kappa\in\mathcal{K}_{k_1}}\|P_\kappa\tilde{P}_{\kappa_1}Q_{\prec k_2-2l_1}u_1\|_{L^2_tL^\infty_x}^2\Big)^{\frac12}\|Q_{\prec k_2-2l_2}\tilde{P}_{\kappa_2}u_2\|_{S[k_2,\kappa_2]}\\ \lesssim&\Big(\sum_{\kappa\in\mathcal{K}_{k_1}}\|P_\kappa Q_{\prec k_2-2l_1}u_1\|_{L^2_tL^\infty_x}^2\Big)^{\frac12}\Big(\sum_{\kappa_2\in\mathcal{K}_{l_2}}\|Q_{\prec k_2-2l_2}\tilde{P}_{\kappa_2}u_2\|_{S[k_2,\kappa_2]}^2\Big)^{\frac12}\\ \lesssim&\,2^{k_1}\|u_1\|_{S^+_{k_1}}\|u_2\|_{S^{+,w}_{k_2}}.\end{aligned}$$

The proof of the claim (5.6) is now complete.

As an immediate consequence of the above argument we obtain

$$\sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}}\|\tilde{P}_{\kappa_1}u_1\overline{\tilde{P}_{\kappa_2}u_2}\|_{L^2}\lesssim 2^{k_1}\|u_1\|_{S^+_{k_1}}\|u_2\|_{S^{+,w}_{k_2}}.\tag{5.7}$$

Now, we turn to the proof of (5.1). Using (5.7) we claim the following

$$\begin{aligned}&\sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}}\|\langle\Pi_+(D)P_{\kappa_1}\psi_1,\beta\Pi_+(D)P_{\kappa_2}\psi_2\rangle\|_{L^2}\\ &\lesssim 2^{k_1-l_1}\|\Pi_+(D)P_{k_1,\kappa_1}\psi_1\|_{S^+_{k_1}}\|\Pi_+(D)P_{k_2,\kappa_2}\psi_2\|_{S^{+,w}_{k_2}},\end{aligned}\tag{5.8}$$

where the sum is restricted to the range $d(\kappa_1,\kappa_2)\approx 2^{-l_1}$ or $d(\kappa_1,\kappa_2)\lesssim 2^{-l_1}$ in the case $|l_1-\min(k_1,k_2)|\le 2$. To prove (5.8), we linearize the operator $\Pi_+(D)$ as follows

$$\Pi_+(D) = \Pi_+(2^{k_j}\omega(\kappa_j)) + \Pi_+(D) - \Pi_+(2^{k_j}\omega(\kappa_j))$$

where $j = 1, 2$. Taking into account (5.7) and (3.5) we obtain

$$\begin{aligned} &\|\langle \Pi_+(2^{k_1}\omega(\kappa_1))P_{\kappa_1}\psi_1, \beta\Pi_+(2^{k_2}\omega(\kappa_2))P_{\kappa_2}\psi_2\rangle\|_{L^2} \\ \lesssim\, & 2^{k_1-l_1}\|P_{\kappa_1}\psi_1\|_{S^+_{k_1}}\|P_{\kappa_2}\psi_2\|_{S^{+,w}_{k_2}} \end{aligned}$$

where we have used $|\angle(\omega(\kappa_1), \omega(\kappa_2))| \lesssim 2^{-l_1}$ and that

$$\mathcal{O}(2^{-k_1} + 2^{-k_2}) \lesssim 2^{-\min(k_1,k_2)} \lesssim 2^{-l_1}.$$

The estimate for the remaining terms follows from using (5.7) and (4.7). By organizing the interacting factors based on their angle of interaction we have

$$\begin{aligned} &\|\langle \Pi_+(D)\psi_1, \beta\Pi_+(D)\psi_2\rangle\|_{L^2} \\ \lesssim & \sum_{|l_1-l_2|\le 2}\ \sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}} \|\langle P_{\kappa_1}\Pi_+(D)\psi_1, \beta P_{\kappa_2}\Pi_+(D)\psi_2\rangle\|_{L^2} \end{aligned}$$

where the first sum is restricted over the range $1 \le l_1, l_2 \le \min(k_1, k_2)$, and the second sum is restricted over the range $\mathrm{d}(\kappa_1, \kappa_2) \approx 2^{-l_1}$ or $\mathrm{d}(\kappa_1, \kappa_2) \lesssim 2^{-l_1}$ in the case $|l_1 - \min(k_1, k_2)| \le 2$. The result for the second sum follows from the (5.8). The first sum, with respect to l_1 (the one with respect to l_2 is redundant), is performed using the factor of 2^{-l_1}.

The proof of (5.2) is entirely similar, expect that in the decomposition above one imposes the range $l \le l_1, l_2 \le \min(k_1, k_2)$ on the first sum and picks up the additional factor of 2^{-l}.

Finally, we turn to the proof of (5.3). Fix l_1, l_2 with $|l_1 - l_2| \le 2$, $1 \le l_1 \le k_1$, $\kappa_1 \in \mathcal{K}_{l_1}$, $\kappa_2 \in \mathcal{K}_{l_2}$ with $\mathrm{d}(\kappa_1, \kappa_2) \approx 2^{-l_1}$ or $\mathrm{d}(\kappa_1, \kappa_2) \lesssim 2^{-l_1}$ in the case $|l_1 - k_1| \le 2$. The proof of (5.7) yields

$$\begin{aligned} &\sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}} \|\tilde{P}_{\kappa_1}u_1\overline{\tilde{P}_{\kappa_2}u_2}\|_{L^2_tL^q_x} \\ \lesssim\, & 2^{(\frac{1}{2}-\frac{1}{q})(3k_2-2l_1)} \sum_{\kappa_1\in\mathcal{K}_{l_1},\kappa_2\in\mathcal{K}_{l_2}} \|\tilde{P}_{\kappa_1}u_1\overline{\tilde{P}_{\kappa_2}u_2}\|_{L^2_tL^2_x} \\ \lesssim\, & 2^{k_1}2^{(\frac{1}{2}-\frac{1}{q})(3k_2-2l_1)}\|u_1\|_{S^+_{k_1}}\|u_2\|_{S^{+,w}_{k_2}}, \end{aligned}$$

where the sum is restricted to caps satisfying $\mathrm{d}(\kappa_1, \kappa_2) \approx 2^{-l_1}$ or $\mathrm{d}(\kappa_1, \kappa_2) \lesssim 2^{-l_1}$ in the case $|l_1 - k_1| \le 2$. Summing this inequality with respect to l_1, l_2 gives (5.3). □

6. The Dirac nonlinearity

In this section we use the theory developed in the previous section to prove the global well-posedness of the Dirac equation with initial data in $H^1(\mathbb{R}^3)$. Throughout this section we abuse notation and set $S_{99}^{\pm} := S_{\leq 99}^{\pm}$, redefine $P_{99} := P_{\leq 99}$, $\tilde{P}_{99} := \tilde{P}_{\leq 99}$, and thus by saying that a function is localized at frequency 2^{99} we mean that it is localized at frequency $\leq 2^{99}$.

The main result of this section is the following

Theorem 6.1. *Choose $s_1, s_2, s_3, s_4 \in \{+,-\}$. Then, for all $\psi_k \in S^{s_k,1}$ satisfying $\psi_k = \Pi_{s_k}(D)\psi_k$ for $k = 1, 2, 3$, we have*

$$\|\Pi_{s_4}(D)(\langle\psi_1, \beta\psi_2\rangle\beta\psi_3)\|_{N^{s_4,1}} \lesssim \|\psi_1\|_{S^{s_1,1}}\|\psi_2\|_{S^{s_2,1}}\|\psi_3\|_{S^{s_3,1}}. \tag{6.1}$$

The rest of this section is devoted to the proof of Theorem 6.1 and the proof of our main result Theorem 1.1. The estimate (6.1) will be derived from similar estimates for frequency localized functions. Our aim will be to identify a function $G(k_1, k_2, k_3, k_4) : \mathbb{N}_{\geq 99}^4 \to (0, \infty)$ such that

$$\sum_{k_1,k_2,k_3,k_4 \in \mathbb{N}_{\geq 99}} G(k_1, k_2, k_3, k_4) a_{k_1} b_{k_2} c_{k_3} d_{k_4} \lesssim \|a\|_{l^2}\|b\|_{l^2}\|c\|_{l^2}\|d\|_{l^2} \tag{6.2}$$

for all sequences $a = (a_j)_{j\in\mathbb{N}_{\geq 99}}$, etc, in l^2. Here $\mathbb{N}_{\geq 99} = \{n \in \mathbb{N} | n \geq 99\}$. We set $\mathbf{k} = (k_1, k_2, k_3, k_4)$.

With these notations, the result of Theorem 6.1 follows from

Proposition 6.2. *There exists a function G satisfying* (6.2) *such that if ψ_j are localized at frequency 2^{k_j}, $k_j \geq 99$ and $\psi_j = \Pi_{s_j}(D)\psi_j$ for $j = 1, \dots, 4$, then the following holds true*

$$2^{k_4}\|P_{k_4}\Pi_{s_4}(D)(\langle\psi_1, \beta\psi_2\rangle\beta\psi_3)\|_{N_{k_4}^{s_4}} \lesssim G(\mathbf{k}) \prod_{j=1}^{3} 2^{k_j}\|\psi_j\|_{S_{k_j}^{s_j}}, \tag{6.3}$$

for any choice of sign $s_1, s_2, s_3, s_4 \in \{+,-\}$.

We break this down into two building blocks:

Lemma 6.3. *Under the assumptions of Proposition 6.2 the following estimate holds true:*

$$2^{\frac{3}{4}k_4}\|P_{k_4}(\langle\psi_1, \beta\psi_2\rangle\beta\psi_3)\|_{L_t^{\frac{4}{3}}L_x^2} \lesssim G(\mathbf{k}) \prod_{j=1}^{3} 2^{k_j}\|\psi_j\|_{S_{k_j}^{s_j}}. \tag{6.4}$$

Lemma 6.4. *Under the assumptions of Proposition 6.2 the following estimates hold true:*

$$\begin{aligned} &\Big| \int \langle \psi_1, \beta\psi_2\rangle \cdot \langle \psi_3, \beta\psi_4\rangle dxdt \Big| \\ \lesssim\ & G(\mathbf{k}) \prod_{j=1}^{3} 2^{k_j} \|\psi_j\|_{S^{s_j}_{k_j}} \cdot 2^{-k_4} \|\psi_4\|_{S^{s_4,w}_{k_4}}, \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} &\Big| \int \langle \psi_1, \beta\psi_2\rangle \cdot \langle \psi_3, \frac{\psi_4}{\langle D\rangle}\rangle dxdt \Big| \\ \lesssim\ & G(\mathbf{k}) \prod_{j=1}^{3} 2^{k_j} \|\psi_j\|_{S^{s_j}_{k_j}} \cdot 2^{-k_4} \|\psi_4\|_{S^{s_4,w}_{k_4}}. \end{aligned} \tag{6.6}$$

Before we provide proofs of Lemma 6.3 and Lemma 6.4, we show how these imply Proposition 6.2.

Proof of Prop. 6.2. Given the structure of the $N^{s_4}_{k_4}$, (6.4) is simply the $L^{\frac{4}{3}}_t L^2_x$ part of (6.3). We owe an explanation for why (6.5) and (6.6) imply the atomic part of (6.3). The nonlinearity

$$\mathcal{N} = P_{k_4}\Pi_{s_4}(D)(\langle \psi_1, \beta\psi_2\rangle \beta\psi_3)$$

satisfies $\mathcal{N} = \Pi_{s_4}(D)\mathcal{N}$ and needs to be estimated in $N^{s_4}_{k_4}$. Using (4.5), it is enough to test $\Pi_{s_4}(D)\mathcal{N}$ against $\psi_4 \in S^{-s_4,w}_{k_4}$ and to prove the bound

$$\int \langle \Pi_{s_4}(D)\mathcal{N}, \psi_4\rangle dxdt \lesssim G(\mathbf{k}) \prod_{j=1}^{3} 2^{k_j} \|\psi_j\|_{S^{s_j}_{k_j}} \cdot 2^{-k_4} \|\psi_4\|_{S^{-s_4,w}_{k_4}}. \tag{6.7}$$

We have

$$\begin{aligned} \int \langle \Pi_{s_4}(D)\mathcal{N}, \psi_4\rangle dx &= \int \langle \hat{\mathcal{N}}(\xi), \Pi_{s_4}(\xi)\hat{\psi}_4(-\xi)\rangle d\xi \\ &= \int \langle \hat{\mathcal{N}}(\xi), (\Pi_{-s_4}(-\xi) - s_4\frac{\beta}{\langle \xi\rangle})\hat{\psi}_4(-\xi)\rangle d\xi \\ &= \int \langle \mathcal{N}, \Pi_{-s_4}(D)\psi_4\rangle dx - s_4 \int \langle \mathcal{N}, \frac{\beta}{\langle D\rangle}\psi_4\rangle dx \end{aligned}$$

The contribution of the first term to (6.7) is

$$\begin{aligned}\int \langle \mathcal{N}, \Pi_{-s_4}(D)\psi_4\rangle dxdt &= \int \langle\langle \psi_1, \beta\psi_2\rangle\beta\psi_3, \Pi_{-s_4}(D)\psi_4\rangle dxdt\\ &= \int \langle \psi_1, \beta\psi_2\rangle\langle\beta\psi_3, \Pi_{-s_4}(D)\psi_4\rangle dxdt\\ &= \int \langle \psi_1, \beta\psi_2\rangle\langle\psi_3, \beta\Pi_{-s_4}(D)\psi_4\rangle dxdt.\end{aligned}$$

By splitting each $\psi_j = \Pi_+(D)\psi_j + \Pi_-(D)\psi_j$, its contribution to (6.7) follows from (6.5). The reason why the contribution of the second term above to (6.7) is provided by (6.6) is similar. □

Proof of Lemma 6.3. We prove the result by using Strichartz type estimates only, thus we can drop all the $\pm$ and simply use scalar functions u_j localized at frequency 2^{k_j} instead. The argument is symmetric with respect to k_1, k_2, k_3, hence we can simply assume that $k_1 \le k_2 \le k_3$. Then, the l.h.s. of (6.4) vanishes unless $k_4 \le k_3 + 10$, and by using (5.3), (5.4) and (5.5) we obtain

$$\begin{aligned}\|u_1u_2u_3\|_{L_t^{\frac{4}{3}}L_x^2} \lesssim &\|u_1\|_{L_t^4L_x^{24}}\|u_2u_3\|_{L_t^2L_x^{\frac{24}{11}}}\\ &\lesssim 2^{\frac{27}{24}k_1+k_2+\frac{1}{8}k_3}\|u_1\|_{S_{k_1}}\|u_2\|_{S_{k_2}}\|u_3\|_{S_{k_3}}.\end{aligned}$$

From this we obtain

$$2^{\frac{3}{4}k_4}\|u_1u_2u_3\|_{L_t^{\frac{4}{3}}L_x^2} \lesssim 2^{\frac{k_1-7k_3+6k_4}{8}}2^{k_1}\|u_1\|_{S_{k_1}}2^{k_2}\|u_2\|_{S_{k_2}}2^{k_3}\|u_3\|_{S_{k_3}}$$

from which (6.4) follows, because the value of $G(\mathbf{k}) = 2^{\frac{k_1-7k_3+6k_4}{8}}$ is acceptable for $k_4 \le k_3 + 10$. □

It remains to prove Lemma 6.4. Before we start to do so, we analyze the modulation of a product of two waves. We consider two functions $\psi_1, \psi_2 \in S^+$ where their native modulation is with respect to the quantity $|\tau - \langle\xi\rangle|$. However, for $\langle\psi_1, \beta\psi_2\rangle$ we quantify the output modulation with respect to $||\tau| - \langle\xi\rangle|$. The following lemma contains the modulation localization claim which will be used several times in the argument.

Lemma 6.5. *Let $k, k_1k_2 \ge 100$ and $l \prec \min(k_1, k_2)$, and let $\kappa_1, \kappa_2 \in \mathcal{K}_l$, with $\mathrm{d}(\kappa_1, \kappa_2) \approx 2^{-l}$, and assume that $u_j = \tilde{P}_{k_j,\kappa_j}\tilde{Q}^+_{\prec m}u_j$, where*

$$m = k_1 + k_2 - k - 2l.$$

Then,

$$\widehat{P_k(u_1\overline{u_2})}(\tau, \xi) = 0 \text{ unless } ||\tau| - \langle\xi\rangle| \approx 2^m.$$

Proof. Since the modulation of the inputs are much less than the claimed modulation of the output it is enough to prove the argument for free solutions. Let $(\xi_1, \langle \xi_1 \rangle)$ be in the support of $\hat{u}_1$ and $(-\xi_2, -\langle \xi_2 \rangle)$ be in the support of $\overline{\hat{u}_2}$. Then, the angle between ξ_1 and ξ_2 is $\approx 2^{-l}$. Let $\xi = \xi_1 - \xi_2$ be of size 2^k and $\tau = \langle \xi_1 \rangle - \langle \xi_2 \rangle$. Our aim is to prove that

$$|\langle \xi_1 - \xi_2 \rangle - |\langle \xi_1 \rangle - \langle \xi_2 \rangle|| \approx 2^m.$$

The claim follows from

$$\begin{aligned}
\langle \xi_1 - \xi_2 \rangle - |\langle \xi_1 \rangle - \langle \xi_2 \rangle| &= \frac{\langle \xi_1 - \xi_2 \rangle^2 - (\langle \xi_1 \rangle - \langle \xi_2 \rangle)^2}{\langle \xi_1 - \xi_2 \rangle + |\langle \xi_1 \rangle - \langle \xi_2 \rangle|} \\
=& \frac{2|\xi_1||\xi_2|(1 - \cos(\angle(\xi_1, \xi_2)))}{\langle \xi_1 - \xi_2 \rangle + |\langle \xi_1 \rangle - \langle \xi_2 \rangle|} + \mathcal{O}(2^{-\min(k,k_1,k_2)}) \\
\approx& 2^{k_1+k_2-k} \angle(\xi_1, \xi_2)^2 + \mathcal{O}(2^{-\min(k,k_1,k_2)}).
\end{aligned}$$

because by assumption we have $2^{k_1+k_2-k-2l} \gg 2^{-\min(k,k_1,k_2)}$. □

Proof of Lemma 6.4. It will be obvious from the proof of (6.5) that the same argument works for (6.6) as well. The basic idea in (6.6) is that the null condition is missing in the term $\langle \psi_3, \frac{1}{\langle D \rangle} \psi_4 \rangle$. On the other hand the factor $\frac{1}{\langle D \rangle}$ brings a gain of 2^{-k_4} in all estimates which is better than all gains from exploiting the null condition in $\langle \psi_3, \beta \psi_4 \rangle$.

Given the choices of sign in (6.5) there are a total of 16 cases. The first major block in the proof is the use of the results in Proposition 5.1 which are symmetric with respect to the choice of $\pm$. The second building block employs frequency and modulation localization, Strichartz and Sobolev estimates and it works again the same way for different choices of $\pm$ in the estimate above. This is why we choose to prove the above estimate for the $+$ choice in all terms. It will become evident from the argument that the same reasoning will work in all other cases. Thus we can drop all the $\pm$ and simply consider $\psi_j \in S^+_{k_j}$ and write $S_{k_j} = S^+_{k_j}$ instead.

For brevity, we denote the l.h.s. of (6.5) as

$$I := \left| \int \langle \psi_1, \beta \psi_2 \rangle \cdot \langle \psi_3, \beta \psi_4 \rangle dx dt \right|$$

and the standard factor on the r.h.s. as

$$J := \prod_{j=1}^{3} 2^{k_j} \|\psi_j\|_{S_{k_j}} \cdot 2^{-k_4} \|\psi_4\|_{S^w_{k_4}}.$$

Since the expression I computes the zero mode of the product $\langle \psi_1, \beta \psi_2 \rangle \cdot \langle \psi_3, \beta \psi_4 \rangle$, it follows that $\langle \psi_1, \beta \psi_2 \rangle$ and $\langle \psi_3, \beta \psi_4 \rangle$ need to be localized at frequencies and modulations of comparable size, where the modulation is computed with respect to $||\tau| - \langle\xi\rangle|$. This will be repeatedly used in the argument below along with the convention that the modulations of $\psi_k, k = 1, \dots, 4$ are with respect to $|\tau - \langle\xi\rangle|$, while the modulations of $\langle\psi_1, \beta\psi_2\rangle$ and $\langle\psi_3, \beta\psi_4\rangle$ are with respect to $||\tau| - \langle\xi\rangle|$.

We also agree that by the angle of interaction in, say, $\langle\psi_1, \beta\psi_2\rangle$ we mean the angle made by the frequencies in the support of $\hat{\psi}_1$ and $\hat{\psi}_2$, where we consider only the supports that bring nontrivial contributions to I.

We organize the argument based on the size of the frequencies.

Case 1: $k_4 \leq \min(k_1, k_2, k_3) + 10.$
Using (5.1) we obtain the bound

$$I \lesssim 2^{k_4 - \max(k_1,k_2)} J,$$

and since $|\max(k_1, k_2) - \max(k_1, k_2, k_3)| \leq 12$ we obtain (6.5) in this case.

Case 2: there are at exactly two $i \in \{1, 2, 3\}$ such that $k_4 \leq k_i + 10.$
Case 2 a) Assume that $k_3 \geq k_4 - 10$. Since the argument is symmetric in k_1 and k_2, it is enough to consider the scenario $k_1 < k_4 - 10 \leq k_2$. Note that $|k_2 - k_3| \leq 12$.

We claim that either the angle of interactions in $\langle\psi_3, \beta\psi_4\rangle$ is $\lesssim 2^{\frac{k_1-k_4}{8}}$ or at least one factor $\psi_j, j = 1, .., 4$ has modulation $\gtrsim 2^{\frac{k_1+3k_4}{4}}$. To see this, suppose that the claim is false. Then, the modulation of $\langle\psi_1, \beta\psi_2\rangle$ is $\lesssim 2^{\frac{k_1+3k_4}{4}}$ while it follows from Lemma 6.5 that the modulation of $\langle\psi_3, \beta\psi_4\rangle$ is $\gg 2^{\frac{k_1+3k_4}{4}}$. This is not possible, hence the claim is true. Note that in using Lemma 6.5 we are assuming that $k_3, k_4 \geq 100$. If this is not the case, that is either $k_3 = 99$ or $k_4 = 99$, the argument in Case 1 can be used to obtain the desired estimate.

In the first subcase, where the angle of interaction in $\langle\psi_3, \beta\psi_4\rangle$ is smaller than $2^{\frac{k_1-k_4}{8}}$, we use (5.2) to obtain $I \lesssim 2^{\frac{k_1-k_4}{8}} 2^{k_4-k_2} J$ and this is fine.

We now consider the second subcase, in which the modulation of the factor ψ_j is $\gtrsim 2^{\frac{k_1+3k_4}{4}} \gtrsim 2^{\frac{k_1+k_4}{2}}$ for some $j \in \{1, 2, 3, 4\}$:

$j = 1$: Since ψ_1 has modulation $\gtrsim 2^{\frac{k_1+k_4}{2}}$, we can use (5.1) to estimate $\|\langle\psi_3, \beta\psi_4\rangle\|_{L^2}$ and the Sobolev embedding for ψ_1 to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^2_t L^\infty_x} \|\psi_2\|_{L^\infty_t L^2_x} 2^{k_3} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{3}{2}k_1} \|\psi_1\|_{L^2} \|\psi_2\|_{L^\infty_t L^2_x} 2^{k_3} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{k_1-k_4}{4}} 2^{k_4-k_2} J. \end{aligned}$$

$j=2$: Since ψ_2 has modulation $\gtrsim 2^{\frac{k_1+k_4}{2}}$, (5.1) and Sobolev embedding for ψ_1 yields

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^\infty}\|\psi_2\|_{L^2}2^{k_3}\|\psi_3\|_{S_{k_3}}\|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{3}{2}k_1}\|\psi_1\|_{L^\infty_t L^2_x}2^{-\frac{k_1+k_4}{4}}\|\psi_2\|_{S_{k_2}}2^{k_3}\|\psi_3\|_{S_{k_3}}\|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{k_1-k_4}{4}}2^{k_4-k_2}J. \end{aligned}$$

$j=3$: Because ψ_3 has modulation $\gtrsim 2^{\frac{k_1+3k_4}{4}}$, we employ (5.4) to estimate $\|\langle\psi_1,\beta\psi_2\rangle\|_{L^2}$ and the Sobolev embedding for ψ_4 to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^4_t L^\infty_x}\|\psi_2\|_{L^4_t L^6_x}\|\psi_3\|_{L^2}\|\psi_4\|_{L^\infty_t L^3_x} \\ &\lesssim 2^{\frac{5}{4}k_1}\|\psi_1\|_{S_{k_1}}2^{\frac{3}{4}k_2}\|\psi_2\|_{S_{k_2}}2^{-\frac{k_1+3k_4}{8}}\|\psi_3\|_{S_{k_3}}2^{\frac{k_4}{2}}\|\psi_4\|_{L^\infty_t L^2_x} \\ &\lesssim 2^{\frac{k_1-k_4}{8}}2^{k_4-k_3}J. \end{aligned}$$

$j=4$: Here, ψ_4 has modulation $\gtrsim 2^{\frac{k_1+k_4}{2}}$ and we use (5.4) and the Sobolev embedding for ψ_1 to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^\infty_t L^{12}_x}\|\psi_2\|_{L^4_t L^{\frac{24}{5}}_x}\|\psi_3\|_{L^4_t L^{\frac{24}{5}}_x}\|\psi_4\|_{L^2} \\ &\lesssim 2^{\frac{5}{4}k_1}\|\psi_1\|_{L^\infty_t L^2_x}2^{\frac{5}{8}k_2}\|\psi_2\|_{S_{k_2}}2^{\frac{5}{8}k_3}\|\psi_3\|_{S_{k_3}}2^{-\frac{k_1+3k_4}{8}}\|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{k_1-k_4}{8}}2^{\frac{k_4-k_2}{2}}J. \end{aligned}$$

Case 2 b) Assume now that $k_3 \le k_4+10$, hence $k_1,k_2 \ge k_4+10$ and $|k_1-k_2|\le 12$. Since $\langle\psi_1,\beta\psi_2\rangle$ is localized at frequency $\approx 2^{k_4}$, the angle of interaction in $\langle\psi_1,\beta\psi_2\rangle$ is $\lesssim 2^{k_4-k_2}$. Moreover, we claim that either the angle of interactions in $\langle\psi_1,\beta\psi_2\rangle$ is $\lesssim 2^{\frac{k_3-k_4}{8}}2^{k_4-k_2}$ or at least one factor $\psi_j, j=1,..,4$ has modulation $\gtrsim 2^{\frac{k_3+3k_4}{4}}$. Indeed, if the claim is false, it follows from Lemma 6.5 that the modulation of $\langle\psi_1,\beta\psi_2\rangle$ is $\gg 2^{\frac{k_3+3k_4}{4}}$ while the modulation of $\langle\psi_3,\beta\psi_4\rangle$ is $\ll 2^{\frac{k_3+3k_4}{4}}$. This is not possible, hence the claim is true. Note that in using Lemma 6.5 we are assuming that $k_1,k_2\ge 100$. If this is not the case, that is either $k_1=99$ or $k_2=99$, the argument in Case 1 can be used to obtain the desired estimate.

In the first subcase the angle of interaction in $\langle\psi_1,\beta\psi_2\rangle$ is smaller than $2^{\frac{k_3-k_4}{8}}2^{k_4-k_2}$. Then, we use (5.2) to obtain $I \lesssim 2^{\frac{k_3-k_4}{8}}2^{2(k_4-k_2)}J$ which is acceptable.

In the second subcase, where at least one modulation is high, we proceed in a similar manner to Case 2b) above. In fact the estimates bring improved factors if one takes into account that the angle of interaction in $\langle\psi_1,\beta\psi_2\rangle$ is $\lesssim 2^{k_4-k_2}$. The details are left to the reader.

Case 3: $|k_2 - k_4| \le 2$ and $k_1, k_3 \le k_4 - 10$. Without restricting the generality of the argument, we may assume that $k_1 \le k_3$.

We claim that either the angle of interaction in $\langle \psi_3, \beta\psi_4 \rangle$ is $\lesssim 2^{\frac{k_1-k_3}{16}}$ or one factor $\psi_j, j = 1,..,4$ has modulation $\gtrsim 2^{\frac{k_1+7k_3}{8}}$. Indeed, if all modulations of the functions involved are $\ll 2^{\frac{k_1+7k_3}{8}}$, then $\langle \psi_1, \beta\psi_2 \rangle$ is localized at modulation $\lesssim 2^{\frac{k_1+7k_3}{8}}$. This forces $\langle \psi_3, \beta\psi_4 \rangle$ to be localized at modulation $\lesssim 2^{\frac{k_1+7k_3}{8}}$, hence the angle of interaction is $\lesssim 2^{\frac{k_1-k_3}{16}}$ by Lemma 6.5. Note that in using Lemma 6.5 we are assuming that $k_3, k_4 \ge 100$. If this is not the case, that is either $k_3 = 99$ or $k_4 = 99$, the argument in Case 1 can be used to obtain the desired estimate.

In the first subcase, when the angle of interaction in $\langle \psi_3, \beta\psi_4 \rangle$ is $\lesssim 2^{\frac{k_1-k_3}{16}}$, we use (5.2) to obtain

$$I \lesssim 2^{\frac{k_1-k_3}{16}} 2^{k_1} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{k_3} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \lesssim 2^{\frac{k_1-k_3}{16}} J.$$

Next, we consider the second subcase when the factor ψ_j has modulation $\gtrsim 2^{\frac{k_1+7k_3}{8}} \gtrsim 2^{\frac{k_1+3k_3}{4}}$ for some $j \in \{1, 2, 3, 4\}$:

$j = 1$: The modulation of ψ_1 is $\gtrsim 2^{\frac{k_1+3k_3}{4}}$, so we use Sobolev embedding for ψ_1 and (5.1) for $\langle \psi_3, \beta\psi_4 \rangle$ to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^2_t L^\infty_x} \|\psi_2\|_{L^\infty_t L^2_x} 2^{k_3} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{3}{2}k_1} \|\psi_1\|_{L^2} \|\psi_2\|_{S_{k_2}} 2^{k_3} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{3}{2}k_1} 2^{-\frac{k_1+3k_3}{8}} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{k_3} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{k_1-k_3}{8}} J. \end{aligned}$$

$j = 2$: Here, the modulation of ψ_2 is $\gtrsim 2^{\frac{k_1+3k_3}{4}}$ and we proceed as above to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^\infty} \|\psi_2\|_{L^2} 2^{k_3} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{3}{2}k_1} \|\psi_1\|_{L^\infty_t L^2_x} 2^{-\frac{k_1+3k_3}{8}} \|\psi_2\|_{S_{k_2}} 2^{k_3} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{k_1-k_3}{8}} J. \end{aligned}$$

$j=3$: The modulation of ψ_3 is $\gtrsim 2^{\frac{k_1+7k_3}{8}}$, so we use (5.4) and the Sobolev embedding for ψ_3 to obtain

$$\begin{aligned}
I &\lesssim \|\psi_1\|_{L_t^4 L_x^\infty} \|\psi_2\|_{L_t^\infty L_x^2} \|\psi_3\|_{L_t^{\frac{4}{3}} L_x^\infty} \|\psi_4\|_{L_t^\infty L_x^2} \\
&\lesssim 2^{\frac{5}{4}k_1} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{\frac{3}{2}k_3} \|\psi_3\|_{L_t^{\frac{4}{3}} L_x^2} \|\psi_4\|_{S^w_{k_4}} \\
&\lesssim 2^{\frac{5}{4}k_1} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{\frac{3}{2}k_3} 2^{\frac{k_3}{4}} 2^{-\frac{k_1+7k_3}{8}} \|\psi_3\|_{S_{k_3}} \|\psi_4\|_{S^w_{k_4}} \\
&\lesssim 2^{\frac{k_1-k_3}{8}} J.
\end{aligned}$$

$j=4$: Since the modulation of ψ_4 is $\gtrsim 2^{\frac{k_1+3k_3}{4}}$, we use (5.4) to obtain

$$\begin{aligned}
I &\lesssim \|\psi_1\|_{L_t^4 L_x^\infty} \|\psi_2\|_{L_t^\infty L_x^2} \|\psi_3\|_{L_t^4 L_x^\infty} \|\psi_4\|_{L^2} \\
&\lesssim 2^{\frac{5}{4}k_1} \|\psi_1\|_{S_{k_1}} \|\psi_2\|_{S_{k_2}} 2^{\frac{5}{4}k_3} \|\psi_3\|_{S_{k_3}} 2^{-\frac{k_1+3k_3}{8}} \|\psi_4\|_{S^w_{k_4}} \\
&\lesssim 2^{\frac{k_1-k_3}{8}} J.
\end{aligned}$$

<u>Case 4: $|k_3-k_4| \leq 2$ and $k_1, k_2 \leq k_4 - 10$</u>. Without loss of generality we assume $k_1 \leq k_2$.

The key observation is that either the angle of interaction between ψ_3 and ψ_4 is $\lesssim 2^{\frac{k_1-k_2}{16}} 2^{k_2-k_3}$ or at least one factor has modulation $\gtrsim 2^{\frac{k_1+7k_2}{8}}$. Indeed, if all modulations are $\ll 2^{\frac{k_1+7k_2}{8}}$, then the modulation of $\langle \psi_1, \beta\psi_2\rangle$ is $\lesssim 2^{\frac{k_1+7k_2}{8}}$ and Lemma 6.5 implies the claim. Note that in using Lemma 6.5 we are assuming that $k_3, k_4 \geq 100$. If this is not the case, that is either $k_3 = 99$ or $k_4 = 99$, the argument in Case 1 can be used to obtain the desired estimate.

In the first subcase, when the angle of interaction between ψ_3 and ψ_4 is $\lesssim 2^{\frac{k_1-k_2}{16}} 2^{k_2-k_3}$, we use (5.2) to obtain

$$I \lesssim 2^{\frac{k_1-k_2}{16}} J.$$

In the second subcase, ψ_j has modulation $\gtrsim 2^{\frac{k_1+7k_2}{8}}$ for some $j \in \{1,2,3,4\}$. Since the output of $\langle \psi_3, \beta\psi_4\rangle$ is localized at frequency $\lesssim 2^{k_2}$ it follows that the angle of interaction is $\lesssim 2^{k_2-k_3}$. This will be used in the following case-by-case analysis:

$j=1$: The modulation of ψ_1 is $\gtrsim 2^{\frac{k_1+7k_3}{8}}$, so we use Sobolev embedding for ψ_1 and (5.2) for $\langle\psi_3,\beta\psi_4\rangle$ to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L_t^2L_x^\infty}\|\psi_2\|_{L_t^\infty L_x^2}\|\langle\psi_3,\beta\psi_4\rangle\|_{L^2} \\ &\lesssim 2^{\frac{3}{2}k_1}\|\psi_1\|_{L^2}\|\psi_2\|_{S_{k_2}}2^{k_3}2^{k_2-k_3}\|\psi_3\|_{S_{k_3}}\|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{\frac{3}{2}k_1}2^{-\frac{k_1+7k_2}{16}}\|\psi_1\|_{S_{k_1}}\|\psi_2\|_{S_{k_2}}2^{k_2}\|\psi_3\|_{S_{k_3}}\|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{-\frac{7}{16}(k_2-k_1)}J. \end{aligned}$$

$j=2$: The modulation of ψ_2 is $\gtrsim 2^{\frac{k_1+7k_3}{8}}$, so we use Sobolev embedding for ψ_1 and (5.2) for $\langle\psi_3,\beta\psi_4\rangle$ to obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L^\infty}\|\psi_2\|_{L^2}\|\langle\psi_3,\beta\psi_4\rangle\|_{L^2} \\ &\lesssim 2^{\frac{3}{2}k_1}\|\psi_1\|_{L_t^\infty L_x^2}2^{-\frac{k_1+7k_2}{16}}\|\psi_2\|_{S_{k_2}}2^{k_3}2^{k_2-k_3}\|\psi_3\|_{S_{k_3}}\|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{-\frac{7}{16}(k_2-k_1)}J. \end{aligned}$$

$j=3$: The modulation of ψ_3 is $\gtrsim 2^{\frac{k_1+7k_3}{8}}$, so we use (5.4) for ψ_1 and ψ_2 and obtain

$$\begin{aligned} I &\lesssim \|\psi_1\|_{L_t^4L_x^\infty}\|\psi_2\|_{L_t^4L_x^\infty}\|\psi_3\|_{L^2}\|\psi_4\|_{L_t^\infty L_x^2} \\ &\lesssim 2^{\frac{5}{4}k_1}\|\psi_1\|_{L_t^\infty L_x^2}2^{\frac{5}{4}k_2}\|\psi_2\|_{S_{k_2}}2^{-\frac{k_1+7k_2}{16}}\|\psi_3\|_{S_{k_3}}\|\psi_4\|_{S^w_{k_4}} \\ &\lesssim 2^{-\frac{3}{16}(k_2-k_1)}J. \end{aligned}$$

$j=4$: The modulation of ψ_4 is $\gtrsim 2^{\frac{k_1+7k_3}{8}}$, so after exchanging the roles of ψ_3 and ψ_4 the same argument as in case $j=3$ applies. □

Based on Theorem 6.1 we can now prove Theorem 1.1 concerning the global well-posedness and scattering of the cubic Dirac equation for small data.

Proof of Theorem 1.1. In Section 3 we reduced the study of the cubic Dirac equation to the study of the system (3.4). In the nonlinearity of (3.4) we split the functions into $\psi=\psi_++\psi_-$ where $\psi_\pm=\Pi_\pm\psi$ and note that $\psi_\pm=\Pi_\pm\psi_\pm$. Using the nonlinear estimate in Theorem 6.1 and the linear estimates in Corollary 4.3, a standard fixed point argument in a small ball in the space $S^{+,1}\times S^{-,1}$ gives global existence, uniqueness and Lipschitz continuity of the flow map for small initial data $(\psi_+(0),\psi_-(0))\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$. Concerning scattering, we simply argue as follows: Let $\psi\in S^1$ be a solution to the cubic Dirac equation constructed above, where S^1 is the space of all ψ such that $\Pi_\pm\psi\in S^{\pm,1}$. Choose initial data $\psi_n(0)\in H^2(\mathbb{R}^3)$ with $\|\psi_n(0)-\psi(0)\|_{H^1(\mathbb{R}^3)}\to 0$ as

$n \to \infty$, and denote the corresponding solutions in S^1 by ψ_n. By continuity we have $\|\psi_n - \psi\|_{S^1} \to 0$ as $n \to \infty$. From the scattering result in [22, Theorem 1] we infer that there exist solutions to the linear Dirac equation $\varrho_n^{\pm\infty}$ such that $\|\psi_n(t) - \varrho_n^{\pm\infty}(t)\|_{H^2} \to 0$ as $t \to \pm\infty$. Let $\varepsilon > 0$. There exists n_0, such that for $n, m \geq n_0$ and sufficiently large $\pm t$ we have

$$\begin{aligned} &\|\varrho_n^{\pm\infty}(0) - \varrho_m^{\pm\infty}(0)\|_{H^1} = \|\varrho_n^{\pm\infty}(t) - \varrho_m^{\pm\infty}(t)\|_{H^1} \\ \leq &\|\varrho_n^{\pm\infty}(t) - \psi_n(t)\|_{H^1} + \|\psi_n(t) - \psi_m(t)\|_{H^1} + \|\psi_m(t) - \varrho_m^{\pm\infty}(t)\|_{H^1} < \varepsilon, \end{aligned}$$

hence the Cauchy-sequence $\varrho^{\pm\infty}(0)$ converges to some $\varrho^{\pm\infty} \in H^1(\mathbb{R}^3)$. Let $\varepsilon > 0$. Then, n can be chosen sufficiently large such that for the corresponding solution $\varrho^{\pm\infty}$ to the linear Dirac equation with $\varrho^{\pm\infty}(0) = \varrho^{\pm\infty}$ it follows that

$$\begin{aligned} \limsup_{t\to\pm\infty} \|\psi(t) - \varrho^{\pm\infty}(t)\|_{H^1} \leq &\sup_{t\in\mathbb{R}} \|\psi(t) - \psi_n(t)\|_{H^1} \\ &+ \lim_{t\to\pm\infty} \|\psi_n(t) - \varrho_n^{\pm\infty}(t)\|_{H^1} + \sup_{t\in\mathbb{R}} \|\varrho_n^{\pm\infty}(t) - \varrho^{\pm\infty}(t)\|_{H^1} < \varepsilon, \end{aligned}$$

which proves the scattering claim. □

Appendix A. Proofs of the decay estimates

Here, we provide proofs of the well-known decay estimates in Section 2, which clearly reveal the frequency dependence and which are self-contained in the important case $k \geq 1$. We do not claim originality here, compare e.g. [28, Section 2.5].

A.1. **Proof of Lemma 2.2 i).** By recaling it suffices to prove the estimate for $k \in \mathbb{Z}$, $k \leq 1$. Let $\zeta \in C_c^\infty(\mathbb{R}^3)$ be a nonnegative, radial function with $\zeta(\xi) = 1$ for $|\xi| \leq 2^4$. We identify the oscillatory integral

$$I(t, x) = \int_{\mathbb{R}^3} e^{i(x,t)\cdot(\xi,\langle\xi\rangle)} \zeta(\xi)\, d\xi$$

as the (inverse) Fourier transform of the surface measure of $\{(\tau, \xi) \in \mathbb{R}^4 : \tau = \langle\xi\rangle\}$ which is induced by $(1 + \frac{|\xi|^2}{\langle\xi\rangle^2})^{-\frac{1}{2}} \zeta(\xi) d\xi$. In the support of ζ the above surface has non-vanishing principal curvatures, and the classical result on Fourier transforms of surface carried measures [34, p. 348, Theorem 1] implies

$$|I(t, x)| \lesssim (1 + |(t, x)|)^{-\frac{3}{2}}.$$

With $f_k(\xi) := \tilde{\chi}_k^2(\xi)$, it holds that $\check{f}_k(x) = 2^{3k} \check{f}_1(2^k x)$, which shows $\|\check{f}_k\|_{L^1(\mathbb{R}^3)} = \|\check{f}_1\|_{L^1(\mathbb{R}^3)}$. For $k \leq 1$ we obtain K_k as the (spatial) convolution of $I(t,\cdot)$ and $\check{f}_k$, which implies

$$|K_k(t,x)| \lesssim (1+|(t,x)|)^{-\frac{3}{2}}$$

by Young's inequality. Estimate (2.7) follows in the case $|(t,x)| > 2^{-2k}$. In the remaining case $|(t,x)| \leq 2^{-2k}$ the estimate (2.7) is trivial.

A.2. **Proof of Lemma 2.2 ii).** Consider

$$P_k(s,y) = \int_{\mathbb{R}^3} e^{iy\cdot\xi} e^{is\langle\xi\rangle_k} \zeta(\xi)\, d\xi. \tag{A.1}$$

We claim that for all $k \in \mathbb{Z}, k \gtrsim 1$ and $s \in \mathbb{R}, y \in \mathbb{R}^3$ the following estimates hold true:

$$|P_k(s,y)| \lesssim (1+|(s,y)|)^{-1}, \tag{A.2}$$

$$|P_k(s,y)| \lesssim 2^k (1+|(s,y)|)^{-\frac{3}{2}}. \tag{A.3}$$

By rescaling $(\tau,\xi) \to 2^k(\tau,\xi)$, we have $K_k(t,x) = 2^{3k} P_k(2^k t, 2^k x)$, where $\zeta(\xi) = \tilde{\chi}_1^2(|\xi|)$. Hence, (2.8) follows from (A.2) and (A.3), which we will prove below. Because of the trivial bound

$$|P_k(s,y)| \leq \|\zeta\|_{L^1(\mathbb{R}^3)} \tag{A.4}$$

it is enough to treat the case $|(s,y)| \geq 1$.

The function $y \mapsto P_k(s,y)$ is radial, so it suffices to consider $y = (|y|,0,0)$. By introducing polar coordinates, we obtain

$$\begin{aligned} P_k(s,(|y|,0,0)) =& 2\pi \int_0^\infty \int_0^\pi e^{ir|y|\cos(\phi)} e^{is\langle r\rangle_k} r^2 \zeta(r) \sin(\phi)\, d\phi dr \\ =& 2\pi \int_0^\infty \int_{-1}^1 e^{i(r|y|z + s\langle r\rangle_k)} r^2 \zeta(r)\, dz dr \end{aligned} \tag{A.5}$$

Case $|s| > 2^4|y|$: For a given $z \in [-1,1]$ let $\phi(r) := r\frac{|y|z}{s} + \langle r\rangle_k$, such that the phase in (A.5) is given by $s\phi(r)$. Notice that $\phi'(r) = \frac{|y|z}{s} + \frac{r}{\langle r\rangle_k}$, so that $|\phi'(r)| \geq c > 0$ and for all $j \geq 2$ it holds $|\phi^{(j)}(r)| \leq c_j$ for all $r \in \operatorname{supp}(\zeta)$, $z \in [-1,1]$, $y \in \mathbb{R}^3$, $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Multiple integration by parts with respect to r yields

$$|P_k(s,(|y|,0,0))| \leq 4\pi \sup_{z\in[-1,1]} \left| \int_0^\infty e^{is\phi(r)} r^2 \zeta(r)\, dr \right| \leq C_N |s|^{-N}$$

for all $N \in \mathbb{N}$ and the claims (A.2) and (A.3) follow in this case.

Case $|s| < 2^{-4}|y|$: The same argument as above applies if we rewrite the phase function as $|y|\tilde{\phi}(r)$ with $\tilde{\phi}(r) = rz + \frac{s}{|y|}\langle r\rangle_k$.

Case $2^{-4}|y| \leq |s| \leq 2^4|y|$: Integrating (A.5) in z yields

$$\begin{aligned} P_k(s,(|y|,0,0)) = & \frac{2\pi}{i|y|}\int_0^\infty e^{i(s\langle r\rangle_k + r|y|)} r\zeta(r)\,dr \\ & - \frac{2\pi}{i|y|}\int_0^\infty e^{i(s\langle r\rangle_k - r|y|)} r\zeta(r)\,dr. \end{aligned} \tag{A.6}$$

which implies

$$|P_k(s,(|y|,0,0))| \leq C|y|^{-1}$$

and the first claim (A.2) follows. We can rewrite (A.6) as

$$\begin{aligned} P_k(s,(|y|,0,0)) &= \frac{2\pi}{i|y|}(I(s,y) - \overline{I(-s,y)}), \\ \text{where } I(s,y) &:= \int_0^\infty e^{i(s\langle r\rangle_k + r|y|)} r\zeta(r)\,dr. \end{aligned} \tag{A.7}$$

Let us define the phase function $\varphi(r) = \langle r\rangle_k + r\frac{|y|}{s}$. We have $\varphi'(r) = \frac{r}{\langle r\rangle_k} + \frac{|y|}{s}$, $\varphi''(r) = \frac{2^{-2k}}{\langle r\rangle_k^3}$ and for $j \geq 2$ $|\varphi^{(j)}(r)| \leq c_j$ for $r \in \operatorname{supp}(\zeta)$, $y \in \mathbb{R}^3$, $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$, see (1.5). Notice that φ' has a unique zero or does not vanish. Let us consider the case where $\varphi'(r_0) = 0$ for some $r_0 \in \operatorname{supp}(\zeta)$. Then, we have $|\varphi'(r)| \geq c2^{-2k}|r - r_0|$ in the support of ζ. Let $\delta := 2^k|s|^{-\frac{1}{2}}$. In case $\delta \geq 2^{-4}$ the claim (A.3) follows from (A.2), so we may assume that $\delta < 2^{-4}$ and we decompose $\int_0^\infty dr = \int_0^{r_0-\delta} dr + \int_{r_0-\delta}^{r_0+\delta} dr + \int_{r_0+\delta}^\infty dr$, in which case we obtain

$$\begin{aligned} \Big|\int_0^{r_0-\delta} e^{i(s\langle r\rangle_k + r|y|)} r\zeta(r)\,dr\Big| &= |s|^{-1}\Big|\int_0^{r_0-\delta} e^{i(s\langle r\rangle_k + r|y|)} \frac{\mathrm{d}}{\mathrm{d}r}\frac{r\zeta(r)}{\varphi'(r)}\,dr\Big| \\ &\leq |s|^{-1}\int_0^{r_0-\delta} \Big|\frac{(r\zeta(r))'}{\varphi'(r)}\Big|\,dr \\ &\quad + |s|^{-1}\int_0^{r_0-\delta} r\zeta(r)\big|(\varphi'(r)^{-1})'\big|\,dr \\ &\leq c2^{2k}(\delta|s|)^{-1} + c|s|^{-1}\int_0^{r_0-\delta} \big|(\varphi'(r)^{-1})'\big|\,dr \\ &\leq c2^{2k}(\delta|s|)^{-1} \end{aligned}$$

where we have used that $(\varphi'(r))^{-1}$ is decreasing in the domain of integration, which implies that

$$\int_0^{r_0-\delta} \big|(\varphi'(r)^{-1})'\big|\,dr \leq |\varphi'(r_0-\delta)^{-1}| \leq c2^{2k}(\delta|s|)^{-1}$$

A similar argument applies to the third part, and the second contribution is trivially bounded by $c\delta$, such that altogether we obtain

$$|I(s,y)| \leq c2^{2k}(\delta|s|)^{-1} + c\delta \leq c2^k|s|^{-\frac{1}{2}}. \tag{A.8}$$

The claim (A.3) follows by combining (A.8) and (A.7). In the remaining case where $\varphi' \neq 0$ in $\operatorname{supp}(\zeta)$, we have $\varphi'(r) \geq c > 0$ for all $r \in \operatorname{supp}(\zeta)$ and we obtain $|I(s,y)| \leq C_N|s|^{-N}$ for every $N \in \mathbb{N}$ by multiple integration by parts with respect to r.

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