THE CUMULATIVE NUMBERS AND THEIR POLYNOMIALS

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In a recent paper [1] the author has shown how the moments of a distribution can be obtained from the last entries of cumulative columns with the use of multiplication by certain numbers. These numbers may be called "cumulative numbers." It is the aim of this paper to show how these numbers can be obtained from the expansion of x^s in terms of factorials of the s-th order and to demonstrate properties of the polynomials of which these numbers are the coefficients.

TABLE 1

Successive Frequency Cumulations

(1) X	(2) x	(3) f_x	(4) C^1	(5) C^2	(6) C^3	(7) C^4	(8) C^5
$a + 6$	6	64	64	64	64	64	64
$a + 5$	5	192	256	320	384	448	512
$a + 4$	4	240	496	816	1200	1648	2160
$a + 3$	3	160	656	1472	2672	4320	6480
$a + 2$	2	60	716	2188	4860	9180	15660
$a + 1$	1	12	728	2916	7776	16956	32616
a	0	1	729	3645	11421	28377	60993

1. **The values $C_i^j(u_x)$.** We use the notation $C_i^j(u_x)$ of the previous paper [1,289] to express the columnar cumulated entries. The j indicates the order of the cumulation while the i indicates the number of the term, counting from the bottom of the column. Thus in Table I, which presents the cumulations of a frequency distribution used in the previous paper [1,289], $C_1^1 = 729$; $C_1^2 = 3645$; $C_2^2 = 2916$; $\cdots$, $C_4^5 = 6480$, etc. Now if $k + 1$ values of x are spaced at unit distances and if the smallest value of x is 0, it can be shown that

$$C_1^1 = \sum_0^k u_x; \quad C_1^2 = \sum_0^k (x+1)u_x; \quad C_2^2 = \sum_0^k x u_x; \quad C_1^3 = \sum_0^k \frac{(x+2)(x+1)}{2!} u_x;$$

$$C_2^3 = \sum_0^k \frac{(x+1)x}{2!} u_x; \quad C_3^3 = \sum_0^k \frac{x(x-1)}{2!} u_x$$

and, in general, $j > 0$ and $j + 1 \geq i$,

$$C_i^{j+1} = \sum_{x=0}^k \frac{(x+j+1-i)^{(j)}}{j!} u_x. \tag{1}$$

Similarly if k values of x are spaced at unit distances and if the smallest value of x is 1, it can be shown that

$$C_1^1 = \sum_1^k u_x; \quad C_1^2 = \sum_1^k xu_x; \quad C_2^2 = \sum_1^k (x-1)u_x; \quad C_1^3 = \sum_1^k \frac{(x+1)x}{2!} u_x;$$

$$C_2^3 = \sum_1^k \frac{x(x-1)}{2!} u_x; \quad C_3^3 = \sum_1^k \frac{(x-1)(x-2)}{2!} u_x$$

and, in general, $j > 0$ and $j + 1 \geq i$,

$$(2) \qquad C_i^{j+1} = \sum_{x=1}^k \frac{(x+j-i)^{(j)}}{j!} u_x.$$

It is to be noted that the coefficients of u_x in (2) could be obtained from the coefficients of u_x in (1) by the substitution $x + 1 = x'$.

2. **The powers in terms of factorials of the s-th order.** If the s-th powers can be expressed in terms of factorials of the s-th order (factorials having s factors) then the moments can be expressed in terms of the cumulations. For example

$$x^2 = \frac{(x+1)x + x(x-1)}{2} \text{ so, from (1)}$$

$$\sum_0^k x^2 f_x = \sum_0^k \frac{(x+1)^{(2)}}{2!} f_x + \sum_0^k \frac{x^{(2)}}{2!} f_x = C_2^3 + C_3^3.$$

And since

$$x^3 = \frac{(x+2)^{(3)} + 4(x+1)^{(3)} + x^{(3)}}{3!}, \text{ we have}$$

$$\sum_0^k x^3 f_x = \sum_0^k \frac{(x+2)^{(3)}}{3!} f_x + 4 \sum_0^k \frac{(x+1)^{(3)}}{3!} f_x + \sum_0^k \frac{x^{(3)}}{3!} f_x = C_2^4 + 4C_3^4 + C_4^4.$$

In general if

$$(3) \quad x^s = \frac{A_{s1}(x+s-1)^{(s)} + A_{s2}(x+s-2)^{(s)} + \cdots + A_{sj}(x+s-j)^{(s)} + \cdots + A_{ss}x^{(s)}}{s!},$$

then

$$(4) \quad \sum_0^k x^s f_x = A_{s1}C_2^{s+1} + A_{s2}C_3^{s+1} + \cdots + A_{sj}C_{j+1}^{s+1} + \cdots + A_{ss}C_{s+1}^{s+1},$$

while if the smallest value of x is 1, we have

$$(5) \quad \sum_1^k x^s f_x = A_{s1}C_1^{s+1} + A_{s2}C_2^{s+1} + \cdots + A_{sj}C_j^{s+1} + \cdots + A_{ss}C_s^{s+1}.$$

These quantities, A_{sj}, in (4) and (5) are simply the coefficients of certain factorials of the s-th order in the expansion of $x^s s!$.

These numbers, for small values of s, are easily obtained. It is possible to use the table and a recursion formula of a previous paper [1,294–295] for larger values of s. It is also possible to obtain these values, without involving cumulative theory, from (3) above.

While doing this we make a more general approach by expanding $(a + x)^s$ in terms of these same factorials with the coefficients now functions of a. This is possible if we add an additional term, $A_{s0}(x + s)^{(s)}$, to the numerator of the right hand side of (3). We have then

$$(6) \quad (a + x)^s = \frac{A_{s0}(x + s)^{(s)} + A_{s1}(x + s - 1)^{(s)} + \cdots + A_{sj}(x + s - j)^{(s)} + \cdots + A_{ss}x^{(s)}}{s!}$$

The determination of the values A_{sj} can be accomplished by purely algebraic means by successive substitution of $x = 0, 1, 2, \cdots s$. In this way we obtain $s + 1$ equations in $s + 1$ unknowns. For example when $s = 2$

$$(a + x)^2 = \frac{A_{20}(x + 2)^{(2)} + A_{21}(x + 1)^{(2)} + A_{22}x^{(2)}}{2!}$$

so that when $x = 0, 1, 2,$ we have

$$a^2 = A_{20} \; ; \; (a + 1)^2 = 3A_{20} + A_{21} \; ; \; (a + 2)^2 = 6A_{20} + 3A_{21} + A_{22} \, .$$

The solution is $A_{20} = a^2$; $A_{21} = 2ab + 1$; $A_{22} = b^2$ where $b = 1 - a$. It follows that

$$(a + x)^2 = a^2 \frac{(x + 2)^{(2)}}{2!} + (2ab + 1) \frac{(x + 1)^{(2)}}{2!} + b^2 \frac{x^{(2)}}{2!} \text{ and hence that}$$

$$\sum_0^k (a + x)^2 f_x = a^2 C_1^3 + (2ab + 1)C_2^3 + b^2 C_3^3$$

as indicated in the previous paper [1,293].

When $a = 0$, then $b = 1$ and we have

$\Sigma x^2 f_x = C_2^3 + C_3^3$ while when $a = 1$, $b = 0$ and the right hand side becomes $C_1^3 + C_2^3$.

It follows that the general cumulative numbers might also be defined as the solutions of the $s + 1$ equations in the $s + 1$ unknowns obtained by placing $x = 0, 1, 2, \cdots , s$ in (6).

3. **The evaluation of the cumulative numbers.** Formal algebraic methods of evaluating equations (6) are somewhat tedious so we use finite difference theory to aid in finding the solution. As in the previous paper [1] we use the notation

$$\nabla v_x = v_x - v_{x-1} \text{ and } \underline{v}_x = \begin{cases} v_x \text{ when } a \leq x \leq a + k \\ 0 \text{ otherwise} \end{cases}.$$ We then write, from (6)

$$(7)\qquad \begin{aligned} s!\underline{(a+x)^s} &= A_{s0}\underline{(x+s)^{(s)}} + A_{s1}\underline{(x+s-1)^{(s)}} \\ &\quad + \cdots + A_{sj}\underline{(x+s-j)^{(s)}} + \cdots + A_{ss}\underline{x^{(s)}}. \end{aligned}$$

We note further that $\nabla^{s+1}\underline{(x+r)^{(s)}} = \begin{Bmatrix} s! \text{ when } r = 0 \\ 0 \text{ when } r \neq 0 \end{Bmatrix}$. We have then

$$(8)\qquad \nabla^{s+1}\underline{(a+j)^s} = A_{sj}.$$

It has been shown in the previous paper [1,292] that

$$(9)\qquad \nabla^{s+1}\underline{(a+j)^s} = \sum_{t=0}^{j} (-1)^t \binom{s+1}{t} (a+j-t)^s$$

and it appears that the cumulative numbers could be defined by (9). A useful recursion formula has been derived from (9)

$$(10)\qquad \nabla^{s+1}\underline{(a+x)^s} = (a+x)\nabla^s\underline{(a+x)^{s-1}} + (s+1-a-x)\nabla^s\underline{(a+x-1)^{s-1}}.$$

4. **The cumulative polynomials.** We define the cumulative polynomials to be the polynomials obtained by using the cumulative numbers as coefficients. Thus when $a = 0$,

$P_1 = y$; $P_2 = y + y^2$; $P_3 = y + 4y^2 + y^3$; $P_4 = y + 11y^2 + 11y^3 + y^4$; etc.

It is possible to derive a recursion formula for these polynomials. We use (10) with s replaced by $s+1$ and $a = 0$ and get

$$(11)\qquad P_{s+1} = \Sigma\nabla^{s+2}\underline{(x)^{s+1}}y^x = \Sigma x\nabla^{s+1}\underline{(x)^s}y^x + \Sigma(s+2-x)\nabla^{s+1}\underline{(x-1)^s}y^x,$$

which becomes, after some manipulation,

$$(12)\qquad P_{s+1} = (1-y)\Sigma x\nabla^{s+1}\underline{(x)^s}y^x + (s+1)yP_s.$$

To illustrate we get P_4 from $P_3 = y + 4y^2 + y^3$. Now $\Sigma x\nabla^4\underline{(x)^3}y^x = y + 8y^2 + 3y^3$ and $P_4 = (1-y)(y + 8y^2 + 3y^3) + 4y(y + 4y^2 + y^3) = y + 11y^2 + 4y^3 + y^4$. The recursion formula (12) can be expressed also in the form of a differential equation, since $P_s' = \frac{d}{d_y}(P_s) = \Sigma x\nabla^{s+1}\underline{(x)^s}y^{x-1}$, as

$$(13)\qquad P_{s+1} = y[(1-y)P_s' + (s+1)P_s].$$

It can be shown more generally that for any a

$P_{a,0} = 1$; $P_{a,1} = a + by$; $P_{a,2} = a^2 + (2ab+1)y + b^2y^2$, etc. with

$$(14)\qquad P_{a,s+1} = y(1-y)P_{a,s}' + [a(1-y) + (s+1)y]P_{a,s}$$

as the recursion formula.

5. **The numerator coefficients in successive derivatives of the logistic function.** Lotka has recently exhibited the coefficients of the numerator terms of suc-

cessive derivatives of the logistic function [2, 160]. These appear to be, aside from sign, the same as the cumulative numbers when $a = 0$. It is shown in this section that these numbers are the cumulative numbers. The scheme is generalized to include the numerator coefficients of the derivatives of a more general function involving the parameter a.

Lotka used the function $\Phi_0 = \frac{1}{1 + e^{rt}}$ and obtained $\Phi_1 = \frac{re^{rt}}{(1 + e^{rt})^2}$, $\Phi_2 = \frac{r^2 e^{rt}(1 - e^{rt})}{(1 + e^{rt})^3}$, etc. The numerical coefficients are the same if $r = 1$ so we might as well use $\Phi_0 = \frac{1}{1 + e^x}$. A more general function is the two parameter function

$$\Phi_{a,c} = \frac{e^{ax}}{1 + ce^x}. \tag{15}$$

Let successive derivatives with respect to x be indicated by $\Phi_{a,c,1}$; $\Phi_{a,c,2}$; $\Phi_{a,c,3}$; etc. Then

$$\Phi_{a,c,1} = \frac{e^{ax}[a + c(1 - a)e^x]}{(1 + ce^x)^2},$$

$$\Phi_{a,c,2} = \frac{e^{ax}[a^2 + (-2a^2 + 2a + 1)ce^x + (1 - a)^2 c^2 e^{2x}]}{(1 + ce^x)^3}.$$

In general,

$$\Phi_{a,c,s} = \frac{e^{ax}Q_{a,c,s}}{(1 + ce^x)^{s+1}} = e^{ax}Q_{a,c,s}(1 + ce^x)^{-s-1}$$

so that

$$\Phi_{a,c,s+1} = \frac{e^{ax}\{(1 + ce^x)[aQ_{a,c,s} + Q'_{a,c,s}] - (s + 1)ce^x Q_{a,c,s}\}}{(1 + ce^x)^{s+2}}$$

and

$$Q_{a,c,s+1} = (1 + ce^x)[aQ_{a,c,s} + Q'_{a,c,s}] - (s + 1)ce^x Q_{a,c,s}\,. \tag{16}$$

The Q functions can be changed to polynomials with the substitution $e^x = y$. Then derivatives are taken with respect to y and

$$P_{a,c,s+1} = (1 + cy)[aP_{a,c,s} + yP'_{a,c,s}] - (s + 1)cyP_{a,c,s}\,. \tag{17}$$

When $c = -1$, this becomes formula (14) and since $P_{a,0} = 1$, it follows that the numbers of the present section are generalized cumulative numbers. When $c = 1$ and $a = 0$ we have the numbers found by Lotka.

It can be shown, further, that the c coefficient of y^j is c^j. It follows that the absolute values of the coefficients, when $c = 1$ and when $c = -1$, are the same.

6. Formulas for Σx^s. A formula for the sums of the s-th powers of the integers from 1 to k is obtained by summing (3). We get

$$\sum_1^k x^s = A_{s1}\sum_1^k \frac{(x+s-1)^{(s)}}{s!} + \cdots + A_{sj}\sum_1^k \frac{(x+s-j)^{(s)}}{s!} + \cdots + A_{ss}\sum_1^k \frac{x^{(s)}}{s!} \tag{18}$$

from which

$$\sum_1^k x^s = A_{s1}\frac{(k+s-1)^{(s+1)}}{(s+1)!} + \cdots + A_{sj}\frac{(k+s-j)^{(s+1)}}{(s+1)!} + \cdots + A_{ss}\frac{k^{(s+1)}}{(s+1)!}, \tag{19}$$

or

$$\sum_1^k x^s = \sum_{j=1}^s A_{sj}\frac{(k+s-j)^{(s+1)}}{(s+1)!} = \frac{1}{(s+1)!}\sum_{j=1}^s (k+s-j)^{(s+1)}\nabla^{s+1}\underline{(j)^s} = \sum_{j=1}^s C_j^{s+1}(1)\nabla^{s+1}\underline{(j)^s}. \tag{20}$$

For example

$$\sum_1^k x^2 = \frac{(k+2)^{(3)} + (k+1)^{(3)}}{3!} = \frac{k(k+1)(2k+1)}{6},$$

$$\sum_1^k x^3 = \frac{(k+3)^{(4)} + 4(k+2)^{(4)} + (k+1)^{(4)}}{4!} = \frac{k^2(k+1)^2}{4}.$$

More generally the values of $\sum_a^{a+k} x^s$ can be evaluated by

$$\sum_a^{a+k} x^s = \frac{1}{(s+1)!}\sum_{j=0}^s (k+s-j)^{(s+1)}\nabla^{s+1}\underline{(a+j)^s} = \sum_{j=0}^s C_{j+1}^{s+1}(1)\nabla^{s+1}\underline{(a+j)^s}. \tag{21}$$

7. **Summary.** It is shown how the cumulative numbers and the cumulative polynomials may be obtained in a variety of ways. Of special interest is the fact that the cumulative numbers can be obtained by expanding powers in terms of factorials and hence they might be called factorial coefficients of a kind. It is also possible, though it is not within the scope of this paper, to establish interesting relations between the cumulative numbers and the multinomial coefficients, the usual factorial coefficients, the difference of 0, etc.

REFERENCES

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[2]. A. J. Lotka. "An Integral Equation in Population Analysis." *Annals of Math. Stat.* Vol. X, no. 2, June 1939, pp. 144–161.

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