The Current Value of the Mathematical Provision: A Financial Risk Prospect

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Abstract

The paper addresses the question of the calculation of the current value of the mathematical provision and moulds it in a deterministic and stochastic scenario, using a proper term structure of interest rates estimated by means of a Cox-Ingersoll-Ross model. It provides a complete and original year-by-year evaluation model for the business performance, and a closed solution for the current evaluation of the reserve, together with a comprehensive insight into the dynamics of the reserve connected to the selection of a defined term structure of interest rates. Moreover, the calculation of the VaR of the mathematical provision is prospected as risk measure useful to appreciate also the evaluation rate risk. Future research prospects concern the selection of the stochastic process used to describe the dynamics of the interest rates and the possible managerial and regulatory application of a VaR measure. The modelling has been applied, as an exemplification, to a life annuity portfolio but it can be easily replicated for any kind of policy and any kind of portfolios even non homogeneous.

Key words: Risk indicators, life insurance, solvency, financial risk, demographic risk.
JEL Classification: G22.

1. Introduction

Life insurance business is traditionally characterised by a complex system of risks that can be essentially split into two main types of drivers: actuarial and financial; within the pricing process, they refer to the insurance company aptitude to select the "right" mortality table, and to apply the "right" discounting process, where accuracy covers forecasting proficiency. Both the aspects can be regarded at the same time as risk drivers and value drivers, since they can give rise to a loss – or a profit – if the ex-ante expected values prove to be higher or lower than the ex-post actual realizations. With special reference to the intermediation portfolio (contingent claims versus corresponding assets), these effects can be enhanced by the specific accounting standard applied for the financial statement; in other words a fair valuation system can substantially modify the disclosure of the economic result and the solvency appraisal in time and space.

At the end of March 2004, the International Accounting Standards Board issued the International Financial Reporting Standard 4 Insurance Contracts. For the first time, it provides guidance on accounting for insurance contracts, and marks the first step in the IASB's project to achieve the convergence of widely varying insurance industry accounting practices around the world. More specifically, the IFRS *permits an insurer to change its accounting policies for insurance contracts only if, as a result, its financial statements present information that is more relevant and no less reliable, or more reliable and no less relevant. Moreover, it permits the introduction of an accounting policy that involves remeasuring designated insurance liabilities consistently in each period to reflect current market interest rates (and, if the insurer so elects, other current estimatesand assumptions),* thus giving rise to a potential reclassification of some or all financial assets as *at fair value through profit or loss*. On the one hand, the compromising solution derives from the recognition of a fair value disclosure requirement, also to comply with a more general tendency concerning financial statements. On the other hand, it stems from the equally widespread and deep worries

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concerning the lack of agreement upon a definition of fair value as well as of any guidance from the Board on how the fair value has to be calculated. According to the majority of commentators including the National Association of Insurance Commissioners, the International Association of Insurance Supervisors, and the Basle Committee on Banking Supervision, this uncertainty may lead to fair value disclosures that are unreliable and inconsistently measured among insurance entities. As also the American Academy of Actuaries (2003) clearly states, market valuations do not exist for many items on the insurance balance sheet and this would lead to the reliance on entity specific measurement for determining insurance contract and asset fair values. However, such values would be unreasonably subject to wide ranges of judgment and to significant abuse, and may provide information that is not at all comparable among companies. The cause for this concern is the risk margin component of the fair value. Risk margins are clearly a part of market values for uncertain assets and liabilities, but with respect to many insurance contracts, their value cannot be reliably calibrated to the market. Hence, a market-based valuation basis for them would produce irrelevant information. This statement gives rise to a wider trouble. If there is an amendment in the evaluation criteria for the reserve from one year to another because of current market yields or even of current mortality tables, there is a possible change in the value of the reserve according to the application of a more stringent or, at the opposite, a more flexible criterion. This may turn into a proper *fair valuation risk*.

In the accounting perspective, the introduction of an accounting policy involving remeasuring designated insurance liabilities consistently in each period to reflect current market interest rates (and, if the insurer so elects, other current estimates and assumptions) implies that the fair value of the mathematical provision is properly a "current value" or a "net present value". Consistently, the fair value of the mathematical provision could be properly defined as the net present value of the residual debt towards the policyholders evaluated "at current interest rates and, eventually, at current mortality rates". In a sense, this is marking to market. However, this is the crucial point. The International Accounting Standards Committee defines the fair value as *the amount for which an asset could be exchanged, or a liability settled, between knowledgeable, willing parties in an arms-length transaction, while the Financial Accounting Standard Board defines the fair value as an estimate of an exit price determined by market interactions*. Hence, the CAS Fair Value Task Force defined fair value as *the market value, if a sufficiently active market exists, or an estimated market value, otherwise*. From the accounting side a fair value is not necessarily an equilibrium price, but merely a market price.

The paper addresses the question of the calculation of the current value of the mathematical provision and moulds it in both a deterministic and stochastic scenario, using a proper term structure of interest rates estimated by means of a Cox-Ingersoll-Ross model. To this end, the paper starts with a complete and original balance-sheet modelling of the insurance business, which consents: to identify all the variables relevant to the capital performance of the insurer (risk and value drivers); to monetarily measure the impact of any risk factor change; to use the result for a proper risk based approach in a value-at-risk context (cf. section 2); and, from a more general point of view, to take specific decision to manage and hedge the risks. A special attention is devoted to the fair valuation risk, defined as the change in the economic results due to a change in the evaluation criteria adopted for insurance liabilities, which proves to be proportional to the value and the duration of the portfolio reserve (cf. section 2.1). The specific question of the current value of the reserve is subsequently put into a stochastic context (section 3) by means of a cash flow analysis, where relevant reserve risk drivers are randomly treated. Finally, the appraisal of the mathematical reserve at current interest rates is prospected by section 4, where the fair valuation of the mathematical provisions is focused on the financial driver while the actuarial components are appreciated by means of an opportunely projected mortality table. In this section the reader will find the details of the term structure (section 4.1), the current values in a continuous approach with a closed form for the analytical solution to the current valuation of the mathematical provision (section 4.2) and a clue on the value-at-risk of the mathematical provision due to a change in the evaluation rate (evaluation rate risk). Research and practical implications are drawn in the last section, since the modelling can give rise to a wide range of managerial, regulatory, and accounting applications.

From a methodological point of view, it is important to stress that, for the sake of clarity, a synthetic representation of the balance sheet, which serves as a guideline, has been employed in the paper although the model, where useful and appropriate, can be extended in order to include also components such as loadings and mark-up margins. At the same time, the whole analysis has been applied to a life annuity portfolio but it can be easily replicated for any kind of policy and any kind of portfolios even non homogeneous.

2. The balance-sheet scheme

Let us consider the premium (P) for an immediate temporary (*n*) unitary annuity:

$$P = \sum_{r=1}^{n} {}_r\hat{p}_x e^{-\eta r} , \tag{1}$$

where η is the instantaneous rate of interest observed at the beginning of the business and applied for premium setting, and ${}_r\hat{p}_x$ identifies the corresponding probability table used to price the policy.

The formulation (1) implies that the *final equilibrium* (time *n*) of a cohort of *c* identical policies is constrained by:

$$cPe^{n\delta_A} - \sum_{r=1}^{n} N^x(r) e^{(n-r)\delta_A} \geq 0 , \tag{2}$$

where δ_A is the instantaneous total rate of return[1] on asset purchased with written premiums observed at the end of the business, and $N^x(r)$ is the actual number of survivors at age x + *r*. The formulation (2) computes the result of the portfolio: if it proves positive, the business produced profit and, of course, losses, if negative, while the null level sets the minimal equilibrium that is to say the insolvency threshold (Cocozza *et al.*, 2001 and 2003).

As showed elsewhere (Cocozza *et al.*, 2005), the final equilibrium given by the (2) is strictly dependent on the result accrued in each time bucket and therefore solvency is conditional upon the ability to keep positive the difference between the capitalised result up to time t (*t*<*n*) and the present value of the residual debt; it can be formally expressed as the probability of respecting permanently – that is to say in each period – the general equilibrium condition expressed by:

$$\text{Prob}\left(\left[cPe^{t\delta_A} - \sum_{r=1}^{t} N^x(r) e^{(t-r)\delta_A}\right] - N^x(t) \sum_{r=1}^{n-t} {}_r p_{x+t}\, e^{-r\delta_{L_t}}\right) \geq 0 = 1 - \varepsilon , \tag{3}$$

where the level of this probability is a political question and sets the amount of capital adequate to keep insolvency within a limit, which is considered bearable, with reference to both capital costs borne by the intermediaries and the risk level faced by policyholders. When the result of the initial capital (K_0) invested by the insurer is explicitly included into the formulation, the value of the intermediation portfolio at the end of the first year is equal to

$$K_1 = (cP + K_0) e^{\delta_{A_0}} - N^x(1)\left[1 + \sum_{r=1}^{n-1} {}_r p_{x+1} e^{-r\delta_{L_1}}\right] \tag{4}$$

from which it can be easily inferred that the net value of the intermediation portfolio depends on the insurer initial capital K_0, on the actual number of contracts existing at the end of the first year ($N^x(1)$), on the table adopted in the valuation of reserve at the end of the first year (${}_r p_{x+1}$), and on

[1] This rate encoppasses interest income, capital gains and losses (Parker, 1997).

the return on assets earned in the first year (δ_{A_0}) as well as on the evaluation rate selected for the mathematical provision at the end of the first year (δ_{L_1}). This net value is directly influenced by both the two interest rates while is inversely affected by an increase in the actual number of survivors at the end of the year, since:

$$\frac{\partial K_1}{\partial \delta_{A_0}} = (cP + K_0)e^{\delta_{A_0}} > 0, \tag{5}$$

$$\frac{\partial K_1}{\partial \delta_{L_1}} = N^x(1)\sum_{r=1}^{n-1} r\,{}_rp_{x+1}e^{-r\delta_{L_1}} > 0, \tag{6}$$

$$\frac{\partial K_1}{\partial N^x(1)} = -\left[1 + \sum_{r=1}^{n-1} {}_r p_{x+1}e^{-r\delta_{L_1}}\right] < 0. \tag{7}$$

Consistently, the selection for the evaluation of the reserve at the end of the year of a mortality table divergent from that applied for any previous evaluation (both for pricing and reserving) produces a variation of the net value of the intermediation portfolio, which is inverse with respect to the “sign” of the variation of the table itself. As a matter of fact, the selection for example of a more prudent table (showing in this case higher probability of surviving) will produce an increase in the present value of future net outflows and therefore a decrease in the net value, while the selection of a table showing lower probability will give rise to an increase in the same value. As far as the intensity of this impact is concerned, it is easy to verify that is filtered by the size of the portfolio under observation and by its implicit financial discounting process.

Formulations (5), (6) and (7) define a complete series of risk filters connected to relevant risk drivers and are able to capture the effect of a change in the relevant variable on the net value of the intermediation portfolio. The impact is measured trough the monetary change and the corresponding formulations can be properly identified as risk indicators. The first two (formulations (5) and (6)) measure *financial risk* and can be addressed respectively as *investment risk* and *evaluation rate risk*, since the first approximates the net value change due to a variation in the return on assets, while the second approximates the net value change due to a modification in the rate applied for the evaluation of the mathematical provision. The last one (cf. formulation (7)) measures *actuarial risk*, since it approximates the net value change due to a variation in the actual number of survivors at the end of the year. As far as the sizes of these impacts are concerned, the most relevant risk drivers can be easily identified in the investment risk and in the evaluation rate risk, since they are respectively filtered by the value of the total assets at the beginning of the year, and proxies of the portfolio mathematical provision.

Similarly, the net value of the intermediation portfolio at the end of the second year can be expressed in one of the two equivalent following formulations

$$\begin{aligned} K_2 &= \left[(cP + K_0)e^{\delta_{A_0}} - N^x(1)\right]e^{\delta_{A_1}} - N^x(2)\left[1 + \sum_{r=1}^{n-2} {}_r p_{x+2}e^{-r\delta_{L_2}}\right] = \\ &= \left[K_1 + N^x(1)\sum_{r=1}^{n-1} {}_r p_{x+1}e^{-r\delta_{L_1}}\right]e^{\delta_{A_1}} - N^x(2)\left[1 + \sum_{r=1}^{n-2} {}_r p_{x+2}e^{-r\delta_{L_2}}\right] \end{aligned} \tag{8}$$

As previously, the value of the K_2 portfolio depends on the value K_1, on the actual number of contracts existing at the end of the second year [$N^x(2)$], on the table adopted in the valuation of reserve at the end of the first and second year (${}_rp_{x+1}$; ${}_rp_{x+2}$) and on the first and second year returns on assets $\left(\delta_{A_0};\delta_{A_1}\right)$ as well as on selected evaluation rates at the end of the first and second year (δ_{L_1}; δ_{L_2}). Also for this the direct dependence on the most recent interest rates holds, since:

$$\frac{\partial K_2}{\partial \delta_{A_1}} = \left[\left(cP + K_0\right)e^{\delta_{A_0}} - N^x(1)\right]e^{\delta_{A_1}} =$$

$$= \left[K_1 + N^x(1)\sum_{r=1}^{n-1} r\,{}_r p_{x+1} e^{-r\delta_{L_1}}\right]e^{\delta_{A_1}} > 0, \tag{9}$$

$$\frac{\partial K_2}{\partial \delta_{L_2}} = N^x(2)\sum_{r=1}^{n-2} r\,{}_r p_{x+2} e^{-r\delta_{L_2}} > 0. \tag{10}$$

Generally, for any intermediate t period, the following relations hold:

$$\begin{aligned} K_t = &\left[K_{t-1} + N^x(t-1)\sum_{r=1}^{n-t-1} {}_r p_{x+t-1} e^{-r\delta_{L_{t-1}}}\right]e^{\delta_{A_{t-1}}} - \\ &- N^x(t)\left[1 + \sum_{r=1}^{n-t} {}_r p_{x+t} e^{-r\delta_{L_t}}\right] \end{aligned} \tag{11}$$

$$\frac{\partial K_t}{\partial \delta_{A_{t-1}}} = \left[K_{t-1} + N^x(t-1)\sum_{r=1}^{n-t-1} r\,{}_r p_{x+t-1} e^{-r\delta_{L_{t-1}}}\right]e^{\delta_{A_{t-1}}} > 0, \tag{12}$$

$$\frac{\partial K_t}{\partial \delta_{L_t}} = N^x(t)\sum_{r=1}^{n-t} r\,{}_r p_{x+t} e^{-r\delta_{L_t}} > 0. \tag{13}$$

Consistently with formulation (2), with reference also to a total instantaneous rate of return on assets (δ_A) set equivalent to the single period rates, the ultimate result equations are:

$$\begin{aligned} K_n = &\left[(cP + K_0)e^{n\delta_A} - \sum_{r=1}^{n} N^x(r)e^{(n-r)\delta_A}\right] = \\ = &\left[K_{n-1} + N^x(n-1)p_{x+n-2}e^{-\delta_{L_{n-1}}}\right]e^{\delta_{A_{n-1}}} - N^x(n) \end{aligned} \tag{14}$$

$$\frac{\delta K_n}{\partial \delta_{A_{n-1}}} = \left[K_{n-1} + N^x(n-1)p_{x+n-2}e^{-\delta_{L_{n-1}}}\right]e^{\delta_{A_{n-1}}} > 0. \tag{15}$$

2.1. An insight on the evaluation rate risk

If we concentrate on the evaluation rate risk it is easy to demonstrate that the dollar change in the net value of the intermediation portfolio is directly proportional to the Macaulay duration of the mathematical provision (D_{R_t}) since:

$$\frac{\partial K_t}{\partial \delta_{L_t}}\frac{1}{R_t} = \frac{\sum_{r=1}^{n-t} r\,{}_r p_{x+t} e^{-r\delta_{L_t}}}{\sum_{r=1}^{n-t} {}_r p_{x+t} e^{-r\delta_{L_t}}} = D_{R_t}, \tag{16}$$

$$\frac{\partial K_t}{\partial \delta_{L_t}} = D_{R_t} R_t \Rightarrow \Delta K_t \cong D_{R_t} R_t \Delta \delta_{L_t} \,. \qquad (17)$$

Therefore, formulation (17) gives the opportunity to state that the evaluation rate risk can be effectively measured by a sort of value at risk of the mathematical provision due to a change in the evaluation rate. It has been proved that the net value change is directly dependent on the current value of the reserve (R_t), on a proportionality factor (proxied by Macaulay duration) and on a potential (adverse) movement in yield. As far as the sign is concerned, an increase (decrease) in the yield will result in a net value rise (fall), corresponding to a lower (higher) present value of the liability. Moreover, since the equity/reseve ratio is by definition less than one, the impact of an evaluation rate modification on the net value of the intermediation portfolio is directly proportional to the level of the financial-insurance leverage, since:

$$\frac{\Delta K_t}{K_t} \cong \frac{R_t}{K_t} D_{R_t} \Delta \delta_{L_t} \,. \qquad (18)$$

Therefore, apart from the single risk factor time evolution, the size of this impact depends directly on the progression of the reserve/equity ratio and of the reserve duration. Since these two quantities tend to reduce alongside the length of the policy, a variation in the evaluation rate is susceptible to produce more relevant effect at the beginning than towards the end of the policy; at the same time, *ceteris paribus*, the impact will reduce alongside the reduction of the entrance age of the insured.

2.2. A risk based approach

If we denote K_t the future (random) value of the net value of the intermediation portfolio and k_0 the corresponding (known) current value at present and $F_t(x)$ the distribution function of K_t, whose density function is $f_t(x)$, the basic structural property of $VaR_\alpha(t)$ then is given by

$$\text{Prob}(K_t - k_0 \le -VaR_\alpha(t)) = \alpha \qquad (19)$$

from which it is easy to infer that the Value-at-Risk at the confidence level α is that realization of the loss amount $k_0 - K_t$ which will not be exceeded with probability $1-\alpha$. By denoting the α-quantile distribution of K_t as

$$K_\alpha(t) = F_t^{-1}(\alpha) \qquad (20)$$

we have that

$$\text{Prob}(K_t \le K_\alpha(t)) = \alpha \qquad (21)$$

and, consequently,

$$VaR_\alpha(t) = k_0 - K_\alpha(t) \,. \qquad (22)$$

The values of the α-quantile distribution of K_t can be split into an expected part $E[K_t]$ and an excess quota $Z_\alpha[K_t] = E[K_t] - K_\alpha(t)$ so that formulation (22) can be re-written as

$$VaR_\alpha(t) = (k_0 - E[K_t]) + Z_\alpha(t) \,. \qquad (23)$$

If we regard the formulation (23) as the summation of the expected result and an additional quota connected to the variance of the result itself, the value-at-risk of the entire net value can be interpreted as the VaR of a portfolio exposed to those risk factors that are the changing parameters of the net value itself and can be appreciated by modelling the risk factors and by using risk filters to proportionate the effect on the net value such as, for example, in the case of the reserve duration. This approach can be applied to evaluate the effect of a variation in the risk drivers with respect to both the initial setting, i.e. pricing hypothesis, and year-by-year eventual revaluation of the earlier hypothesis. Moreover, it can be directly employed with reference to managerial goals and to institutional targets (risk based capital requirements) and can give rise to a firm specific evaluation if it is possible to obtain a complete dataset of correlations among risk factors or a standardised evaluation if these correlations are supposed to be pre-set to fixed values.

3. The cash flow analysis

Let us consider a portfolio consisting of c insureds aged exactly x, each of whom having a deferred life annuity policy with premiums payable at the beginning of the first t years and benefits payable at the beginning of each year after t, in the case the insured is alive.

Aim of the section is to provide a current valuation of the portfolio reserve in a stochastic environment for the interest rate, by means of a gradual construction of the valuation formula.

With the following notation (e.g. Cocozza et al., 2004):

- j_i = curtate future lifetime ($j_i = 0,1,2,...$) of the i^{th} insured ($i=1,2,...,c$),
- $F_{i,s}$ = flow of cash at time s ($s=0,1,2,...$) related to the i^{th} policy in portfolio at time 0,
- $B_{i,s}$ = benefit payable to the i^{th} insured at time s,
- $P_{i,s}$ = premium payable by the i^{th} insured at time s,

we can define the loss of each policy and of the portfolio.

3.1. A deterministic scenario

As a first step, we start from a deterministic scenario, considering, for each policy coming in portfolio at time 0 and for a given value of j_i, the following scheme for the cash flow:

$$\ell_{i,s}(j_i) = \begin{cases} F_{i,s} & \text{if } j_i > \text{s} \\ 0 & \text{if } j_i \le s \end{cases},$$

where $F_{i,s} = \begin{cases} -P_{i,s} & \text{if } s < \text{t} \\ B_{i,s} & \text{if } s \ge t \end{cases}$.

The loss in $k\ (k = 0,1,2,...)$ connected to the *i-th* policy is the sum of the discounted net cash flows (cf. Frees, 1990):

$$L_i(j_i,k) = \sum_{s=k}^{j_i+k} L_{i,s}(j_i,k) = \sum_{s=k}^{j_i+k} \ell_{i,s}(j_i)v(s-k) \qquad (25)$$

in which, according to the usual notation, $v(s-k)$ represents the present value at time k of one monetary unit at time s.

3.2. Inserting the randomness in the future lifetime

Replacing the future lifetime j_i by a random variable, the projected cash flow results:

$$\ell_{i,s} = \begin{cases} F_{i,s} & \text{with probability } {}_sp_x \\ 0 & \text{with probability } 1-{}_sp_x \end{cases} \tag{26}$$

having indicated by ${}_s p_x$ the probability for a policy at time 0 to be still in portfolio at time *s*.

In *k* each policy incurs the expected random loss, with respect to the future lifetime, given by the expression:

$$L_i(k) = \sum_{s=k}^{+\infty} F_{i,s} v(s-k)_{s-k} p_{x+k} . \tag{27}$$

3.3. Inserting the randomness in the number of policies in portfolio

Indicating by $N^{(x)}(k)$ the number of survivors at time k belonging to the initial group of *c* insureds at time 0, we can write the expression of the loss in k of the portfolio existing at this time, on the basis of the information the insurer owns at time 0:

$$L(k) = \sum_{i=1}^{N^{(x)}(k)} L_i(k), \tag{28}$$

Supposing the j_i's mutually independent and identically distributed random variables, the random loss at time k of the entire portfolio existing at that time is given by the sum of the losses referred to each policy:

$$L(k) = N^{(x)}(k) L_i(k) \tag{29}$$

3.4. Inserting the randomness in the interest rate used for the valuations

Now we extend what said above to a stochastic interest rate scenario. The discounting factor is described by a stochastic process $\{v(h)\}_{h=0}^{+\infty}$ modelling the evolution in time of the interest rate and this scenario is shared by all the policies in portfolio. Two more assumptions hold: the random variables $L_i(k)$ are independent and identically distributed given the sequence $\{v(h)\}_{h=0}^{+\infty}$ and the j_s's and $\{v(h)\}$ are mutually independent.

Within a more general formal scheme, let us denote by $(\Omega, \Im', P')$, $(\Omega, \Im'', P'')$ respectively, two probability spaces, with $\Im'$ the σ-algebra containing the financial events and $\Im''$ the σ-algebra containing the survival events. Therefore in the probability space $(\Omega, \Im, P)$, generated by the preceding two, the σ-algebra $\Im = \Im' \cup \Im''$ contains both the information about mortality and financial history, by means of the filtration $\{\Im_k\} \subset \Im$ where $\Im_k = \Im'_k \cup \Im''_k$, with $\{\Im'_k\} \subset \Im'$ and $\{\Im''_k\} \subset \Im''$.

In the following we will assume a frictionless market with continuous trading, no restrictions on borrowing or short-sales, zero-bonds and stocks both infinitely divisible. So we take the expectation in equations (27) and (28) under the risk-neutral probability measure, following a canonical procedure well stated in the financial literature:

$$E[L_i(k) \mid \Im_k] = \sum_{s=k}^{+\infty} F_{i,s\ s-k} p_{x+k} E[v(s-k) \mid \Im_k]. \tag{30}$$

Then we obtain the expected value in k of the loss referred to the entire portfolio:

$$E\left[L_k \mid \Im_k\right] = E\left[N^x(k)L_i(k) \mid \Im_k\right] = c_k p_x \sum_{s=k}^{+\infty} F_{i,s\ s-k}\ p_{x+k} E\left[v(s-k) \mid \Im_k\right]. \tag{31}$$

The problem of the hypothesis of market completeness in the demographic framework is well known in literature; some papers present the construction of an opportune probability measure, in order to guarantee the appropriate properties of the price function (cf. De Felice *et al.*, 2004 for a brief survey on the subject). As a matter of fact, by virtue of the intrinsic characteristics of the reserves, we consider reasonable to express the current valuations by means of the expectation framed within the *best prediction* of the demographic scenario. In this order of ideas, we obtain proxies of the reserve market values, which are deduced consistently with the model, so that we can say they are "marked to model".

4. Applications

4.1. Background hypotheses

A possible application is given by the calculus of (16) and (17) for the case of an immediate temporary (n = 10 years) unitary annuity for a male policyholder (x = 40), by means of a term structure of interest rates and of a projected mortality table. The term structure was based on a Cox-Ingersoll-Ross square root model according to a simple discretisation (Chan et al., 1992; Deelstra et al., 1995), in which the continuous centred interest rate is defined by the stochastic differential equation

$$dr_t = -kr_t dt + \sigma\sqrt{r_t + \gamma}\, dB_t, \tag{32}$$

where k and σ are positive constants, γ is the long term mean, B_t is a Brownian motion and r_t is the shifted interest rate set equal to the single observation net of the historical mean. The solution of (32) is given by

$$r_t = e^{-kt} r_0 + \sigma e^{-kt} \int_0^t e^{ku} \sqrt{r_u + \gamma}\, dB_u \tag{33}$$

for which expected value, covariance and variance are respectively (Deelstra et al., 1995, 734-735):

$$E\left[r_t\right] = r_0 e^{-kt} \tag{34}$$

$$\operatorname{var}\left[r_t\right] = -\frac{\sigma^2}{2k}(2r_0 + \gamma)e^{-2kt} + \frac{\sigma^2}{k} r_0 e^{-2kt} + \frac{\gamma\sigma^2}{2k} \tag{35}$$

$$\operatorname{cov}\left[r_t, r_s\right] = \sigma^2 \frac{e^{-kt} - e^{-k(s+t)}}{k} r_0 + \sigma^2 \gamma \frac{e^{-k(t-s)} - e^{-k(s+t)}}{2k} \qquad \forall s \le t\ . \tag{36}$$

As far as data are concerned, interest rates derive from Bank of Italy official statistics and consist of annualised net interest rate of Government 3-month T-Bill rate covering the period from January 1996 until January 2004; mortality rates have been derived from the RG48 Italian Male table. The following simple discretisation

$$r_t = \phi r_{t-1} + \sigma_a \sqrt{r_{t-1} + \gamma}\, a_t \tag{37}$$

estimated as shown by Deelstra et al. (1995, 741) gave the results shown by Table 1.

Table 1

CIR parameters and corresponding term structure of interest rates

r_0 starting value	γ long term mean	φ initial value	σ_α volatility coeff. (discrete case)	κ drift coff. (discrete case)	σ diffusion coeff.
1,72%	4,52%	97,40%	0,52%	2,63%	0,53%

Time	0	1	2	3	4	5	6	7	8	9	10
Expected rate	1,72%	2,48%	3,03%	3,43%	3,73%	3,94%	4,10%	4,21%	4,29%	4,35%	4,40%
Standard deviation	0,00%	0,23%	0,31%	0,37%	0,40%	0,43%	0,44%	0,46%	0,47%	0,47%	0,48%

4.2. Current values of projected cash-flows (a continuous approach)

As Chan et al. (1992), we adopt the same parameters in both the discrete and the continuous processes, obtaining the estimation for k posing $k = 1 - \phi$ and using the equation $\sigma_a = \sigma$.

We calculate the expected values of the current value of the reserves. Our example of application refers to the case of an immediate temporary (10) unitary annuity for an insured aged 40. On the basis of equation (31) we get the current values of the reserves at the beginning of each of the ten years constituting the policy duration.

In Table 2 we report the results and compare them with the corresponding values calculated at the contractual annual rate 0.04.

The current values of the reserves are less than the contractual premiums. The influence of the financial risk due to the evaluation rate can be observed in the behaviour of the difference between the current value premium and the contractual one, always negative but decreasing when the reference time increases.

Table 2

Current values of the mathematical provision

Evaluation time	0	1	2	3	4	5	6	7	8	9	10
Fixed value 4%	8,06	7,39	6,69	5,97	5,22	4,43	3,61	2,77	1,88	0,96	0,00
Current value (CIR-discrete)	8,04	7,30	6,59	5,87	5,14	4,37	3,57	2,74	1,87	0,96	0,00
Current value (CIR-continuous)	7,61	7,02	6,39	5,73	5,03	4,30	3,53	2,71	1,85	0,95	0,00
Difference (4%-CIR discr.)	0,02	0,09	0,11	0,10	0,08	0,06	0,04	0,03	0,01	0,00	0,00
Difference (4%-CIR cont.)	0,45	0,37	0,30	0,24	0,18	0,13	0,09	0,05	0,03	0,01	0,00

4.3. Risk indicators for projected cash-flows (a discrete approach)

The calculations give us the opportunity to evaluate not only a synthetic risk indicator such as the duration but also the VaR, based on the cash flows mapping, as shown by Table 3. Setting apart from the actuarial risk components, a first order approximation of the variation due to a change in the evaluation interest rate can be obtained through the cash flow mapping of the portfolio reserve. The mathematical provision can be seen as a portfolio of *n* zero coupon bonds whose nominal value is the certain-equivalent of the conditional payment with maturity equal the each single critical (remaining) instant r and with portfolio contribution equal to w. The corresponding duration (formulation (16)) can be calculated as

$$D_{R_t} = \sum_{s=1}^{n-t} w_s s, \; with \; w_s = \frac{{}_s p_{x+t} e^{-s\delta_{L_t}}}{\sum_{s=1}^{n-t} {}_s p_{x+t} e^{-s\delta_{L_t}}} . \tag{38}$$

Table 3

VaR of the mathematical provision at issue time

Time/item	Total	1	2	3	4	5	6	7	8	9	10
Certain equivalent (RG48)		0,9991	0,9981	0,9970	0,9959	0,9946	0,9931	0,9916	0,9899	0,9881	0,9863
Current value (CIR discrete)	8,0413	0,9749	0,9402	0,9010	0,8602	0,8198	0,7805	0,7429	0,7072	0,6733	0,6413
Portfolio weights	100%	12,12%	11,69%	11,20%	10,70%	10,19%	9,71%	9,24%	8,79%	8,37%	7,98%
Modified duration		0,9758	1,9411	2,9004	3,8562	4,8104	5,7638	6,7171	7,6706	8,6245	9,5787
Individual item change		0,0022	0,0057	0,0095	0,0133	0,0168	0,0200	0,0228	0,0253	0,0274	0,0293
Individual item VaR99%	0,4008	0,0051	0,0133	0,0222	0,0309	0,0391	0,0464	0,0530	0,0587	0,0638	0,0682
Var Report (99%)	Value	8,041	Undiversified		0,4008	Portfolio		0,2788	Correlation		-0,1220
		100%			4,98%			3,47%			-1,52%

In this perspective, the value of the reserve, as a portfolio, depends on the passage of time and on a vector of n risk factors $f = \{f_1, f_2, ..., f_n\}$ made up by the interest rate on each single node of the term structure[1]. Then first order approximation is given by

$$\Delta R_t \cong \sum_{s=1}^{n-t} w_s \frac{\delta R_s(f;t)}{\delta t} \Delta t + \sum_{s=1}^{n-t} w_s \frac{\delta R_s(f;t)}{\delta f_s} \Delta f_s = \mu_{R_T} + \sum_{s=1}^{n-t} \varphi_s \Delta f_s , \tag{39}$$

where μ_{R_t} is the change in value resulting from the passage of time, and φ_s is the so-called aggregate delta-factor.

Once we recall that the vector of changes in the underlying factors follows a multivariate normal with known mean and covariance matrix (cf. section 4.1 formulations (34), (35) and (36)), the first derivative of reserve value with respect to "bond-equivalent" prices can be calculated from the deltas of the individual instruments that is to say through the duration components of the reserve. More specifically, at issue time the first addendum of the formulation (39) is null while the second can be expressed for chosen level of confidence linked to the parameter α as

$$VaR_{R_t} = -\alpha R_t \sigma_{R_t} = -\alpha R_t \sqrt{w \Sigma w^T} = \sqrt{\mathbf{VaR} \cdot \mathbf{C} \cdot \mathbf{VaR}^{\mathbf{T}}} , \tag{40}$$

[1] The risk factor vector is made up exactly of n elements, equal to the number of assets constituting the portfolio, since each of them is exposed to a specific interest rate.

where **VaR** is the n-vector of individual undiversified VaRs, $\mathbf{VaR}^T$ is its transpose, and **C** is the $n \times n$ complete matrix of return correlations. Formulation (40) shows that the value-at-risk of the mathematical provision depends on underlying "primitive" factors (i.e. the variance-covariance matrix Σ) and relevant scale factors (i.e. the weights embodied in w and overall reserve size R_t).

A reserve VaR calculation at issue time is given by Table 3, which in the VaR report section shows at a 99% confidence level the maximum increase in the reserve value we are likely to experiment if, immediately after the issue of the policy, there is an interest rate shock for both market and accounting reasons.

5. Conclusions

The paper addressed the question of the calculation of the current value of the mathematical provision in a deterministic and stochastic scenario, using a proper term structure of interest rates and providing an original model of year-by-year evaluation of relevant risk factors. At a first level, it has been showed that the evaluation of relevant risk factors can be usefully performed through a complete sensitivity analysis, which can be split in a number of segments reflecting the numbers of relevant risk factors. As can be appreciated from calculations reported by Table 2, the current value is strictly dependent on the evaluation parameters. As far as the interest rate is concerned, the use of a term structure expressing a long term mean slightly higher (4,52%) then the corresponding fixed rate (4%) results in a systematically lower value of the reserve, both in the continuous and in the discrete approach.

The evaluations in the stochastic continuous framework in section 3 support the results obtained in the other sections, by means of closed formulations. The continuous scheme tries to provide a unitary view of the main risk components affecting the *current valuations* of the insurance business, according to the recent suggestions from IASB.

In this order of ideas, treating reserves, because of their specific nature, cannot be set within a reference market. So the liability exchange is virtual and the mathematical model is aimed to a pricing system according to a coherent representation of the insurance realm. This implies the choice of models which better represent the evolution in time of interest and mortality rates, mainly focusing on the demographic reference system. In fact this one does not present the fundamental characteristics required for a risk neutral valuation leading to market values; in this sense it is reasonable to consider an appropriate proxy of properly defined market value, referring to evaluations consistent with the model.

At a second level, the application to the case of the evaluation rate risk has shown that it is possible to measure the impact also of a change in the evaluation criteria and to appreciate precisely the performance modification. Apart from the numerical difference, there is a further question: the use of a rising term structure $(r_0 < \gamma)$ implies the inclusion of future profits arising from roll-over strategies, although these margins are definitely uncertain both for the realization of the interest rates and for the application of roll-over strategies. This implies that the application of a rising term structure may give rise to an under-evaluation of the reserve. Consistently, the use of a falling term structure $(r_0 > \gamma)$ may give rise to an over-evaluation of the reserve itself, since it is based on decreasing interest rates. Given these problems, the possibility of calculate a reserve VaR can provide for an insight into its evaluation dynamic. In this perspective, the knowledge of a maximum likelihood value change can provide for a benchmark of the evaluation itself. For example, with reference to the numerical application, it could be considered a maximum variation of the 3,47% or, alternatively, the corresponding amount could be set apart in a special provision to offset future unrealized profit margins.

Other implications, opening future research prospects, concern the selection of the stochastic process used to describe the dynamics of the interest rates and the possible managerial and regulatory application of a VaR measure. With reference to the first point, the selection of a CIR model seems to be particularly relevant, since the evaluation rate is by definition a non-negative rate. Other processes, such as the Ornstein-Uhlenbeck collection for example, may be more appropriate for the investigation of the investment risk (cf. section 2), since they embody even negative returns. With reference to the second aspect, the managerial and regulatory targets could be defined through VaR measures for both solvency and strategic decisions.

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