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The Curvature Tensor of g-Natural Metrics on Unit Tangent Sphere Bundles

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Abstract

We calculate the curvature tensor of an arbitrary Riemannian g-natural metric on the unit tangent sphere bundle T_1M of a Riemannian manifold M. This calculation is the fundamental tool to generalize classical theorems on the unit tangent sphere bundle, equipped with either the Sasaki metric or the standard contact metric structure.

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1 Introduction

The study of the geometry of a Riemannian manifold (M, g) through the properties of its unit tangent sphere bundle T_1M, represents a well known and interesting research field in Riemannian geometry. Traditionally, T_1M has been equipped with one of the following Riemannian metrics:

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- either the *Sasaki metric* $\widetilde{g_S}$, induced by the Sasaki metric g_S of the tangent bundle TM (or the metric $\bar{g} = \frac{1}{4} g_S$ of the *standard contact metric structure* $(\eta, \bar{g})$ of T_1M), or

- the metric $\widetilde{g_{CG}}$, induced by the *Cheeger-Gromoll metric* g_{CG} on TM.

Since $\bar{g}$ is homothetic to $\widetilde{g_S}$, these Riemannian metrics share essentially the same curvature properties. As concerns $(T_1M, \widetilde{g_{CG}})$, it is isometric to the tangent sphere bundle T_rM, with radius $r = \frac{1}{\sqrt{2}}$, equipped with the metric induced by the Sasaki metric of TM, the isometry being explicitly given by $\Phi : T_1M \to T_{\frac{1}{\sqrt{2}}}M$: $(x, u) \mapsto (x, u/\sqrt{2})$.

Several curvature properties on T_1M, equipped with one of the metrics above, turn out to correspond to very rigid properties for the base manifold M. We can refer to [9] for a survey on the geometry of $(T_1M, \widetilde{g_S})$. A survey on the contact metric geometry of $(T_1M, \eta, \bar{g})$ was made by the second author in [10].

In [8], the first author and M. Sarih investigated geometric properties of the tangent bundle TM, equipped with the most general "g-natural" metric. On unit tangent sphere bundles, the restrictions of g-natural metrics possess a simpler form. Precisely, it was proved in [4] that for every Riemannian metric $\tilde{G}$ on T_1M induced by a Riemannian g-natural metric G on TM, there exist four constants a, b, c and d, with

$$a > 0, \ \alpha := a(a + c) - b^2 > 0, \text{ and } \phi := a(a + c + d) - b^2 > 0, \qquad (1.1)$$

such that $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$, where

* k is the natural F-metric on M defined by

$$k(u; X, Y) = g(u, X)g(u, Y), \quad \text{for all} \quad (u, X, Y) \in TM \oplus TM \oplus TM,$$

* $\widetilde{g^s}$, $\widetilde{g^h}$, $\widetilde{g^v}$ and $\widetilde{k^s}$ are the metrics on T_1M induced by the three lifts g^s, g^h, g^v and k^v, respectively *(we refer to Section* 2 *for the definitions of F-metrics and their lifts).*

In Section 3 of this paper, we shall give the explicit expression of the curvature tensor of any Riemannian g-natural metric $\tilde{G}$ of T_1M. This calculation is an essential step for further investigations about Riemannian geometry of $(T_1M, \tilde{G})$. For example, it will be used in [2] to completely classify all $(T_1M, \tilde{G})$ with constant sectional curvature, and in [3] to investigate curvature conditions on g-natural contact metric structures introduced by the authors in [1]. Moreover, we announce here some of the results we can obtain by applying these curvature equations.

Theorem 1. **[2]** *$(T_1M, \tilde{G})$ has constant sectional curvature $\tilde{K}$ if and only if the base manifold is a Riemannian surface (M^2, g) of constant Gaussian curvature $\bar{c}$ and one of the following cases occurs:*

(i) $d = 0$ and $\bar{c} = 0$. In this case, $\tilde{K} = 0$.

(ii) $b = 0$ and $\bar{c} = \dfrac{d}{a}$. In this case, $\tilde{K} = \dfrac{d}{a\varphi}$.

(iii) $b = 0$, $d = a + c$ and $\bar{c} = \dfrac{a+c}{a} > 0$. In this case, $\tilde{K} = \dfrac{1}{2a} > 0$.

Theorem 2. **[2]** *Let (M^2, g) be a Riemannian surface. The following properties are equivalent:*

(i) $(M^2 g)$ has constant Gaussian curvature $\bar{c}$,

(ii) The scalar curvature

$$\widetilde{\tau} = \frac{1}{2\alpha\varphi}\left\{-a^2\bar{c}^2 + 2\left[\alpha + \phi + \frac{b^4}{\alpha}\right]\bar{c} - d^2\right\}. \tag{1.2}$$

of (T_1M^2, G) is constant,

(iii) (T_1M^2, G) is curvature homogeneous.

Moreover, when one of the properties above is satisfied, then all *g-natural Riemannian metrics on T_1M^2 are curvature homogeneous.*

Theorem 3. **[3]** *The g-natural contact metric structure $(T_1M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature $\widetilde{K}$ if and only if the base manifold (M, g) has constant sectional curvature $\bar{c}$ either equal to $\dfrac{d}{a}$ or to $\dfrac{a+c}{a} > 0$.*

Theorem 4. **[3]** *If the g-natural contact metric structure $(T_1M, \tilde{\eta}, \tilde{G})$ has constant φ-sectional curvature, then the base manifold (M, g) is locally isometric to a two-point homogeneous space.*

Theorem 5. **[3]** *A g-natural contact metric structure $(\tilde{\eta}, \tilde{G})$ on T_1M is locally symmetric if and only if $(\tilde{\eta}, \tilde{G}) = (\bar{\eta}, \bar{g})$ is the standard contact metric structure of T_1M and (M, g) is flat.*

2 Basic formulae on tangent bundles

Let (M, g) be a Riemannian manifold, ∇ its Levi-Civita connection and R its curvature tensor, taken with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

We write $p_M : TM \to M$ for the natural projection and F for the natural bundle with $FM = p_M^*(T^* \otimes T^*)M \to M$. Then, $Ff(X_x, g_x) = (Tf.X_x, (T^* \otimes T^*)f.g_x)$ for all manifolds M, local diffeomorphisms f of M, $X_x \in T_xM$ and

$g_x \in (T^* \otimes T^*)_x M$. The sections of the canonical projection $FM \to M$ are called *F-metrics* in literature. So, if we denote by $\oplus$ the fibered product of fibered manifolds, then the F-metrics are mappings $TM \oplus TM \oplus TM \to I\!R$ which are linear in the second and the third argument.

For a given F-metric δ on M, there are three distinguished constructions of metrics on the tangent bundle TM [12]:

(a) If δ is symmetric, then the *Sasaki lift* δ^s of δ is defined by

$$\begin{cases} \delta^s_{(x,u)}(X^h,Y^h) = \delta(u;X,Y), & \delta^s_{(x,u)}(X^h,Y^v) = 0, \\ \delta^s_{(x,u)}(X^v,Y^h) = 0, & \delta^s_{(x,u)}(X^v,Y^v) = \delta(u;X,Y), \end{cases}$$

for all X, $Y \in M_x$. When δ is non degenerate and positive definite, so is δ^s.

(b) The *horizontal lift* δ^h of δ is a pseudo-Riemannian metric on TM, given by

$$\begin{cases} \delta^h_{(x,u)}(X^h,Y^h) = 0, & \delta^h_{(x,u)}(X^h,Y^v) = \delta(u;X,Y), \\ \delta^h_{(x,u)}(X^v,Y^h) = \delta(u;X,Y), & \delta^h_{(x,u)}(X^v,Y^v) = 0, \end{cases}$$

for all X, $Y \in M_x$. If δ is positive definite, then δ^s is of signature (m,m).

(c) The *vertical lift* δ^v of δ is a degenerate metric on TM, given by

$$\begin{cases} \delta^v_{(x,u)}(X^h,Y^h) = \delta(u;X,Y), & \delta^v_{(x,u)}(X^h,Y^v) = 0, \\ \delta^v_{(x,u)}(X^v,Y^h) = 0, & \delta^v_{(x,u)}(X^v,Y^v) = 0, \end{cases}$$

for all X, $Y \in M_x$. The rank of δ^v is exactly that of δ.
If $\delta = g$ is a Riemannian metric on M, then these three lifts of δ coincide with the three well-known classical lifts of the metric g to TM.

The three lifts above of *natural* F-metrics generate the class of g-natural metrics on TM. These metrics were first introduced by Kowalski and Sekizawa in [12] (see also [7] for the definition of g-natural metrics and [11] for the general definition of naturality). As we already mentioned in the Introduction, on unit tangent sphere bundles the restrictions of g-natural metrics possess the simpler form $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$. Notice that such a metric $\tilde{G}$ on T_1M is necessarily induced by a metric on TM of the form $G = a.g^s + b.g^h + c.g^v + \beta.k^v$, where a, b, c are constants and $\beta : [0,\infty) \to I\!R$ is a C^∞-function depending on the norm of $u \in TM$, such that $a > 0$, $\alpha := a(a+c) - b^2 > 0$, and $\phi(t) := a(a+c+t\beta(t)) - b^2 > 0$, for all $t \in [0,\infty)$. Inequalities (1.1) express the fact that G is Riemannian (cf. [6]).

The Levi-Civita connection of a Riemannian metric on TM of the form $G = a.g^s + b.g^h + c.g^v + \beta.k^v$ is given by :

Proposition 1 ([7]). *Let $G = a.g^s + b.g^h + c.g^v + \beta.k^v$, be a g-natural Riemannian metric on TM. Then, the Levi-Civita connection $\bar{\nabla}$ of (TM,G) is characterized by*

$$(i)\,(\bar\nabla_{X^h}Y^h)_{(x,u)} = \Big\{(\nabla_X Y)_x - \frac{ab}{2\alpha}[R(X_x,u)Y_x + R(Y_x,u)X_x] + \frac{b\beta}{2\alpha}[g(X_x,u)Y_x$$
$$+g(Y_x,u)X_x] + \frac{b}{\alpha\phi}[a^2\beta\, g(R(X_x,u)Y_x,u) + (\alpha\beta' - a\beta^2)\, g(X_x,u)g(Y_x,u)]u\Big\}^h$$
$$+\Big\{\frac{b^2}{\alpha}\, R(X_x,u)Y_x - \frac{a(a+c)}{2\alpha}\, R(X_x,Y_x)u - \frac{(a+c)\beta}{2\alpha}\,[g(Y_x,u)\,X_x + g(X_x,u)\,Y_x]$$
$$+\frac{1}{\alpha\phi}[-ab^2\beta\, g(R(X_x,u)Y_x,u) + (-\alpha(a+c+t\beta)\beta' + b^2\beta^2)\, g(Y_x,u)g(X_x,u)]u\Big\}^v,$$

$$(ii)\,(\bar\nabla_{X^h}Y^v)_{(x,u)} = \Big\{-\frac{a^2}{2\alpha}\, R(Y_x,u)X_x + \frac{a\beta}{2\alpha}\, g(X_x,u)\,Y_x + \frac{a}{2\alpha\phi}\,[a^2\beta\, g(R(X_x,u)Y_x,u)$$
$$+\alpha\beta\, g(X_x,Y_x) +\ (2\alpha\beta' - a\beta^2)\, g(X_x,u)g(Y_x,u)]u\Big\}^h$$
$$+\Big\{(\nabla_X Y)_x + \frac{ab}{2\alpha}\, R(Y_x,u)X_x - \frac{b\beta}{2\alpha}\, g(X_x,u)\,Y_x + \frac{b}{2\alpha\phi}\,[-\alpha\beta\, g(X_x,Y_x)$$
$$-a^2\beta\, g(R(X_x,u)Y_x,u) -\ (2\alpha\beta' - a\beta^2)\, g(X_x,u)g(Y_x,u)]u\Big\}^v,$$

$$(iii)(\bar\nabla_{X^v}Y^h)_{(x,u)} = \Big\{-\frac{a^2}{2\alpha}\, R(X_x,u)Y_x + \frac{a\beta}{2\alpha}\, g(Y_x,u)\,X_x + \frac{a}{2\alpha\phi}\,[a^2\beta\, g(R(X_x,u)Y_x,u)$$
$$+\alpha\beta\, g(X_x,Y_x) +\ (2\alpha\beta' - a\beta^2)\, g(X_x,u)g(Y_x,u)]u\Big\}^h$$
$$+\Big\{\frac{ab}{2\alpha}\, R(X_x,u)Y_x - \frac{b\beta}{2\alpha}\, g(Y_x,u)\,X_x + \frac{b}{2\alpha\phi}\,[-a^2\beta\, g(R(X_x,u)Y_x,u)$$
$$-\alpha\beta\, g(X_x,Y_x) -\ (2\alpha\beta' - a\beta^2)\, g(X_x,u)g(Y_x,u)]u\Big\}^v,$$

$$(iv)(\bar\nabla_{X^v}Y^v)_{(x,u)} = 0$$

for all vector fields X, Y *on* M *and* $(x,u) \in TM$.

Substituting from Proposition 1 into the general form for the Riemannian curvature of an arbitrary Riemannian g-natural metric, some standard but long calculations lead to the following result:

Proposition 2 ([7]). *Let* (M,g) *be a Riemannian manifold and let* $G = a.g^s + b.g^h + c.g^v + \beta.k^v$, *where* a, b *and* c *are constants and* $\beta : [0,\infty) \to \mathbb{R}$ *is a function satisfying* (1.1). *Denote by* ∇ *and* R *the Levi-Civita connection and the Riemannian curvature tensor of* (M,g), *respectively. If we denote by*

$\bar{R}$ the Riemannian curvature tensor of (TM, G), then:

$$
\begin{aligned}
&\bar{R}(X^h, Y^h)Z^h \\
&= \Big\{ R(X,Y)Z + \frac{ab}{2\alpha}\,[2(\nabla_u R)(X,Y)Z - (\nabla_Z R)(X,Y)u] + \frac{a^2}{4\alpha}\,[R(R(Y,Z)u,u)X \\
&- R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z] + \frac{a^2b^2}{4\alpha^2}\,[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z \\
&+ R(X,u)R(Z,u)Y - R(Y,u)R(Z,u)X] + \frac{a\beta(\alpha-b^2)}{4\alpha^2}\,[g(Z,u)R(X,Y)u + g(Y,u)R(X,u)Z \\
&- g(X,u)R(Y,u)Z] - \frac{ab^2}{2\alpha^2\phi}\,[a^2\beta\, g(R(Y,u)Z,u) + (\alpha\beta' - a\beta^2)\, g(Y,u)g(Z,u)]R(X,u)u \\
&+ \frac{ab^2}{2\alpha^2\phi}\,[a^2\beta\, g(R(X,u)Z,u) + (\alpha\beta' - a\beta^2)\, g(X,u)g(Z,u)]R(Y,u)u \\
&+ \frac{\beta}{2\alpha\phi}\,[-a^2 b\, g(R(Y,u)Z,u) + (b^2(\beta + r^2\beta') + \frac{\beta\phi}{2})\, g(Y,u)g(Z,u)]X \\
&- \frac{\beta}{2\alpha\phi}\,[-a^2 b\, g(R(X,u)Z,u) + (b^2(\beta + r^2\beta') + \frac{\beta\phi}{2})\, g(X,u)g(Z,u)]Y \\
&+ \frac{a\beta}{\alpha\phi}\{-abg((\nabla_u R)(X,Y)Z,u) + \frac{a^2}{4}\,[g(R(Y,Z)u, R(X,u)u) \\
&- g(R(X,Z)u, R(Y,u)u) - 2g(R(X,Y)u, R(Z,u)u)] - \frac{3a(a+c)}{4}\, g(R(X,Y)Z,u) \\
&+ \frac{a^2b^2}{4\alpha}\,[g(R(Y,u)Z + R(Z,u)Y, R(X,u)u) - g(R(X,u)Z + R(Z,u)X, R(Y,u)u)] \\
&+ \frac{a\beta(\alpha-b^2)}{4\alpha}\,[g(X,u)g(R(Y,u)Z,u) - g(Y,u)g(R(X,u)Z,u)] \\
&+ \frac{(a+c)\beta}{4}\,[g(X,u)g(Y,Z) - g(Y,u)g(X,Z)]\}u \Big\}^h \\
&+ \Big\{ -\frac{b^2}{\alpha}\,(\nabla_u R)(X,Y)Z + \frac{a(a+c)}{2\alpha}(\nabla_Z R)(X,Y)u - \frac{ab}{4\alpha}\,[R(R(Y,Z)u,u)X \\
&- R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z - R(X, R(Y,u)Z)u \\
&- R(X, R(Z,u)Y)u + R(Y, R(X,u)Z)u + R(Y, R(Z,u)X)u] \\
&- \frac{b\beta(3\alpha - b^2)}{4\alpha^2}\,[g(Z,u)R(X,Y)u + g(Y,u)R(X,u)Z - g(X,u)R(Y,u)Z] \\
&+ \frac{b(b^2-\alpha)}{2\alpha^2\phi}\,[a^2\beta\, g(R(Y,u)Z,u) + (\alpha\beta' - a\beta^2)\, g(Y,u)g(Z,u)]R(X,u)u \\
&- \frac{b(b^2-\alpha)}{2\alpha^2\phi}\,[a^2\beta\, g(R(X,u)Z,u) + (\alpha\beta' - a\beta^2)\, g(X,u)g(Z,u)]R(Y,u)u \\
&+ \frac{(a+c)b\beta}{2\alpha\phi}\,[a\, g(R(Y,u)Z,u) - (\beta + r^2\beta')\, g(Y,u)g(Z,u)]X - \frac{(a+c)b\beta}{2\alpha\phi}\,[a\, g(R(X,u)Z,u) \\
&- (\beta + r^2\beta')\, g(X,u)g(Z,u)]Y + \frac{b}{\alpha\phi}\,\{ab\beta\, g((\nabla_u R)(X,Y)Z,u) \\
&- \frac{a^2\beta}{4}\,[g(R(Y,Z)u, R(X,u)u) - g(R(X,Z)u, R(Y,u)u) - 2g(R(X,Y)u, R(Z,u)u)] \\
&- \frac{a^2b^2\beta}{4\alpha}\,[g(R(Y,u)Z + R(Z,u)Y, R(X,u)u) - g(R(X,u)Z + R(Z,u)X, R(Y,u)u)]
\end{aligned}
$$

$$+\frac{3a(a+c)\beta}{4}\,g(R(X,Y)Z,u)+[\alpha\beta'-\frac{a\beta^2(3\alpha-b^2)}{4\alpha}]\,[g(X,u)g(R(Y,u)Z,u)$$
$$-g(Y,u)g(R(X,u)Z,u)]-\frac{(a+c)\beta^2}{4}\,[g(X,u)g(Y,Z)-g(Y,u)g(X,Z)]\Big\}u\Big\}^v,$$

$$\bar{R}(X^h,Y^v)Z^h$$
$$=\Big\{-\frac{a^2}{2\alpha}\,(\nabla_X R)(Y,u)Z+\frac{ab}{2\alpha}[R(X,Y)Z+R(Z,Y)X]+\frac{a^3b}{4\alpha^2}\,[R(X,u)R(Y,u)Z$$
$$-R(Y,u)R(X,u)Z-R(Y,u)R(Z,u)X]+\frac{a^2b\beta}{4\alpha^2}\,[g(X,u)R(Y,u)Z-g(Z,u)R(X,Y)u]$$
$$-\frac{a^2b}{4\alpha^2\phi}\,[a^2\beta\,g(R(Y,u)Z,u)+\alpha\beta\,g(Y,Z)+(2\alpha\beta'-a\beta^2)\,g(Y,u)g(Z,u)]R(X,u)u$$
$$+\frac{a^2b}{2\alpha^2\phi}\,[a^2\beta\,g(R(X,u)Z,u)+(\alpha\beta'-a\beta^2)\,g(X,u)g(Z,u)]R(Y,u)u$$
$$+\frac{b}{4\alpha\phi}\,[-a^2\beta\,g(R(Y,u)Z,u)-(\alpha+\phi)\beta\,g(Y,Z)-(2(\alpha+\phi)\beta'-a\beta^2)\,g(Y,u)g(Z,u)]X$$
$$-\frac{b}{2\alpha\phi}\,[a^2\beta\,g(R(X,u)Z,u)+((\alpha+\phi)\beta'-a\beta^2)\,g(X,u)g(Z,u)]Y-\frac{b}{2\alpha}[\beta\,g(X,Y)$$
$$+2\beta'\,g(X,u)g(Y,u)]Z+\frac{1}{\alpha\phi}\{\frac{a^3\beta}{2}\,g((\nabla_X R)(Y,u)Z,u)+\frac{a^4b\beta}{4\alpha}\,[g(R(Y,u)Z,R(X,u)u)$$
$$-g(R(X,u)Z+R(Z,u)X,R(Y,u)u)]-a^2b\,[\frac{a\beta^2}{4\alpha}\,g(X,u)g(R(Y,u)Z,u)$$
$$+\frac{\alpha\beta'-a\beta^2}{\phi}\,g(Y,u)g(R(X,u)Z,u)]-\frac{b(2\alpha\beta'-a\beta^2)}{2}\,[g(X,u)g(Y,Z)+g(Z,u)g(X,Y)]$$
$$-\frac{a^2b\beta}{2}\,[2g(R(X,Y)Z,u)+g(R(Z,Y)X,u)]$$
$$+b[-2\alpha\beta''+a((2\beta(1+\frac{\alpha}{\phi})+\frac{r^2\alpha}{\phi}\beta')\beta'-\frac{a\beta^3}{\phi})]\,g(X,u)g(Y,u)g(Z,u)\}u\Big\}^h$$
$$+\Big\{\frac{ab}{2\alpha}\,(\nabla_X R)(Y,u)Z+\frac{a^2}{4\alpha}\,R(X,R(Y,u)Z)u-\frac{a^2b^2}{4\alpha^2}\,[R(X,u)R(Y,u)Z-R(Y,u)R(X,u)Z$$
$$-R(Y,u)R(Z,u)X]-\frac{b^2}{\alpha}\,R(X,Y)Z+\frac{a(a+c)}{2\alpha}R(X,Z)Y+\frac{a\beta(\alpha-b^2)}{4\alpha^2}\,[g(X,u)R(Y,u)Z$$
$$-g(Z,u)R(X,Y)u]-\frac{a(\alpha-b^2)}{4\alpha^2\phi}\,[a^2\beta\,g(R(Y,u)Z,u)+\alpha\beta\,g(Y,Z)$$
$$+(2\alpha\beta'-a\beta^2)\,g(Y,u)g(Z,u)]R(X,u)u-\frac{ab^2}{2\alpha^2\phi}\,[a^2\beta\,g(R(X,u)Z,u)$$
$$+(\alpha\beta'-a\beta^2)\,g(X,u)g(Z,u)]R(Y,u)u+\frac{a+c}{4\alpha\phi}\,[a^2\beta\,g(R(Y,u)Z,u)+(\alpha+\phi)\beta\,g(Y,Z)$$
$$+(2(\alpha+\phi)\beta'-a\beta^2)\,g(Y,u)g(Z,u)]X+\frac{1}{4\alpha\phi}\,[2ab^2\beta\,g(R(X,u)Z,u)+(\frac{2}{a}(2\alpha\phi$$
$$+b^2(\alpha+\phi))\beta'-\beta^2(\phi+2b^2))\,g(X,u)g(Z,u)]Y+\frac{a+c}{2\alpha}[\beta\,g(X,Y)+2\beta'\,g(X,u)g(Y,u)]Z$$
$$+\frac{1}{\alpha\phi}\{-\frac{a^2b\beta}{2}\,g((\nabla_X R)(Y,u)Z,u)-\frac{a^3b^2\beta}{4\alpha}\,[g(R(Y,u)Z,R(X,u)u)$$

$$- g(R(X,u)Z, R(Y,u)u)] + \frac{ab^2\beta}{2}[2g(R(X,Y)Z,u) + g(R(Z,Y)X,u)]$$
$$+ [\frac{a\alpha\beta'}{2} + \frac{a^2(b^2-\alpha)\beta^2}{4\alpha}]\, g(X,u)g(R(Y,u)Z,u) + \frac{ab^2\beta}{\phi}(\alpha\beta' - a\beta^2)g(Y,u)g(R(X,u)Z,u)$$
$$+ \frac{1}{4}[2\alpha(2(a+c)+\beta r^2)\beta' - (\alpha+2b^2)\beta^2]\, g(X,u)g(Y,Z) + [\alpha(a+c+\beta r^2)\beta'$$
$$- \frac{b^2\beta^2}{2}]\, g(Z,u)g(X,Y)] + [2\alpha(a+c+\beta r^2)\beta'' + \frac{1}{\phi}(\alpha(b^2-\phi)r^2\beta'$$
$$-\phi(\alpha+2b^2)\beta)\beta' + \frac{a(\phi+4b^2)\beta^3}{4\phi}]\, g(X,u)g(Y,u)g(Z,u)\}u\Big\}^v,$$

$$\bar{R}(X^v, Y^v)Z^v = 0,$$

for all $x \in M$ and X, Y, $Z \in M_x$.

3 Riemannian g-natural metrics on T_1M and their curvature tensor

As it is well known, the *tangent sphere bundle of radius* $\rho > 0$ over a Riemannian manifold (M,g), is the hypersurface $T_\rho M = \{(x,u) \in TM | g_x(u,u) = \rho^2\}$. The tangent space of $T_\rho M$, at a point $(x,u) \in T_\rho M$, is given by

$$(T_\rho M)_{(x,u)} = \{X^h + Y^v / X \in M_x, Y \in \{u\}^\perp \subset M_x\}. \tag{3.1}$$

When $\rho = 1$, T_1M is called *the unit tangent (sphere) bundle.*

Let $G = a.g^s + b.g^h + c.g^v + \beta.k^v$ be a Riemannian g-natural metric on TM and $\tilde{G}$ the metric on T_1M induced by G. Then, $\tilde{G}$ only depends on a, b, c and $d := \beta(1)$, and these coefficients satisfy (1.1) (see also [4]).

Using the Schmidt's orthonormalization process, a simple calculation shows that the vector field on TM defined by

$$N^G_{(x,u)} = \frac{1}{\sqrt{(a+c+d)\phi}}[-b.u^h + (a+c+d).u^v], \tag{3.2}$$

for all $(x,u) \in TM$, is normal to T_1M and unitary at any point of T_1M. Here ϕ is, by definition, the quantity $\phi(1) = a(a+c+d) - b^2$.

Now, we define the "tangential lift" X^{t_G} –with respect to G– of a vector $X \in M_x$ to $(x,u) \in T_1M$ as the tangential projection of the vertical lift of X to (x,u) –with respect to N^G–, that is,

$$X^{t_G} = X^v - G_{(x,u)}(X^v, N^G_{(x,u)})\, N^G_{(x,u)} = X^v - \sqrt{\frac{\phi}{a+c+d}} g_x(X,u)\, N^G_{(x,u)}. \tag{3.3}$$

If $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$.

The tangent space $(T_1M)_{(x,u)}$ of T_1M at (x,u) is spanned by vectors of the form X^h and Y^{t_G}, where X, $Y \in M_x$. Hence, the Riemannian metric $\tilde{G}$ on T_1M, induced from G, is completely determined by the identities

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)g_x(X,Y) + dg_x(X,u)g_x(Y,u), \\ \tilde{G}_{(x,u)}(X^h, Y^{t_G}) &= bg_x(X,Y), \\ \tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) &= ag_x(X,Y) - \frac{\phi}{a+c+d}g_x(X,u)g_x(Y,u), \end{cases} \tag{3.4}$$

for all $(x,u) \in T_1M$ and X, $Y \in M_x$. It should be noted that, by (3.4), horizontal and vertical lifts are orthogonal with respect to $\tilde{G}$ if and only if $b = 0$.

CONVENTION 1. Notice that, for $(x,u) \in T_1M$, the tangential lift to (x,u) of the vector u is given by $u^{t_G} = \frac{b}{a+c+d}\, u^h$, that is, it is a horizontal vector. It follows that the tangent space $(T_1M)_{(x,u)}$ coincides with the set

$$\{X^h + Y^{t_G} / X \in M_x, Y \in \{u\}^\perp \subset M_x\}. \tag{3.5}$$

Hence, the operation of tangential lift from M_x to a point $(x,u) \in T_1M$ will be always applied only to vectors of M_x which are orthogonal to u.

Now, the Riemannian curvature of $(T_1M, \tilde{G})$ is determined by the metric $\tilde{G}$ and the three components of the Riemannian curvature tensor, given in the following

Proposition 3. *Let (M,g) be a Riemannian manifold and let $G = a.g^s + b.g^h + c.g^v + \beta.k^v$, where a, b and c are constants and $\beta : [0,\infty) \to \mathbb{R}$ is a function satisfying* (1.1)*. Denote by ∇ and R the Levi-Civita connection and the Riemannian curvature tensor of (M,g), respectively. If we denote by $\tilde{R}$ the*

Riemannian curvature tensor of $(T_1M, \tilde{G})$, then:

$$
\begin{aligned}
&(i)\tilde{R}(X^h, Y^h)Z^h \\
&= \Big\{ R(X,Y)Z + \frac{ab}{2\alpha}\,[2(\nabla_u R)(X,Y)Z - (\nabla_Z R)(X,Y)u] + \frac{a^2}{4\alpha}\,[R(R(Y,Z)u,u)X \\
&- R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z] + \frac{a^2b^2}{4\alpha^2}\,[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z \\
&+ R(X,u)R(Z,u)Y - R(Y,u)R(Z,u)X] + \frac{ad(\alpha-b^2)}{4\alpha^2}\,[g(Z,u)R(X,Y)u \\
&+ g(Y,u)R(X,u)Z - g(X,u)R(Y,u)Z] + \frac{ab^2}{2\alpha^2}\,\Big[-\frac{ad+b^2}{a+c+d}\,g(R(Y,u)Z,u) \\
&+d\,g(Y,u)g(Z,u)]\,R_uX - \frac{ab^2}{2\alpha^2}\,\Big[-\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\Big]\,R_uY \\
&+ \frac{d}{4\alpha}\,\Big[-\frac{2b^2}{a+c+d}\,g(R(Y,u)Z,u) + d\,g(Y,u)g(Z,u)\Big]\,X \\
&- \frac{d}{4\alpha}\,\Big[-\frac{2b^2}{a+c+d}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\Big]\,Y \\
&+ \frac{d}{4\alpha(a+c+d)}\{-4abg((\nabla_u R)(X,Y)Z,u) + a^2\,[g(R(Y,Z)u, R(X,u)u) \\
&- g(R(X,Z)u, R(Y,u)u) - 2g(R(X,Y)u, R(Z,u)u)] + \frac{a^2b^2}{\alpha}\,[g(R(Y,u)Z \\
&+ R(Z,u)Y, R(X,u)u) - g(R(X,u)Z + R(Z,u)X, R(Y,u)u)] \\
&- \Big[\frac{ad(b^2-\alpha)}{\alpha} + \frac{2b^2d(\phi+2b^2)}{\phi(a+c+d)} + \frac{4b^2\alpha}{\phi}\Big]\,[g(X,u)g(R(Y,u)Z,u) \\
&- g(Y,u)g(R(X,u)Z,u)] - 3a(a+c)\,g(R(X,Y)Z,u) \\
&+(a+c)d\,[g(X,u)g(Y,Z) - g(Y,u)g(X,Z)]\}u\Big\}^h \\
&+ \Big\{-\frac{b^2}{\alpha}\,(\nabla_u R)(X,Y)Z + \frac{a(a+c)}{2\alpha}(\nabla_Z R)(X,Y)u - \frac{ab}{4\alpha}\,[R(R(Y,Z)u,u)X \\
&- R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z - R(X,R(Y,u)Z)u - R(X,R(Z,u)Y)u \\
&+ R(Y,R(X,u)Z)u + R(Y,R(Z,u)X)u] - \frac{ab^3}{4\alpha^2}\,[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z \\
&+ R(X,u)R(Z,u)Y - R(Y,u)R(Z,u)X] - \frac{bd(3\alpha-b^2)}{4\alpha^2}\,[g(Z,u)R(X,Y)u \\
&+ g(Y,u)R(X,u)Z - g(X,u)R(Y,u)Z] + \frac{b(b^2-\alpha)}{2\alpha^2}\,\Big[\frac{ad+b^2}{a+c+d}\,g(R(Y,u)Z,u)
\end{aligned}
$$

$$-d\,g(Y,u)g(Z,u)]\,R_uX - \frac{b(b^2-\alpha)}{2\alpha^2}\left[\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) - d\,g(X,u)g(Z,u)\right]R_uY$$
$$\left.+\frac{(a+c)bd}{2\alpha(a+c+d)}\,[g(R(Y,u)Z,u)X - g(R(X,u)Z,u)Y]\right\}^{t_G},$$

$(ii)\tilde{R}(X^h, Y^{t_G})Z^h$

$$= \left\{-\frac{a^2}{2\alpha}\,(\nabla_X R)(Y,u)Z + \frac{ab}{2\alpha}[R(X,Y)Z + R(Z,Y)X] + \frac{a^3b}{4\alpha^2}\,[R(X,u)R(Y,u)Z\right.$$
$$- R(Y,u)R(X,u)Z - R(Y,u)R(Z,u)X] + \frac{a^2bd}{4\alpha^2}\,[g(X,u)R(Y,u)Z - g(Z,u)R(X,Y)u]$$
$$- \frac{ab}{4\alpha^2(a+c+d)}\,[a(ad+b^2)\,g(R(Y,u)Z,u) + \alpha d\,g(Y,Z)]R_uX$$
$$+ \frac{a^2b}{2\alpha^2}\left[\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) - d\,g(X,u)g(Z,u)\right]R_uY$$
$$- \frac{bd}{4\alpha(a+c+d)}\,[a\,g(R(Y,u)Z,u) + (2(a+c)+d)\,g(Y,Z)]X$$
$$+ \frac{b}{\alpha}\left[-\frac{ad+b^2}{2(a+c+d)}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\right]Y$$
$$-\frac{bd}{2\alpha}\,g(X,Y)Z + \frac{d}{4\alpha(a+c+d)}\left\{2a^2\,g((\nabla_X R)(Y,u)Z,u) + \frac{a^3b}{\alpha}\,[g(R(Y,u)Z, R(X,u)u)\right.$$
$$- g(R(X,u)Z + R(Z,u)X, R(Y,u)u)] + ab\left[-\frac{\alpha+\phi}{\alpha} + \frac{d}{a+c+d}\right]g(X,u)g(R(Y,u)Z,u)$$
$$- 2ab\,[2g(R(X,Y)Z,u) + g(R(Z,Y)X,u)]$$
$$\left.\left.+bd\left[\left(3 - \frac{d}{a+c+d}\right)g(X,u)g(Y,Z) + 2\,g(Z,u)g(X,Y)\right]\right\}u\right\}^h$$

$$+\left\{\frac{ab}{2\alpha}\,(\nabla_X R)(Y,u)Z + \frac{a^2}{4\alpha}\,R(X,R(Y,u)Z)u - \frac{a^2b^2}{4\alpha^2}\,[R(X,u)R(Y,u)Z\right.$$
$$- R(Y,u)R(X,u)Z - R(Y,u)R(Z,u)X] - \frac{b^2}{\alpha}\,R(X,Y)Z + \frac{a(a+c)}{2\alpha}R(X,Z)Y$$
$$+ \frac{ad(\alpha-b^2)}{4\alpha^2}\,[g(X,u)R(Y,u)Z - g(Z,u)R(X,Y)u]$$
$$- \frac{\alpha-b^2}{4\alpha^2(a+c+d)}\,[a(ad+b^2)\,g(R(Y,u)Z,u) + \alpha d\,g(Y,Z)]R_uX$$
$$+ \frac{ab^2}{2\alpha^2}\left[-\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\right]R_uY$$
$$+ \frac{(a+c)d}{4\alpha(a+c+d)}\,[a\,g(R(Y,u)Z,u) + (2(a+c)+d)\,g(Y,Z)]X$$
$$+ \frac{1}{4\alpha}\left[2b^2\left(2 - \frac{d}{a+c+d}\right)g(R(X,u)Z,u) - d(4(a+c)+d)\,g(X,u)g(Z,u)\,\right]Y$$
$$\left.+\frac{(a+c)d}{2\alpha}\,g(X,Y)Z\right\}^{t_G},$$

$$(iii)\tilde{R}(X^{t_G},Y^{t_G})Z^{t_G} = \frac{1}{2\alpha(a+c+d)}\Big\{\Big\{a^2b\ [g(Y,Z)R_uX - g(X,Z)R_uY] - b(\alpha+\phi)[g(Y,Z)X - g(X,Z)Y]\Big\}^h + \Big\{-ab^2\ [g(Y,Z)R_uX - g(X,Z)R_uY] + [(a+c)(\alpha+\phi)+\alpha d]\,[g(Y,Z)X - g(X,Z)Y]\,\Big\}^{t_G}\Big\},$$

for all $x \in M$, $(x,u) \in T_1M$ and all arbitrary vectors $X,Y,Z \in M_x$ satisfying Convention 1, *where $R_uX = R(X,u)u$ denotes the* Jacobi operator *associated to u.*

Proof. Denoting by $\bar{R}$ and $\tilde{R}$ the Riemannian curvature tensors of (TM,G) and $(T_1M,\tilde{G})$, respectively, from the Gauss equation for hypersurfaces we deduce that the tangential component $(\bar{R}(V,W)Z)^t$ of $\bar{R}(V,W)Z$ satisfies

$$\tilde{R}(V,W)Z = (\bar{R}(V,W)Z)^t - \theta(V,Z).\widetilde{S}_GW + \theta(W,Z).\widetilde{S}_GV, \tag{3.6}$$

for all $(x,u) \in T_1M$ and V, W and Z in $(T_1M)_{(x,u)}$, where $\widetilde{S}_G$ is the shape operator of T_1M in (TM,G) derived from N^G, and θ is the second fundamental form of T_1M (as a hypersurface immersed in TM), associated to N^G on T_1M. For all $Z \in T(T_1M)$, $\widetilde{S}_GZ$ is, by definition, the tangential component $(-\bar{\nabla}_ZN^G)^t$ of $-\bar{\nabla}_ZN^G$, with respect to the pointwise decomposition

$$(TM)_{(x,u)} = (T_1M)_{(x,u)} \oplus \langle N^G_{(x,u)}\rangle. \tag{3.7}$$

Then, using Proposition 1, we obtain

$$\widetilde{S}_GX^h = \frac{1}{\sqrt{(a+c+d)\phi}}\Big\{\Big\{\tfrac{b^2}{2\alpha}[-a\,R_uX + d\,X] - \Big[\beta'(1) + d\Big(1+\tfrac{db^2}{2(a+c+d)\alpha}\Big)\Big]\,g(X,u)u\Big\}^h + \tfrac{b}{2\alpha}\left\{(b^2-\alpha)\,R_uX\ - (a+c)d\,X\right\}^{t_G}\Big\}, \tag{3.8}$$

$$\widetilde{S}_GX^{t_G} = \frac{1}{\sqrt{(a+c+d)\phi}}\Big\{\big\{\tfrac{b}{2\alpha}[-ab\,R_uX + (\alpha+\phi)\,X] - \tfrac{b}{a+c+d}\,\big[\beta'(1) + \tfrac{d(2\alpha+\phi)}{2\alpha}\big]\,g(X,u)u\Big\}^h + \tfrac{1}{2\alpha}\left\{ab^2\,R_uX\ - [(a+c)(\alpha+\phi)+\alpha d]\,X\right\}^{t_G}\Big\}, \tag{3.9}$$

for all $(x,u) \in T_1M$ and $X \in M_x$.

On the other hand, the second fundamental form $\theta : \mathfrak{X}(T_1M)\times\mathfrak{X}(T_1M) \to C^\infty(T_1M)$, associated to N^G, is defined by $\bar{\nabla}_VW = \tilde{\nabla}_VW + \theta(V,W).N^G$, for all vector fields V and W on T_1M. So, $\theta(V,W) = G(\bar{\nabla}_VW, N^G)$, for all V, $W \in \mathfrak{X}(T_1M)$ and from Proposition 1 we deduce the following identities:

$$\theta(X^h,Y^h)_{(x,u)} = -\frac{1}{\sqrt{(a+c+d)\phi}}\{-b^2\,g(R(X,u)Y,u) + (a+c+d)\,g(X_x,u)g(Y_x,u)\}, \tag{3.10}$$

$$\begin{aligned}\theta(X^h, Y^{t_G})_{(x,u)} = &\ \ \theta(X^{t_G}, Y^h)_{(x,u)} \\ = &- \frac{b}{2\sqrt{(a+c+d)\phi}}\{-a\, g(R(X,u)Y,u) + d\, g(X,Y) \\ &+ (2\beta'(1)+d)\, g(X_x,u)g(Y_x,u)\},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\theta(X^{t_G}, Y^{t_G})_{(x,u)} = &\frac{1}{\sqrt{(a+c+d)\phi}}\Big\{ - \phi\, g(X,Y) \\ &+ \left[\phi - \frac{b^2(\beta'(1)+d)}{a+c+d}\right] g(X_x,u)g(Y_x,u)\Big\},\end{aligned} \tag{3.12}$$

for all vector fields X and Y on M and $(x,u) \in T_1M$.

It now suffices to apply (3.6) to various classical lifts to T_1M of vectors on M. Using Proposition 2 and (3.8)-(3.12), we obtain the required formulae for the curvature tensor. Note that formulae above can also be obtained calculating directly $\tilde{R}$ from the Levi Civita connection $\tilde{\nabla}$ of $(T_1M, \tilde{G})$ given in [1] □

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