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THE CUT LOCUS OF A TWO-SPHERE OF REVOLUTION AND TOPONOGOV'S COMPARISON THEOREM

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Abstract. We determine the structure of the cut locus of a class of two-spheres of revolution, which includes all ellipsoids of revolution. Furthermore, we show that a subclass of this class gives a new model surface for Toponogov's comparison theorem.

1. Introduction. Let $\gamma : [0, t_0] \to M$ be a minimal geodesic segment on a complete Riemannian manifold M. The endpoint $\gamma(t_0)$ of the geodesic segment is called a *cut point* of $p := \gamma(0)$ along γ if any extended geodesic segment $\widetilde{\gamma} : [0, t_1] \to M$ of γ, where $t_1 > t_0$, is not a minimizing arc joining p to $\tilde{\gamma}(t_1)$ anymore. The *cut locus* C_p of the point p is defined by the set of the cut points along all geodesic segments emanating from p. It is known that the cut locus of a point p on a complete 2-dimensional Riemannian manifold is a *local tree* (see [7] or [15]), i.e., for any $q \in C_p$ and any neighborhood U around q in M, there exists an open neighborhood $V \subset U$ around q such that any two cut points in V can be joined by a unique rectifiable Jordan arc in $V \cap C_p$. Here a *Jordan arc* is an arc homeomorphic to the interval $[0, 1]$. Furthermore the cut locus of a point on a compact and simply connected 2-dimensional Riemannian manifold is arcwise connected and a *tree*, i.e., a local tree without a circle (see Theorems 4.2.1 and 4.3.1 in [16]).

A compact Riemannian manifold (M, g) homeomorphic to a 2-sphere is called a *2-sphere of revolution* if M admits a point p such that for any two points q_1, q_2 on M with $d(p, q_1) = d(p, q_2)$, where $d(\ ,\ )$ denotes the Riemannian distance function, there exists an isometry f on M satisfying $f(q_1) = q_2$, and $f(p) = p$. The point p is called a *pole* of M. Let (r, θ) denote geodesic polar coordinates around a pole p of (M, g). The Riemannian metric g can be expressed as $g = dr^2 + m(r)^2 d\theta^2$ on $M \setminus \{p, q\}$, where q denotes the unique cut point of p and

$$m(r(x)) := \sqrt{g\left(\left(\frac{\partial}{\partial\theta}\right)_x, \left(\frac{\partial}{\partial\theta}\right)_x\right)}\,.$$

It will be proved in Lemma 2.1 that each pole of a 2-sphere of revolution M has a unique cut point. A pole and its unique cut point are called a *pair of poles.* Any geodesic emanating from a pole is a periodic geodesic through its cut point. Each periodic geodesic through a pair of

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poles is called a *meridian.* In this paper we prove the following structure theorem of the cut locus of a 2-sphere of revolution.

MAIN THEOREM. *Let* $(M, dr^2 + m(r)^2 d\theta^2)$ *be a* 2*-sphere of revolution with a pair of poles* p, q *satisfying the following two properties.*

(1.1) $(M, dr^2 + m(r)^2 d\theta^2)$ *is symmetric with respect to the reflection fixing* $r = a$, *where* $2a$ *denotes the distance between* p *and* q.

(1.2) *The Gaussian curvature* G *of* M *is monotone along a meridian from the point* p *to the point on* $r = a$.

Then the cut locus of a point $x \in M \setminus \{p, q\}$ *with* $\theta(x) = 0$ *is a single point or a subarc of the opposite half meridian* $\theta = \pi$ (*resp. the parallel* $r = 2a - r(x)$) *when* G *is monotone non-increasing* (*resp. non-decreasing*) *along a meridian from* p *to the point on* $r = a$. *Furthermore, if the cut locus of a point* $x \in M \setminus \{p, q\}$ *is a single point, then the Gaussian curvature is constant.*

An ellipsoid of revolution defined by

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,, \quad b\,,\ c > 0\,, \tag{1.3}$$

is a typical example satisfying (1.1) and (1.2) in the Main Theorem. As a corollary to the Main Theorem we have,

COROLLARY. *The cut locus of a point* $(x_0, 0, z_0)$ *with* $x_0 > 0$ *on the ellipsoid defined by* (1.3) *is a subarc of* $\{(-b\sin\theta, 0, c\cos\theta)\ ;\ 0 \le \theta \le \pi\}$ (*resp. the parallel* $z = -z_0$) *if* $c > b$ (*resp.* $c < b$).

Recently we learned that Itoh and Kiyohara ([8]) determined the cut loci and conjugate loci of all triaxial ellipsoids with unequal axes. Thus the corollary above is also a corollary to their result. Here we want to emphasize that a 2-sphere of revolution satisfying (1.1) and (1.2) does not always have positive Gaussian curvature. For example, the surface of revolution (see Figure 1) generated by the (x, z)-plane curve $(m(t), 0, z(t))$, where

$$m(t) := \frac{\sqrt{3}}{10}\left(9\sin\frac{\sqrt{3}}{9}t + 7\sin\frac{\sqrt{3}}{3}t\right), \quad z(t) := \int_0^t \sqrt{1 - m'(t)^2}\,dt\,,$$

satisfies (1.1) and (1.2). Moreover the Gaussian curvature is -1 on the equator $r = 3\sqrt{3}\pi/2$.

The example constructed by Gluck and Singer (see [4]) shows that one cannot impose any strong restriction on the structure of the cut locus of a point on a surface, even if the surface is assumed to be a surface of revolution with positive Gaussian curvature. The conditions (1.1) and (1.2) in the Main Theorem are thus reasonable and yet quite flexible in the sense that they are satisfied for a larger family of 2-spheres of revolution. We note that the structure of the cut locus of a very familiar surface of revolution such as a 2-sheeted hyperboloid, a paraboloid and a standard torus in 3-dimensional Euclidean space has been determined (see [3], [5] and [17]). The property (1.1) is equivalent to the property

$$m(r) = m(2a - r) \quad \text{for any } r \in (0, 2a)\,. \tag{1.4}$$

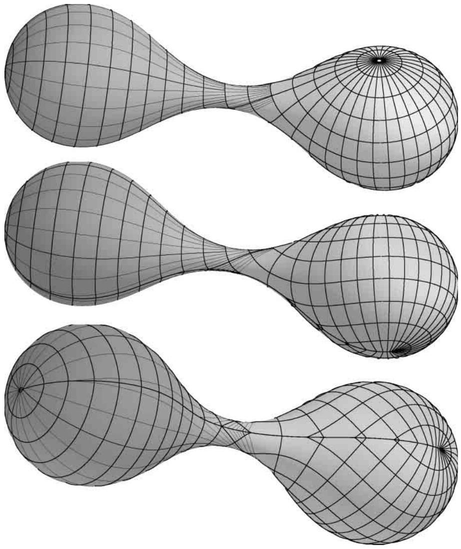

FIGURE 1. A surface of revolution satisfying (1.1) and (1.2) viewed from three different viewpoints. The two families of curves on the surface denote geodesic circles and geodesics emanating from a point (on the upper right in the first illustration), respectively.

Since the Gaussian curvature G of a 2-sphere of revolution $(M, dr^2 + m(r)^2 d\theta^2)$ with a pair of poles p, q is equal to

$$G(x) = -\frac{m''(r(x))}{m(r(x))} \tag{1.5}$$

for each $x \in M \setminus \{p, q\}$, the property (1.2) is equivalent to the monotonicity of the function $m''(r)/m(r)$ on $(0, a]$.

As is well-known, a complete and simply connected surface with constant Gaussian curvature is used as a model surface in Toponogov's comparison theorem (see [14] for example). The comparison theorem was first generalized to model surfaces with non-constant Gaussian curvature by Elerath ([3]). His model surface is a surface of revolution generated by a function $y = f(x)$, whose Gaussian curvature is monotone non-increasing along a meridian. Furthermore he proves in the paper above that the cut locus of each point on such a surface is empty or a subset of the meridian opposite to the point. After his work, some surfaces of revolution have been introduced as model surfaces for the comparison theorem, e.g., an Hadamard surface of revolution with finite total curvature (see [1, 2]), von Mangoldt surfaces of revolution and compact von Mangoldt surfaces of revolution with a singular point (see [12, 11]). The comparison theorem for a compact von Mangoldt surface with a singular point is applied in the proof of a sphere theorem by Kondo and Ohta ([13]). A new model surface, which we will introduce in Section 6, is a 2-sphere of revolution whose cut locus is a subarc of a meridian or a single point. For example, the ellipsoid defined by (1.3) with $c > b$ and the surface in Figure 1 are model surfaces which are distinct from the model surfaces above.

We refer to [14] for basic tools in Riemannian geometry, and [16] or [18] for some properties of geodesics on a surface of revolution.

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2. Preliminaries. Let (M, g) be a 2-sphere of revolution. As is defined in the introduction, the manifold (M, g) is homeomorphic to a 2-sphere and admits a point p such that for any points $q_1, q_2 \in M$ with $d(p, q_1) = d(p, q_2)$, where $d(\ ,\ )$ denotes the Riemannian distance function, there exists an isometry f satisfying $f(p) = p$, $f(q_1) = q_2$. The point p is called a *pole* of M.

LEMMA 2.1. *Each pole p of M has a unique cut point. Moreover, the unique cut point of p is also a pole of M.*

PROOF. Let q be a farthest point from a pole p. Suppose that there exists a cut point x of p such that $d(p, x) < d(p, q)$. Let $\gamma : [0, l_0] \to M$ denote a unit speed minimal geodesic segment joining p to q. Since there exists an isometry f on M such that $f(p) = p$, $f(x) = \gamma(t_0)$, where $t_0 := d(p, x)$, the point $\gamma(t_0)$ is a cut point of p. This is a contradiction. Thus any cut point of p is a farthest point from p. Since M is simply connected, the cut locus of p has an endpoint q (see Theorem 4.2.1 in [16]). Hence any cut point q_1 of p is also an endpoint, since there exists an isometry f such that $f(q) = q_1$, $f(p) = p$. Suppose that p has two cut points q_1 and q_2. It follows from Theorems 4.2.1 and 4.2.3 in [16] that q_1 and q_2 can be joined in the cut locus of p by a rectifiable Jordan arc c. This is a contradiction, because each interior point on the curve c is not an endpoint of the cut locus of p. Thus the point p has a unique cut point q. Let p_1, p_2 be any points on M with $d(p_1, q) = d(p_2, q)$. Since q is the unique cut point of p, it is clear that $d(p, p_1) = d(p, p_2)$. Hence there exists an isometry f such that $f(p_1) = f(p_2)$, $f(p) = p$. Furthermore, since q is the unique cut point of p, $f(q) = q$. This implies that the unique cut point of the pole p is also a pole of M. □

Let (r, θ) denote geodesic polar coordinates around a pole p of (M, g). The Riemannian metric g is expressed as $g = dr^2 + m(r)^2 d\theta^2$ on $M \setminus \{p, q\}$, where q denotes the unique cut point of p and

$$m(r(x)) := \sqrt{g\left(\left(\frac{\partial}{\partial\theta}\right)_x, \left(\frac{\partial}{\partial\theta}\right)_x\right)}\,.$$

From now on we fix a pair of poles p, q and geodesic polar coordinates (r, θ) around p. Notice that both functions $m(r)$ and $m(2a - r)$ are extensible to a C^∞ odd function around $r = 0$, where $2a := d(p, q)$, and $m'(0) = 1 = -m'(2a)$. By Lemma 2.1, any geodesic emanating from p (resp. q) passes through q (resp. p). It is easily checked that any unit speed geodesic emanating from $p = \mu(0)$ is a periodic geodesic, i.e., $\mu(t + 4a) = \mu(t)$ for any real t. Each geodesic passing through p is called a *meridian*. Each curve $r = c \in (0, 2a)$ is called a *parallel*.

For technical reasons, we introduce the Riemannian universal covering manifold $\tilde{M} := ((0, 2a) \times \boldsymbol{R}, d\tilde{r}^2 + m(\tilde{r})^2 d\tilde{\theta}^2)$ of $(M \setminus \{p, q\}, dr^2 + m(r)^2 d\theta^2)$. Let $\tilde{\gamma}(s) = (\tilde{r}(s), \tilde{\theta}(s))$ be a unit speed geodesic on $\tilde{M}$. There exists a constant ν such that

$$m(\tilde{r}(s))^2 \tilde{\theta}'(s) = m(\tilde{r}(s)) \cos \eta(s) = \nu \tag{2.1}$$

holds for any s, where $\eta(s)$ denotes the angle $\angle(\dot{\tilde{\gamma}}(s), (\partial/\partial\tilde{\theta})_{\tilde{\gamma}(s)})$ made by $\dot{\tilde{\gamma}}(s) := d\tilde{\gamma}_t(\partial/\partial s)$ and $(\partial/\partial\tilde{\theta})_{\tilde{\gamma}(s)}$. This relation is a well-known formula, which is called the *Clairaut relation.* The constant ν is called the *Clairaut constant* of $\tilde{\gamma}$. Since $\tilde{\gamma}$ is unit speed, we have by (2.1),

$$\tilde{r}'(s) = \pm \frac{\sqrt{m(\tilde{r}(s))^2 - \nu^2}}{m(\tilde{r}(s))} . \tag{2.2}$$

In particular, $\tilde{r}'(s) = 0$ if and only if $m(\tilde{r}(s)) = |\nu|$. It follows from (2.1) and (2.2) that for a unit speed geodesic $\tilde{\gamma}(s) = (\tilde{r}(s), \tilde{\theta}(s))$, $s_1 \le s \le s_2$, with the Clairaut constant ν,

$$\tilde{\theta}(s_2) - \tilde{\theta}(s_1) = \varepsilon(\tilde{r}'(s)) \int_{\tilde{r}(s_1)}^{\tilde{r}(s_2)} \frac{\nu}{m(x)\sqrt{m(x)^2 - \nu^2}} dx \tag{2.3}$$

holds if $\tilde{r}'(s) \neq 0$ on (s_1, s_2), where $\varepsilon(\tilde{r}'(s))$ denotes the sign of $\tilde{r}'(s)$, and moreover the length $L(\tilde{\gamma})$ of $\tilde{\gamma}|_{[s_1, s_2]}$ equals

$$L(\tilde{\gamma}) = \varepsilon(\tilde{r}'(s)) \int_{\tilde{r}(s_1)}^{\tilde{r}(s_2)} \frac{m(x)}{\sqrt{m(x)^2 - \nu^2}} dx \tag{2.4}$$

if $\tilde{r}'(s) \neq 0$ on (s_1, s_2). Since

$$\frac{m}{\sqrt{m^2 - \nu^2}} = \frac{\sqrt{m^2 - \nu^2}}{m} + \frac{\nu^2}{m\sqrt{m^2 - \nu^2}} ,$$

we have

$$L(\tilde{\gamma}) = \varepsilon(\tilde{r}'(s)) \int_{\tilde{r}(s_1)}^{\tilde{r}(s_2)} \frac{\sqrt{m(x)^2 - \nu^2}}{m(x)} dx + \nu(\tilde{\theta}(s_2) - \tilde{\theta}(s_1)) \tag{2.5}$$

if $\tilde{r}'(s) \neq 0$ on (s_1, s_2). Notice that the equations (2.1), (2.2) and (2.4) also hold for a unit speed geodesic on $(M, dr^2 + m(r)^2 d\theta^2)$.

Hereafter $(M, dr^2 + m(r)^2 d\theta^2)$ *denotes a* 2*-sphere of revolution satisfying the two properties* (1.1) *and* (1.2) *stated in the introduction.*

LEMMA 2.2. *If the derivative function m' of m is zero at some $c \in (0, a)$, then there exists $c_1 \in (c, a)$ such that $G \circ \mu(c_1) = 0$ and $G \circ \mu$ is monotone non-increasing on* $[0, a]$. *Here $\mu : [0, 4a] \to M$ denotes a unit speed meridian emanating from $p = \mu(0)$ and G denotes the Gaussian curvature of M.*

PROOF. Since $m'(c) = m'(a) = 0$ by the assumption and (1.4), it follows from the mean value theorem that there exists a number $c_1 \in (c, a)$ with $m''(c_1) = 0$. On the other hand, by (1.5), we have

$$m''(t) + G(\mu(t))m(t) = 0 . \tag{2.6}$$

Thus $G \circ \mu(c_1) = 0$. Since $m'(0) = 1 > m'(c) = 0$, m'' must be negative at some $c_2 \in (0, c)$. Hence, by (2.6), $G \circ \mu(c_2) > 0 = G \circ \mu(c_1)$. Since $G \circ \mu$ is monotone on $[0, a]$ and $c_2 < c < c_1$, it is monotone non-increasing on $[0, a]$. □

LEMMA 2.3. *The Gaussian curvature G is negative on $r = a$ if and only if $m'(r)$ is negative for some $r \in (0, a)$. Furthermore, if $m'(r_0) < 0$ for some $r_0 \in (0, a)$, then there exist two numbers $b_1 \leq b_2$ in $(0, a)$ such that $m' > 0$ on $[0, b_1) \cup (a, 2a - b_2)$, $m' = 0$ on $[b_1, b_2] \cup [2a - b_2, 2a - b_1]$ and $m' < 0$ on $(b_2, a) \cup (2a - b_1, 2a]$. In particular, $m(b_1) = m(b_2)$ is the maximum of $m[0, a]$ which is greater than $m(a)$, and the function $G \circ \mu$ is monotone non-increasing on* $[0, a]$.

PROOF. Suppose that G is negative on $r = a$. By (2.6), $m''(a) > 0$. Since $m'(a) = 0$, it is trivial that $m'(r)$ is negative for any $r \in (0, a)$ sufficiently close to a. Suppose that $m'(r_0) < 0$ for some $r_0 \in (0, a)$. Since $m'(0) = 1$ and $m'(r_0) < 0$, there exists $c \in (0, r_0)$ such that $m'(c) = 0$. Thus by Lemma 2.2, $G \circ \mu$ is monotone non-increasing on $[0, a]$ and zero at some $c_1 \in (c, a)$. By supposing that G is non-negative on $r = a$, we will get a contradiction. Since $G \circ \mu$ is monotone non-increasing, $G \circ \mu$ is non-negative on $[0, a]$. It follows from (2.6) that m' is monotone non-increasing on $[0, a]$.

Thus $m'(r) \geq m'(a) = 0$ for any $r \in [0, a]$. This contradicts the existence of the number r_0. Hence G is negative on $r = a$. Since the proof of the first claim is complete, we will prove the latter claim. Let $r_1 \in (0, a)$ denote the minimal number $r \in (0, a)$ such that $G \circ \mu < 0$ on $(r, a]$. Since $G \circ \mu \geq 0$ on $[0, r_1]$ and $G \circ \mu < 0$ on $(r_1, a]$, it follows from (2.6) that $m'' \leq 0$ on $[0, r_1]$ and $m'' > 0$ on $(r_1, a]$. Thus for any $r \in (r_1, a)$

$$m'(r_1) < m'(r) < m'(a) = 0 .$$

Therefore the function $m|_{[0,a]}$ attains a maximum at $r = b \in (0, r_1)$, which is greater than $m(a)$. Let b_1, b_2 denote the minimum and maximum of $m^{-1}(m(b))$, respectively. Choose any $r_2 \in (0, a)$ with $m'(r_2) = 0$. Since $m'' \leq 0$ on $[0, r_1]$, for any r, s with $0 < r < r_2 < s < r_1$, $m'(r) \geq m'(r_2) = 0 \geq m'(s)$. This means that $m(r_2)$ is the maximum of $m[0, a]$. Thus $b_1 \leq r_2 \leq b_2$. Now it is clear that the numbers b_1 and b_2 have the required property in our lemma. □

LEMMA 2.4. *If m' is non-negative on* $[0, a]$, *then m' is positive on $(0, b_3)$, where b_3 denotes the minimum of $m^{-1}(m(a))$. Furthermore, m attains the maximum $m(a)$ of $m[0, 2a]$ at each point of* $[b_3, 2a - b_3]$.

PROOF. Suppose that $m'(c) = 0$ for some $c \in (0, b_3)$. By Lemma 2.2 and Lemma 2.3, $G \circ \mu$ is non-negative on $[0, a]$. Thus, by (2.6), m' is monotone non-increasing on $[0, a]$. Since $m'(c) = m'(a) = 0$, $m' = 0$ on $[c, a]$. In particular, $m(c) = m(a)$. This contradicts the assumption of b_3. The latter claim is clear from (1.4). □

By Lemmas 2.3 and 2.4, $m' > 0$ on $[0, b_3)$, where b_3 denotes the minimum of $m^{-1}(m(a))$. Thus the inverse function $\xi(\nu)$ on $(0, m(a))$ can be defined by

$$\xi(\nu) := (m|_{[0,b_3)})^{-1}(\nu) .$$

Let $\tilde{p}_u$ denote the point $\tilde{r}^{-1}(u) \cap \tilde{\theta}^{-1}(0)$, where $u \in (0, 2a)$. For each $\nu \in (0, m(u)]$ let $\tilde{\beta}_\nu{}^{(u)}$ and $\tilde{\gamma}_\nu{}^{(u)}$ denote the geodesics with Clairaut constant ν emanating from $\tilde{p}_u = \tilde{\beta}_\nu{}^{(u)}(0) = \tilde{\gamma}_\nu{}^{(u)}(0)$ such that

$$(\tilde{r} \circ \tilde{\beta}_\nu{}^{(u)})'(0) \geq 0\,, \quad (\tilde{r} \circ \tilde{\gamma}_\nu{}^{(u)})'(0) \leq 0\,.$$

For simplicity, put $\tilde{c}_\nu(s) := \tilde{\gamma}_\nu^{(a)}(s)$. If $\nu \in (0, m(a))$, then $\tilde{c}_\nu$ is tangent to the arc $\tilde{r} = \xi(\nu)$ at a point $\tilde{c}_\nu(t_1(\nu))$, and $\tilde{c}_\nu$ intersects the arc $\tilde{r} = a$ again at $\tilde{c}_\nu(t_0(\nu))$, where

$$t_0(\nu) := \min\{\, t > 0; \tilde{r}(\tilde{c}_\nu(t)) = a\}\,.$$

By (2.3), we get

$$\tilde{\theta}(\tilde{c}_\nu(t_0(\nu))) - \tilde{\theta}(\tilde{c}_\nu(0)) = 2(\tilde{\theta}(\tilde{c}_\nu(t_1(\nu))) - \tilde{\theta}(\tilde{c}_\nu(0))) = \varphi(\nu)\,, \tag{2.7}$$

where $\varphi : (0, m(a)) \to R$ is the function defined by

$$\varphi(\nu) := 2\int_{\xi(\nu)}^{a} \frac{\nu}{m(t)\sqrt{m(t)^2 - \nu^2}}\, dt\,. \tag{2.8}$$

LEMMA 2.5. *If $\nu \in (0, m(a))$, then any $t \in R$,*

$$\tilde{T}_\nu(\tilde{c}_\nu(t)) = \tilde{c}_\nu(t + t_0(\nu))$$

holds, where $\tilde{T}_\nu$ is an isometry on $\tilde{M}$ defined by

$$\tilde{T}_\nu(\tilde{r}, \tilde{\theta}) := (2a - \tilde{r}, \tilde{\theta} + \varphi(\nu))\,.$$

PROOF. It is clear from (1.4) that $\tilde{T}_\nu$ is an isometry. Hence $\beta(t) := \tilde{T}_\nu(\tilde{c}_\nu(t))$ is a geodesic on $\tilde{M}$ with Clairaut constant ν. By (2.1) and (2.2), we have $\dot{\beta}(0) = d\tilde{T}_\nu(\dot{\tilde{c}}_\nu(0)) = \dot{\tilde{c}}_\nu(t_0(\nu))$ and $\beta(0) = \tilde{c}_\nu(t_0(\nu))$. By uniqueness, we have $\beta(t) = \tilde{c}_\nu(t + t_0(\nu))$. □

The next lemma is a direct consequence of Lemma 7.3.2 in [16] (or Lemma 1.3 in [17]), (2.1) and the first variational formula.

LEMMA 2.6. *Let q_1, q_2 be two points on M satisfying $r(q_1) = r(q_2) \in (0, 2a)$. If $0 \leq \theta(q_1) < \theta(q_2) \leq \pi$, then $d(x, q_1) < d(x, q_2)$ for any point $x \in M \setminus \{p, q\}$ with $\theta(x) = 0$. Moreover, for any two points q_1, q_2 in D_ν, $\nu \in (0, m(a))$, satisfying $\tilde{r}(q_1) = \tilde{r}(q_2)$ and $0 < \tilde{\theta}(q_1) < \tilde{\theta}(q_2)$, $d_\nu(\tilde{c}_\nu(0), q_1) < d_\nu(\tilde{c}_\nu(0), q_2)$ holds. Here D_ν denotes the domain bounded by the two geodesic segments $\tilde{r}^{-1}(a) \cap \tilde{\theta}^{-1}[0, \tilde{\theta}(\tilde{c}_\nu(t_0(\nu)))]$ and $\tilde{c}_\nu[0, t_0(\nu)]$, and d_ν denotes the Riemannian distance function on the closure of $(D_\nu, d\tilde{r}^2 + m(\tilde{r})^2 d\tilde{\theta}^2)$.*

Since both geodesics $\tilde{\beta}_\nu{}^{(u)}$ and $\tilde{\gamma}_\nu{}^{(u)}$ depend smoothly on $\nu \in (0, m(u))$, we get Jacobi fields X_ν, Y_ν defined by

$$X_\nu(t) := \frac{\partial}{\partial \nu}(\tilde{\beta}_\nu{}^{(u)}(t))\,, \quad Y_\nu(t) := \frac{\partial}{\partial \nu}(\tilde{\gamma}_\nu{}^{(u)}(t))\,. \tag{2.9}$$

It is clear that $X_\nu(0) = 0 = Y_\nu(0)$. Both Jacobi fields above can be explicitly expressed in terms of the function m (see [16] or [18]). By Corollary 7.2.1 in [16] we have

LEMMA 2.7. *Let $\gamma : [0, s] \to \tilde{M}$ be a geodesic segment such that $(\tilde{r} \circ \gamma)'(t) \neq 0$ on $[0, s)$. Then $\gamma|_{[0,s]}$ has no conjugate point of $\gamma(0)$.*

Let $\gamma : R \to \tilde{M}$ be a unit speed geodesic emanating from $\tilde{p}_u$ with Clairaut constant $\nu \in (0, m(u))$. It follows from (2.2) that the geodesic γ lies in the strip $\{(\tilde{r}, \tilde{\theta})\,;\, \xi_1(\nu) \le \tilde{r} \le \xi_2(\nu)\}$, where $(\xi_1(\nu), \xi_2(\nu))$ denotes the maximal open interval containing u such that $m > \nu$ on $(\xi_1(\nu), \xi_2(\nu))$. Furthermore, it follows from (1.4), Lemmas 2.3 and 2.4 that $\xi_1(\nu) = \xi(\nu)$ and $\xi_2(\nu) = 2a - \xi(\nu)$ if $\nu < m(a)$. If $m'(\xi_1(\nu)) \neq 0$ and $m'(\xi_2(\nu)) \neq 0$, then γ is tangent to both parallel arcs $\tilde{r} = \xi_1(\nu)$ and $\tilde{r} = \xi_2(\nu)$ infinitely many times and interchangeably, i.e., γ is tangent to $\tilde{r} = \xi_1(\nu)$ (resp. $\tilde{r} = \xi_2(\nu)$), right after it is tangent to $\tilde{r} = \xi_2(\nu)$ (resp. $\tilde{r} = \xi_1(\nu)$). If $m'(\xi_i(\nu)) = 0$ for some i, then γ is asymptotic to the parallel arc $\tilde{r} = \xi_i(\nu)$ as t goes to infinity.

LEMMA 2.8. *If a unit speed geodesic $\gamma : \boldsymbol{R} \to \tilde{M}$ is tangent to two distinct arcs $\tilde{r} = \xi_1$ and $\tilde{r} = \xi_2$, then a pair of points on γ that is tangent to $\tilde{r} = \xi_1$ or $\tilde{r} = \xi_2$ is mutually conjugate along γ.*

PROOF. For each $\tau \in \boldsymbol{R}$, let $f_\tau : \tilde{M} \to \tilde{M}$ denote the isometry defined by

$$f_\tau(\tilde{r}, \tilde{\theta}) := (\tilde{r}, \tilde{\theta} + \tau)\,.$$

The vector field $X(t)$ defined by

$$X(t) := \left.\frac{\partial}{\partial \tau}\right|_0 f_\tau(\gamma(t))$$

is a Jacobi field along $\gamma(t)$. For each τ the geodesic $f_\tau(\gamma(t))$ is tangent to both $\tilde{r} = \xi_1$ and $\tilde{r} = \xi_2$. Thus $Y(t) := X(t) - \tilde{g}(X(t), \dot{\gamma}(t))\dot{\gamma}(t)$, where $\tilde{g}$ denotes the Riemannian metric on $\tilde{M}$, is a Jacobi field that vanishes at each point on γ which is tangent to $\tilde{r} = \xi_1$ or $\tilde{r} = \xi_2$. The Jacobi field Y is non-zero and vanishes at each point on γ tangent to $\tilde{r} = \xi_1$ or $\tilde{r} = \xi_2$. Therefore the proof is complete. □

LEMMA 2.9. *If $u \in (0, 2a)$ and $\nu \in (0, m(a)) \cap (0, m(u))$, then there exists a first conjugate point $\tilde{\gamma}_\nu{}^{(u)}(t_c(u, \nu))$ of $\tilde{\gamma}_\nu{}^{(u)}(0)$ along $\tilde{\gamma}_\nu{}^{(u)}$ and*

$$\tilde{\theta}(\tilde{\gamma}_\nu{}^{(u)}(t_c(u, \nu))) = \psi(\tilde{r}(\tilde{\gamma}_\nu{}^{(u)}(t_c(u, \nu))), u, \nu)$$

holds, where

$$\psi(r, u, \nu) := \varphi(\nu) - \int_u^a f(t, \nu)dt + \int_a^r f(t, \nu)dt = \varphi(\nu) - \int_r^{2a-u} f(t, \nu)dt\,,$$

$$f(t, \nu) := \frac{\nu}{m(t)\sqrt{m(t)^2 - \nu^2}}\,.$$

Furthermore, $\tilde{r}(\tilde{\gamma}_\nu{}^{(u)}(t_c(u, \nu)))$ equals the unique solution r of

$$\frac{\partial \psi}{\partial \nu}(r, u, \nu) = 0\,.$$

PROOF. Since $\nu > 0$ is less than $m(a)$ and $(\tilde{r} \circ \tilde{\gamma}_\nu{}^{(u)})'(0) < 0$, $\tilde{\gamma}_\nu{}^{(u)}$ is tangent to $\tilde{r} = \xi(\nu)$ at a point $\tilde{\gamma}_\nu{}^{(u)}(s_0)$ first, and then it is tangent to $\tilde{r} = 2a - \xi(\nu)$ at a point $\tilde{\gamma}_\nu{}^{(u)}(s_1)$. The numbers $s_0 < s_1$ are solutions t of $(\tilde{r} \circ \tilde{\gamma}_\nu{}^{(u)})'(t) = 0$ and s_0 is the unique solution t of

$(\tilde{r}\circ\tilde{\gamma}_\nu{}^{(u)})'(t)=0$ on $(0,s_1)$. It follows from (1.17) in [18] (or (7.2.21) in [16]) that the Jacobi field Y_ν defined by (2.9) is equal to

$$Y_\nu(t)=\frac{\partial\psi}{\partial\nu}(\tilde{r}(t),u,\nu)\left(-\frac{\nu m(\tilde{r}(t))}{\sqrt{m(\tilde{r}(t))^2-\nu^2}}\left(\frac{\partial}{\partial\tilde{r}}\right)_{\tilde{\gamma}_\nu{}^{(u)}(t)}+\left(\frac{\partial}{\partial\tilde{\theta}}\right)_{\tilde{\gamma}_\nu{}^{(u)}(t)}\right)$$

on (s_0,s_1). Here $\tilde{r}(t):=\tilde{r}(\tilde{\gamma}_\nu{}^{(u)}(t))$. Since $\psi(r,u,\nu)=\varphi(\nu)-\int_u^a f(t,\nu)dt+\int_a^r f(t,\nu)dt$, we have

$$\frac{\partial\psi}{\partial\nu}=\varphi'(\nu)-\int_u^a\frac{\partial f}{\partial\nu}(t,\nu)dt+\int_a^r\frac{\partial f}{\partial\nu}(t,\nu)dt\,.$$

Hence $\lim_{r\to\xi(\nu)^+}\partial\psi/\partial\nu=-\infty$ and $\lim_{r\to(2a-\xi(\nu))^-}\partial\psi/\partial\nu=+\infty$. From the continuity of $\partial\psi/\partial\nu$ it follows that there exists a solution $r=h(u,\nu)$ of $(\partial\psi/\partial\nu)(r,u,\nu)=0$. The uniqueness of the solution is clear, since $\partial^2\psi/\partial r\partial\nu=m(r)/\sqrt{m(r)^2-\nu^2}^3>0$. By (1.4) and Lemma 2.7, it is trivial that

$$\tilde{\gamma}_\nu{}^{(u)}(t_c(u,\nu)):=(\tilde{r},\tilde{\theta})^{-1}(h(u,\nu),\psi(h(u,\nu),u,\nu))$$

is the first conjugate point of $\tilde{\gamma}_\nu{}^{(u)}(0)$ along $\tilde{\gamma}_\nu{}^{(u)}$ and $\psi(r,u,\nu)=\varphi(\nu)-\int_r^{2a-u}f(t,\nu)dt$. □

3. The case where the Gaussian curvature is monotone non-increasing. Throughout this section, we assume that the Gaussian curvature G of M is monotone non-increasing on a meridian from the pole p to the point on $r=a$, which is called the *equator.*

LEMMA 3.1. *The cut locus C_{p_0} of a point p_0 on $r=a$ is a subset of the opposite half meridian to p_0. Thus $\varphi(\nu)\geq\pi$ for any $\nu\in(0,m(a))$.*

PROOF. Without loss of generality, we may assume $\theta(p_0)=0$. First, we will prove that the cut locus of p_0 is a subset of the union of the opposite half meridian $\theta=\pi$ to p_0 and the equator. We will then get a contradiction by assuming the existence of a cut point of p_0 which does not lie on the union of $r=a$ and $\theta=\pi$.

Since M is a 2-sphere of revolution satisfying (1.2), we may assume that there exists a cut point q_0 of p_0 in $r^{-1}(0,a)\cap\theta^{-1}(0,\pi)$. The cut point q_0 is not a unique cut point of p_0, for example the point p_π on $r=a$ with $\theta(p_\pi)=\pi$ is a cut point of p_0. Let $c:[0,\delta]\to C_{p_0}$ be a unit speed rectifiable Jordan arc joining $q_0=c(0)$ and $p_\pi=c(\delta)$. It follows from Corollary 4.2.1 in [16] or Lemmas 2, 3 and 4 in [9] that the set of all *normal* cut points in $c[0,\delta]$ forms an open and dense subset in the cut locus C_{p_0}. Here a cut point q of p_0 is said to be *normal* if q is not conjugate to p_0 along any minimal geodesic segment joining q and p_0 and there exist exactly two minimal unit speed geodesic segments α and β joining $p_0=\alpha(0)=\beta(0)$ and q. Since $q_0\in\theta^{-1}(0,\pi)\cap r^{-1}(0,a)$, we may choose a positive $\delta_0<\delta$ such that $c[0,\delta_0]\subset\theta^{-1}(0,\pi)\cap r^{-1}(0,a)$ and $c(\delta_0)$ is a normal cut point of p_0. Thus we get the 2-disc domain $D(\alpha,\beta)$ bounded by α and β in $\theta^{-1}(0,\pi)\cap r^{-1}(0,a)$. Notice that neither α nor β intersects $r=a$ except at $p_0=\alpha(0)=\beta(0)$ by the property (1.2). Since there exists an endpoint of C_{p_0} in $D(\alpha,\beta)$, we may assume that q_0 is an endpoint of C_{p_0}. Thus q_0 is a conjugate point of p_0 along any minimizing geodesic segment joining p_0 and q_0.

Since $d(p, c(t))$ is a Lipschitz function, it follows from Theorem 7.29 in [20] that the function $d(p_0, c(t))$ is differentiable for almost all t and

$$d(p_0, c(\delta_0)) - d(p_0, q_0) = \int_0^{\delta_0} \frac{d}{dt} d(p_0, c(t))dt \tag{3.1}$$

holds. For each normal cut point $c(t)$, $t \in [0, \delta_0]$ on the curve, there exists a pair of two minimal unit speed geodesic segments α_t and β_t joining p_0 to $c(t)$. The angle $\angle(\dot{\alpha}_t(d(p_0, c(t))), \dot{\beta}_t(d(p_0, c(t))))$ is less than π, since $(\theta \circ \alpha_t)'(s)$ and $(\theta \circ \beta_t)'(s)$ are always positive by (2.1). Furthermore, it follows from the first variational formula that the curve c bisects the angle $\angle(\dot{\alpha}_t(d(p_0, c(t))), \dot{\beta}_t(d(p_0, c(t))))$ at $c(t)$ for each normal cut point $c(t)$ and hence $(d/dt)d(p_0, c(t))$ is positive for any normal cut points $c(t)$. Therefore by (3.1), we get $d(p_0, c(\delta_0)) > d(p_0, q_0)$, i.e., γ is shorter than β. Here $\gamma : [0, d(p_0, q_0)] \to M$ denotes a minimal geodesic segment joining p_0 to q_0. Since the geodesic segment γ lies in $D(\alpha, \beta)$, without loss of generality, we may assume that

$$\angle(\dot{\beta}(0), (\partial/\partial r)_{p_0}) < \angle(\dot{\gamma}(0), (\partial/\partial r)_{p_0}) < \angle(\dot{\alpha}(0), (\partial/\partial r)_{p_0}) . \tag{3.2}$$

From the first variational formula, we have

$$(r \circ \beta)'(0) > (r \circ \gamma)'(0) > (r \circ \alpha)'(0) .$$

Thus the number $s_0 := \sup\{s \in (0, d(p_0, q_0)] \,;\, r(\alpha(t)) < r(\gamma(t)) < r(\beta(t)) \; for \; any \; t \in (0, s]\}$ is positive. By supposing $s_0 < d(p_0, q_0)$, we will get a contradiction. Since $(r \circ \gamma)(t)$, $(r \circ \alpha)(t)$ and $(r \circ \beta)(t)$ are continuous,

$$r(\gamma(s_0)) = r(\beta(s_0)) \quad \text{or} \quad r(\gamma(s_0)) = r(\alpha(s_0)) \tag{3.3}$$

holds. Hence, by Lemma 2.6, we have

$$\theta(\gamma(s_0)) = \theta(\beta(s_0)) \quad \text{or} \quad \theta(\gamma(s_0)) = \theta(\alpha(s_0)) , \tag{3.4}$$

since $d(p_0, \gamma(s_0)) = d(p_0, \beta(s_0)) = d(p_0, \alpha(s_0)) = s_0$. By (3.3) and (3.4), we get $\gamma(s_0) = \beta(s_0)$ or $\gamma(s_0) = \alpha(s_0)$, which is impossible. Therefore $s_0 = d(p_0, q_0)$, i.e.,

$$r(\alpha(s)) \leq r(\gamma(s)) \leq r(\beta(s)) \tag{3.5}$$

for any $s \in [0, d(p_0, q_0)]$. Since the Gaussian curvature is monotone non-increasing along the meridian through p_0 from the pole p to p_0, by (3.5), we get

$$G(\alpha(s)) \geq G(\gamma(s)) \geq G(\beta(s)) \tag{3.6}$$

for any $s \in [0, d(p_0, q_0)]$. Thus, by the Rauch comparison theorem, there exists a conjugate point $\alpha(t_R)$, $t_R \in [0, d(p_0, q_0)]$, of p_0 along α. This contradicts the minimality of α. Therefore we have proved that the cut locus of p_0 is a subset of the union of $r = a$ and $\theta = \pi$.

We have to prove that C_{p_0} is a subset of $\theta = \pi$. Suppose there exists a cut point q_1 of p_0 on $r = a$ with $\theta(q_1) \neq \pi$. We may assume that $q_1 \in \theta^{-1}(0, \pi) \cap r^{-1}(a)$ and q_1 is the nearest cut point of p_0 from p_0, since M is a 2-sphere of revolution. Let $\gamma_1 : [0, d(p_0, q_1)] \to M$ be the unit speed subarc of $r = a$ joining p_0 to q_1. Since $r = a$ is a geodesic and q_1 is the nearest cut point of p_0 from p_0, γ_1 is a minimal geodesic segment. Let $c_1 : [0, d(q_1, p_\pi)] \to M$ be a unit speed rectifiable curve in C_{p_0} joining q_1 to p_π. Since C_{p_0} is a subset of the union of $\theta = \pi$ and $r = a$, the curve c_1 is a subarc of $r = a$, and hence a geodesic segment. Let $\alpha_1 : [0, d(p_0, p_\pi)] \to M$ be a unit speed minimal geodesic segment joining p_0 to p_π. By applying the same argument as above for γ_1, c_1 and α_1, we get a contradiction. Therefore the cut locus of p_0 is a subset of the opposite half meridian to p_0. If $\varphi(\nu_0) < \pi$ for some $\nu_0 \in (0, m(a))$, then the geodesic $\tilde{c}_{\nu_0}$ meets $\tilde{r} = a$ at a point $\tilde{p}$ in $\tilde{\theta}^{-1}(0, \pi)$. This means there exists a cut point of p_0 in $\theta^{-1}(0, \pi) \cap r^{-1}(0, a]$, which is a contradiction. □

Let D_ν, $\nu \in (0, m(a))$, denote the domain bounded by the two geodesic segments $\tilde{r}^{-1}(a) \cap \tilde{\theta}^{-1}[0, \tilde{c}_\nu(t_0(\nu))]$ and $\tilde{c}_\nu[0, t_0(\nu)]$, and d_ν the Riemannian distance function on the closure $(\bar{D}_\nu, d\tilde{r}^2 + m(\tilde{r})^2 d\tilde{\theta}^2)$ of $(D_\nu, d\tilde{r}^2 + m(\tilde{r})^2 d\tilde{\theta}^2)$. It is clear that any minimizing arc in $\bar{D}_\nu$ with respect to d_ν is a geodesic segment on $(\tilde{M}, d\tilde{r}^2 + m(\tilde{r})^2 d\tilde{\theta}^2)$. Let $\gamma : [0, t_0] \to \bar{D}_\nu$ denote a minimal geodesic segment with respect to d_ν. The endpoint $\gamma(t_0)$ is called a *cut point* of $\gamma(0)$ if any extended geodesic segment of γ lying in $\bar{D}_\nu$ is not minimizing with respect to d_ν anymore. In the same way as in the proof of Lemma 3.1, we get the following two lemmas.

LEMMA 3.2. *For each* $\nu \in (0, m(a))$, D_ν *has no cut point of* $\tilde{c}_\nu(0)$ *with respect to* d_ν. *Furthermore, if there is no conjugate point of* $\tilde{c}_\nu(0)$ *on* $\tilde{c}_\nu[0, t]$ *along* $\tilde{c}_\nu$ *for some* $t < t_0(\nu)$, *then* $\tilde{c}_\nu[0, t]$ *is a unique minimal geodesic segment in* $\bar{D}_\nu$ *joining* $\tilde{c}_\nu(0)$ *to* $\tilde{c}_\nu(t)$.

LEMMA 3.3. *For each point* x *on* $r^{-1}(0, a) \cap \theta^{-1}(0)$ *there does not exist a cut point of* x *in* $r^{-1}[0, a] \cap \theta^{-1}[0, \pi)$.

LEMMA 3.4. *The function* φ *is monotone non-decreasing on* $(0, m(a))$. *Furthermore, if* $\varphi'(\nu_0) = 0$ *for some* $\nu_0 \in (0, m(a))$, *then* $G \circ \mu = G \circ \mu(a) > 0$ *on* $[\xi(\nu_0), a]$ *and* $\varphi = \varphi(m(a)) := \lim_{\nu \to m(a)^-} \varphi(\nu)$ *on* $[\nu_0, m(a)]$. *Recall that* μ *is a meridian defined in Lemma* 2.2.

PROOF. It follows from Lemma 2.9 that for each $\nu \in (0, m(a))$, there exists a first conjugate point $\tilde{c}_\nu(t_c(a, \nu))$ of $\tilde{c}_\nu(0)$ along $\tilde{c}_\nu$. Since $\psi(r, a, \nu) = \varphi(\nu) + \int_a^r f(t, \nu) dt$, we have

$$\varphi'(\nu) + \int_a^{l(\nu)} f_\nu(t, \nu) dt = 0\,, \tag{3.10}$$

where $l(\nu) := \tilde{r}(\tilde{c}_\nu(t_c(a, \nu)))$ and $f_\nu(t, \nu) = m(t)(m(t)^2 - \nu^2)^{-3/2}$. First we will prove that $l(\nu) \le a$ for any $\nu \in (0, m(a))$. Then the first claim of Lemma 3.4 is clear from (3.10). By supposing $l(\nu_3) > a$ for some $\nu_3 \in (0, m(a))$, we get a contradiction.

It follows from (2.1) and Lemma 3.2 that for each $\nu \in (\nu_3, m(a))$, the geodesic $\tilde{c}_\nu|_{(0, t_0(\nu))}$ lies in D_{ν_3} and cannot meet $\tilde{c}_{\nu_3}|_{(0, t_0(\nu_3))}$. Furthermore, by Lemma 3.2, any two geodesics $\tilde{c}_{\nu_1}$, $\tilde{c}_{\nu_2}$, $\nu_3 < \nu_1 < \nu_2 < m(a)$, cannot meet in the domain D_{ν_3}. Hence $\varphi(\nu_1) \ge \varphi(\nu_2)$. This implies that φ is monotone non-increasing on $[\nu_3, m(a))$. Since geodesic segments $\tilde{c}_\nu|_{[0, t_0(\nu)]}$ converge to the geodesic segment $\tilde{c}_{m(a)}|_{[0, t_0(m(a))]}$ as ν converges to $m(a)$, where $t_0(m(a)) := \lim_{\nu \to m(a)^-} t_0(\nu)$, the point $\tilde{c}_{m(a)}(t_0(m(a)))$ is a conjugate point of $\tilde{p}_a := \tilde{c}_{m(a)}(0)$ along

$\tilde{c}_{m(a)}$. It follows from (2.5) that the length $L(\tilde{c}_\nu)$ of the geodesic segment $\tilde{c}_\nu|_{[0,t_0(\nu)]}$, $\nu \in [\nu_3, m(a))$, is

$$L(\tilde{c}_\nu) = 2\int_{\xi(\nu)}^{a} \frac{\sqrt{m(t)^2 - \nu^2}}{m(t)} dt + \nu\varphi(\nu)\,.$$

Since

$$\frac{d}{d\nu} L(\tilde{c}_\nu) = \nu\varphi'(\nu)\,,$$

the function $L(\tilde{c}_\nu)$ is monotone non-increasing on $[\nu_3, m(a))$. Thus, the geodesic segment $\tilde{c}_{m(a)}|_{[0,t_0(m(a))]}$ is not longer than $\tilde{c}_{\nu_3}|_{[0,t_0(\nu_3)]}$. Since the Gaussian curvature $\tilde{G}$ of $\tilde{M}$ attains a minimum on $\tilde{r} = a$,

$$\tilde{G}(\tilde{c}_{\nu_3}(t)) \geq \tilde{G}(\tilde{c}_{m(a)}(t)) = \tilde{G}(\tilde{p}_a)$$

on $[0, t_0(\nu_3)]$. It follows from the Rauch comparison theorem that there exists a conjugate point $\tilde{c}_{\nu_3}(t_2)$, $t_2 \leq t_0(m(a))$, of $\tilde{p}_a$ along $\tilde{c}_{\nu_3}$. This is a contradiction. Thus we have proved that $l(\nu) \leq a$ for any $\nu \in (0, m(a))$.

Suppose that $\varphi'(\nu_0) = 0$ for some $\nu_0 \in (0, m(a))$. Let $\nu \in [\nu_0, m(a))$ be any fixed number. It follows from Lemma 3.2 that the geodesic $\tilde{c}_\nu$ cannot meet the geodesic segment $\tilde{c}_{\nu_0}(0, t_0(\nu_0))$. Since φ is monotone non-decreasing, $\tilde{c}_\nu$ passes through $\tilde{c}_{\nu_0}(t_0(\nu_0))$. Hence, for any $\nu \in [\nu_0, m(a))$, $\tilde{c}_\nu$ passes through the common point $\tilde{c}_{\nu_0}(t_0(\nu_0))$. The function φ is constant on $[\nu_0, m(a))$ and the point $\tilde{c}_{\nu_0}(t_0(\nu_0))$ is the first conjugate point of $\tilde{p}_a$ along both geodesics $\tilde{r} = a$ and $\tilde{c}_{\nu_0}$. Since

$$\tilde{G}(\tilde{p}_a) = \tilde{G}(\tilde{c}_{m(a)}(t)) \leq \tilde{G}(\tilde{c}_{\nu_0}(t))$$

for any $t \in [0, t_0(\nu_0)]$, it follows from the Rauch comparison theorem that $\tilde{G}(\tilde{c}_{\nu_0}(t)) = \tilde{G}(\tilde{p}_a)$ for any $t \in [0, t_0(\nu_0)]$. Hence $\tilde{G}$ is a positive constant on $[\xi(\nu_0), a]$. □

LEMMA 3.5. *For any $\nu \in (0, m(a))$ and any $u \in (\xi(\nu), 2a - \xi(\nu))$, $2a - u \geq h(u, \nu)$ holds. Furthermore, if $2a - u = h(u, \nu)$, then $\varphi'(\nu) = 0$ and $\psi(h(u, \nu), u, \nu) = \varphi(\nu) \geq \pi$. Recall that $h(u, v)$ is defined in the proof of Lemma* 2.9.

PROOF. It follows from Lemma 2.9 that

$$\varphi'(\nu) = \int_{h(u,\nu)}^{2a-u} f_\nu(t, \nu)dt\,. \tag{3.11}$$

Hence, by Lemma 3.4, we get $2a - u \geq h(u, \nu)$. From Lemmas 2.9 and 3.1, and (3.11), it follows that $\varphi'(\nu) = 0$ and

$$\psi(h(u, \nu), u, \nu) = \varphi(\nu) \geq \pi$$

if $2a - u = h(u, \nu)$. □

LEMMA 3.6. *If $u \in (0, a)$ satisfies $m(u) \geq m(a)$, then for any $\nu \in (0, m(u)]$ and any $b > 0$ with $\tilde{\gamma}_\nu{}^{(u)}(b) \in \tilde{\theta}^{-1}(0, \pi)$, $\tilde{\gamma}_\nu{}^{(u)}|_{[0,b]}$ has no conjugate points of $\tilde{\gamma}_\nu{}^{(u)}(0)$ along $\tilde{\gamma}_\nu{}^{(u)}$.*

PROOF. Let $u \in (0, a)$ be any number satisfying $m(u) \geq m(a)$. By (2.1), $\tilde{\gamma}_{m(u)}{}^{(u)}$ does not meet $\tilde{r} = a$. Thus the geodesic $\tilde{\gamma}_{m(u)}{}^{(u)}$ meets $\tilde{\theta} = \pi$ at a point $\tilde{\gamma}_{m(u)}{}^{(u)}(t_\pi)$ in $\tilde{r}^{-1}(0, a)$. Let $\tilde{D}_u$ denote the domain bounded by $\tilde{\gamma}_{m(u)}{}^{(u)}[0, t_\pi]$, $\tilde{\theta}^{-1}(0) \cap \tilde{r}^{-1}(0, u)$ and

$\tilde{\theta}^{-1}(\pi) \cap \tilde{r}^{-1}(0, \tilde{r}(\tilde{\gamma}_{m(u)}{}^{(u)}(t_\pi)))$, which is a subset of $\tilde{\theta}^{-1}(0, \pi) \cap \tilde{r}^{-1}(0, a)$. It follows from (2.1) and Lemma 3.3 that for any $\nu \in (0, m(u))$ and any $b > 0$ with $\tilde{\gamma}_\nu{}^{(u)}(b) \in \tilde{\theta}^{-1}(0, \pi)$, $\tilde{\gamma}_\nu{}^{(u)}|_{(0,b)}$ lies in the domain $\tilde{D}_u$. Therefore by Lemma 3.3, $\tilde{\gamma}_\nu{}^{(u)}|_{[0,b]}$ has no conjugate points of $\tilde{\gamma}_\nu{}^{(u)}(0)$ along $\tilde{\gamma}_\nu{}^{(u)}$ for any $\nu \in (0, m(u))$ and any $b > 0$ with $\tilde{\gamma}_\nu{}^{(u)}(b) \in \tilde{\theta}^{-1}(0, \pi)$. From Lemma 2.7 and Lemma 3.1, it is clear that $\tilde{\gamma}_{m(u)}{}^{(u)}|_{[0,b]}$ has no conjugate points of $\tilde{\gamma}_{m(u)}{}^{(u)}(0)$ along $\tilde{\gamma}_{m(u)}{}^{(u)}$. □

LEMMA 3.7. *Let $\nu_0 \in (0, m(a))$ be fixed. If the function $\psi(h(u, \nu_0), u, \nu_0)$ attains a local minimum at $u = u_0 \in (\xi(\nu_0), a)$, then*

$$m(u_0) = m(h(u_0, \nu_0)) \tag{3.12}$$

holds.

PROOF. By differentiating the equation (3.11) with respect to u, we have

$$\frac{\partial h}{\partial u}(u, \nu_0) = -\frac{f_\nu(u, \nu_0)}{f_\nu(h(u, \nu_0), \nu_0)} \tag{3.13}$$

for any $u \in (\xi(\nu_0), a)$. By Lemma 2.9, we have

$$\frac{d}{du}(\psi(h(u, \nu_0), u, \nu_0)) = f(u, \nu_0) + f(h(u, \nu_0), \nu_0)\frac{\partial h}{\partial u}(u, \nu_0) \tag{3.14}$$

for any $u \in (\xi(\nu_0), a)$. By combining (3.13) and (3.14), we get

$$\frac{d}{du}(\psi(h(u, \nu_0), u, \nu_0)) = \frac{\nu_0^3 f_\nu(u, \nu_0)}{m(u)^2 m(h(u, \nu_0))^2}(m(u)^2 - m(h(u, \nu_0))^2). \tag{3.15}$$

Since the function $\psi(h(u, \nu_0), u, \nu_0)$ attains a local minimum at $u = u_0 \in (\xi(\nu_0), a)$,

$$\left.\frac{d}{du}\right|_{u=u_0} \psi(h(u, \nu_0), u, \nu_0) = 0\,.$$

Hence, by (3.15), the equation (3.12) holds. □

LEMMA 3.8. *For any $u \in (0, a)$, $\nu \in (0, m(u)]$, and $b > 0$ with $\tilde{\gamma}_\nu{}^{(u)}(b) \in \tilde{\theta}^{-1}(0, \pi)$, $\tilde{\gamma}_\nu{}^{(u)}|_{[0,b]}$ has no conjugate points of $\tilde{\gamma}_\nu{}^{(u)}(0)$ along $\tilde{\gamma}_\nu{}^{(u)}$.*

PROOF. By supposing that the conclusion is false, we will get a contradiction. Thus there exist some $u_1 \in (0, a)$, $\nu_0 \in (0, m(u_1)]$ and $t_c > 0$ with $\tilde{\gamma}_{\nu_0}{}^{(u_1)}(t_c) \in \tilde{\theta}^{-1}(0, \pi)$ such that $\tilde{\gamma}_{\nu_0}{}^{(u_1)}(t_c)$ is conjugate to $\tilde{\gamma}_{\nu_0}{}^{(u_1)}(0)$ along $\tilde{\gamma}_{\nu_0}{}^{(u_1)}$. Since $\nu_0 \le m(u_1) < m(a)$ by Lemma 3.6, it follows from Lemmas 2.3, 2.4 and 2.9 that $\xi(\nu_0) \le u_1 < a$ and

$$\tilde{\theta}(\tilde{\gamma}_{\nu_0}{}^{(u_1)}(t_c(u_1, \nu_0))) = \psi(h(u_1, \nu_0), u_1, \nu_0) < \pi\,.$$

The existence of $u_1 \in [\xi(\nu_0), a)$ implies that the function $\psi(h(u, \nu_0), u, \nu_0) = \tilde{\theta}(\tilde{\gamma}_{\nu_0}{}^{(u)}(t_c(u, \nu_0)))$ attains a local minimum, which is less than π, at $u = u_0 \in [\xi(\nu_0), a]$. Hence we have $u_0 \in (\xi(\nu_0), a)$ by Lemmas 2.7, 2.8 and 3.1. From Lemmas 3.6 and 3.7, we have

$$m(u_0) = m(h(u_0, \nu_0)) < m(a)\,.$$

Hence by Lemmas 2.3, 2.4 and 3.5, $u_0 = h(u_0, \nu_0)$. This means that $\tilde{\gamma}_{\nu_0}{}^{(u_0)}|_{[0,t_c(u_0,\nu_0)]}$ lies in $\tilde{r}^{-1}[0, a] \cap \tilde{\theta}^{-1}[0, \pi)$. Therefore by Lemma 3.3, $\tilde{\gamma}_{\nu_0}{}^{(u_0)}|_{[0,t_c(u_0,\nu_0)]}$ has no conjugate points of $\tilde{\gamma}_{\nu_0}{}^{(u_0)}(0)$. This is a contradiction. □

THEOREM 3.1. *Let $(M, dr^2 + m^2(r)d\theta^2)$ be a 2-sphere of revolution with a pair of poles p, q satisfying the following two properties.*

(3.16) *$(M, dr^2 + m^2(r)d\theta^2)$ is symmetric with respect to $r = a$, where $2a$ denotes the distance between p and q.*

(3.17) *The Gaussian curvature G of M is monotone non-increasing along a meridian from the point p to the point on $r = a$.*

Then, for any $x \in M\backslash\{p, q\}$, the cut locus of x is a single point or a subarc of the opposite half meridian. Furthermore, if the cut locus of a point $x \in M \setminus \{p, q\}$ is a single point, then the Gaussian curvature is constant.

PROOF. Suppose that for some point $x_0 \in M\backslash\{p, q\}$ there exists a cut point y_0 of x_0 which does not lie in the opposite half meridian to x_0. We may assume that the geodesic polar coordinates (r, θ) around p are chosen in such a way that $\theta(x_0) = 0 < \theta(y_0) < \pi$.

Furthermore, from Lemma 3.1 and (3.16), we may assume $r(x_0) \in (0, a)$. Let $\alpha : [0, d(x_0, y_0)] \to M$ be a unit speed minimal geodesic segment joining x_0 to y_0. We may assume that y_0 is conjugate to x_0 along α. Indeed, suppose that y_0 is not a conjugate point of $\alpha(0)$ along α. Then there exists a minimal geodesic β joining x_0 to y_0, which is distinct from α. Since the cut locus has an endpoint z_0 in the domain bounded by α and β, z_0 is a conjugate point of x_0 along any minimal geodesic segment joining x_0 to z_0. Hence we may assume that y_0 is conjugate to x_0 along α. Since $0 = \theta(x_0) < \theta(y_0)$, the Clairaut constant ν_1 of α is positive by (2.1). Thus $\alpha = \Pi \circ \tilde{\gamma}_{\nu_1}{}^{(u)}|_{[0,d(x_0,y_0)]}$ or $\Pi \circ \tilde{\beta}_{\nu_1}{}^{(u)}|_{[0,d(x_0,y_0)]}$, where $u := r(x_0)$ and $\Pi : \tilde{M} \to M \setminus \{p, q\}$ denotes the covering projection. Suppose that $r(y_0) \le a$. It follows form Lemma 3.8 that $\alpha = \Pi \circ \tilde{\beta}_{\nu_1}{}^{(u)}|_{[0,d(x_0,y_0)]}$ and α meets $r = a$ twice. Hence by Lemma 3.1, $\theta(y_0) \ge \varphi(\nu_1) \ge \pi$, which is a contradiction. Therefore, $r(y_0) > a$. By (3.16), we may assume that $\alpha = \Pi \circ \tilde{\gamma}_{\nu_1}{}^{(u)}|_{[0,d(x_0,y_0)]}$. This implies that there exists a conjugate point $\tilde{\gamma}_{\nu_1}{}^{(u)}(d(x_0, y_0)) \in \tilde{\theta}^{-1}(0, \pi)$ of $\tilde{\gamma}_{\nu_1}{}^{(u)}(0) \in \tilde{\theta}^{-1}(0)$ along $\tilde{\gamma}_{\nu_1}{}^{(u)}$. This contradicts Lemma 3.8. Thus we have proved the first claim of Theorem 3.1. Suppose that the cut locus of a point $x \in M \setminus \{p, q\}$ is a single point. We may assume that $\theta(x) = 0$ and $r(x) \in (0, a]$. It is clear that the single cut point of x is $(r, \theta)^{-1}(2a - u, \pi)$, where $u := r(x)$. Thus for each $\nu \in (0, m(u))$, $\tilde{\gamma}_\nu{}^{(u)}$ passes through the common point $(\tilde{r}, \tilde{\theta})^{-1}(2a - u, \pi)$. This means that $\varphi(\nu) = \pi$ for each $\nu \in (0, m(u))$. By Lemma 3.4, the Gaussian curvature is constant. □

4. The case where the Gaussian curvature is monotone non-decreasing.

Throughout this section, we assume that the Gaussian curvature G of M is monotone non-decreasing on a meridian from the pole p to the point on the equator. From Lemmas 2.2 and 2.3, $m' > 0$ on $(0, a)$. By the same argument as the proof of Lemma 3.1, we may prove the following lemma.

LEMMA 4.1. *The cut locus C_{p_0} of a point p_0 on the equator is a subset of the equator. Therefore $\varphi(\nu) \le \pi$ for any $\nu \in (0, m(a))$.*

LEMMA 4.2. *The function φ is monotone non-increasing on $(0, m(a))$ and $\varphi(0) := \lim_{\nu\to 0^+} \varphi(\nu) = \pi$. Moreover, if $\varphi(\nu_0) = \pi$ for some $\nu_0 \in (0, m(a))$, then $\varphi = \pi$ on $(0, m(a))$ and the Gaussian curvature is a positive constant.*

PROOF. It is trivial from geometrical observation that $\lim_{\nu\to 0^+} \varphi(\nu) = \pi$. Choose any numbers $\nu_1 < \nu_2$ from the interval $(0, m(a))$. Since $\arccos(\nu_2/m(a)) < \arccos(\nu_1/m(a))$, it follows from (2.1) and Lemma 4.1 that the geodesic segment $\tilde{c}_{\nu_1}|_{[0,t_0(\nu_1)]}$ does not enter the domain D_{ν_2} bounded by $\tilde{c}_{\nu_2}[0, t_0(\nu_2)]$ and the geodesic segment $\tilde{r}^{-1}(a)\cap\tilde{\theta}^{-1}[0, \tilde{c}_{\nu_2}(t_0(\nu_2))]$. Thus, by (2.7), $\varphi(\nu_1) \ge \varphi(\nu_2)$. Furthermore suppose that $\varphi(\nu_1) = \varphi(\nu_2)$. This means that the two geodesics $\tilde{c}_{\nu_1}$ and $\tilde{c}_{\nu_2}$ meet at $q_0 := \tilde{c}_{\nu_1}(t_0(\nu_1))$. From Lemma 4.1, for each $\nu \in [\nu_1, \nu_2]$, the geodesic $\tilde{c}_\nu$ passes through the point q_0. Thus q_0 is a conjugate point of $\tilde{c}_{\nu_1}(0)$ along $\tilde{c}_\nu$ for each $\nu \in [\nu_1, \nu_2]$. By repeating the proof of the equation (3.5), we get the inequality $\tilde{r}(\tilde{c}_{\nu_1}(t)) < \tilde{r}(\tilde{c}_{\nu_2}(t)) < a$ for any $t \in (0, t_0(\nu_1))$. Thus

$$\tilde{G}(\tilde{c}_{\nu_1}(t)) \le \tilde{G}(\tilde{c}_{\nu_2}(t))$$

for any $t \in [0, t_0(\nu_1)]$, where $\tilde{G}$ denotes the Gaussian curvature of $\tilde{M}$. Since q_0 is a conjugate point of $\tilde{c}_{\nu_1}(0)$ along $\tilde{c}_\nu$ for each $\nu \in [\nu_1, \nu_2]$, it follows from the Rauch comparison theorem that $\tilde{G}(\tilde{c}_{\nu_1}(t)) = \tilde{G}(\tilde{c}_{\nu_2}(t))$ holds for any $t \in [0, t_0(\nu_1)]$. Thus from the monotonicity of $\tilde{G}$, $\tilde{G}$ is constant on $\tilde{r}^{-1}(I_t)$, for each $t \in (0, t_0(\nu_1))$, where I_t denotes the closed interval $I_t := [\tilde{r}(\tilde{c}_{\nu_1}(t)), \tilde{r}(\tilde{c}_{\nu_2}(t))]$. If we fix $t_\nu \in (0, t_0(\nu_1))$, then for any t sufficiently close to t_ν, $I_t \cap I_{t_\nu}$ is non-empty. Therefore $\tilde{G}$ is constant on $\tilde{r}^{-1}[\xi(\nu_1), a]$. Notice that $\xi(\nu_1)$ equals $\min\{\tilde{r}(\tilde{c}_{\nu_1}(t))\,;\; t \in [0, t_0(\nu_1)]\}$ by (2.2). This means that the function φ is constant on $[\nu_1, m(a)]$. Suppose that $\varphi(\nu_0) = \pi = \lim_{\nu\to 0^+} \varphi(\nu)$ for some $\nu_0 \in (0, m(a))$. From the monotonicity of φ, $\varphi(\nu) = \varphi(\nu_0) = \pi$ for any $\nu \in (0, \nu_0]$. By the argument above, φ is constant on $(0, m(a))$. □

LEMMA 4.3. *For each $u \in (0, 2a)$ and $\nu \in (0, m(u))$, the geodesic segments $\tilde{\beta}_\nu{}^{(u)}|_{[0,t_0(\nu)]}$ and $\tilde{\gamma}_\nu{}^{(u)}|_{[0,t_0(\nu)]}$ bound a 2-disk domain $\tilde{D}(u, \nu)$. Furthermore, if $\varphi(\nu) > \varphi(m(u))$, then*

$$\tilde{\beta}_{m(u)}{}^{(u)}(0, t_0(m(u))] \cup \{2a - u\} \times [\varphi(m(u)), \varphi(\nu)) \subset \tilde{D}(u, \nu)\,,$$

where $\varphi(m(a)) := \lim_{\nu\to m(a)^-} \varphi(\nu)$ and $t_0(m(a)) := \lim_{\nu\to m(a)^-} t_0(\nu)$.

PROOF. Let $\nu \in (0, m(u))$ and $u \in (0, 2a)$ be fixed. Since $\tilde{M}$ is symmetric with respect to $\tilde{r} = a$, we may assume that $u \in (0, a]$. Furthermore, we may assume $u \in (0, a)$, because the conclusion of Lemma 4.3 is clear if $u = a$. Let $S : \tilde{M} \to \tilde{M}$ denote an isometry defined by $S(\tilde{r}, \tilde{\theta}) := (\tilde{r}, \tilde{\theta} - \tilde{\theta}(\tilde{c}_\nu(t_1)))$, where t_1 denotes a parameter value of $\tilde{c}_\nu$ such that $\tilde{r} \circ \tilde{c}_\nu(t_1) = u$, $(\tilde{r}\circ\tilde{c}_\nu)'(t_1) < 0$. Since the geodesic $\bar{\gamma}(t) := S(\tilde{c}_\nu(t+t_1))$ satisfies $\bar{\gamma}(0) = \tilde{\gamma}_\nu{}^{(u)}(0)$, $\dot{\bar{\gamma}}(0) = \dot{\tilde{\gamma}}_\nu{}^{(u)}(0)$, we have $S(\tilde{c}_\nu(t + t_1)) = \bar{\gamma}(t) = \tilde{\gamma}_\nu{}^{(u)}(t)$ by uniqueness. Since $T_\nu \circ S = S \circ T_\nu$, it follows from Lemma 2.5 that $T_\nu(\tilde{\gamma}_\nu{}^{(u)}(t)) = \tilde{\gamma}_\nu{}^{(u)}(t + t_0(\nu))$. Hence it is clear that

$$(2a - u, \varphi(\nu)) = \tilde{T}_\nu(\tilde{\gamma}_\nu{}^{(u)}(0)) = \tilde{\gamma}_\nu{}^{(u)}(t_0(\nu))\,. \tag{4.1}$$

Similarly, we get

$$(2a - u, \varphi(\nu)) = \tilde{T}_\nu(\tilde{\beta}_\nu{}^{(u)}(0)) = \tilde{\beta}_\nu{}^{(u)}(t_0(\nu))\,. \tag{4.2}$$

By (4.1) and (4.2), both geodesics $\tilde{\gamma}_\nu{}^{(u)}$ and $\tilde{\beta}_\nu{}^{(u)}$ meet at $(2a - u, \varphi(\nu))$ again. Fix any $s \in (\tilde{r} \circ \tilde{\gamma}_\nu{}^{(u)})^{-1}(u, 2a - u)$. Let $\tilde{\beta}_\nu{}^{(u)}(t_\nu(s))$ (resp. $\tilde{\beta}_{m(u)}{}^{(u)}(t_{m(u)}(s))$) denote the intersection of the geodesic segment $\tilde{\beta}_\nu{}^{(u)}|_{[0,t_0(\nu)]}$ (resp. $\tilde{\beta}_{m(u)}{}^{(u)}|_{[0,t_0(m(u))]}$) and $\tilde{r} = \tilde{r}(s) := \tilde{r} \circ \tilde{\gamma}_\nu{}^{(u)}(s)$. By (2.3),

$$\tilde{\theta} \circ \tilde{\beta}_\nu{}^{(u)}(t_\nu(s)) = \int_u^{\tilde{r}(s)} f(t, \nu)dt \tag{4.3}$$

and

$$\tilde{\theta} \circ \tilde{\beta}_{m(u)}{}^{(u)}(t_{m(u)}(s)) = \int_u^{\tilde{r}(s)} f(t, m(u))dt\,. \tag{4.4}$$

Since $f(t, \nu) < f(t, m(u))$, we get

$$\tilde{\theta} \circ \tilde{\beta}_\nu{}^{(u)}(t_\nu(s)) < \tilde{\theta} \circ \tilde{\beta}_{m(u)}{}^{(u)}(t_{m(u)}(s))\,. \tag{4.5}$$

By (1.4) and (2.3),

$$\tilde{\theta} \circ \tilde{\gamma}_\nu{}^{(u)}(s) = \varphi(\nu) - \int_{\tilde{r}(s)}^{2a-u} f(t, \nu)dt\,. \tag{4.6}$$

Since $f(t, \nu) < f(t, m(u))$ and $\varphi(\nu) \geq \varphi(m(u))$,

$$\begin{aligned} \varphi(\nu) - \int_{\tilde{r}(s)}^{2a-u} f(t, \nu)dt &> \varphi(m(u)) - \int_{\tilde{r}(s)}^{2a-u} f(t, m(u))dt \\ &= \int_u^{\tilde{r}(s)} f(t, m(u))dt\,. \end{aligned} \tag{4.7}$$

Hence, by (4.3), (4.5), (4.6) and (4.7),

$$\tilde{\theta} \circ \tilde{\beta}_\nu{}^{(u)}(t_\nu(s)) < \tilde{\theta} \circ \tilde{\beta}_{m(u)}{}^{(u)}(t_{m(u)}(s)) < \tilde{\theta} \circ \tilde{\gamma}_\nu{}^{(u)}(s) \tag{4.8}$$

holds for any $s \in (\tilde{r} \circ \tilde{\gamma}_\nu{}^{(u)})^{-1}(u, 2a - u)$. Therefore, the proof of Lemma 4.3 is complete. □

THEOREM 4.1. *Let $(M, dr^2 + m(r)^2 d\theta^2)$ be a 2-sphere of revolution with a pair of poles p, q satisfying the following two properties.*

(4.9) *$(M, dr^2 + m(r)^2 d\theta^2)$ is symmetric with respect to $r = a$, where $2a$ denotes the distance between p and q.*

(4.10) *The Gaussian curvature of M is monotone non-decreasing along a meridian from p to the point on $r = a$.*

Then for any $x \in M \setminus \{p, q\}$, the cut locus of x is a point or a subarc of the parallel $r = 2a - r(x)$. Furthermore, if the cut locus of $x \in M \setminus \{p, q\}$ is a single point, then the Gaussian curvature is constant.

PROOF. Fix any point $x \in M \setminus \{p, q\}$. We may choose geodesic polar coordinates (r, θ) around p satisfying $\theta(x) = 0$. Put $r(x) =: u \in (0, 2a)$. If $\varphi(\nu) = \pi$ for some $\nu \in (0, m(a))$, then the conclusion of our theorem is trivial by Lemma 4.2. Thus we may assume that $\varphi(\nu) < \pi$ for any ν.

For each $\nu \in [-m(u), m(u)]$, let $\beta_\nu{}^{(x)}$ and $\gamma_\nu{}^{(x)}$ denote the unit speed geodesic with Clairaut constant ν emanating from $x = \beta_\nu{}^{(x)}(0) = \gamma_\nu{}^{(x)}(0)$ such that $(r \circ \beta_\nu{}^{(x)})'(0) \geq 0$ and $(r \circ \gamma_\nu{}^{(x)})'(0) \leq 0$. It is trivial that $\Pi \circ \tilde{\beta}_\nu{}^{(u)} = \beta_\nu{}^{(x)}$ and $\Pi \circ \tilde{\gamma}_\nu{}^{(u)} = \gamma_\nu{}^{(x)}$ for each $\nu \in (0, m(u)]$, where $\Pi : \tilde{M} \to M \setminus \{p, q\}$ denotes the covering projection. Since $\varphi(\nu) < \pi$ for any $\nu \in (0, m(a))$, the geodesic segments $\tilde{\beta}_\nu{}^{(u)}[0, t_0(\nu)]$ and $\tilde{\gamma}_\nu{}^{(u)}[0, t_0(\nu)]$ lie in $\tilde{\theta}^{-1}[0, \pi)$, where $\tilde{\theta} := \theta \circ \Pi$. Since $\Pi : \tilde{\theta}^{-1}(-\pi, \pi) \to \theta^{-1}(-\pi, \pi)$ maps homeomorphically, it follows from Lemma 4.3 that $\beta_\nu{}^{(x)}[0, t_0(\nu)]$ and $\gamma_\nu{}^{(x)}[0, t_0(\nu)]$ bound a 2-disk domain for each $\nu \in (0, m(u))$. Fix any $\nu_1 \in (0, m(u))$ with $\varphi(\nu_1) \in (\varphi(m(u)), \pi)$. Let $\alpha : [0, l_1] \to M$ denote a unit speed minimal geodesic segment joining x to $\gamma_{\nu_1}{}^{(x)}(t_0(\nu_1))$. Since $\theta(x) = 0$ and $0 < \theta(\alpha(l_1)) = \varphi(\nu_1)$, the Clairaut constant ν_2 of α is positive and $\alpha = \beta_{\nu_2}{}^{(x)}|_{[0,l_1]}$ or $\gamma_{\nu_2}{}^{(x)}|_{[0,l_1]}$. Suppose that $\varphi(\nu_2) < \varphi(\nu_1)$. Then $\alpha|_{[0,l_1]}$ meets the curve $\theta = \varphi(\nu_2)$ at the point $\alpha(t_0(\nu_2)) = \beta_{\nu_2}{}^{(x)}(t_0(\nu_2)) = \alpha_{\nu_2}{}^{(x)}(t_0(\nu_2))$. Thus $l_1 > t_0(\nu_2)$. If $\nu_2 = m(u)$, then by Lemma 2.8, $\alpha(t_0(\nu_2))$ is conjugate to x along $\alpha = \beta_{m(u)}{}^{(x)} = \alpha_{m(u)}{}^{(x)}$. This is impossible, since $\alpha|_{[0,l_1]}$ is minimal. Therefore $\nu_2 < m(u)$. Since both minimal geodesic segments $\beta_{\nu_2}{}^{(x)}|_{[0,t_0(\nu_2)]}$ and $\gamma_{\nu_2}{}^{(x)}|_{[0,t_0(\nu_2)]}$ have the same length and meet at $\gamma_{\nu_2}{}^{(x)}(t_0(\nu_2)) = \beta_{\nu_2}{}^{(x)}(t_0(\nu_2))$, α cannot be minimal. Thus $\varphi(\nu_2) \geq \varphi(\nu_1)$. Suppose that $\varphi(\nu_2) > \varphi(\nu_1)$. Thus $\nu_2 < \nu_1$ by Lemma 4.2. By (2.1) and Lemma 4.3, α does not pass through $r^{-1}(2a - u) \cap \theta^{-1}[\varphi(m(u)), \varphi(\nu_2))$. In particular, α does not pass through the point $r^{-1}(2a - u) \cap \theta^{-1}(\varphi(\nu_1)) = \gamma_{\nu_1}{}^{(x)}(t_0(\nu_1)) = \alpha(t_1)$. This is impossible. Hence $\varphi(\nu_1) = \varphi(\nu_2)$. Therefore for any $\nu \in (0, m(u))$ with $\varphi(\nu) \in (\varphi(m(u)), \pi)$, both geodesic segments $\beta_\nu{}^{(x)}|_{[0,t_0(\nu)]}$ and $\gamma_\nu{}^{(x)}|_{[0,t_0(\nu)]}$ are minimal. This implies that any point of $r^{-1}(2a - u) \cap \theta^{-1}[\varphi(m(u)), 2\pi - \varphi(m(u))]$ is a cut point of x.

Since a limit geodesic segment of a sequence of minimal geodesic segments is also minimal, both geodesic segments $\beta_0{}^{(x)}|_{[0,t_0(0)]}$ and $\gamma_0{}^{(x)}|_{[0,t_0(0)]}$, where $t_0(0) := \lim_{\nu \to 0^+} t_0(\nu) = a$, are minimal. Now it is clear that the cut locus of x is $r^{-1}(2a - u) \cap \theta^{-1}[\varphi(m(u)), 2\pi - \varphi(m(u))]$, because any minimal geodesic segment emanating from x is a subarc of $\beta_\nu{}^{(x)}|_{[0,t_0(\nu)]}$ or $\gamma_\nu{}^{(x)}|_{[0,t_0(\nu)]}$ for some $\nu \in [-m(u), m(u)]$. In particular, the cut locus of x is a subarc of the parallel $r = 2a - r(x)$. The latter claim is clear from Lemma 4.2. □

5. Examples. Let t_0 be any value in $(\pi/2, 3\pi/4)$. It is clear that $\sin t_0 > -\cos t_0 > 0$. For each $\delta \in (0, \pi - t_0)$, let $G_\delta : [0, \infty) \to R$ be a C^∞ monotone non-increasing function such that $G_\delta = 1$ on $[0, t_0]$ and $G_\delta = -1$ on $[t_0 + \delta, \infty)$.

LEMMA 5.1. *There exists* $\delta_0 > 0$ *such that for any* $\delta \in (0, \delta_0)$

$$m_\delta > -m_\delta{}' > 0$$

on $[t_0, t_0 + \delta_0]$, *where* m_δ *denotes the solution of the differential equation*

$$m''_\delta(t) + G_\delta(t)m_\delta(t) = 0 \tag{5.1}$$

with initial condition $m_\delta(0) = 0$, $m'_\delta(0) = 1$. *In particular*, $m_\delta > 0$ *on* $(0, t_0 + \delta_0]$.

PROOF. Let $(R^2, \langle\ ,\ \rangle)$ denote a 2-dimensional vector space R^2 with canonical inner product $\langle\ ,\ \rangle$. Let $F : R \to R$ denote the function defined by

$$F(t) := \sqrt{\langle X(t), X(t)\rangle}\,,$$

where $X(t) := (m_\delta(t) - \sin t, m'_\delta(t) - \cos t)$. Since $G_\delta = 1$ on $[0, t_0]$, $F(t) = 0$ on $[0, t_0]$. Since

$$X'(t) = (m'_\delta(t) - \cos t, -G_\delta(t)(m_\delta(t) - \sin t)) + (0, (1 - G_\delta(t))\sin t)\,,$$

by (5.1), it follows from the triangle inequality that

$$|X'(t)| \le F(t) + 2\,, \tag{5.2}$$

where $|X'(t)| := \sqrt{\langle X'(t), X'(t)\rangle}$, $|X(t)| := \sqrt{\langle X(t), X(t)\rangle}$. Thus, by the Schwarz inequality, we get

$$F'(t) \le |X'(t)| \le F(t) + 2 \tag{5.3}$$

for any t with $F(t) \ne 0$. Since

$$(e^{-t}F(t))' = e^{-t}F'(t) - e^{-t}F(t)\,,$$

we have, by (5.3),

$$F(t) = e^t\int_{t_1}^{t}(e^{-t}F(t))'dt \le 2e^t\int_{t_1}^{t} e^{-t}dt = 2(e^{(t-t_1)} - 1) \tag{5.4}$$

for any $t > t_1$, where $t_1(\ge t_0)$ denotes the maximum of $F^{-1}(0)$. In particular,

$$|m_\delta(t) - \sin t| \le F(t) \le 2(e^{(t-t_0)} - 1) \tag{5.5}$$

and

$$|m'_\delta(t) - \cos t| \le F(t) \le 2(e^{(t-t_0)} - 1) \tag{5.6}$$

hold for any $t \ge t_0$. It is clear that

$$\sin t - |m_\delta(t) - \sin t| \le m_\delta(t) \tag{5.7}$$

and

$$\cos t - |m'_\delta(t) - \cos t| \le m'_\delta(t) \le \cos t + |m'_\delta(t) - \cos t|\,. \tag{5.8}$$

By combining (5.5), (5.6), (5.7) and (5.8), we may get the conclusion of Lemma 5.1. □

LEMMA 5.2. *If* $\delta \in (0, \delta_0)$, *then there exist constants* a $(> t_0 + \delta)$, *and* $C > 0$ *such that* $m_\delta(t) = C\cosh(t - a)$ *on* $[t_0 + \delta, \infty)$.

PROOF. Since m_δ is a solution of $m''_\delta(t) - m_\delta(t) = 0$ on $[t_0+\delta, \infty)$, there exist constants A, B such that $m_\delta(t) = Ae^t + Be^{-t}$ on $[t_0 + \delta, \infty)$. By Lemma 5.1, A and B are positive constants and $m_\delta > 0$ on $(0, \infty)$. Since $m'_\delta(t_0+\delta) < 0$ and $\lim_{t\to\infty} m_\delta(t) = +\infty$, there exists a constant $a > t_0 + \delta$ such that $m'_\delta(a) = 0$. Hence $B = e^{2a}A$ and $m_\delta(t) = 2Ae^a \cosh(t - a)$ on $[t_0 + \delta, \infty)$. □

PROPOSITION 5.1. *There exists a C^∞ Riemannian metric g on a 2-sphere S^2 such that (S^2, g) is a 2-sphere of revolution satisfying* (1.1) *and* (1.2), *whose Gaussian curvature is -1 on the equator.*

PROOF. Choose any $\delta \in (0, \delta_0)$ and fix it. Let $m_0 : [0, 2a] \to R$ be the function defined by $m_0(t) = m_\delta(t)$ for $t \in [0, a]$ and $m_0(t) = m_\delta(2a - t)$ for $t \in [a, 2a]$. Here a denotes the constant guaranteed in Lemma 5.2. By Lemmas 5.1 and 5.2, m_0 is C^∞ on $[0, 2a]$. It is clear that m_0 satisfies (1.1). Let (S^2, g_0) denote a 2-sphere with radius $2a/\pi$ and (r_0, θ_0) geodesic polar coordinates around a point on (S^2, g_0). The new Riemannian metric $dr_0^2 + m_0(r_0)^2 d\theta_0^2$ defines a C^∞ metric on S^2. It is clear from (1.5) and Lemma 5.2 that the Gaussian curvature of $(S^2, dr_0^2 + m_0(r_0)^2 d\theta_0^2)$ is -1 on the equator. □

REMARK. By imitating the above, we can construct a 2-sphere of revolution whose Gaussian curvature is -1 (resp. 1) at the poles (resp. on the equator).

6. Toponogov's comparison theorem. Let M be a complete n-dimensional Riemannian manifold with a base point $p \in M$. The manifold M is said to have *radial sectional curvature at p bounded from below by a function $K : [0, l] \to R$* if for every unit speed minimal geodesic $\gamma : [0, b] \to M$, with $b \le l$, emanating from $p = \gamma(0)$, the sectional curvature K_M of M satisfies

$$K_M(\sigma_t) \ge K(t)$$

for any $t \in [0, b]$ and any 2-dimensional linear space σ_t spanned by $\dot\gamma(t)$ and a tangent vector to M at $\gamma(t)$. The following is the Toponogov comparison theorem for a 2-sphere of revolution whose cut locus is a subarc of a meridian or a single point.

THEOREM 6.1. *Let M be a complete Riemannian n-manifold with a base point p such that the radial sectional curvature of M at p is bounded from below by $G \circ \mu : [0, 2a] \to R$. Here, G denotes the Gaussian curvature of a* 2*-sphere of revolution $\tilde M$, μ a meridian of $\tilde M$ and* $2a$ *the distance between its two poles. Suppose that the cut locus of any point on $\tilde M$ distinct from its two poles is a subarc of the half meridian opposite to the point. Then for each geodesic triangle $\triangle(pxy) \subset M$, there exists a geodesic triangle $\triangle(\tilde p\tilde x\tilde y) \subset \tilde M$, where $\tilde p = \mu(0)$, such that*

$$d(p, x) = d(\tilde p, \tilde x), \quad d(p, y) = d(\tilde p, \tilde y), \quad d(x, y) = d(\tilde x, \tilde y) \tag{6.1}$$

and such that

$$\angle(pxy) \ge \angle(\tilde p\tilde x\tilde y), \quad \angle(pyx) \ge \angle(\tilde p\tilde y\tilde x), \quad \angle(xpy) \ge \angle(\tilde x\tilde p\tilde y). \tag{6.2}$$

Here, $\angle(pxy)$ denotes the angle at the vertex x of the geodesic triangle $\triangle(pxy)$.

PROOF. Let $\mathrm{x}, \mathrm{y} : [0, 1] \to M$ be any two distinct minimal geodesics emanating from the base point p. Put $x := \mathrm{x}(1)$ and $y := \mathrm{y}(1)$. We will prove the existence of a geodesic triangle $\triangle(\tilde{p}\tilde{x}\tilde{y})$ corresponding to the geodesic triangle $\triangle(pxy)$ satisfying (6.1) and (6.2). We may assume that $\angle(xpy) < \pi$, because the case where $\angle(xpy) = \pi$ is reduced to this case by the limit argument. Let T be the set of all positive numbers t such that, for any $u \in (0, t)$, there exists a geodesic triangle $\triangle(\tilde{p}\tilde{\mathrm{x}}(u)\tilde{\mathrm{y}}(u))$ corresponding to $\triangle(p\mathrm{x}(u)\mathrm{y}(u))$ satisfying (6.1) and (6.2) with $x = \mathrm{x}(u)$ and $y = \mathrm{y}(u)$.

First we will prove that the set T is non-empty. From the Rauch comparison theorem, for any sufficiently small positive t, there exists a geodesic triangle $\triangle(\tilde{p}\tilde{\mathrm{x}}(t)\tilde{\mathrm{y}}(t))$ *corresponding*, i.e., satisfying (6.1) with $x = \mathrm{x}(t)$ and $y = \mathrm{y}(t)$, to $\triangle(p\mathrm{x}(t)\mathrm{y}(t))$ such that $\angle(\mathrm{x}(t)p\mathrm{y}(t)) \geq \angle(\tilde{\mathrm{x}}(t)\tilde{p}\tilde{\mathrm{y}}(t))$. On the other hand, it follows from Theorem 1.2 in [12] that there exists a geodesic triangle $\triangle(\tilde{p}\hat{\mathrm{x}}(t)\hat{\mathrm{y}}(t))$ corresponding to $\triangle(p\mathrm{x}(t)\mathrm{y}(t))$ for any sufficiently small t such that

$$\angle(p\mathrm{x}(t)\mathrm{y}(t)) \geq \angle(\tilde{p}\hat{\mathrm{x}}(t)\hat{\mathrm{y}}(t)), \quad \angle(p\mathrm{y}(t)\mathrm{x}(t)) \geq \angle(\tilde{p}\hat{\mathrm{y}}(t)\hat{\mathrm{x}}(t)).$$

Since the corresponding geodesic triangle to $\triangle(p\mathrm{x}(t)\mathrm{y}(t))$ exists uniquely, up to a rotation about $\tilde{p}$ for sufficiently small positive t, we may assume that $\triangle(\tilde{p}\tilde{\mathrm{x}}(t)\tilde{\mathrm{y}}(t)) = \triangle(\tilde{p}\hat{\mathrm{x}}(t)\hat{\mathrm{y}}(t))$. Hence T is non-empty. By assuming the supremum t_0 of the set T to be less than 1, we will get a contradiction. Hence there exists a geodesic triangle $\triangle(\tilde{p}\tilde{\mathrm{x}}(t_0)\tilde{\mathrm{y}}(t_0))$ corresponding to $\triangle(p\mathrm{x}(t_0)\mathrm{y}(t_0))$ such that equality holds in one of the inequalities (6.2) for $x = \mathrm{x}(t_0)$ and $y = \mathrm{y}(t_0)$. From the corollary to GACT-I in [12], there exists a piece of totally geodesic surface bounded by $\triangle(p\mathrm{x}(t_0)\mathrm{y}(t_0))$ which is isometric to the interior of $\triangle(\tilde{p}\tilde{\mathrm{x}}(t_0)\tilde{\mathrm{y}}(t_0))$. Therefore, by the Rauch comparison theorem, there exists a positive δ such that for any $t \in (t_0, t_0 + \delta)$ there exists a geodesic triangle $\triangle(\tilde{p}\tilde{\mathrm{x}}(t)\tilde{\mathrm{y}}(t))$ corresponding to $\triangle(p\mathrm{x}(t)\mathrm{y}(t))$ satisfying (6.1) with $x = \mathrm{x}(t)$, $y = \mathrm{y}(t)$ and $\angle(\mathrm{x}(t)p\mathrm{y}(t)) \geq \angle(\tilde{\mathrm{x}}(t)\tilde{p}\tilde{\mathrm{y}}(t))$. Since $\pi > \angle(xpy) = \angle(\mathrm{x}(t_0)p\mathrm{y}(t_0)) \geq \angle(\tilde{\mathrm{x}}(t_0)\tilde{p}\tilde{\mathrm{y}}(t_0))$, we may assume that the edge $\tilde{\mathrm{x}}(t)\tilde{\mathrm{y}}(t)$ does not intersect the cut locus of $\tilde{\mathrm{x}}(t)$ for any $t \in (t_0, t_0+\delta)$. Fix any $t_1 \in (t_0, t_0+\delta)$. From the unique existence of a geodesic triangle corresponding to $\triangle(p\mathrm{x}(t_1)\mathrm{y}(t_1))$ satisfying (6.1) with $x = \mathrm{x}(t_1)$ and $y = \mathrm{y}(t_1)$, we get the geodesic triangle $\triangle(\tilde{p}\tilde{\mathrm{x}}(t_1)\tilde{\mathrm{y}}(t_1))$ satisfying (6.1) and (6.2) with $x = \mathrm{x}(t_1)$ and $y = \mathrm{y}(t_1)$. This contradicts the definition of t_0. Hence the supremum of the set T is 1. □

COROLLARY 6.1. *The perimeter of any geodesic triangle on the manifold M is at most $4a$. Furthermore, if equality holds in any one of the inequalities* (6.2), *then there exists a piece of totally geodesic surface bounded by $\triangle(pxy)$ which is isometric to the interior of $\triangle(\tilde{p}\tilde{x}\tilde{y})$.*

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