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The cycle space of an infinite graph

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Abstract

Finite graph homology may seem trivial, but for infinite graphs things become interesting. We present a new 'singular' approach that builds the cycle space of a graph not on its finite cycles but on its topological *circles*, the homeomorphic images of S^1 in the space formed by the graph together with its ends.

Our approach permits the extension to infinite graphs of standard results about finite graph homology – such as cycle-cocycle duality and Whitney's theorem, Tutte's generating theorem, MacLane's planarity criterion, the Tutte/Nash-Williams tree packing theorem – whose infinite versions would otherwise fail. A notion of end degrees motivated by these results opens up new possibilities for an 'extremal' branch of infinite graph theory.

Numerous open problems are suggested.

Introduction

This is an expository paper describing a new line of research started jointly with Kühn [10, 11, 12], Bruhn [3], and Bruhn and Stein [4], as well as some work of the latter two authors inspired by this approach [2, 5, 6] and a few new observations. This paper has three aims: to describe informally the main underlying ideas of the new approach; to compile a list of all the results known to date; and to draw attention to the problems and conjectures that remain open.

As our starting point we take the well-known facts and theorems describing the cycle space of a finite graph. When we try to extend these verbatim to infinite graphs, a curious phenomenon occurs: while the easy ones among them remain true (and easy), all the deeper ones become false.

As a case in point, consider Tutte's theorem that the 'peripheral' (= non-separating induced) cycles of a 3-connected finite graph generate all other cycles. (When the graph is planar, these are precisely its face boundaries.) As two easy counterexamples will show, this is no longer true for infinite graphs, even planar ones: to be able to generate all cycles, one needs a larger generating set than just the (finite) peripheral cycles, and one also needs to allow infinite summing as a more powerful generating mechanism.

As our counterexamples are planar, it will be obvious which additional generators to admit: those sets of edges that bound a region of the plane, even when they do not form a cycle. We shall call these additional generators 'infinite cycles', a term compellingly suggested by their position in the plane. But which additional generators, or 'infinite cycles', should we admit in general to make Tutte's theorem true for infinite graphs, i.e., when the graph is not planar and we can no longer refer to regions and their boundaries?

Starting out from our examples, we are led to consider first simple and then increasingly complicated structures as suitable candidates for infinite cycles. The problem is where to stop: while each step towards this increasing complication is just as natural as the previous, the end result seems more of a mess than, as had been the aim, the 'right' setting in which to describe the homology aspects of an infinite graph.

At this point, the decisive step is to change the viewpoint from combinatorial to topological. Viewing the graph as a 1-complex and compactifying this by adding its *ends* as extra points at infinity, we obtain a topological space in which we can define a 'cycle' simply as a homeomorphic image of the circle S^1. This definition is certainly natural, and it comprises both the usual finite cycles in the graph and the substructures previously considered as candidates for infinite cycles. Put another way: it was not the objects that we found should qualify as infinite cycles that were so complicated, but only their combinatorial description.

On the other hand, once we adopt such a topological definition of a cycle, it is no longer clear what combinatorial structures other than the above might now qualify as infinite cycles as well. And indeed, these can be more complicated combinatorially than even our wildest earlier examples: we shall see that a very simple locally finite graph can have a cycle passing through uncountably many ends and containing a dense set of double rays, in the sense that between any two there lies another!

All the same, our topological definition of a cycle introduces no unnecessary complication, even combinatorially: we shall prove that no smaller set of infinite cycles allows simultaneous extensions to infinite graphs of the basic facts and theorems about finite cycle spaces that we started out from. And we do indeed get these extensions (and much more) – in a cycle space that admits both infinite cycles and infinite sums.

This paper is organized as follows. We begin in Section 1 with a summary of all the known facts and theorems about the cycle space of a finite graph that we wish to (but cannot easily) extend to infinite graphs. In Sections 2–3 we look at a sequence of examples showing the need for a more general notion of a cycle space in infinite graphs, and what features this should have in order to make the desired generalizations possible. In Section 4 we define this infinite cycle space rigorously. Section 5 surveys the results obtained in the papers cited earlier, as well as some new ones; these include extensions of all the finite facts and theorems from Section 1. Section 6, finally, is devoted to open problems and conjectures. Several of these suggest infinite 'topological' extensions of finite results that would otherwise fail to generalize (or fail to do so in an interesting way). Moreover, we shall see how a new notion of end-degrees introduced earlier might allow us to consider extremal-type problems also for infinite graphs. We conclude with a brief discussion of how our new cycle space of an infinite graph relates to the standard singular homology of its Freudenthal compactification, as a basis for possible generalizations of our approach to higher-dimensional complexes.

1 The cycle space of a finite graph

Let $G = (V, E)$ be a finite graph or multigraph. As the *edge space* $\mathcal{E} = \mathcal{E}(G)$ we take the vector space $\{0, 1\}^E$ over $\mathbb{F}_2$, which we view as the power set of E with symmetric difference as addition. A set $F \subseteq E$ is a *circuit* in G if there is a cycle $C \subseteq G$ with $E(C) = F$. The *cycle space* $\mathcal{C} = \mathcal{C}(G)$ of G is the subspace of $\mathcal{E}$ generated by the circuits in G. If $T \subseteq G$ is a spanning tree of G, then for every edge $e \in E \setminus E(T)$ there is a unique cycle C_e in $T + e$. The edge sets D_e of these cycles are the *fundamental circuits* of T in G. A set $F \subseteq E$ is a *cut* (or *cocycle*) of G if there is a partition of V into two non-empty sets A, B such that F consists of all the edges of G between A and B. A multigraph $G^* = (V^*, E^*)$ is called a *dual* of G if $E^* = E$ and the circuits in G coincide with the minimal non-empty cuts in G^*.

The following facts summarize the best known properties of the cycle space $\mathcal{C}$ and related aspects. Our aim will be to extend them all in a unified way to infinite graphs.

(1.1) *The following statements are equivalent for a set $F \subseteq E$:*

(i) *$F \in \mathcal{C}$;*

(ii) *F is a disjoint union of circuits;*

(iii) *Every vertex of the graph (V, F) has even degree.*

Proof. (i)$\to$(iii): Induction along the inductive definition of $\mathcal{C}$.

(iii)$\to$(ii): Since the deletion of a circuit does not affect the validity of (iii), the circuits for (ii) can be found greedily.

(ii)$\to$(i) is trivial. $\square$

(1.2) *The fundamental circuits of any fixed spanning tree T in G generate $\mathcal{C}$.*

Proof. For any circuit D, the set $D + \sum_{e \in D} D_e$ is a subset of $E(T)$ that lies in $\mathcal{C}$, and hence must be empty by (1.1.ii). Thus, $D = \sum_{e \in D} D_e$. $\square$

(1.3) *Let $\emptyset \neq F \subseteq E$.*

(i) *$F \in \mathcal{C}$ if and only if F meets every cut in an even number of edges.*

(ii) *F is a cut if and only if F meets every element of $\mathcal{C}$ in an even number of edges.*

Proof. (i) Clearly every circuit, and hence every element of $\mathcal{C}$, meets every cut in an even number of edges. Conversely, if $F \subseteq E$ meets every cut in an even number of edges, then in particular all the degrees of (V, F) are even. Hence $F \in \mathcal{C}$ by (1.1).

(ii) For the backward implication, let H be the multigraph obtained from G by contracting every edge in $E \setminus F$. If F meets every circuit of G in an even number of edges, then H contains no odd cycle, and hence is bipartite. Its bipartition defines a bipartition of G crossed by precisely the edges in F, so F is a cut. $\square$

Recall that an *Euler tour* of a graph is a closed walk in it that contains every edge exactly once. Euler's theorem below can be rephrased as an extension of (1.1), and hence belongs in our context. But for simplicity we state it in its usual form:

Theorem 1.4 (Euler 1736)
G admits an Euler tour if and only if G is connected and every vertex has even degree.

The following pretty theorem seems to be less well known than it deserves; see Lovász [19, Problem 5.17] for a short proof due to Pósa.

Theorem 1.5 (Gallai, unpublished)
There exists a partition of V into two sets, possibly empty, each of which induces a subgraph whose edge set lies in $\mathcal{C}$.

One of the earliest applications of the cycle space is the following planarity criterion due to MacLane. Call a set $\mathcal{F} \subseteq \mathcal{C}$ *simple* if no edge of G lies in more than two elements of $\mathcal{F}$.

Theorem 1.6 (MacLane 1937)
G is planar if and only if $\mathcal{C}$ has a simple generating subset.

Call a circuit $D = E(C)$ in G *peripheral* if C is an induced and non-separating cycle in G. (The term is due to Tutte. It comes from the fact that, if G is plane and 3-connected, these cycles are precisely its face boundaries.)

Theorem 1.7 (Tutte 1963)
If G is 3-connected, then its peripheral circuits span $\mathcal{C}$.

Together with MacLane's theorem, Theorem 1.7 implies what has become known as Tutte's planarity criterion:

Corollary 1.8 (Tutte 1963)
If G is 3-connected, then G is planar if and only if every edge lies in at most two peripheral circuits.

Another classical result is Whitney's duality theorem. It is often thought of as a planarity criterion, but can equally be viewed as a topological characterization of the graphs that admit a dual:

Theorem 1.9 (Whitney 1933)
G has a dual if and only if it is planar.

By colouring-flow duality (see [8]), the four colour theorem can be rephrased as follows:

Theorem 1.10 (4CT)
If G is planar and bridgeless, then E is a union of two elements of $\mathcal{C}$.

The last theorem in our list is also intimately linked to graph homology, even if the relationship is less obviously visible. Call an edge of G a *cross-edge* with respect to a given partition of V if it has its two vertices in different partition sets.

Theorem 1.11 (Tutte 1961; Nash-Williams 1961)
Given any $k \in \mathbb{N}$, the graph G contains k edge-disjoint spanning trees if and only if for every partition of V, into ℓ sets say, it has at least $k(\ell - 1)$ cross-edges.

2 Finite cycles in infinite graphs

If desired, the cycle space of an infinite graph $G = (V, E)$ can be defined exactly as in Section 1. Then its elements are finite subsets of E, and the three basic facts from Section 1 carry over verbatim (as long as we insert 'finite' before 'set' throughout), complete with proofs. Curiously, though, none of Theorems 1.4–1.11 remains true.

As a case in point, let us look at Theorem 1.7. Here are two counterexamples, due to Halin [15] and Bruhn [2] respectively. In the graph of Figure 1,

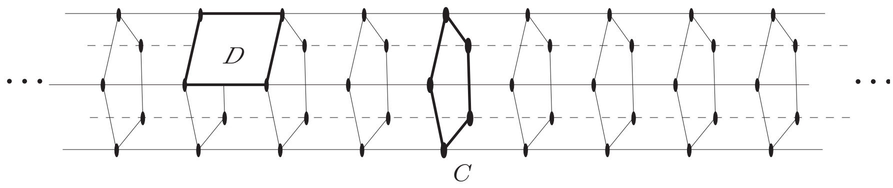

Figure 1: $E(C)$ is not a finite sum of peripheral circuits

the edge set of the separating cycle C is not a finite sum (mod 2) of peripheral circuits. However, $E(C)$ is an infinite sum of such circuits, eg. of all the peripheral circuits D 'to its left'. So it appears that, if we wish to extend Tutte's theorem to infinite graphs, we ought to allow at least certain infinite sums as well as finite sums.

Allowing infinite sums alone does not help, however, with the counterexample shown in Figure 2: the edge e there lies in no peripheral circuit at all, so the circuits of the separating cycles containing e cannot be sums, finite or infinite, of peripheral circuits.

Since the counterexample of Figure 2 is planar, we might try to mend it by appealing to the original planar version of Tutte's theorem rather than the later more general version. The planar version says that the cycle space of a plane 3-connected graph is generated by the ciruits of its face boundaries, which in a finite graph happen to be its induced non-separating cycles. In Figure 2, not all the face boundaries are cycles: for the two faces incident with the edge e, the points of G on their boundary form 2-way infinite paths,

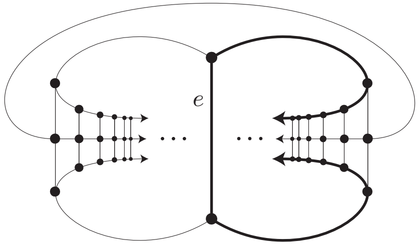

Figure 2: The edge e lies on no peripheral circuit

or *double rays*. However, the face boundaries in this graph do generate (by possibly infinite summing) all its circuits. So if we admit the edge sets of those two double rays to the generating set of $\mathcal{C}$ (as 'infinite circuits'), then Tutte's theorem holds for this graph.

For an arbitrary infinite graph G, we are thus left with the following two questions:

- Which infinite sums should we allow in the definition of $\mathcal{C}$?
- Which edge sets should we admit to the generating set of $\mathcal{C}$ as 'infinite circuits'?

Regarding the first of these questions, a necessary requirement on the sums to be allowed is that they must be well defined, ie. that *every edge of G lies in at most finitely many of the circuits that are terms in the sum.* From now on, we shall allow infinite sums in $\mathcal{E}(G)$ that satisfy this condition, and will use the term "generate" accordingly.

Our next topic is the second question above.

3 Infinite cycles

The two infinite face boundaries in Figure 2, which we found we had to admit as 'infinite cycles', are double rays: infinite connected 2-regular graphs. Since cycles are precisely the finite connected 2-regular graphs, one might be tempted to define a 'cycle' – be it finite or infinite – as just that: a connected 2-regular graph. However, common sense tells us that this can hardly be right: shouldn't cycles be round? Fortunately perhaps, there are also technical grounds on which this definition can be rejected: although it would make (1.1) true for infinite graphs, (1.2) and (1.3) would fail beyond repair. (Consider as G a single double ray.)

The two double rays that bound a face in Figure 2, however, are indeed 'round' in a sense that can be made precise: for each of them, their two ends join up in a common end[1] of G. So let us admit as generators of the cycle space of an infinite graph G those double rays in G whose two ends are subsets of a common end of G.

But if a cycle is allowed to contain one end as a point at infinity, why not two or more? More precisely, should we not define an infinite cycle as a sequence

$$\omega_1 D_1 \omega_2 D_2 \omega_3 \ldots \omega_k D_k \omega_1 ,$$

where $\omega_1, \ldots, \omega_k$ are distinct ends of G and the D_i are disjoint double rays whose two ends are subsets of ω_i and ω_{i+1}, respectively? Then the two double rays in Figure 2 would be infinite loops in this sense, and Figure 3 shows a graph whose three ends are joined by double rays to form an infinite 3-cycle.

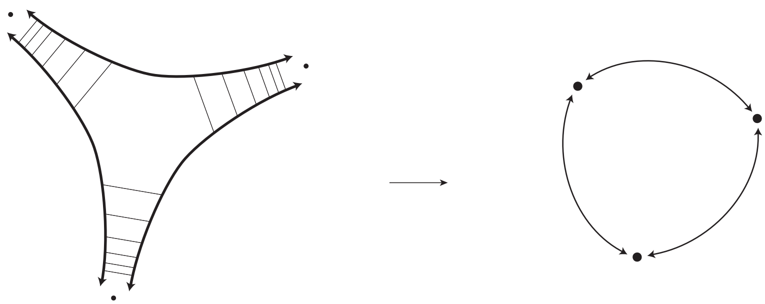

Figure 3: An infinite cycle consisting of three ends and three double rays

If we want our cycle space $\mathcal{C}$ to be closed under well-defined infinite sums, and if $\mathcal{C}$ is to contain all finite circuits, then the edge set of the 'infinite 3-cycle' in Figure 3 will certainly have to be in $\mathcal{C}$, because it is the sum of all the (finite) peripheral cycles of that graph. However, as our next example will show, even admitting such infinite cycles as above will not be enough.

The infinite cycles above can be viewed as the result of replacing the vertices of a finite cycle by ends and its edges by double rays, so that vertex-edge incidences are preserved as end-ray incidences (where a ray is *incident* with an end of G if it belongs to it). But if this makes sense once, why not do it again? For example, if we take one of the three double rays in

[1]An *end* of a graph is an equivalence class of *rays* (1-way infinite paths) in it, where two rays are *equivalent* if no finite set of vertices separates them. Intuitively, an end is thought of as a 'point at infinity' to which its rays 'converge', and we shall make this precise in Section 4. The graphs in Figures 1 and 2 have two ends; the graph in Figure 3 (left) has three.

Figure 3 and replace *its* vertices and edges with ends and double rays, we get a transfinite cyclic order of ends and double rays as shown in Figure 4.

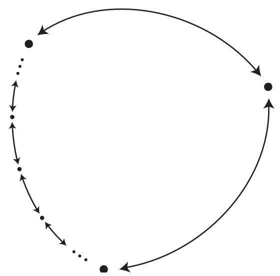

Figure 4: A transfinite cyclic pattern of double rays

The question is: can such a pattern really occur as an incidence pattern of ends and double rays in a graph? Figure 5 suggests that it can. But what exactly does the figure mean: in what sense can a 'ray of double rays' be incident with, ie. in some sense converge to, an end of an ordinary ray?

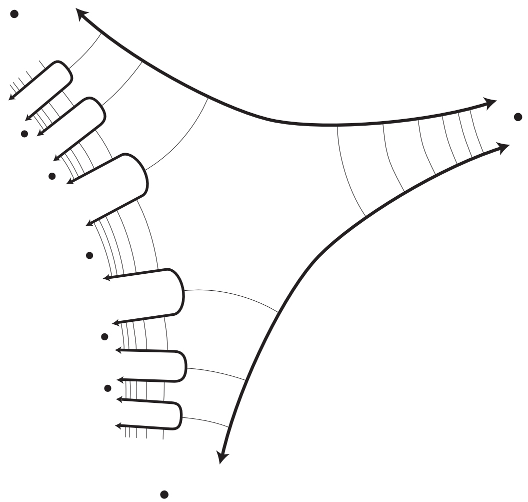

Figure 5: A graph realizing the transfinite 3-cycle

It may not be hard to give a sensible ad-hoc definition for such an incidence in this particular case. But in general, this is not an easy problem. Indeed, if we allow in principle that the replacement of vertices and edges by ends and double rays may be iterated, perhaps even transfinitely, we shall get a corresponding hierarchy of double rays, double rays of double rays etc., with 'ends' at each level, and the task would be to define end-ray incidences across the levels of this hierarchy. On the other hand, we cannot simply opt out of this iteration: just as in Figure 3, the union of the edge sets of the double rays shown in Figure 4 will be an element of the cycle space of the graph in Figure 5, because it is the sum of its peripheral circuits. And if we want (1.1) to generalize to this graph, then this element of $\mathcal{C}$ must be a disjoint union of circuits, which it will not be unless we admit it as a circuit in its own right.

So the situation looks quite hopeless. And yet: there is a very simple solution. This is an immediate consequence of changing our view of these problems from combinatorial to topological, which is the topic of the next section.

4 Circles: a topological solution

There have been two places in our discussion above where we already appealed to topological (or even geometric) intuition: we were looking for a definition of when a meta-ray 'converges' to the end of an ordinary ray (or of another meta-ray of some unknown or mixed level), and we said that clearly not all double rays could be admitted as infinite cycles 'because they may fail to be round'.

If we take this intuition seriously, the following radically different approach comes to mind:

> *Put a topology on the graph and its ends, and define a 'cycle' (finite or infinite) simply as a circle in this space, a homeomorphic image of the unit circle in the Euclidean plane.*

This topology should satisfy the following basic requirements:

(i) it should be Hausdorff and natural, also for finite graphs;

(ii) every ray should converge to its end;

(iii) every circle should be identifiable by the edges it contains.

Such a topology can indeed be found. In fact, there are several that fit the bill [7, 10]; we describe the simplest of the most natural ones.

To define this topology (which is not at all new: its origins go back to Jung [16] and Freudenthal [14]), consider a graph $G = (V, E)$ with its set $\Omega = \Omega(G)$ of ends. Let G itself carry the topology of a 1-complex.[2] To

[2]Every edge is homeomorphic to the real interval $[0, 1]$, the basic open sets around an inner point being just the open intervals on the edge. The basic open neighbourhoods of a vertex x are the unions of any half-open intervals $[x, z)$, one from every edge $[x, y]$ at x.

extend this topology to Ω, let us define for each end $\omega \in \Omega$ a basis of open neighbourhoods. As we want our topology to be Hausdorff, we have to be able to exclude any other end $\omega' \neq \omega$. By definition, $\omega' \neq \omega$ means that G has a finite set S of vertices that separates a ray in ω from a ray in ω'. Given any finite set $S \subseteq V$, let $C = C(S, \omega)$ denote the component of $G - S$ that contains some (and hence a subray of every) ray in ω, and let $\Omega = \Omega(S, \omega)$ denote the set of all ends of G that contain a ray from C. As our basis of open neighbourhoods of ω we now take all sets of the form

$$C(S, \omega) \cup \Omega(S, \omega) \cup E'(S, \omega),$$

where S ranges over the finite subsets of V and $E'(S, \omega)$ is any union of non-empty partial edges $(z, y]$, one from every S–C edge $[x, y]$ with $x \in S$ and $y \in C$.

Let $|G|$ denote the topological space of $G \cup \Omega$ endowed with the topology generated by these open sets together with those of the 1-complex G. It is not difficult to see that $|G|$ is compact if G is (connected and) locally finite.[3]

Note that $|G|$ satisfies our basic requirements (i)–(iii). Indeed, (i) and (ii) are built into its definition.[4] For (iii), it is straightforward to show that if C is a circle in $|G|$ (a subset homeomorphic to S^1), then every edge with an inner point in C lies entirely in C. So every circle 'has' a well-defined set D of edges, which we call its *circuit*. It is not difficult to show [10, Lemma 4.3] that the point set $\bigcup D$ is dense in C, so every circle is uniquely identified by its circuit (as its closure). We can therefore study the circles in terms of their circuits, as planned. (For future reference we remark that the corresponding statements hold also for *arcs* in $|G|$, the homeomorphic images of the real unit interval $[0, 1]$.)

But which subsets of $|G|$ are circles? Clearly, all finite cycles are circles, and our earlier examples of 'infinite cycles' turn out to be circles too. But, despite our sensibility requirements (i)–(iii), there can be much wilder circles than these – whose existence will make $\mathcal{C}(G)$ larger, and extending the facts from Section 1 harder, than expected. We wind up this section with an example of such a 'wild' circle in quite a harmless looking graph.

Let T be the set of finite 0–1 sequences, including the empty sequence $\emptyset$. Define a tree on T by joining every $\ell \in T$ to its two one-digit extensions, the sequences $\ell 0$ and $\ell 1$. For every $\ell \in T$, add another edge e_ℓ between

[3] For graphs with infinite degrees, $|G|$ is not compact in the topology just defined. However there are natural variants of this topology that can make $|G|$ compact; see [7].

[4] As the precise meaning of (ii) let us take the assertion that every ray in G has a subray in every open neighbourhood of its end.

the vertices $\ell 01$ and $\ell 10$, and let D_ℓ denote the double ray consisting of e_ℓ and the two rays starting at e_ℓ whose vertices have the form $\ell 1000\dots$ and $\ell 0111\dots$ (Fig. 6). Finally, let D be the double ray whose vertices are $\emptyset$ and the all-zero and the all-one sequences. Then D, all the double rays D_ℓ, and all the (continuum many) ends of this graph together form a circle C in it [10]. Every neighbourhood of every end contains infinitely many of the D_ℓ, so for every two ends at least one of the two arcs on C between them contains infinitely many double rays.

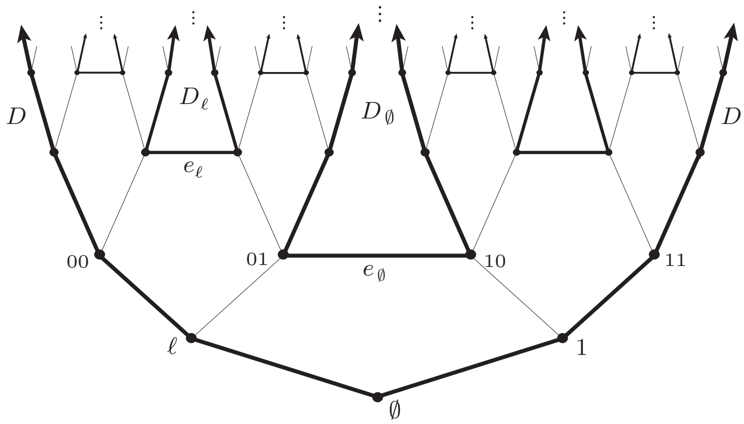

Figure 6: A circle formed by the double rays D_ℓ ($\ell \in T$) and D

5 The cycle space of an infinite graph

In this section we define the cycle space of an infinite graph formally, and state all the known theorems about it. For simplicity we consider only locally finite graphs.

Let $G = (V, E)$ be a locally finite graph, fixed throughout this section. Let $|G|$ be the associated topological space consisting of G and its ends. A subset of $|G|$ is a *circle* if it is homeomorphic to S^1, the unit circle in $\mathbb{R}^2$. A subset D of E is a *circuit* if there is a circle C in $|G|$ such that $D = \{e \in E \mid e \subseteq C\}$.

Call a family $(D_i)_{i \in I}$ of subsets of E *thin* if no edge lies in D_i for infinitely many i. Let the *sum* $\sum_{i \in I} D_i$ of this family be the set of all edges that lie in D_i for an odd number of indices i, and let the *cycle space* $\mathcal{C} = \mathcal{C}(G)$ of G be the set of all sums of (thin families of) circuits, with symmetric difference as addition. (Note that $\mathcal{C}$ is closed under such addition: just combine the two thin families into one.) Clearly, this definition of $\mathcal{C}$ coincides with that from Section 1 when G is finite.

Note that an infinite union of thin families need not be thin, even if the family of their sums is. Therefore $\mathcal{C}$ is not obviously closed under infinite (thin) summing. But in fact it is [10, Cor. 5.2], and so we may say that a set $\mathcal{F} \subseteq \mathcal{C}$ *generates* $\mathcal{C}$ if every element of $\mathcal{C}$ is the sum of some thin family of elements of $\mathcal{F}$.

We now generalize all the basic facts and theorems from Section 1 to our locally finite graph G, with its new definition of the cycle space $\mathcal{C} = \mathcal{C}(G)$.

Our first basic fact, the implication (i)→(ii) in (1.1), generalizes verbatim:

Theorem 5.1 [11] *Every element of $\mathcal{C}$ is a disjoint union of circuits.*

The proof of Theorem 5.1 is not easy, although the approach is similar to the finite case: we find a single circuit in a given set $D \in \mathcal{C}$ of edges, delete it, and iterate. Finding this circuit, which is trivially done in the finite case just by moving along consecutive edges in D until we hit a vertex previously visited, is now the hardest part: moving along the edges in D may simply take us into some end, and it is not clear how to re-emerge from this end to complete the circuit. If D is the edge set of our 'wild' circle C in Figure 6, for example, then D is the first (and only) circle that the proof of Theorem 5.1 has to find – and it is obviously not easy to do that just by 'moving along' the edges of D.

Our second basic fact, the equivalence of (i)↔(iii) in (1.1), will be treated below together with Euler's theorem.

Let us now try to generalize (1.2). At first glance, it appears that the *double ladder* depicted on the left in Figure 7 shows a counterexample: every fundamental circuit of its spanning tree T_1 contains the edge e, so every thin family of fundamental circuits is finite, and hence the infinite circuit consisting of all horizontal edges is not a sum of fundamental circuits. On the other hand, the spanning tree T_2 shown on the right works well: its fundamental circuits do span the whole cycle space, including all infinite circuits.

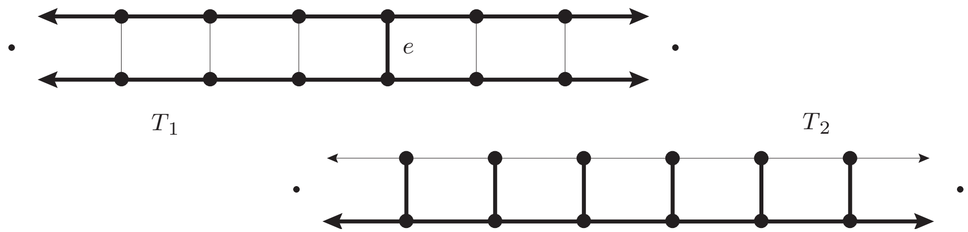

Figure 7: The fundamental circuits of T_1 span no infinite circuit

So what is wrong with T_1? The answer is disarmingly simple: T_1 itself contains infinite circuits (of G, not of T_1), and hence is no longer a 'proper tree' now that we have extended the notion of a cycle!

The following adapted notion of a spanning tree corresponds naturally to our topological definition of the cycle space, and it remedies the situation. Given a subgraph T of G containing all its vertices, call its closure $\overline{T}$ in $|G|$ (the set $V \cup \bigcup E(T) \cup \Omega(G)$) a *topological spanning tree* of $|G|$ if $\overline{T}$ is path-connected but contains no circle. As before, for every edge e not in T there is a unique circle in $\overline{T} \cup e$, whose circuit D_e we call a *fundamental circuit* of $\overline{T}$.

The following result is a special case of [12, Thm. 6.1]:

Theorem 5.2 [12] *The fundamental circuits of any fixed topological spanning tree of $|G|$ generate $\mathcal{C}$.*

Our ladder example shows that an arbitrary spanning tree of G need not define a topological spanning tree, but one can show that if G is connected then it has a spanning tree T that does [12]. For such T, the fundamental circuits of $\overline{T}$ in $|G|$ are the same as those of T in G, so they are all finite. Hence, Theorem 5.2 has the following consequence:

Cororollary *The finite circuits of G generate $\mathcal{C}$.*

Conversely, when $\overline{T}$ is a topological spanning tree of $|G|$ the graph T need not be a spanning tree of G, because it need not be connected. For example, if G is the half-grid (ie., $V = \mathbb{N} \times \mathbb{Z}$ and $E = \{xy \mid d(x,y) = 1\}$), its horizontal rays $R_y = (1,y)(2,y)\dots$ and the unique end of G together form a topological spanning tree whose fundamental circuits are all infinite. (Think of this 'tree' as a topological star with the end at its centre.)

Let us now turn to (1.3), and ask what happens when F is infinite. The example of the ladder shows that an infinite cut may well meet an infinite circuit in an odd number of edges, eg. in exactly one edge. However, if one of the two sets is finite, one can show that their intersection must be even. Moreover, this property characterizes the elements of $\mathcal{C}$ and the cuts in G:

Theorem 5.3 *Let $\emptyset \neq F \subseteq E$.*

(i) *$F \in \mathcal{C}$ if and only if F meets every finite cut in an even number of edges.*

(ii) *F is a cut if and only if F meets every finite element of $\mathcal{C}$ in an even number of edges.*

The proof of (ii) is the same as that in (1.3); part (i) is proved in [10].[5] To get a feel for the proof, let us show the simplest case of the forward implication in (i), that a circle C – even a 'wild' one like that of Figure 6 – cannot meet a finite cut D in exactly one edge. Let (X, Y) be the partition of V corresponding to D; thus, D is the set of all X–Y edges of G. Suppose C meets D in exactly the edge $e = xy$, with $x \in X$ and $y \in Y$ say. Deleting the inner points of e from C, we obtain an x–y arc A in the subspace Z of $|G|$ obtained by deleting the edges of D. Since D is finite, Z is the disjoint union of the closures $\overline{X}$ and $\overline{Y}$ of the subgraphs of G spanned by X and by Y, respectively. Hence A, which is a subset of Z meeting both $\overline{X}$ and $\overline{Y}$, is topologically disconnected, which contradicts the definition of an arc (since $[0, 1]$ is connected).

Next, let us see how Theorems 1.4–1.11 extend to locally finite graphs.[6] Theorem 1.4, along with (1.1.iii), has a particularly pretty extension. Clearly, neither of these generalizes to infinite graphs verbatim, because the even-degrees condition is too weak to guarantee the existence of anything like an 'infinite Euler tour' or membership in $\mathcal{C}$. (For example, if we identify two double rays in one vertex, we obtain a graph with all degrees even that contains nothing resembling an Euler tour, and neither does the edge set lie in $\mathcal{C}$.) But there is an obvious topological version of an Euler tour on which a generalization of Theorem 1.4 might be based: call a continuous (but not necessarily injective) map $\sigma\colon S^1 \to |G|$ a *topological Euler tour* of $|G|$ if every inner point of an edge of G is the image of exactly one point of S^1. (Thus, every edge is traversed exactly once, and in a 'straight' manner.) Using Theorem 5.1, one can indeed prove that if G is connected and its entire edge set lies in its cycle space, then $|G|$ contains such a topological Euler tour [10] – an assertion that should be a corollary of any common generalization of (1.1) and Theorem 1.4.

Hence our question becomes: can we strengthen the even-degrees condition in Theorem 1.4 so as to make it equivalent to both the assertion that $E \in \mathcal{C}(G)$ and the existence of a topological Euler tour of $|G|$, while retaining the spirit of a degree condition?

The two 4-regular graphs in Figure 8 suggest that the solution might lie in an additional kind of degree condition on ends. Note that while in G_1 four edge-disjoint rays can approach each end simultaneously, only three can do so in G_2. And if we, accordingly, assign 'degrees' of 4 and 3 to the ends

[5] See [11] for a generalization to graphs with infinite degrees.

[6] The extensions typically fail for graphs with infinite degrees; see the references for counterexamples.

in G_1 and G_2, respectively, we obtain a straightforward generalization of Theorem 1.4 to these graphs: a topological Euler tour exists if and only if all vertices and all ends have even degree.

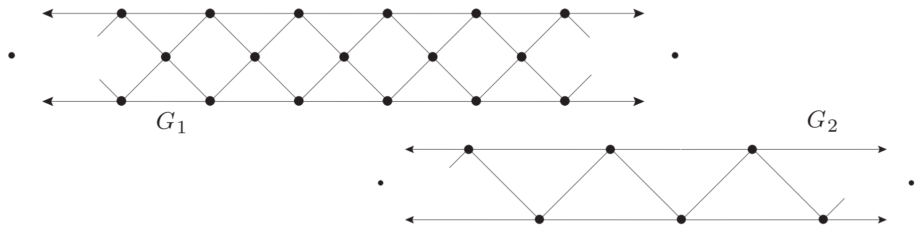

Figure 8: G_1 has a topological Euler tour; G_2 does not

Building on ideas of Laviolette [18], Bruhn and Stein [6] succeeded in generalizing this idea into a comprehensive definition for the degrees of ends that makes the simultaneous extensions of Theorem 1.4 and (1.1) possible. Given an end ω, if there exists an integer k such that in ω we can find k but not $k+1$ edge-disjoint rays, let k be the *degree* of ω. If there is no such k, then there are infinitely many edge-disjoint rays in ω (this is a non-trivial result of Halin), but it is still possible to classify such ends into 'odd' and 'even' degrees. See [6] for precise definitions.

Theorem 5.4 (Bruhn & Stein [6]; [10])
If G is connected, the following assertions are equivalent:

(i) *$E \in \mathcal{C}(G)$;*

(ii) *Every vertex and every end of G has even degree;*

(iii) *$|G|$ admits a topological Euler tour.*

Gallai's theorem extends verbatim with our new cycle space $\mathcal{C}$:

Theorem 5.5 [4] *There exists a partition of V into two sets, possibly empty, each inducing a subgraph whose edge set lies in $\mathcal{C}$.*

It is not obvious in Theorem 5.5 that infinite circuits, rather than merely infinite elements of $\mathcal{C}$ obtained as infinite sums of finite circuits, are needed to make the result true. But they are: in [4], we exhibit an example of a locally finite graph that admits only one partition as required in Theorem 5.5, and one of the two partition sets induces a double ray (whose closure is a circle).

Next, our extension of MacLane's theorem. Here, the need for infinite circuits is demonstrated by Figure 2. Solving a long-standing problem of Wagner [33, p. 128], Bruhn and Stein [5] showed that no more is needed:

Theorem 5.6 (Bruhn & Stein [5])
If G is countable (eg., connected), then G is planar if and only if $\mathcal{C}$ has a simple generating subset.

For an extension of Tutte's theorem, call a (finite or infinite) circuit in G *peripheral* if its closure C in $|G|$ is a circle that includes every edge of G whose endvertices both lie in C and for which $C \cap V$ does not separate G. (By [11, Lemma 3.4], these two conditions together are equivalent to saying that the subspace $|G| \setminus C$ is path-connected.)

Theorem 5.7 (Bruhn [2])
If G is 3-connected, then its peripheral circuits generate $\mathcal{C}$.

Corollary 5.8 [5] *If G is 3-connected, then G is planar if and only if every edge lies in at most two peripheral circuits.*

Let us turn now to cycle-cocycle duality. In an infinite graph, there is an obvious disparity between cuts, which can be infinite, and cycles, which are (usually) finite. This disparity has to be resolved before any extension of duality is attempted.

Thomassen [28, 29] proposed to call a multigraph $G^* = (V^*, E^*)$ a dual of G if $E^* = E$ and every *finite* set $F \subseteq E$ is a circuit in G if and only if it is a minimial non-empty cut in G^*. In our context, where infinite sets of edges can be circuits as well as cuts, it seems unnatural to restrict F to finite sets. Thus, let us call G^* a *finitary dual* of G if it satisfies Thomassen's definition, and a (full) *dual* of G if in addition it also satisfies the above requirement for infinite sets F.

Note that a (finitary or full) dual G^* of our locally finite graph G need not be locally finite. If it is not, then G^* need not have a dual, not even a finitary one. Thus, the property of finite graphs that G is always a dual of G^* fails for finitary duals.

However, one can show that if G^* is a full dual of G, locally finite or not, then G is also a dual of G^* – provided we take the circles (and hence the circuits) of G^* in the correct topology when G^* is not locally finite. The correct space is obtained from $|G^*|$ by identifying vertices that *dominate* an end (send an infinite fan to one of its rays), and hence are not topologically separable from it, with that end. Thus, in this space any ray dominated by a vertex converges to it. One can show that in a dual G^* no end can be dominated by more than one vertex, so the identification above does not lead to the identification of distinct vertices: it changes the topology on G^* and hence its circuits, but not the graph G^* itself. (See [3] for details.)

Thus, in the proper topological setting for graphs that are not locally finite, duality extends in all its main aspects. These include Whitney's theorem, with the following extension of Thomassen's results to full duals:

Theorem 5.9 [3]

(i) *G has a dual if and only if it is planar. If G is 3-connected, then its dual is unique.*

(ii) *If G^* is a dual of G, then G is a dual of G^*.*

(iii) *If G is 3-connected, it has a locally finite dual if and only if it is planar and all its peripheral circuits are finite.*

By compactness, a graph is 4-colourable as soon as all its finite subgraphs are. Combining this fact with Theorems 5.3 and 5.9, one can use colouring-flow duality to extend Theorem 1.10:

Corollary 5.10 [3] *If G is bridgeless and planar, then E is a union of two elements of $\mathcal{C}$.*

For finite graphs, duality can also be expressed in terms of spanning trees. Indeed, if G and G^* are a pair of duals and T is a spanning tree of G, then the edges of G^* that correspond to the edges of $G - E(T)$ form a spanning tree in G^*. Conversely, if there is a bijection * between the edge sets of G and G^* such that the spanning trees of G and G^* complement each other in this way, then G and G^* form a pair of duals.

For infinite graphs, this fails as long as arbitrary spanning trees are allowed. Indeed, a spanning tree T of G might contain an infinite circuit D of G in its edge set, in which case D^* would be a cut in G^*, and $G^* - D^*$ could not contain a spanning tree of G^*. However, with spanning trees whose closures do not contain circles (ie. whose closures are topological spanning trees in the sense defined earlier), this finite duality theorem does extend [3].

It remains to generalize the Tutte/Nash-Williams packing theorem. Oxley [26] showed that Theorem 1.11 does not extend to infinite graphs with finite k,[7] even locally finite ones, although this had been conjectured by Nash-Williams [24]. (See [1] for another counterexample.) What Tutte thought about this is not recorded. In his paper [32], he does not claim to have an infinite counterexample to the finite statement, but he does prove

[7] For $k = \aleph_0$ and arbitrary countable G, the statement is easily seen to be true. The cross-edge condition then implies infinite edge-connectedness, and the required trees can be constructed simultaneously in ω steps by Menger's theorem.

a locally finite version of a weaker statement. When read by itself, this weaker statement appears awkward and ad-hoc. The reason is that it simply expresses what a straightforward application of compactness to the finite version of the theorem happens to produce: that the existence of $k(\ell-1)$ cross-edges for every vertex partition into ℓ (say) sets is equivalent to the existence in G of k edge-disjoint *semiconnected factors*, spanning subgraphs $H \subseteq G$ such that every partition of V is crossed either by an edge of H or by infinitely many edges of G. Note that this definition depends not only on H but also on G.

In our context, however, Tutte's notion of a 'semiconnected factor' immediately makes sense: it is not hard to show that a spanning subgraph of G is semiconnected if and only if its closure in $|G|$ is topologically connected (equivalently [12], path-connected) as a subspace of $|G|$! Moreover, since the intersection of nested closed connected subsets of a compact Hausdorff space is again connected, we can use Zorn's Lemma to find in the closure of any semiconnected factor of G a topological spanning tree of $|G|$. We thus have our desired extension of Theorem 1.11:

Theorem 5.11 *$|G|$ has k edge-disjoint topological spanning trees if and only if for every finite partition of V, into ℓ sets say, G has at least $k(\ell-1)$ cross-edges.*

For graphs with infinite degrees, $K_{2,\aleph_0}$ is a counterexample to the statement of Theorem 5.11 with $k=2$. [26]

A final remark. The idea of defining the cycle space of an infinite graph via its topological circles was presented here as an unexpected topological cure to some concrete combinatorial problems posed by concrete examples. We have not explicitly addressed the question of whether it is the only possible cure. Indeed, we have seen that circles can have very complicated edge sets even in a locally finite graph, and it is legitimate to ask whether fewer circuits than these, or a smaller cycle space, might still generalize the facts and theorems of Section 1 appropriately to infinite graphs.

Surprisingly, this is not the case. To see this, recall that if we wish to extend Tutte's theorem to infinite graphs, then any 'alternative cycle space' $\mathcal{C}'$ of G with a chance of achieving this should be closed under infinite (thin) sums, at least of finite circuits. By our Corollary of Theorem 5.2, this cycle space contains all of $\mathcal{C}$. Furthermore, if we wish to generate it from a set of circuits that satisfies (1.1.ii), then this set must contain all circuits (however 'wild'), because no homeomorphic copy of S^1 can be a non-trivial union of other such copies. We thus have the following uniqueness theorem.

Theorem 5.12 (Uniqueness theorem)

(i) *If $\mathcal{C}' \subseteq \mathcal{C}$ contains all finite circuits of G as well as their sums, then $\mathcal{C}' = \mathcal{C}$.*

(ii) *If $\mathcal{D}$ is a set of circuits such that every element of $\mathcal{C}$ is a disjoint union of elements of $\mathcal{D}$, then $\mathcal{D}$ contains all circuits of G.* □

6 Further problems

The approach to infinite graph homology presented in the preceding sections appears to be new, so it may be fitting to wind up with a few open problems that suggest themselves for further research.

The most obvious source of problems, of course, will be a search for more theorems about cycles in finite graphs that should, but do not, generalize to infinite graphs, or fail to do so in an interesting and non-trivial way. We shall list several such problems below. More generally, we shall see how the new notion of end-degrees might allow us to consider extremal-type problems also for infinite graphs. We conclude with a brief discussion of how our cycle space of G relates to the standard singular homology of $|G|$, as a basis for possible generalizations of our approach to higher-dimensional complexes.

Perhaps the most prominent problem of the first kind above is to extend Tutte's theorem that 4-connected finite planar graphs have Hamilton cycles [31]. Nash-Williams [25] conjectured that an infinite 4-connected planar graph contains a spanning double ray unless it has more than two ends.[8] This restriction is clearly necessary. But maybe it is just an indication that spanning double rays are not the 'right' generalization of Hamilton cycles?

Bruhn (personal communication) suggested that we might instead ask for a *Hamilton circle* in $|G|$, a circle that contains every vertex (and hence, since it is closed, also every end).

Conjecture 6.1 *Every locally finite 4-connected planar graph admits a Hamilton circle.*

Similarly, one can ask whether Fleischner's theorem (see [8]) extends in this way:

[8] Nash-Williams's original condition is that no finite set of vertices should separate the graph into more than two infinite parts. If the graph is 3-connected and contains no subdivision of $K_{3,3}$, however, then no finite separator can leave infinitely many components. This implies that any infinite part left by a finite separator contains a ray, so the two versions are equivalent. A proof of Nash-Williams's conjecture has been given by Yu [34].

Conjecture 6.2 *The square of every 2-connected locally finite graph admits a Hamilton circle.*

Note that, since a circle is compact, a graph in which the deletion of finitely many vertices leaves infinitely many components cannot admit a Hamilton circle. The statement of Conjecture 6.2, therefore, does not hold for arbitrary countable graphs.

A problem of the opposite kind, where the naive non-topological extension of a finite result is true but too easy, is the following. Nash-Williams's *arboricity theorem* [23] says that a finite graph has an edge-partition into at most k forests if and only if every set of $\ell \geq 1$ vertices induces at most $k(\ell - 1)$ edges. (The condition is obviously necessary.) Since an infinite graph is acyclic as soon as its finite subgraphs are, this statement easily extends to infinite graphs by compactness.

A topological extension, however, might be more interesting. Call the closure $\overline{H}$ in $|G|$ of a subgraph H of G a *topological forest* if it contains no circle. (In particular, H must be a forest.) Perhaps surprisingly, the finite arboricity theorem does not in general extend with topological forests. However, the constructions of all the known counterexamples use ends of large degree. Perhaps the following is true:[9]

Conjecture 6.3 *Let G be locally finite, and let every end of G have degree smaller than $2k$. Then $|G|$ is the union of at most k topological forests if and only if no set of ℓ vertices in G induces more than $k(\ell - 1)$ edges.*

A more fundamental consequence of taking the ends of a graph into account when describing its structure, and in particular of the new notion of end degrees, might be the feasibility of an 'extremal' branch of infinite graph theory. In finite extremal graph theory, one asks what kind of 'dense' substructures can be forced by assuming a certain minimum edge density, or average or minimum degree. Without ends, even a large minimum degree does not force any 'dense' substructures in an infinite graph; consider, say, a k-regular tree for large k. Assuming large degrees for both vertices and ends, however, might:

Problem 6.4 *Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$ and every locally finite graph G, if all the vertices and ends of G have degree at least $f(k)$ then G has a k-connected subgraph?*

[9] Conjecture 6.3 has recently been proved by Maya Stein.

If H is a k-connected subgraph of G, then its closure $\overline{H}$ in $|G|$ is *topologically k-connected*: for every set X of fewer than k vertices or ends (of G), the space obtained from $\overline{H}$ by deleting X and any edges incident with vertices in X is connected. (The converse of this is not generally true: if G is the double ladder and H is obtained from G by deleting all the rungs, then $\overline{H}$ is a circle in $|G|$ and hence topologically 2-connected, but H is disconnected as a graph.)

The following topological version of Problem 6.4 is perhaps more natural in our setting, and may be easier to prove. As pointed out above, it would follow from the graph version; I do not know whether it easily implies it.

Problem 6.5 *Is there a function $f: \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$ and every locally finite graph G, if all the vertices and ends of G have degree at least $f(k)$ then G has a subgraph H whose closure in $|G|$ is topologically k-connected?*

Note that high minimum degree, or even high connectivity, does not force large complete minors in an infinite graph: one can easily construct one-ended k-connected planar graphs whose end has infinite degree, for arbitrarily large k.

Our next four problems should have positive answers at least for locally finite graphs G. The corresponding finite statements are known to be true[10] (see [1] for references), but Aharoni and Thomassen [1] showed that their naive extensions to infinite graphs (with 'cycle' replacing 'circle' and 'path' replacing 'arc') all fail.

When $X \subseteq |G|$ is an arc or a circle, we write $G - X$ and $|G| - X$ for the subgraph of G (respectively, the subspace of $|G|$) obtained by deleting the vertices in X and their incident edges, as well as any ends in X (in the case of $|G| - X$). Similarly, we write $G - E(X)$ for the subgraph of G obtained by deleting all the edges contained in X, and $|G| - E(X)$ for the subspace of $|G|$ obtained by deleting all inner points of edges contained in X. Let us call $|G|$ *topologically k-edge-connected* if deleting the inner points of fewer than k edges always leaves $|G|$ topologically connected.

Problem 6.6 *If G is $(k+3)$-connected, does $|G|$ contain a circle C such that $G - C$ is k-connected or $|G| - C$ is topologically k-connected?*

Problem 6.7 *If G is $(k+2)$-edge-connected, does $|G|$ contain a circle C such that $G - E(C)$ is k-edge-connected or $|G| - E(C)$ is topologically k-edge-connected?*

[10] Except that the finite version of Problem 6.8 is an open conjecture of Lovász.

Problem 6.8 *Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$ and $x, y \in V(G) \cup \Omega(G)$, if G is $f(k)$-connected then $|G|$ contains an arc A from x to y such that $G - A$ is k-connected or $|G| - A$ is topologically k-connected?*

Problem 6.9 *If $x, y \in V(G) \cup \Omega(G)$ and G is $(k+2)$-edge-connected, does $|G|$ contain an arc A from x to y such that $G - A$ is k-edge-connected or $|G| - A$ is topologically k-edge-connected?*

The problems we have listed so far are all designed to *apply* our new notions, to make infinite extensions of standard theorems about finite graphs possible or more interesting. However, there are also a number of natural intrinsic questions raised by our new definitions that I have been unable to answer. One embarrassingly simple problem, which bears on almost every proof in the field, is this:

Problem 6.10 *When G is locally finite, is every connected subset of $|G|$ also path-connected?*

Clearly, the answer to this question should be positive.[11] It is easy to confirm for open subsets of $|G|$, and was proved in [12] for closed subsets. For graphs with infinite degrees the statement is easily seen to be false.

Another intrinsic problem concerns our new notion of a topological spanning tree and their relationship to 'end-faithful' spanning trees [12]. The existence problem for end-faithful spanning trees has a long history and was eventually settled negatively [17, 27, 30]. However, all the known 'minimal' counterexamples are infinitely connected, and in particular have only one end. For such graphs, however, it is easy to find a topological spanning tree: start with a maximal set of disjoint rays, and join a spanning tree from each of the remaining components to one of those rays by an edge.

As we remarked earlier, locally finite connected graphs have topological spanning trees, and so do all countable connected graphs. But the general existence problem is open:

Problem 6.11 *Are there connected graphs G such that $|G|$ has no topological spanning tree?*

We remark that the closure in $|G|$ of a *normal* spanning tree of G is always a topological spanning tree in $|G|$. The connected graphs that have normal spanning trees were characterized in [13]; they include all countable connected graphs.

[11] Problem 6.10 has recently been settled negatively: Agelos Georgakopoulos constructed a locally finite graph G together with a connected set $X \subseteq |G|$ that is not path-connected.

Finally, readers with a topological background may have wondered about possible generalizations of our approach to higher dimensions. Indeed, it is not difficult to extend our definitions formally to arbitrary dimensions, based on standard singular homology; note that our circles, finite or infinite, are (images of) singular 1-cycles in $|G|$. The only essential difference is that we allow infinite sums, either explicitly or, as in our formal definition of $\mathcal{C}$, 'built into' individual elements of $\mathcal{C}$ (of which we then take finite sums, to make $\mathcal{C}$ into a group).

However, there is a way to introduce infinite sums through the back door: by using singular homology 'with cancellation'. Define a *singular 1-cycle with cancellation* as the set of edges traversed by a simple closed curve in $|G|$ (which we assume to be a local homeomorphism around inner points of edges, so that any edge is traversed a well-defined finite number of times, and each time in a well-defined direction), where edges are counted with multiplicities (especially if we work over $\mathbb{Z}$ rather than $\mathbb{F}_2$), and are counted negatively when traversed in the opposite direction. It can be shown that every thin sum of circuits in a locally finite graph (more generally: in precisely those connected graphs whose normal spanning trees are locally finite) can be expressed as a single 'singular 1-cycle with cancellation', and conversely. (In a finite graph, cancellation happens automatically when Z_1 is factored over B_1, because every path consisting of a single edge traversed once in each direction bounds a singular disc. However, we cannot factor over infinitely many disc boundaries at once, unless we admit infinite sums in B_1. Hence for infinite graphs we need to allow either infinite sums or cancellation explicitly.)

On the other hand, allowing infinite sums in one way or another is essential: recall that, without them, the finite theorems we set out to extend do not generalize to infinite graphs, even when infinite cycles are allowed.

The question thus becomes the following. Are there homology aspects of higher-dimensional non-compact manifolds that have well-known equivalents for compact manifolds, but are neither trivial extensions of these nor expressable in terms of the homology of their end-compactifications?[12] If so, might an approach as outlined here help as naturally as it does for graphs?

[12] See Freudenthal [14], or [9], for the definition of ends in arbitrary topological spaces, and conditions on those spaces that ensure adding the ends makes them compact.

References

[1] R. Aharoni and C. Thomassen, Infinite highly connected digraphs with no two arc-disjoint spanning trees, *J. Graph Theory* **13** (1989), 71–74.

[2] H. Bruhn, The cycle space of a 3-connected locally finite graph is generated by its finite and infinite peripheral circuits, *J. Combin. Theory B* (to appear).
`http://www.math.uni-hamburg.de/math/research/preprints/hbm.html`

[3] H. Bruhn and R. Diestel, Duality in infinite graphs, preprint 2004.

[4] H. Bruhn, R. Diestel and M. Stein, Cycle-cocycle partitions and faithful cycle covers for locally finite graphs, preprint 2004.
`http://www.math.uni-hamburg.de/math/research/preprints/hbm.html`

[5] H. Bruhn and M. Stein, MacLane's planarity criterion for locally finite graphs, preprint 2003.
`http://www.math.uni-hamburg.de/math/research/preprints/hbm.html`

[6] H. Bruhn and M. Stein, On end degrees and infinite circuits in locally finite graphs, in preparation.

[7] R. Diestel, End spaces and spanning trees, preprint 2004.

[8] R. Diestel, *Graph Theory* (2nd edition), Springer-Verlag 2000.
`http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/download.html`

[9] R. Diestel and D. Kühn, Graph-theoretical versus topological ends of graphs, *J. Combin. Theory B* **87** (2003), 197–206.
`http://www.math.uni-hamburg.de/math/research/preprints/hbm.html`

[10] R. Diestel and D. Kühn, On infinite cycles I, *Combinatorica* **24** (2004), 69–89.
`http://www.math.uni-hamburg.de/math/research/preprints/hbm.html`

[11] R. Diestel and D. Kühn, On infinite cycles II, *Combinatorica* **24** (2004), 91–116.
`http://www.math.uni-hamburg.de/math/research/preprints/hbm.html`

[12] R. Diestel and D. Kühn, Topological paths, cycles and spanning trees in infinite graphs, *Europ. J. Combinatorics* **25** (2004), 835–862.
`http://www.math.uni-hamburg.de/math/research/preprints/hbm.html`

[13] R. Diestel and I. Leader, Normal spanning trees, Aronszajn trees and excluded minors, *J. London Math. Soc.* (2) **63** (2001), 16–32. `http://www.math.uni-hamburg.de/math/research/preprints/hbm.html`

[14] H. Freudenthal, Über die Enden topologischer Räume und Gruppen, *Math. Zeitschr.* **33** (1931), 692–713.

[15] R. Halin, Miscellaneous problems on infinite graphs, *J. Graph Theory* **35** (2000), 128–151.

[16] H.A. Jung, Connectivity in infinite graphs, in (L. Mirsky, ed.) *Studies in Pure Mathematics*, Academic Press 1971.

[17] P. Komjáth, Martin's axiom and spanning trees of infinite graphs, *J. Combin. Theory B* **56** (1992), 141–144.

[18] F. Laviolette, Decompositions of infinite graphs I & II, preprints 2003.

[19] L. Lovász, *Combinatorial Problems and Exercises* (2nd edition), North-Holland 1993.

[20] S. MacLane, A combinatorial condition for planar graphs, *Fund. Math* **28** (1937), 22–32.

[21] C.St.J.A. Nash-Williams, Decompositions of graphs into closed and endless chains, *Proc. London Math. Soc.* (3) **10** (1960), 221–238.

[22] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961), 445–450.

[23] C.St.J.A. Nash-Williams, Decompositions of finite graphs into forests, *J. London Math. Soc.* **39** (1964), 12.

[24] C.St.J.A. Nash-Williams, Infinite graphs – a survey, *J. Combin. Theory B* **3** (1967), 286–301.

[25] C.St.J.A. Nash-Williams, Unexplored and semi-explored territories in graph theory, in (F. Harary, ed.): *New Directions in Graph Theory*, Academic Press (1973), 149–186.

[26] J.G. Oxley, On a packing problem for infinite graphs and independence spaces, *J. Combin. Theory B* **26** (1979), 123–130.

[27] P.D. Seymour and R. Thomas, An end-faithful spanning tree counterexample, *Proc. Am. Math. Soc.* **113** (1991), 1163–1171.

[28] C. Thomassen, Planarity and duality of finite and infinite graphs, *J. Combin. Theory B* **29** (1980), 244–271.

[29] C. Thomassen, Duality of infinite graphs, *J. Combin. Theory B* **33** (1982), 137–160.

[30] C. Thomassen, Infinite connected graphs with no end-preserving spanning trees, *J. Combin. Theory B* **54** (1992), 322–324.

[31] W.T. Tutte, A theorem on planar graphs, *Trans. Am. Math. Soc* **82** (1956) 99–116.

[32] W.T. Tutte, On the problem of decomposing a graph into n connected factors, *J. London Math. Soc.* **36** (1961), 221–230.

[33] K. Wagner, *Graphentheorie*, BI-Hochschultaschenbücher, Bibliographisches Institut, Mannheim 1970.

[34] X. Yu, Infinite Paths in Planar Graphs I–V, preprints 1999-2004.

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