Invent. math. 111, 465–540 (1993)

Inventiones mathematicae

The cyclotomic trace and algebraic K-theory of spaces

M. Bökstedt [1,*], W.C. Hsiang [2], and I. Madsen [1]

[1] Department of Mathematics, Aarhus University, DK-8000 Aarhus, Denmark
[2] Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

Oblatum VI-1990 & 3-VII-1992

Introduction

The cyclotomic trace is a map from algebraic K-theory of a group ring to a certain topological refinement of cyclic homology. The target is naturally mapped to topological Hochschild homology, and the cyclotomic trace lifts the topological Dennis trace. Our cyclic homology can be defined also for "group rings up to homotopy", and in this setting the cyclotomic trace produces invariants of Waldhausen's A-theory.

Our main applications go in two directions. We show on the one hand that the K-theory assembly map is rationally injective for a large class of discrete groups, including groups which have finitely generated Eilenberg-MacLane homology in each degree. This is the analogue in algebraic K-theory of Novikov's conjecture about homotopy invariance of higher signatures. It implies for Quillen's K-groups the inclusion

(0.1) $$H_i(\Gamma;\mathbb{Q}) \oplus \sum_{k\geqq 1}^{\oplus} H_{i-4k-1}(\Gamma;\mathbb{Q}) \subset K_i(\mathbb{Z}\Gamma)\otimes\mathbb{Q}\,.$$

On the other hand, the cyclotomic trace gives information about $A(*)$. We show that its p-adic completion contains $\Omega^\infty S^\infty(\Sigma BO(2))\times\Omega^\infty S^\infty$ as a direct factor, at least if p is a regular prime (in terms of number theory). This in turn gives

(0.2) $$\underrightarrow{\mathrm{holim}}\, BC^{\mathrm{Diff}}(D^n)_p^\wedge \simeq \Omega^\infty S^\infty(BO(2))_p^\wedge \times T_p$$

(after p-adic completion, p regular) where $C^{\mathrm{Diff}}(D^n)$ denotes the space of differentiable pseudo-isotopies of the n dimensional disc, and T_p is a torsion space (possibly contractible), cf. [W4].

The topological cyclic homology space $TC(F, p)$ can be defined for any "functor with smash product" in the sense of [B] and for any prime p. Such functors include group rings, $R\Gamma$ and homotopy group rings, $\Omega^\infty S^\infty(\Gamma_+)$. At the time of writing only limited information is available about $TC(R\Gamma, p)$ in the group ring case, and anyhow this is not the subject of the present paper; here we give, for any group-like topological momoid Γ, an explicit calculation of $TC(\Omega^\infty S^\infty(\Gamma_+), p)$ in terms of more familiar objects in homotopy theory.

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* Partially supported by an NSF-Grant

Fixing the prime p, let $TC(X,p)^\wedge_p$ be the infinite loop space defined by the following homotopy Cartesian diagram:

$$\begin{array}{ccc} TC(X,p)^\wedge_p & \xrightarrow{\alpha} & \Omega^\infty S^\infty(\Sigma_+(ES^1\times_{S^1}\Lambda X))^\wedge_p \\ \downarrow \beta & & \downarrow \mathrm{Trf} \\ \Omega^\infty S^\infty(\Lambda X_+)^\wedge_p & \xrightarrow{1-\Delta_p} & \Omega^\infty S^\infty(\Lambda X_+)^\wedge_p \end{array} \tag{0.3}$$

In (0.3), $\Sigma_+(Y)$ denotes the suspension of $Y_+ = Y \amalg \{+\}$, ΛX is the free loop space of X, Trf is the S^1-transfer and Δ_p is the p-fold power map, i.e. $\Delta_p(\lambda)(z) = \lambda(z^p)$ for the loop $\lambda(z)$ in X.

For a topological group-like monoid Γ (i.e. $\pi_0\Gamma$ a group) we show in Sect. 5 that

$$TC(\Omega^\infty S^\infty(\Gamma_+),p)^\wedge_p \simeq TC(B\Gamma,p)^\wedge_p\,. \tag{0.4}$$

Given a "functor with smash product", the cyclotomic trace is an infinite loop map

$$\mathrm{Trc}\colon K(F)\to TC(F,p)$$

from its algebraic K-theory. In the special case where $F(U) = U_+\wedge\Omega X$, $K(F) = A(X)$ and

$$\mathrm{Trc}\colon A(X)^\wedge_p \to TC(X,p)^\wedge_p\,.$$

This is a highly non-trivial invariant.

To analyse it in the basic case when X consists of only one point we use a modified version of Soulé's construction of the Borel regulators in $K_i(\mathbb{Z})$, [S2], to get a map

$$\varepsilon^\#\colon \Omega^\infty S^\infty(\Sigma_+\mathbb{C}P^\infty)^\wedge_p \to A(*)^\wedge_p \tag{0.5}$$

for each $\varepsilon\in\varprojlim(RC_{p^m})^\times$. Here $R=\mathbb{Z}[1/g]$, g a generator of the units modulo p^2 and the inverse limit is over the norm maps, and C_k denotes the cyclic group of order k.

The composition $\alpha\circ\mathrm{Trc}_p\circ\varepsilon^\#$ is a self map of $\Omega^\infty S^\infty(\Sigma_+\mathbb{C}P^\infty)^\wedge_p$, and since it is an infinite loop map it is determined by its induced self map of the suspension spectrum of $\Sigma_+(\mathbb{C}P^\infty)$. With the aid of [BM] we show (for a specific choice of ε) that

$$\alpha\circ\mathrm{Trc}_p\circ\varepsilon^\#\colon H_{1+2n}(\Sigma_+\mathbb{C}P^\infty;\hat{\mathbb{Z}}_p)\to H_{1+2n}(\Sigma_+\mathbb{C}P^\infty;\hat{\mathbb{Z}}_p) \tag{0.6}$$

is multiplication by $(g^{-n}-1)L_p(1+n;\omega^{-n})$ where $L_p(-;\omega^{-n})$ is the p-adic L-function and ω is the Teichmüller character. (For $p=2$, the number should be interpreted to be 2). One may factor $\varepsilon^\#$ over $\Omega^\infty S^\infty(\Sigma BO(2))$ and can use the realification map $\mathbb{C}P^\infty\to BO(2)$ in the target to deduce (0.2).

We note in passing that the reduced functor $\widetilde{\mathrm{Trc}}_p$ from $\tilde{A}(X)^\wedge_p$ to $\widetilde{TC}(X,p)$ is a homotopy equivalence for X simply connected by [BCCGHM].

There is a version of A-theory based on p-completed spheres; we denote it $A(B\Gamma;\hat{\mathbb{Z}}_p)$. It maps to $K(\hat{\mathbb{Z}}_p\Gamma)$ by linearization. For this theory one can in (0.5) use $\hat{\varepsilon}\in\varprojlim(\hat{\mathbb{Z}}_pC_{p^n})^\times$. It is possible to choose $\hat{\varepsilon}$ in such a way that the number theory disappears from (0.6): For $n\not\equiv -1(p-1)$ the composite is an isomorphism. For $n\equiv -1(\mathrm{mod}\,p-1)$ it multiplies by $(1+n)p$. This is in good agreement with [S2], and one may speculate about an explicit connection between $TC(\hat{\mathbb{Z}}_p, p)$ and the étale cohomology of Spec $\mathbb{Q}_p$, and between the étale chern character and the cyclotomic trace.

The paper is divided up into two parts. The first part, consisting of five sections, contains the construction of the topological cyclic homology and of the cyclotomic trace, and is to a large extent equivariant generalizations of results from [B]. The second part of the paper examines the cyclotomic trace invariant for $A(*)$ and derives the K-theory analogue of Novikov's conjecture.

A couple of notational comments are in order. Throughout the paper, $Q(X)$ denotes the unreduced stable homotopy space of X, i.e.

$$Q(X) = \Omega^\infty S^\infty(X_+), \quad X_+ = X \amalg \{+\}.$$

For based spaces X the reduced version is $\tilde{Q}(X) = \Omega^\infty S^\infty(X)$. We have used $X_p^\wedge$ to denote the completion at p of X in the sense of Bousfield and Kan. The paper is written in the language of infinite loop spaces (rather than the equivalent notion of connected spectra). This has at certain places some funny looking notational consequences. For example, $X \wedge \tilde{Q}(Y)$ is identified with $\tilde{Q}(X \wedge Y)$.

The cyclotomic trace is very much inspired by work of T. Goodwillie. In fact it is one way of making precise his ideas of epicyclic spaces as explained in a celebrated letter from him to F. Waldhausen. We are indebted to G. Carlsson for drawing our attention to Soulé's paper [S2]. We thank F. Waldhausen for valuable philosophical as well as practical suggestions.

The proof of the K-analogue of Novikov's conjecture is inspired by ideas of R. Cohen, J. Jones and M. Karoubi. We sincerely thank J. McClure who read the entire manuscript. His detailed comments made us change the exposition at many places, and in fact rewrite several sections completely. In particular he pointed out a serious mistake we had made in our definition of the Γ-space structure on $TC(X, p)$. Finally, J. Rognes and L. Hesselholt have made valuable comments on the present version.

Contents

1 Edgewise subdivision and cyclic spaces

Given a simplicial set $X_\bullet$ and a natural number r there is an edgewise subdivision $sd_r X_\bullet$ whose topological realization is homeomorphic to that of $X_\bullet$, cf. [Se2]. We present a variant of this construction.

Let Δ be the simplicial category with objects $[n]$ of ordered sets, $[n] = \{0, 1, \ldots, n\}$, and order preserving maps as morphisms. Consider the functor

$$sd_r: \Delta \to \Delta$$

with $sd_r[m-1] = [mr-1]$ and $sd_r(f) = f \amalg \ldots \amalg f$ (i.e. $sd_r(f)(am+b) = an + f(b)$, when $f: [m-1] \to [n-1]$ and $0 \leqq a < r$, $0 \leqq b < m$).

The r-fold edgewise subdivision of a simplicial set (or space) $X_\bullet: \Delta^{op} \to$ sets is the composition $sd_r X_\bullet = X_\bullet \circ sd_r$ with $sd_r X_n = X_{(n+1)r-1}$.

Observe that the face and degeneracy operators in $sd_r X_\bullet$ are given by

$$\bar{d}_i: sd_r X_n \to sd_r X_{n-1}$$

$$\bar{s}_i: sd_r X_n \to sd_r X_{n+1}$$

with

$$\bar{d}_i = d_i \circ d_{i+(n+1)} \circ \ldots \circ d_{i+(r-1)(n+1)}$$

$$\bar{s}_i = s_{i+(r-1)(n+2)} \circ \ldots \circ s_{i+(n+2)} \circ s_i$$

where d_i and s_i are the face and degeneracy operators for $X_\bullet$.

The standard simplex Δ^{rm-1} is the r-fold join of Δ^{m-1} with itself, and we have the diagonal embedding d_r: $\Delta^{m-1} \to \Delta^{rm-1}$, $d_r(u) = \frac{1}{r} u \oplus \ldots \oplus \frac{1}{r} u$, $\Delta^n = \{(u_0, \ldots, u_n) \mid \sum u_i = 1\}$

Lemma 1.1 *The map D_r: $|sd_r(X_\bullet)| \to |X_\bullet|$ of topological realizations induced from $1 \times d_r$: $X_{rm-1} \times \Delta^{m-1} \to X_{rm-1} \times \Delta^{rm-1}$ is a homeomorphism.*

Proof. This is easily checked when $X_\bullet$ is the simplicial 1-simplex $\Delta[1]$. It follows for the (diagonal) of any product $\Delta[1]^k_\bullet$ and then for the simplicial k-simplex $\Delta[k]_\bullet$ upon using the retraction $\Delta[k]_\bullet \subsetneq \Delta[1]^k_\bullet \to \Delta[k]_\bullet$ to check that D_r is both injective and surjective. The case of a general simplicial set is now obvious. □

The second edgewise subdivision of the standard 2-dimensional simplex and Segal's original subdivision can be pictured as

Recall A. Connes' extension Λ of the simplicial category Δ. It has the same objects, but the morphisms are extended by the 'cyclic permutation' $\tau_n: [n] \to [n]$, and one has the extra relations

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1}, \quad 1 \leqq i \leqq n$$

$$\tau_n \delta_0 = \delta_n$$

$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \quad 1 \leqq i \leqq n$$

$$\tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \,. \tag{1.2}$$

Moreover, the $(n+1)$'st power of τ_n is the identity:

$$\tau_n^{n+1} = \mathrm{id}\ . \tag{1.3}$$

If $X_\bullet$ is a cyclic object, i.e. a functor from Λ^{op}, then the r-fold edgewise subdivision has a simplicial action of the cyclic group C_r of order r. Indeed, the $(m-1)$-simplices of $sd_r X_\bullet$ is equal to X_{rm-1} and τ_{rm-1}^m generates the C_r-action.

More generally, observe that (1.2) alone implies that τ_n^{n+1} commutes with δ_i and σ_i in that

$$\begin{aligned} \tau_n^{n+1}\delta_i &= \delta_i \tau_{n-1}^n \\ \tau_n^{n+1}\sigma_i &= \sigma_i \tau_{n+1}^{n+2}\ . \end{aligned} \tag{1.4}$$

Definition 1.5 Let Λ_r $(1 \leqq r \leqq \infty)$ be the category which contains Δ and morphisms τ_n: $[n] \to [n]$ subject to the relations (1.2) and the relation $\tau_n^{r(n+1)} = \mathrm{id}$ (when $r < \infty$).

A Λ_r^{op}-object is a functor from Λ_r^{op}, so is for $r = 1$ a cyclic object in the sense of Connes.

By (1.4) every Λ_r^{op}-space has a simplicial C_r-action. Actually, the topological realization of a Λ_r^{op}-space has a continuous circle action which restricts to the simplicial C_r-action. Precisely, let $\Lambda_r[n]_\bullet$ be the Λ_r^{op}-set of morphisms

$$[m] \to \Lambda_r([m], [n])$$

and let Λ_r^n be the realization of its underlying simplicial set. The functor $[n] \to \Lambda_r^n$ is a Λ_r-space.

Lemma 1.6 *There is a homeomorphism $\Lambda_r^n \cong \mathbb{R}/r\mathbb{Z} \times \Delta^n$, and the action of τ_n on Λ_r^n is given by $\tau_n(\theta; u_0, \ldots, u_n) = (\theta - u_0; u_1, \ldots, u_n, u_0)$.*

Proof. This follows from [J, Theorem 3.4] or from [DHK]. Indeed, the usual triangulation of $\mathbb{R} \times \Delta^n$ with vertices (i, v) for $i \in \mathbb{Z}$ and $v \in \mathrm{Vertex}(\Delta^n)$, ordered lexicographically, gives a model for $\Lambda_r[n]_\bullet$. The identification of the layers $t \times \Delta^n$ and $(t+r) \times \Delta^n$ corresponds precisely to the extra relation $\tau_m^{(m+1)r} = \mathrm{id}$. Thus $|\Lambda_r[n]_\bullet| \cong \mathbb{R}/r\mathbb{Z} \times \Delta^n$. The action of τ_n is equally clear. □

It follows from 1.6 that the realization of any Λ_r^{op}-space has a canonical action of $\mathbb{R}/r\mathbb{Z}$, hence a circle action upon identifying $\theta + r\mathbb{Z}$ with $e^{2\pi i\theta/r}$. There are two possible realizations of such an $X_\bullet$, namely

$$\begin{aligned} |X_\bullet| &= \amalg \Delta^n \times X_n/\sim; \quad (f_* t, x) \sim (t, f^* x) \quad \text{for } f \in \Delta \\ |X_\bullet|_{\Lambda_r} &= \amalg \Lambda_r^n \times X_n/\approx; \quad (f_* \lambda, x) \approx (\lambda, f^* x) \quad \text{for } f \in \Lambda_r\ . \end{aligned} \tag{1.7}$$

The first one is the usual realization of the underlying simplicial set. The second has the $\mathbb{R}/r\mathbb{Z}$ action from 1.6.

Lemma 1.8 *The inclusion $\Delta^n \subset \Lambda_r^n$ induces a homeomorphism of $|X_\bullet|$ onto $|X_\bullet|_{\Lambda_r}$.*

There are functors

$$P_r\colon \Lambda_{rs} \to \Lambda_s, \quad sd_r\colon \Lambda_{rs} \to \Lambda_s\ . \tag{1.9}$$

The first one is the identity on objects and on morphisms from Δ, and is the surjection on $\langle \tau_n \rangle$ (replacing the relation $\tau_n^{rs(n+1)} = 1$ by $\tau_n^{s(n+1)} = 1$). The second functor extends the subdivision functor on Δ by $sd_r(\tau_{n-1}) = \tau_{rn-1}$.

By P_r each Λ_s^{op}-space $X_\bullet$ becomes a Λ_{rs}^{op}-space, denoted $P_r X_\bullet$ or just $X_\bullet$, and the identifications in 1.8 makes $|X_\bullet|$ both an $\mathbb{R}/rs\mathbb{Z}$-space and a $\mathbb{R}/s\mathbb{Z}$-space. The two actions μ_{rs} and μ_s can be compared.

Lemma 1.10 *There is a commutative diagram*

$$\begin{array}{ccc} \mathbb{R}/rs\mathbb{Z} \times |X_\bullet| & \xrightarrow{\mu_{rs}} & |X_\bullet| \\ & {\scriptstyle p \times \mathrm{id}} \searrow \quad \nearrow {\scriptstyle \mu_s} & \\ & \mathbb{R}/s\mathbb{Z} \times |X_\bullet| & \end{array}$$

where p is the projection induced from the identity on $\mathbb{R}$.

Proof. It is direct from 1.6 that P_r induces a commutative diagram

$$\begin{array}{ccc} \Lambda_{rs}^n & \xrightarrow{P_r} & \Lambda_s^n \\ \| & & \| \\ \mathbb{R}/rs\mathbb{Z} \times \Delta^n & \xrightarrow{p \times 1} & \mathbb{R}/s\mathbb{Z} \times \Delta^n . \end{array}$$

The rest follows from the diagram

$$\begin{array}{ccc} & |X_\bullet| & \\ {\scriptstyle \cong} \swarrow & & \searrow {\scriptstyle \cong} \\ (\coprod \Lambda_{rs}^n \times P_r X_n)/\approx & \xrightarrow{P_r \times id} & \coprod \Lambda_s^n \times X_n/\approx \\ \| & & \| \\ |P_r X_\bullet|_{\Lambda_{rs}} & \longrightarrow & |X_\bullet|_{\Lambda_s} . \end{array} \quad \square$$

The functor $sd_r\colon \Lambda_{rs} \to \Lambda_s$ associates to each Λ_s-space $X_\bullet$ a Λ_{rs}-space $sd_r X_\bullet$, and we have the homeomorphism

$$D_r\colon |sd_r X_\bullet| \to |X_\bullet|$$

of 1.1; $R/rs\mathbb{Z}$ acts on the domain and $\mathbb{R}/s\mathbb{Z}$ on the range. We have

Lemma 1.11 *The following diagram is commutative*

$$\begin{array}{ccccc} \mathbb{R}/rs\mathbb{Z} \times |sd_r X_\bullet| & \xrightarrow{\mathrm{id} \times D_r} & \mathbb{R}/rs\mathbb{Z} \times |X_\bullet| & \xrightarrow{1/r \times \mathrm{id}} & \mathbb{R}/s\mathbb{Z} \times |X_\bullet| \\ \downarrow {\scriptstyle \mu_{rs}} & & & & \downarrow {\scriptstyle \mu_s} \\ |sd_r X_\bullet| & & \xrightarrow{D_r} & & |X_\bullet| \end{array}$$

where $1/r\colon \mathbb{R}/rs\mathbb{Z} \to \mathbb{R}/s\mathbb{Z}$ *is induced from division by* r.

Proof. The argument is similar to the one in 1.10 except this time, since sd_r is not the identity on objects, we get a simplicial map

$$sd_r\colon \Lambda_{rs}[n-1] \to sd_r \Lambda_s[rn-1]$$

whose realization we must identify. We claim there is a commutative diagram of realizations:

$$\begin{array}{ccccc} \Lambda_{rs}^{n-1} & \longrightarrow & |sd_r \Lambda_s[rn-1]_\bullet| & \xrightarrow{D_r} & \Lambda_s^{rn-1} \\ \downarrow = & & & & \downarrow = \\ \mathbb{R}/rs\mathbb{Z} \times \Delta^{n-1} & & \xrightarrow{1/r \times d_r} & & \mathbb{R}/s\mathbb{Z} \times \Delta^{rn-1} \end{array}$$

where d_r is the diagonal map (cf. 1.1). This uses the description of Λ_k^m from [J] (cf. 1.6). We leave the details for the reader. □

If we identify $\mathbb{R}/m\mathbb{Z}$ with S^1 in the standard fashion $(\theta \leftrightarrow e^{2\pi i\theta/m})$ then 1.11 simply says that

$$D_r\colon |sd_r X_\bullet| \to |X_\bullet|$$

is an S^1-homeomorphism.

Let us finally remark that the subdivision functor of course can be iterated and that

$$sd_r sd_s X_\bullet = sd_{rs} X_\bullet\,.$$

Moreover, the diagram

(1.12)
$$\begin{array}{ccc} |sd_{rs}X_\bullet| & \xrightarrow{D_r} & |sd_s X_\bullet| \\ & {\scriptstyle D_{rs}}\searrow \quad \swarrow{\scriptstyle D_s} & \\ & |X_\bullet| & \end{array}$$

is commutative.

2 The cyclic bar construction

Given a topological, group-like monoid G and a two-sided G-space E we can form the cyclic bar construction $N_\bullet^{cy}(E; G)$, cf. [W1]. It is the simplicial space:

$$N_n^{cy}(E; G) = E \times G^n$$

$$d_0(e, g_1, \ldots, g_n) = (eg_1, g_2, \ldots, g_n)$$

$$d_n(e, g_1, \ldots, g_n) = (g_n e, g_1, \ldots, g_{n-1})$$

$$d_i(e, g_1, \ldots, g_n) = (e, g_1, \ldots, g_i g_{i+1}, \ldots, g_n), \quad 0 < i < n$$

$$s_i(e, g_1, \ldots, g_n) = (e, g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n)\,.$$

If $E = G$, considered as a two-sided G-space via multiplication, we write $N_\bullet^{cy}(G)$ instead of $N_\bullet^{cy}(G, G)$. Setting

$$t_n(g_0, \ldots, g_n) = (g_n, g_0, \ldots, g_{n-1})$$

it becomes a cyclic space (with t_n corresponding to τ_n).

The r-fold edgewise subdivision $sd_r N_\bullet^{cy}(G)$ is again a cyclic bar construction, namely

(2.1)
$$sd_r N_\bullet^{cy}(G) \cong N_\bullet^{cy}(t(G^r), G^r)$$

where G^r is the r-fold Cartesian product of G and $t(G^r) = G^r$ but with a twisted two-sided G^r-structure:

$$(e_1, \ldots, e_r)\cdot(g_1, \ldots, g_r) = (e_1 g_1, \ldots, e_r g_r)$$

$$(g_1, \ldots, g_r)\cdot(e_1, \ldots, e_r) = (g_r e_1, g_1 e_2, \ldots, g_{r-1} e_r)\,.$$

Let $\pi\colon N_\bullet^{cy}(G) \to N_\bullet(G)$ be the simplicial map into the usual (one-sided) bar construction which in simplicial degree n maps $(g_0, \ldots, g_n)$ to $(g_1, \ldots, g_n)$. Combining with the S^1-action on $|N_\bullet^{cy}(G)|$ we have the map

$$S^1 \times |N_\bullet^{cy}(G)| \xrightarrow{\mu} |N_\bullet^{cy}(G)| \xrightarrow{\pi} |N_\bullet(G)|$$

whose adjoint is a map

$$f\colon |N_\bullet^{cy}(G)| \to \Lambda BG \tag{2.2}$$

into the free loop space of $BG = |N_\bullet(G)|$. Observe that f is an S^1-map when we give ΛBG the S^1-action from rotating the loops. It is well-known that f is a non-equivariant homotopy equivalence, [BF, G], since $\pi_0 G$ is assumed to be a group. We want to prove a corresponding equivariant statement. Let

$$\Delta_{r,\bullet}\colon N_\bullet^{cy}(G) \to sd_r N_\bullet^{cy}(G) \cong N_\bullet^{cy}(t(G^r), G^r)$$

be the diagonal map which sends a k-simplex $(g_0, \ldots, g_n)$ into $(\vec{g}_0, \ldots, \vec{g}_n)$ with $\vec{g}_i = (g_i, \ldots, g_i) \in G^r$.

The simplicial action of the cyclic group C_r on $sd_r N_\bullet^{cy}(G)$, generated by $t_{(n+1)r-1}^{n+1}$ on the n-simplices, corresponds under the identification (2.1) to the permutation action on G^r. Thus

$$\Delta_{r,\bullet}\colon N_\bullet^{cy}(G) \to (sd_r N_\bullet^{cy}(G))^{C_r}\,; \tag{2.3}$$

this is a simplicial isomorphism, whose realization is denoted Δ_r. When G is a group there is the injection

$$i\colon N_\bullet(G) \to N_\bullet^{cy}(G)_{(1)}$$
$$i(g_1, \ldots, g_n) = ((\textstyle\prod g_i)^{-1}, g_1, \ldots, g_n)\,, \tag{2.4}$$

split by the map π used in (2.2). The realization of i corresponds to the inclusion of BG in ΛBG as the point loops.

Proposition 2.5 *For a topological group G,*

$$\begin{array}{ccccc}
|N_\bullet(G)| & \xrightarrow{\ i\ } & |N_\bullet^{cy}(G)| & \xrightarrow{\ \Delta_{rs}\ } & |sd_{rs} N_\bullet^{cy}(G)|^{C_{rs}} \\
\downarrow i & & & & \downarrow \\
|N_\bullet^{cy}(G)| & \xrightarrow{\ \Delta_s\ } & |sd_s N_\bullet^{cy}(G)| & \xleftarrow{\ C_s D_r\ } & |sd_{rs} N_\bullet^{cy}(G)|^{C_s}
\end{array}$$

is homotopy commutative by a homotopy $D_{rs,t}^s$ which is natural in G. Moreover, $D_{rs,t}^s = \Delta_s \circ D_{r,t}^1$

Proof. Suppose first that $s = 1$. Consider the homotopy

$$d_{r,t}\colon \Delta^n \to \Delta^n * \ldots * \Delta^n \quad (r \text{ factors})$$
$$d_{r,t}(u) = tu/r \oplus \ldots \oplus tu/r \oplus (tu/r + (1-t)u) \quad 0 \leqq t \leqq 1\,.$$

For $t = 1$ this is the map d_r used in 1.1. Let

$$D_{r,t}^1\colon |sd_r N_\bullet^{cy}(G)| \to |N_\bullet^{cy}(G)|$$

be the corresponding homotopy of the map D from 1.1. Since

$$d_{r,0}\colon \Delta^n \to \Delta^{(n+1)r-1}$$

is the $(n+1)(r-1)$-th iterate of the 0-th (co)face we have commutativity in

$$\begin{array}{ccc} \Delta^n \times [sd_r N^{cy}(G)]_n & \xrightarrow{1 \times d_0^{(n+1)(r-1)}} & \Delta^n \times N_n^{cy}(G) \\ \downarrow & & \downarrow \\ |sd_r N_\bullet^{cy}(G)| & \xrightarrow{D_{r,0}^1} & |N_\bullet^{cy}(G)| \,. \end{array}$$

A direct calculation shows that

$$d^{(n+1)(r-1)} \Delta_r(g_0, \ldots, g_n) = ((\textstyle\prod g_i)^{r-1} g_0, g_1, \ldots, g_n) \,.$$

But $\prod g_i = 1$ when $(g_0, \ldots, g_n) \in \mathrm{Im}(i)$. Hence $D_{r,0}^1 \circ \Delta_r \circ i = \mathrm{id}$ which proves the claim. For $s > 1$ one takes $D_{rs,t}^r = \Delta_s \circ D_{r,t}^1$. □

Proposition 2.6 *For a group-like topological monoid G and for each finite subgroup C of* S^1,

$$f\colon |N_\bullet^{cy}(G)|^C \to (\Lambda BG)^C$$

is a homotopy equivalence.

Proof. First observe that

$$\Delta_{r,\bullet}\colon P_r N_\bullet^{cy}(G) \to sd_r N_\bullet^{cy}(G)$$

is a map of Λ_r^{op}-spaces with P_r from (1.9). It follows from 1.10 and 1.11 that the composition

$$\bar{\Delta}_r\colon |N_\bullet^{cy}(G)| \xrightarrow{\Delta_r} |sd_r N_\bullet^{cy}(G)| \xrightarrow{D_r} |N_\bullet^{cy}(G)|$$

has the following equivariance property for the S^1-action:

(2.7) $$\bar{\Delta}_r(z^r \cdot x) = z \cdot \bar{\Delta}_r(x)$$

for $z \in S^1$, $x \in |N_\bullet^{cy}(G)|$. Let $g\colon |N_\bullet^{cy}(G)| \to \Lambda |N_\bullet^{cy}(G)|$ be the adjoint of the S^1-action. We have the diagram

(2.8) $$\begin{array}{ccccc} |N_\bullet^{cy}(G)|^{C_r} & \xrightarrow{g^{C_r}} & (\Lambda |N_\bullet^{cy}(G)|)^{C_r} & \xrightarrow{\Lambda\pi} & (\Lambda |N_\bullet(G)|)^{C_r} \\ \uparrow \bar{\Delta}_r & & \uparrow P_r & & \uparrow P_r \\ |N_\bullet^{cy}(G)| & \xrightarrow{g} & \Lambda |N_\bullet^{cy}(G)| & \xrightarrow{\Lambda\pi} & \Lambda |N_\bullet(G)| \end{array}$$

with P_r the power map, $P_r(\sigma)(z) = \sigma(z^r)$. We claim that the outer diagram in (2.8) is homotopy commutative. For $x \in |N_\bullet^{cy}(G)|$, $(P_r \circ g)(x)$ is the loop $\sigma(z) = z^r \cdot x$ and $g \circ \bar{\Delta}_r(x)$ is the loop $\bar{\sigma}(z) = z \cdot \bar{\Delta}_r(x) = \bar{\Delta}_r(z^r \cdot x)$. So we have left to show that the maps

$$\pi,\ \pi \circ \bar{\Delta}_r\colon |N_\bullet^{cy}(G)| \to |N_\bullet G|$$

are homotopic. The homotopy is $\pi \circ D_{r,t}^1 \circ \Delta_r$ with $D_{r,t}^1$ the homotopy from 2.5. Clearly

$$P_r\colon \Lambda BG \to (\Lambda BG)^{C_r}$$

is a homeomorphism. By (2.3) the same is true for $\bar{\Delta}_r$, so f^{C_r} is conjugate to f. Since f is a homotopy equivalence, so is f^{C_r}. □

Remark 2.9 The equivariant Whitehead theorem asserts that an equivariant map is an equivariant homotopy equivalence if and only if the induced maps on all fixed sets are homotopy equivalences, at least if the transformation group in question is compact. Thus $f\colon |N^{cy}_{\bullet}(G)| \to \Lambda BG$ is a C-homotopy equivalence for each finite $C \subset S^1$. However, f is not an S^1-equivalence, since

$$|N^{cy}_{\bullet}(G)|^{S^1} = \{g \in G \mid s_0(g) = t_1 s_0(g)\} = \{1\}\,,$$
$$(\Lambda BG)^{S^1} = BG\,.$$

We need a blown-up version of $N^{cy}_{\bullet}(G)$, e.g. the bi-simplicial set used in [B1], to get an S^1-equivalence.

The homotopies specified in Proposition 2.5 gives a well-defined map

$$I\colon |N_{\bullet}(G)| \to \underset{\overleftarrow{D}}{\operatorname{holim}}\, |sd_r N^{cy}_{\bullet}(G)|$$

with the limits running over the compositions

$$D\colon |sd_{rs} N^{cy}_{\bullet}(G)|^{C_{rs}} \to |sd_{rs} N^{cy}_{\bullet}(G)|^{C_r} \xrightarrow{D_r} |sd_r N^{cy}_{\bullet}(G)|^{C_r}\,.$$

The reader is referred to [BK, Chap. XI] for the definition and general properties of homotopy inverse limits. For the purpose of this paper it suffices to replace the limit system above by the simpler system where r runs over the powers of a fixed prime number p. In this case there is a more well-known description of homotopy inverse limits which we now recall. Given a string of spaces

$$\dots \to S_n \xrightarrow{\sigma_n} S_{n-1} \to \dots \to S_0$$

we can replace it by a string of fibrations

$$\dots \to f(S_n) \xrightarrow{f(\sigma_n)} f(S_{n-1}) \to \dots \to f(S_0)$$

by iterating the usual mapping path space construction which converts a map into a fibration, and one has

$$\underset{\longleftarrow}{\operatorname{holim}}\, S_n \simeq \underset{\longleftarrow}{\lim}\, f(S_n)\,.$$

This follows from [BK, XI, 4.1 and 5.6].

The homotopy groups of $\underset{\longleftarrow}{\operatorname{holim}}\, S_n$ can be calculated from the exact sequence

$$0 \to \underset{\longleftarrow}{\lim}^{(1)} \pi_{k+1} S_n \to \pi_k \underset{\longleftarrow}{\operatorname{holim}}\, S_n \to \underset{\longleftarrow}{\lim}\, \pi_k S_n \to 0$$

cf. [BK, XI, 7.4] or [Mi]. If we assume that each S_n is an infinite loop space and σ_n an infinite loop map, and this will always be the case for us, then one has an exact sequence

$$0 \to \underset{\longleftarrow}{\lim}^{(1)} [\Sigma X, S_n] \to [X, \underset{\longleftarrow}{\operatorname{holim}}\, S_n] \to \underset{\longleftarrow}{\lim}\, [X, S_n] \to 0$$

for every X. The $\underset{\longleftarrow}{\lim}^{(1)}$-term is in general non-zero.

We shall use at several places below that $\varprojlim^{(1)} A_n = 0$ if A_n is a string of compact abelian groups. Indeed, $\varprojlim^{(1)} A_n$ is the cokernel of the map $\prod A_n \to \prod A_n$ which sends (a_n) to $(\sigma_n(a_n) - a_{n-1})$. This has dense image and $\prod A_n$ is compact; thus it is onto.

Lemma 2.10 *Suppose (S_n, σ_n) is a string of infinite loop spaces, and that $f_n\colon X \to S_n$ are maps so that $\sigma_n \circ f_n \simeq f_{n-1}$. There is a homotopy class $[f]$, $f\colon X \to \operatorname{holim}_{\leftarrow} S_n$, inducing $[f_n]$. Its p-adic completion $[\hat{f}_p]$ is well-defined, when each $[\Sigma X, S_n]$ is finitely generated.*

Let us return to the inverse limit system at hand. It is direct from the definitions to check that there are commutative diagrams

$$
\text{(2.11)} \qquad
\begin{array}{ccc}
|sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}} & \xrightarrow{\Delta_p} & |sd_{p^{n+1}} N_\bullet^{cy}(G)|^{C_{p^{n+1}}} \\
\downarrow D & & \downarrow D \\
|sd_{p^{n-1}} N_\bullet^{cy}(G)|^{C_{p^{n-1}}} & \xrightarrow{\Delta_p} & |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}}
\end{array}
$$

where Δ_p is the homeomorphism induced by (2.3). There is an induced homeomorphism

$$\Delta_p\colon \operatorname{holim}_{\leftarrow} |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}} \to \operatorname{holim}_{\leftarrow} |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}}$$

and it is clear from (2.5) that $\Delta_p \circ I = I$. Let $\Phi_p = \Delta_p^{-1}$ be the inverse homeomorphism. We have

$$\text{(2.12)} \qquad I\colon |N_\bullet(G)| \to (\operatorname{holim}_{\leftarrow} |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}})^{\Phi_p}$$

Our preference of Φ_p over Δ_p in (2.12) will become apparent in Sect. 5 below.

In our definition of the cyclotomic trace (Sect. 5) we need to apply the above in a situation where G is a group-like monoid (a topological monoid with $\pi_0 G$ a group). The map I is not a priori defined for monoids, since it uses strict inverses. However, there is a well-known trick to get around this difficulty.

There is a functor $G \to G^\wedge$ which replaces a topological monoid by a group, and another functor $G \to G^\vee$ (the free group) together with natural transformations

$$G \leftarrow G^\vee \to G^\wedge$$

cf. [BF, p. 311] or [G, Sect. I, 1.8].

When G is group-like the induced maps

$$|N_\bullet(G)| \leftarrow |N_\bullet(G^\vee)| \to |N_\bullet(G^\wedge)|$$

$$|sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}} \leftarrow |sd_{p^n} N_\bullet^{cy}(G^\vee)|^{C_{p^n}} \to |sd_{p^n} N_\bullet^{cy}(G^\wedge)|^{C_{p^n}}$$

are all homotopy equivalences. Since homotopy inverse limits, and in particular homotopy fixed points are homotopy invariant notions, we get a well-defined homotopy class

$$\text{(2.13)} \qquad I\colon |N_\bullet(G)| \to (\operatorname{holim}_{\leftarrow} |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}})^{h\Phi_p}$$

for every group-like topological monoid. (Here $h\Phi_p$ indicates homotopy fixed set, i.e. the homotopy equalizer of the self-maps Φ_p and id.)

One may prove that (2.12) is equivalent up to homotopy to the map

$$I^{top}\colon BG \to (\underleftarrow{\text{holim}}\,(\Lambda BG)^{C_{p^n}})^{h\Delta_p^{-1}}$$

which embeds BG as the constant loops, in the homotopy limit over the inclusions of fixed sets. Basically this is a consequence of (1.11) and (2.6).

3 The equivariant topological Hochschild space

Given a ring R and a bi-module E we can form the simplicial space $N^{cy}_{\otimes}(E, R)_\bullet$, analogous to the cyclic bar-construction of Sect. 2. It is the simplicial space

$$N^{cy}_{\otimes}(E, R)_\bullet\colon [n] \to E \otimes R^{\otimes n}$$

with the evident face and degeneracy operators. If $E = R$ it has a cyclic structure.

In [B] this construction was generalized to the category of infinite loop spaces (spectra), replacing R with a "ring up to homotopy" and tensor product with smash product. We need equivariant versions.

Recall that a functor with smash product, an FSP, is a functor from pointed spaces to itself together with two natural transformations

$$\mathbb{1}_X\colon X \to F(X)$$

$$\mu_{X,Y}\colon F(X) \wedge F(Y) \to F(X \wedge Y)$$

such that

$$\text{(i)}\quad \mu_{X,Y}(\mathbb{1}_X \wedge \mathbb{1}_Y) = \mathbb{1}_{X \wedge Y}$$

$$\text{(ii)}\quad \mu_{X \wedge Y, Z}(\mu_{X,Y} \wedge \text{id}_{F(Z)}) = \mu_{X, Y \wedge Z}(\text{id}_{F(X)} \wedge \mu_{Y,Z})$$

(3.1) $$\text{(iii)}\quad F(T) \circ \mu_{X,Y} \circ \mathbb{1}_X \wedge \text{id}_{F(Y)} = \mu_{Y,X} \circ (\text{id}_{F(Y)} \wedge \mathbb{1}_X) \circ T$$

We shall always assume F is convergent in the sense that the limit system

$$\pi_r \Omega^i F(S^i X) \to \pi_r(\Omega^{i+1} F(S^{i+1} X)),$$

given by product with $\mathbb{1}_{S^1}$, stabilizes for every given r.

Example 3.2 (i) Our basic examples will be of the form $F(X) = \underline{\underline{\Gamma}}(X) = X \wedge \Gamma_+$ where Γ is a topological (group-like) monoid. (ii) Given one FSP we may construct the associated matrix functor by $M_{n,m}(F)(X) = \text{Map}([m], [n] \wedge F(X))$.

Here $[m] = \{0, \ldots, m\}$ with 0 as base-point, and Map denotes the set of base-point preserving maps. There are associative pairings

$$M_{n,m}(F)(X) \wedge Y \to M_{n,m}(F)(X \wedge Y)$$

$$M_{n,m}(F)(X) \wedge M_{m,k}(F)(Y) \to M_{n,k}(F)(X \wedge Y).$$

For $n = m$, write $M_n(F)$ or F_n instead of $M_{n,n}(F)$; it is an FSP. When $n \neq m$, $M_{n,m}(F)$ is not an FSP, but there is still a limit system, and convergence is defined as above. (iii) Given any ring R there is an FSP $\underline{\underline{R}}$ defined by $\underline{\underline{R}}(X) = RX_\bullet/R(*)$, the reduced simplicial abelian group of the singular complex X.

Let I be the category whose objects are the natural numbers, considered as ordered sets $n = (1, \ldots, n)$ and whose morphisms $I(n, m)$ are the injective (not necessarily order preserving) maps.

The standard inclusion $n \to m$ induces a map

$$\Omega^n F(S^n) \to \Omega^m F(S^m)$$

upon taking the product with $1\!\!1_{S^{m-n}}$ on the right. The symmetric group Σ_m acts on $(1, \ldots, m)$ and hence on S^m and $F(S^m)$, and on $\Omega^m F(S^m)$ by conjugation. Every morphism $f \in I(n, m)$ can be decomposed as $f = \sigma \circ i$ with $\sigma \in \Sigma_m$ and i the standard inclusion. One gets a functor on I with $f_\# = \sigma_\# \circ i_\#$,

$$f_\#: \Omega^n F(S^n) \to \Omega^m F(S^m) .$$

Indeed, $\sigma_\# \circ i_\# = i_\#$ when $\sigma \in \Sigma_{m-n}$.

The category I is not filtering, but we can still take the homotopy direct limit.

Definition 3.3. $QF = \underset{\overrightarrow{I}}{\operatorname{holim}}\, (n \mapsto \Omega^n F(S^n))$.

It is proved in [B, Theorem 1.5] that the above homotopy limit is a good one in the sense that $\Omega^n F(S^n)$ approximates QF. For the functor $\underline{\underline{\Gamma}}(X) = X \wedge \Gamma_+$,

$$Q\underline{\underline{\Gamma}} = Q(\Gamma) = \Omega^\infty S^\infty(\Gamma_+) .$$

Roughly speaking, the construction $THH_\bullet(F)$ is the $N^{cy}_\otimes$-construction for the "ring up to homotopy" QF. Precisely, define the simplicial space $THH_\bullet(F)$ to be

$$[p] \mapsto \underset{\overrightarrow{I^{p+1}}}{\operatorname{holim}}\, \operatorname{Map}(S^{i_0} \wedge \ldots \wedge S^{i_p}, F(S^{i_0}) \wedge \ldots \wedge F(S^{i_p})) . \tag{3.4}$$

The face operators are induced from functors $I^{p+1} \to I^p$ associated to concatenation of sets, and for the last one, with cyclic permutation followed by concatenation. The degeneracy operators are similar, and the cyclic structure is induced from cyclic permutation.

Often we shall shorten notation and denote the mapping space in (3.4) by $\Omega^{|\underline{i}|} F(S^{i_0}) \wedge \ldots \wedge F(S^{i_p})$ where $|\underline{i}| = \sum i_\nu$ and $\underline{i} = (i_0, \ldots, i_p)$.

The topological realization of (3.4) is the topological Hochschild space, denoted

$$THH(F) = |THH_\bullet(F)| . \tag{3.5}$$

Let R denote the regular representation of the cyclic group C_r and let $iR = R \oplus \ldots \oplus R$ (i summands). Its one point compactification S^{iR} is, as a C_r-space, equal to the r-fold smash product of the i-sphere. In general $X^{(r)}$ denotes r-fold smash product.

With these notions the subdivision $sd_r THH_\bullet(F)$ can be rewritten as the simplicial space

$$[p] \mapsto \underset{\overrightarrow{I^{p+1}}}{\operatorname{holim}}\, \operatorname{Map}(S^{i_0 R} \wedge \ldots \wedge S^{i_p R}, F(S^{i_0})^{(r)} \wedge \ldots \wedge F(S^{i_p})^{(r)}) . \tag{3.6}$$

The simplicial action of the group C_r is induced from conjugation in the mapping space, with cyclic action on the r-fold smash products.

Given a C_r-space Λ, we write

$$Q_{C_r}(\Lambda) = \underset{\overrightarrow{k}}{\operatorname{holim}}\, \operatorname{Map}(S^{kR}, S^{kR} \wedge \Lambda_+)$$

(and *not*, as is unfortunately more customary, $Q_C(\Lambda_+)$).

Proposition 3.7 *Let Γ be a group-like topological monoid and $\underline{\underline{\Gamma}}$ the functor of* 3.2(i). *Then there is a C_r-homotopy equivalence*

$$|sd_r THH_\bullet(\underline{\underline{\Gamma}})| \simeq_{C_r} Q_{C_r}(\Lambda B\Gamma)\,.$$

Here the free loop space $\Lambda B\Gamma$ has its usual C_r-action.

Proof. Consider the bi-simplicial set

$$H_{p,q} = \underset{I^{p+1}\times I^{q+1}}{\underrightarrow{\text{holim}}} \operatorname{Map}(S^{i_0 R} \wedge \ldots \wedge S^{i_p R}, S^{i_0 R} \wedge \ldots \wedge S^{i_p R} \wedge (\Gamma^r)^q_+)\,.$$

The diagonal complex $\delta H_\bullet$ is precisely $sd_r THH_\bullet(F)$. The realization of the diagonal complex is homeomorphic with the realization divided into two steps, by first realizing each column and then realizing the resulting simplicial space.

We have

$$\begin{aligned}|H_{p,\bullet}| &= \underset{I^{p+1}}{\underrightarrow{\text{holim}}} \operatorname{Map}(S^{i_0 R} \wedge \ldots \wedge S^{i_p R}, S^{i_0 R} \wedge \ldots \wedge S^{i_p R} \wedge |sd_r N^{cy}_\bullet(\Gamma)|_+)\\ &\simeq \underset{I^{p+1}}{\underrightarrow{\text{holim}}} \operatorname{Map}(S^{i_0 R} \wedge \ldots \wedge S^{i_p R}, S^{i_0 R} \wedge \ldots \wedge S^{i_p R} \wedge \Lambda B\Gamma_+)\,.\end{aligned}$$

By (2.6) this is a C_r-homotopy equivalence. In particular

$$|H_{0,\bullet}| \simeq Q_{C_r}(\Lambda B\Gamma)\,,$$

a homotopy equivalence of C_r-spaces. The face operators

$$\partial_i\colon |H_{p,\bullet}| \to |H_{p-1,\bullet}|$$

are all equivariant homotopy equivalences, and give an equivariant homotopy equivalence

$$\|H_{\bullet,\bullet}\| = |[p] \mapsto |H_{p,\bullet}|| \simeq |H_{0,\bullet}|\,.$$

Indeed by the equivariant Whitehead theorem it suffices to check that the fixed sets are homotopy equivalent. Now there is a map from the simplicial space $[p] \mapsto |H_{p,\bullet}|^C$ to the constant simplicial set $|H_{0,\bullet}|^C$ whose levelwise homotopy fibres are contractible. Hence the homotopy fibre of the map from $|[p] \mapsto |H_{p,\bullet}|^C|$ to $|H_{0,\bullet}|^C$ is contractible, cf. [G, I.1.3]. □

Morita invariance, in one formulation, gives a homotopy equivalence of the cyclic bar construction for rings

$$N^{cy}_\otimes(R, R) \simeq N^{cy}_\otimes(M_a(R), M_a(R)) \tag{3.8}$$

where $M_a(R)$ is the full matrix ring of $a \times a$ matrices. The proof of this, given in [W2] can be generalized to $THH(F)$ as explicated in [B]. We want an equivariant extension.

Proposition 3.9 *There is a C_r-equivariant homotopy equivalence between the realizations*

$$|sd_r THH_\bullet(F)| \simeq_{C_r} |sd_r THH_\bullet(M_a(F))|$$

with $M_a(F)$ defined in 3.2.

The proof of 3.9 is based upon three easy lemmas in equivariant homotopy theory. We first state and prove these lemmas and then return to the proof of 3.9.

Lemma 3.10 *Let X be a $(k+N-1)$-connected space, and let σ be the C_r-map*

$$\sigma\colon (\Omega^k X)^{(r)} \to \mathrm{Map}(S^{kR}, X^{(r)})$$

given by smash product. For each $C_s \subseteq C_r$, the map σ^{C_s} of fixed sets is $((r/s+1)N-1)$-connected.

Proof. Consider the fibration sequence

$$\mathrm{Map}_{C_s}(S^{kR}/S^{kd}, X^{(r)}) \to \mathrm{Map}_{C_s}(S^{kR}, X^{(r)}) \to \mathrm{Map}(S^{kd}, X^{(d)})$$

induced from the inclusion of the fixed set $S^{kd} = (S^{kR})^{C_s}$ into S^{kR}, $d = r/s$.

The j-th homotopy group of the fiber

$$\pi_j \mathrm{Map}_{C_s}(S^{kR}/S^{kd}, X^{(r)}) = [S^j \wedge S^{kR}/S^{kd}, X^{(r)}]^{C_s}$$

is zero if $j + k(r/t) < (r/t)(k+N)$ for all proper cyclic subgroup C_t of C_s. This follows by elementary obstruction theory. In particular, it vanishes for $j < 2(r/s)N$.

The non-equivariant map

$$\sigma\colon (\Omega^k X)^{(d)} \to \mathrm{Map}(S^{kd}, X^{(d)})$$

is $(d+1)N$-connected. It follows that σ^{C_s} is always $(r/s+1)N-1$ connected. □

Lemma 3.11 *Let $f\colon X \to Y$ be a C-map with f^{C_s} $N(s)$-connected for each subgroup C_s with $N(s) \geqq k(r/s) + N(1) - kr$. Then the induced map*

$$f_*\colon \mathrm{Map}_{C_s}(S^{kR}, X) \to \mathrm{Map}_{C_s}(S^{kR}, Y)$$

is $(N(s) - k(r/s))$-connected.

Proof. The homotopy fiber F of f is a C_r-space with $F^{C_s}(N(s)-1)$-connected. The fiber of the induced map f_* of the mapping spaces is $\mathrm{Map}_{C_s}(S^{kR}, F)$, which is $(N(s) - k(r/s) - 1)$ connected, again by elementary obstruction theory. □

Lemma 3.12 *Suppose X is $(k-1)$-connected and $f\colon X \to Y$ is $(k+N)$-connected. The C_r-map $f^{(r)}\colon X^{(r)} \to Y^{(r)}$ induces a $(k(r/s)+N)$-connected map on C_s-fixed sets.*

Proof of Proposition 3.9 We follow the outline from [W2] and define a certain bi-simplicial space, which maps both to $sd_r THH_\bullet(F)$ and to $sd_r THH_\bullet(M_a(F))$. Let us use the shorthand notation F_a instead of $M_a(F)$, and $F_{a,b} = M_{a,b}(F)$. For $\underline{i} = (i_0, \ldots, i_p)$ and $\underline{j} = (j_0, \ldots, j_q)$ define

$$H(\underline{i}, \underline{j}) = F(S^{i_0}) \wedge \ldots \wedge F(S^{i_{p-1}}) \wedge F_{1,a}(S^{i_p}) \wedge F_a(S^{j_0}) \wedge \ldots \wedge F_a(S^{j_{q-1}}) \wedge F_{a,1}(S^{j_q}) .$$

It induces a functor $\Omega^{|\underline{i}| + |\underline{j}|} H(\underline{i}, \underline{j})$ on $I^{p+1} \times I^{q+1}$ where $|\underline{i}| = \sum i_v$, $|\underline{j}| = \sum j_v$. We are interested in the homotopy limit, or rather in its r-fold subdivision. First,

$$(\underline{i}, \underline{j}) \mapsto \mathrm{Map}(S^{|\underline{i}|R} \wedge S^{|\underline{j}|R}, H(\underline{i}, \underline{j})^{(r)})$$

is a functor on $I^{p+1} \times I^{q+1}$, and we can define a bi-simplicial space with a C_r-action,

$$X_{p,q}(r) = \underset{I^{p+1} \times I^{q+1}}{\underrightarrow{\text{holim}}} \text{Map}(S^{|\underline{i}|R} \wedge S^{|\underline{j}|R}, H(\underline{i}, \underline{j})^{(r)}) .$$

The two sets of face and degeneracy operators are similar to the case $r = 1$ (d_0' uses the action $F_{a,1}(S^{j_q}) \wedge F(S^{i_0}) \to F_{a,1}(S^{j_q} \wedge S^{i_0})$; d_p' and d_q'' use a twisted multiplication similar to (2.1) etc.). Define

$$G'(j_0, \ldots, j_{q+1}) = F_{1,a}(S^{j_0}) \wedge F_a(S^{j_1}) \wedge \ldots \wedge F_a(S^{j_q}) \wedge F_{a,1}(S^{j_{q+1}})$$

$$G''(i_0, \ldots, i_{p+1}) = F_{a,1}(S^{i_0}) \wedge F(S^{i_1}) \wedge \ldots \wedge F(S^{i_p}) \wedge F_{1,a}(S^{i_{p+1}})$$

so that

$$H(\underline{i}, \underline{j}) = F(S^{i_0}) \wedge \ldots \wedge F(S^{i_{p-1}}) \wedge G'(i_p, \underline{j})$$

$$H(\underline{i}, \underline{j}) = F_a(S^{j_0}) \wedge \ldots \wedge F_a(S^{j_{q-1}}) \wedge G''(j_q, \underline{i}) . \tag{3.9.1}$$

Multiplication defines maps

$$G'(j_0, \ldots, j_{q+1}) \to F(j_0, \ldots, j_{q+1})$$

$$G''(i_0, \ldots, i_{p+1}) \to F_a(i_0, \ldots, i_{p+1}) \tag{3.9.2}$$

where we have used the notation

$$F(\underline{j}) = F(S^{j_0} \wedge \ldots \wedge S^{j_{q+1}}) = F(S^{|\underline{j}|})$$

$$F_a(\underline{i}) = F_a(S^{i_0} \wedge \ldots \wedge S^{i_{p+1}}) = F_a(S^{|\underline{i}|}) .$$

We can subdivide and get induced maps from $X_{\bullet,\bullet}(r)$ to bi-simplicial spaces which are constant in one direction, namely

$$([p], [q]) \mapsto sd_r THH_p(F) \qquad \text{(constant in } q\text{-direction)}$$

$$([p], [q]) \mapsto sd_r THH_q(F) \qquad \text{(constant in } p\text{-direction)}$$

and hence maps of realizations

$$|sd_r THH_\bullet(F_a)| \leftarrow \| X_{\bullet,\bullet}(r) \| \to |sd_r THH_\bullet(F)| . \tag{3.9.3}$$

(The double bar indicates two-fold realization: First realize in one direction, and then in the other direction).

We will argue that the maps in (3.9.3) are C_r-homotopy equivalences. This is the case non-equivariantly (or equivalently for $r = 1$) by [B], [W1].

Consider the simplicial spaces

$$[q] \mapsto \underset{I^{q+2}}{\underrightarrow{\text{holim}}}\, \Omega^{|\underline{j}|R} G'(\underline{j})^{(r)} = sd_r B_q'$$

$$[p] \mapsto \underset{I^{p+2}}{\underrightarrow{\text{holim}}}\, \Omega^{|\underline{i}|R} G''(\underline{i})^{(r)} = sd_r B_p'' \tag{3.9.4}$$

with simplicial C_r-action, analogous to (subdivisions) of the 2-sided bar-construction for rings (cf. [W1]).

The maps of (3.9.2) define maps into simplicial objects

$$[q] \mapsto \underset{I^{q+2}}{\underrightarrow{\text{holim}}}\, \Omega^{|\underline{j}|R} F(\underline{j})^{(r)}$$

$$[p] \mapsto \underset{I^{p+2}}{\underrightarrow{\text{holim}}}\, \Omega^{|\underline{i}|R} F_a(\underline{i})^{(r)}$$

which in turn are equivalent to the constant ones

$$[q] \mapsto \underset{I}{\underrightarrow{\text{holim}}}\, \Omega^{jR} F(S^j)^{(r)}$$

(3.9.5) $$[p] \mapsto \underset{I}{\underrightarrow{\text{holim}}}\, \Omega^{iR} F_a(S^i)^{(r)} .$$

The key point in the proof is to show the above maps, induce C_r-homotopy equivalences

$$|sd_r B'_\bullet| \to \underset{I}{\underrightarrow{\text{holim}}}\, \Omega^{jR} F(S^j)^{(r)}$$

(3.9.6) $$|sd_r B''_\bullet| \to \underset{I}{\underrightarrow{\text{holim}}}\, \Omega^{iR} F_a(S^i)^{(r)}$$

or equivalently, homotopy equivalences of every fixed point set. As mentioned above, this is the case for $r = 1$, and will be proved in general by rewriting the spaces in question, using the Lemmas 3.10, 3.11 and 3.12.

In the rest of the proof we assume for notational convenience that

$$F(S^i) \to \Omega F(S^{i+1})$$

is $(2i - 1)$-connected.

Let us write G for either one of the four objects G', G'' or F or F_a of (3.9.2). Suppose $\underline{i} = (i_0, \ldots, i_{p+1})$ satisfies $i_v > N$ for all v, and let $f\colon \underline{i} \to \underline{j}$ be any morphism in I^{p+2}. Then we have:

Sublemma 3.9.7 *The induced C_r-equivariant map*

$$f\colon \Omega^{|\underline{i}|R} G(\underline{i})^{(r)} \to \Omega^{|\underline{j}|R} G(\underline{j})^{(r)}$$

is equivariantly $(N - 1)$-connected in the sense that each fixed set map f^{C_s} is $(N - 1)$-connected.

Proof. We may assume f is a product of standard inclusions, and let $\underline{j} = \underline{i} + \underline{k}$. Then $G(\underline{i}) \to \Omega^{|\underline{k}|} G(\underline{j})$ is $(|\underline{i}| + N)$ – connected, and its r-fold smash power is $(r/s|i| + N)$-connected on C_s-fixed sets by (3.12).

Since $(\Omega^{|\underline{k}|} G(\underline{j}))^{(r)} \to \Omega^{|\underline{k}|R} G(\underline{j})^{(r)}$ is $((r/s + 1)|\underline{i}| - 1)$-connected on C_s-fixed sets by (3.10), the composition is $(r/s|\underline{i}| + N - 1)$-connected on C_s-fixed sets. Apply (3.11) to finish.

Next, consider the subcategory $\tilde{I}^{p+2} \subset I^{p+2}$ of sequences $(i_0, i_1, \ldots, i_{p+1})$ with $i_0 = m + i'_0$. We claim to have a C_r-homotopy equivalence

(3.9.8) $$\underset{\tilde{I}^{p+2}}{\underrightarrow{\text{holim}}}\, \Omega^{(i)R} G(\underline{i})^{(r)} \xrightarrow{\simeq} \underset{I^{p+2}}{\underrightarrow{\text{holim}}}\, \Omega^{|i|R} G(\underline{i})^{(r)} .$$

This is contained in [B, Lemma 1.4] when $r = 1$. The proof of the C_r-equivariant statement is completely similar, based on (3.9.7).

The assignment $(i_0, \ldots, i_{p+1}) \mapsto (m + i_0, i_1, \ldots, i_{p+1})$ is a bijection of categories, $m + (\cdot)\colon I^{p+2} \to \tilde{I}^{p+2}$, and in view of (3.9.8) we obtain a C_r-equivariant homotopy equivalence

$$\text{(3.9.10)} \qquad \underset{I^{p+2}}{\underrightarrow{\operatorname{holim}}}\, \Omega^{|\underline{i}|R} G(\underline{i})^{(r)} \to \underset{I^{p+2}}{\underrightarrow{\operatorname{holim}}}\, \Omega^{mR} \Omega^{|\underline{i}|R} G(m + \underline{i})^{(r)} .$$

In the target, Ω^{mR} can be moved outside the limit. Thus we have obtained a degreewise, equivariant delooping of the simplicial spaces in question, namely

$$\text{(3.9.11)} \qquad [p] \mapsto \underrightarrow{\operatorname{holim}}\, \Omega^{|\underline{i}|R} G(m + i)^{(r)}$$

with the obvious face and degeneracy operators (corresponding to the two-sided bar-construction). Its topological realization is also an equivariant delooping, cf. [May 1, Sect. 12].

The deloopings (3.9.11) apply to all four functors in (3.9.2). We can now study the first map in (3.9.6). There is a diagram of simplicial spaces, which in simplicial degree q has the form

$$\begin{array}{ccc}
\underset{I^{q+2}}{\underrightarrow{\operatorname{holim}}}\, (\Omega^{|\underline{i}|} G'(m + \underline{i}))^{(r)} & \longrightarrow & \underset{I^{q+2}}{\underrightarrow{\operatorname{holim}}}\, \Omega^{|\underline{i}|R} G'(m + \underline{i})^{(r)} \\
\Big\downarrow \phi_q^{(r)} & & \Big\downarrow \psi_q \\
\underset{I^{p+2}}{\underrightarrow{\operatorname{holim}}}\, (\Omega^{|\underline{i}|} F(m + |\underline{i}|))^{(r)} & \longrightarrow & \underset{I^{q+2}}{\underrightarrow{\operatorname{holim}}}\, \Omega^{|\underline{i}|R} F(m + \underline{i})^{(r)} .
\end{array}$$

The vertical maps are induced from (3.9.2) and the horizontal ones from smash product. The topological realization of the left hand $\phi_\bullet^{(r)}$ is an r-fold smash product of a homotopy equivalence, by [B, Lemma 2.5], so is a C_r-equivariant homotopy equivalence. The horizontal maps induce $(r/s + 1)m$-connected maps on C_s-fixed sets by 3.10. We conclude that the realization $|\psi_\bullet|^{C_s}$ is $((r/s + 1)m - 1)$-connected, and can finally apply Lemma 3.11 to see that $(\Omega^{mR} |\psi_\bullet|)^{C_s}$ is $(m - 1)$-connected for each subgroup $C_s \subseteq C_r$.

In the above m was arbitrary, so letting $m \to \infty$ we see that the first map in (3.9.6) becomes a C_r-equivariant homotopy equivalence. It follows that the right-hand map in (3.9.3) is a C_r-homotopy equivalence. This ends the proof of Proposition 3.9. □

We finally have to compare the various subdivisions under the subdivision maps D_s, cf. 1.1 and 1.12. We state the necessary result below, and leave the proof to the reader.

Proposition 3.13 *With the notation of* (3.9.3) *there exists a C_r-equivariant homeomorphism*

$$\bar{D}_s\colon \|X_{\bullet,\bullet}(sr)\| \to \|X_{\bullet,\bullet}(r)\|$$

such that the diagram

$$\begin{array}{ccccc} |sd_{sr}THH_\bullet(M_a(F))| & \leftarrow & \|X_{\bullet,\bullet}(sr)\| & \rightarrow & |sd_{sr}THH_\bullet(F)| \\ \downarrow D_s & & \downarrow \bar{D}_s & & \downarrow D_s \\ |sd_r THH_\bullet(M_a(F))| & \leftarrow & \|X_{\bullet,\bullet}(r)\| & \rightarrow & |sd_r THH_\bullet(F)| \end{array}$$

is homotopy commutative in the category of C_r-spaces.

In particular we obtain a homotopy equivalence

$$(3.14) \qquad \underset{\overleftarrow{D}}{\operatorname{holim}} |sd_{p^n} THH_\bullet(M_a(F))|^{C_{p^n}} \rightarrow \underset{\overleftarrow{D}}{\operatorname{holim}} |sd_{p^n} THH_\bullet(F)|^{C_{p^n}}$$

cf. Sect. 2 and [BK, XI.5.6].

4 The topological Hochschild spectrum

The topological Hochschild space $THH(F)$, discussed in Sect. 3, turns out to be the the zero'th part of an Ω-spectrum $tHH(F)$. This is true even equivariantly, with respect to the group action of any finite cyclic group induced from the cyclic structure. We use the theory of equivariant Γ-spaces to construct the deloopings. In particular, we obtain deloopings of the fixed sets $THH(F)^{C_r}$; and this is what we are really after.

Let Γ_G^{op} be the category of finite based G-sets. We use the model where the objects are pairs $([n], \rho)$, with $[n] = \{0, \ldots, n\}$ and ρ is an action of G on $[n]$ which keeps 0 fixed. Following Segal (unpublished) and Shimakawa [Sh], a *special* G-equivariant Γ-space (or Γ_G-space) is a functor T_G from Γ_G^{op} to G-spaces with the property that $T_G([0])$ is G-contractible, and such that for each object $A = ([n], \rho)$, the natural map,

$$(4.1) \qquad P_A\colon T_G(A) \rightarrow \operatorname{Map}_0(A, T_G([1]))\,,$$

is a G-homotopy equivalence. In (4.1) $P_A(t)(a) = P_a(t)$ and $P_a\colon [1] \rightarrow A$ is the based G-map with $P_a(1) = a$.

For $G = C_r$, the cyclic group of order r we will show that the r-fold subdivision $|sd_r THH_\bullet(F)|$ is an equivariant Γ-space.

The basis of the construction is a certain map

$$\theta\colon E\Sigma_k \times THH(F_k) \rightarrow THH(F_k)$$

where $F_k = M_k(F)$ is the $(k \times k)$-matrix FSP associated with F.

Since the actual construction is rather technical, we first outline the main steps. The sum operation is produced by wedge $F_a \times F_b \rightarrow F_{a+b}$ corresponding to direct sum of matrices. It is not strictly commutative, but as usual it is commutative up to a permutation. The fact that this permutation is "irrelevant" is expressed by the existence of this θ, satisfying certain relations made precise in (4.4). We have to show that we can construct an equivariant Γ-space from these data.

We first rewrite Segals construction of categories with sum diagrams in a form which is uglier than his, but convenient for writing explicit formulas. This gives us an equivariant version of the Eilenberg-Maclane spectrum $H\mathbb{Z}$. The components of this Γ-space are contractible, but carry free actions of various symmetric groups. The point of this is that these components of $E(\underline{k})$ can act as operations on spaces like $THH(F_k)$, using the map θ.

A Γ-space is a functor from the opposite of the category of finite, based sets. The functor which is going to give THH an equivariant Γ-space structure is first defined on objects. We take its values to be disjoint unions of products. One of the factors is a component of the simple Γ-space we just defined, and the other is of the form $THH(\prod F_{k_i})$. The main problem is to define the value of the functor on morphisms in such a way that we obtain a functor. For instance, given an equivariant, pointed map ϕ: $\underline{k} \to \underline{l}$, we have to produce maps

$$E(\underline{k})_{(a_1,\ldots,a_k)} \times THH(F_{a_1} \times F_{a_2} \times \ldots \times F_{a_k}) \to E(\underline{l})_{(b_1,\ldots,b_l)} \times THH(F_{b_1} \times \ldots \times F_{b_l}) .$$

After taking the union over all components of $E(\underline{k})$, this is to be a map over the already given map $E(\underline{k}) \to E(\underline{l})$. In a certain sense, this says that E acts on THH. The functoriality means, among other things, that if ϕ is invariant under some permutation of its source, this map is also invariant under the same permutation. The wedge sum defines maps

$$THH(F_{a_1} \times F_{a_2} \times \ldots \times F_{a_k}) \to THH(F_{b_1} \times \ldots \times F_{b_l})$$

which do not have this property. Composing with θ we obtain a space of such maps, that is a map

$$F(\phi, \underline{l}) \times THH(F_{a_1} \times F_{a_2} \times \ldots \times F_{a_k}) \to THH(F_{b_1} \times \ldots \times F_{b_l})$$

for some suitable space F. The permutations of the source of ϕ acts on F, so that at least this map is invariant with respect to the diagonal action. The main problem left is book keeping. We do this by specifying maps $E(\underline{k}) \to F(\phi)$, and declare that the action of E on THH is via these maps.

Now we have to make this outline precise. We work simplicially (with cyclic sets). For each group Σ, $E_\bullet\Sigma$ is the cyclic set with

$$E_p\Sigma = \Sigma^{p+1}$$

$$\partial_i(\sigma_0, \ldots, \sigma_p) = (\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_p)$$

$$s_i(\sigma_0, \ldots, \sigma_p) = (\sigma_0, \ldots, \sigma_i, \sigma_i, \ldots, \sigma_p)$$

$$t(\sigma_0, \ldots, \sigma_p) = (\sigma_p, \sigma_0, \ldots, \sigma_{p-1}) . \tag{4.2}$$

It is contractible and so is each of its fixed set $(E_\bullet\Sigma)^{C_r}$. Indeed,

$$|(E_\bullet\Sigma)^{C_r}| \underset{\simeq}{\overset{D_r}{\to}} |(sd_r E_\bullet\Sigma)^{C_r}| = |E_\bullet\Sigma| . \tag{4.3}$$

Hence $|E_\bullet\Sigma|$ with its induced C_r-action is a model for the equivariant $E_{C_r}\Sigma$, and its quotient (by the diagonal C_r-action) is model for the C_r-equivariant classifying space $B_{C_r}(\Sigma)$.

The symmetric group Σ_k acts on $F_k = M_k(F)$ by conjugation, hence on $THH_\bullet(F_k)$, and there is a simplicial map

$$\theta_\bullet\colon E_\bullet\Sigma_k \times THH_\bullet(F_k) \to THH_\bullet(F_k)$$

with the following properties:

$$\theta_\bullet(g_1 e g_2, t) = g_2^{-1}\theta_\bullet(e, g_1^{-1} t g_1) g_2$$

(4.4) $$\theta_\bullet(e_1, \theta_\bullet(e_2, t)) = \theta_\bullet(e_1 e_2, t) .$$

Here and below the product of two simplicial spaces means (without further indication in notation) the diagonal simplicial space. The multiplication in $(E_p\Sigma_k) = \Sigma_k^{p+1}$ is component wise and the right and left action of Σ_k on Σ_k^{p+1} is multiplication on each factor Σ_k. The map $\theta_\bullet$ is defined as follows: Let $(\sigma_0, \ldots, \sigma_p) \in E_p\Sigma_k$ and let

$$f\colon S^{i_0} \wedge \ldots \wedge S^{i_p} \to F_k(S^{i_0}) \wedge \ldots \wedge F_k(S^{i_p})$$

represent an element of $THH_p(F_k)$. Then

(4.5) $$\theta_p(\sigma_0, \ldots, \sigma_p, f)(u) = (\sigma_p^{-1} f_0(u)\sigma_0, \sigma_0^{-1} f_1(u)\sigma_1, \ldots, \sigma_{p-1}^{-1} f_p(u)\sigma_p) .$$

Let $\mathbb{N}_0$ denote the non-negative integers and write $P_0[n]$ for the subsets of $[n]$ which contain 0, the basepoint. The set of functions

$$\underline{k}\colon P_0[n] \to \mathbb{N}_0$$

which are additive,

$$\underline{k}(S \cup T) = \underline{k}(S) + \underline{k}(T) \text{ if } S \cap T = \{0\} ,$$

will be denoted $\mathrm{Hom}(P_0[n], \mathbb{N}_0)$. A G-action on $[n]$ implies a G-action on $\mathrm{Hom}(P_0[n], \mathbb{N}_0)$, and a based G-map $\phi\colon [m] \to [n]$ induces a G-map

$$\phi_\sharp\colon \mathrm{Hom}(P_0[m], \mathbb{N}_0) \to \mathrm{Hom}(P_0[n], \mathbb{N}_0)$$

by the rule

$$\phi_\sharp(\underline{k})(S) = \underline{k}(\phi^\sharp(S))$$

where $\phi^\sharp(S) = \phi^{-1}(S - \{0\}) \cup \{0\}$.

We shall use $\theta_\bullet$ to exhibit an equivariant Γ-structure on $THH(F)$ for each finite cyclic group C. But first let us recall the general method for constructing Γ-spaces. To each $\underline{k} \in \mathrm{Hom}(P_0[n], \mathbb{N}_0)$ one associates a space $X(\underline{k})$, and to each morphism ϕ: $[m] \to [n]$ a map $\phi_*\colon X(\underline{k}) \to X(\phi_\sharp(\underline{k}))$ such that $(\phi\psi)_* = \phi_*\psi_*$. The n'th space in the associated Γ-structure is then

$$X_n = \coprod \{X(\underline{k}) \mid \underline{k} \in \mathrm{Hom}(P_0[n], \mathbb{N}_0)\} .$$

In the equivariant situation one has pairs $A = ([n], \rho)$ where ρ is a C-action on $[n]$ inducing a C-action $\underline{k} \mapsto \underline{k}^g$ on $\mathrm{Hom}(P_0[n], \mathbb{N}_0)$, and one further needs maps from $X(\underline{k})$ to $X(\underline{k}^g)$ to define a C-action on

$$X_A = \coprod \{X(\underline{k}) \mid \underline{k} \in \mathrm{Hom}(P_0[n], \mathbb{N}_0)\} .$$

Let us first consider (non-equivariantly) a Γ-space which only involves permutation groups and whose underlying infinite loop space is homotopy equivalent to the integers. For two based sets of equal cardinality, let $\Sigma(S_1, S_2)$ denote the set of based bijections. It generates a contractible cyclic set (cf. (4.2)) with

$$E_p(S_1, S_2) = \Sigma(S_1, S_2)^{p+1} .$$

Given based ordered sets S and T we write $S \coprod T$ for the based concatenation of S and T; it starts with the elements of S and then lists the elements of $T' = T - \{0\}$. We are going to identify totally ordered sets of equal cardinality. In particular $[k] \coprod [l] = [k+l]$ and the bijection $[k] \coprod [l] \to [l] \coprod [k]$ induces the permutation of $[k+l]$ which fixes 0 and has $\sigma(i) = l + i$ for $0 < i \leqq k$ and $\sigma(i) = i - k$ for $k < i \leqq k + l$.

Consider an additive function $\underline{k}: P_0[m] \to \mathbb{N}_0$ as above, and let us write $k_i = \underline{k}\{0, i\}$. A subset $S \in P_0[m]$ has an ordering induced from the standard ordering of $[m] = \{0, 1, \ldots, m\}$, so we have an induced identification

$$\coprod_{i \in S} [k_i] = [\underline{k}(S)] .$$

We define the subspace

$$E_\bullet([m], \underline{k}) \subseteq \prod_{S \in P_0[m]} E_\bullet\left(\coprod_{i \in S} [k_i], [\underline{k}(S)]\right) \tag{4.6}$$

by the condition that $x = (x_S)$ belongs to $E_\bullet([m]; \underline{k})$ if

$$x_S = 1 \text{ when } \operatorname{card}(S_k) = 1,\ S_k = \{i \in S \mid k_i \neq 0\} . \tag{$*$}$$

We make $E_\bullet([\bullet], \underline{k})$ a functor on Γ^{op} as follows. Let ϕ: $[m] \to [n]$ be a morphism in Γ^{op} with $\phi_\sharp(\underline{k}) = \underline{l}$. Then $l_j = \Sigma\{k_i \mid i \in \phi^{-1}(j)\}$ for $j > 0$. For $x \in E_\bullet([m], \underline{k})$ and $T \in P_0[n]$ we have elements

$$\begin{gathered} x_{\phi^\sharp(T)} \in E_\bullet\left(\coprod_{i \in \phi^\sharp(T)} [k_i], \underline{l}(T)\right) \\ x_{\phi^\sharp\{0, j\}} \in E_\bullet\left(\coprod_{i \in \phi^\sharp\{0, j\}} [k_i], [l_j]\right), \quad j > 0 \end{gathered} \tag{4.7}$$

where the orderings of $\phi^\sharp(T)$ and $\phi^\sharp\{0, j\}$ both are induced from $[m]$. Let

$$\mu\colon E_\bullet([k_1], [k_2]) \times E_\bullet([l_1], [l_2]) \to E_\bullet([k_1] \textstyle\coprod [l_1], [k_2] \coprod [l_2])$$

be the obvious sum of permutations. We form

$$\mu((x^{-1}_{\phi^\sharp\{0, j\}})_{j \in T}) \in E_\bullet\left(\coprod_{j \in T} [l_j], \coprod_{j \in T} \coprod_{i \in \phi^\sharp\{0, j\}} [k_i]\right) .$$

We can identify the indexing set of pairs

$$(j, i) \in T \times \phi^\sharp(T), \quad \phi(i) = j$$

with $\phi^\sharp(T)$ via projection onto the second factor. This gives rise to a bijection

$$\sigma_{\phi, T}\colon \coprod_{j \in T} \coprod_{i \in \phi^\sharp\{0, j\}} [k_i] \to \coprod_{i \in \phi^\sharp(T)} [k_i]$$

or in other words, a permutation of $[\underline{l}(T)]$. We define

$$E_\bullet(\phi)(x)_T = x_{\phi^\sharp(T)} \circ \sigma_{\phi, T} \circ \mu((x^{-1}_{\phi^\sharp\{0, j\}})_{j \in T}) \tag{4.8}$$

and note that $E_{\bullet}(\phi)(x)$ satisfies (*) above when x does, so that

$$E_{\bullet}(\phi)\colon E_{\bullet}([m], \underline{k}) \to E_{\bullet}([n], \phi_{\#}(\underline{k})) .$$

We must still check that $E(\psi) \circ E(\phi) = E(\psi \circ \phi)$ when $\psi\colon [n] \to [r]$ is a further morphism in Γ^{op}. We leave the verification to the reader, but notice that it uses the commutative diagram (where $U \in P_0[r]$):

$$\begin{array}{ccc}
\coprod\limits_{\nu \in U} \coprod\limits_{\mu \in \psi^{\#}\{0,\nu\}} \coprod\limits_{i \in \phi^{\#}\{0,\mu\}} [k_i] & \xrightarrow{\coprod \sigma_{\phi,\psi^{\#}\{0,\nu\}}} & \coprod\limits_{\nu \in U} \coprod\limits_{i \in \phi^{\#}\psi^{\#}\{0,\nu\}} [k_i] \\
\downarrow \sigma_{\psi, U} & & \downarrow \sigma_{\psi\phi, U} \\
\coprod\limits_{\mu \in \psi^{\#}(U)} \coprod\limits_{i \in \phi^{\#}\{0,\mu\}} [k_i] & \xrightarrow{\sigma_{\phi,\psi^{\#}(U)}} & \coprod\limits_{i \in \phi^{\#}\psi^{\#}(U)} [k_i] .
\end{array}$$

We can now define a Γ-space by letting $X(\underline{k})$ be the geometric realization of $E_{\bullet}([m], \underline{k})$, i.e.

$$E_n = \coprod \{ |E_{\bullet}([n]; \underline{k})| : \underline{k} \in \mathrm{Hom}(P_0[n], \mathbb{N}_0)\} .$$

This is precisely the classifying space of G. Segal's category of sum diagrams, [Se]; its associated spectrum is just the Eilenberg-Maclane spectrum $H\mathbb{Z}$. Let C be a finite cyclic group. Then there is a C-equivariant version of the above construction upon making use of the fact that $E_{\bullet}([m], \underline{k})$ is a cyclic set so that its subdivision $sd_{|C|} E_{\bullet}([m], \underline{k})$ is a simplicial C-set. More precisely, suppose $A = ([m], \rho)$ with $\rho\colon C \to Aut_0([m])$ and let $x = (x_S)$ with

$$x_S \in sd_{|C|} E_{\bullet}\left(\coprod_{i \in S} [k_i], [\underline{k}(S)] \right), \quad S \in P_0[m] .$$

For $g \in C$ and $\underline{k}\colon P_0[m] \to \mathbb{N}_0$, let $\underline{k}^g\colon P_0[m] \to \mathbb{N}_0$ be $\underline{k}^g(S) = \underline{k}(g^{-1}S)$, so that $k_i^g = k_{g^{-1}i}$. Then

$$x_{g^{-1}S} \in sd_{|C|} E_{\bullet}\left(\coprod_{i \in S} [k_i^g], \underline{k}^g(S) \right)$$

and we have

$$g_*\colon sd_{|C|} E_{\bullet}([m], \underline{k}) \to sd_{|C|} E_{\bullet}([m], \underline{k}^g)$$

by setting $g_*(x) = y$ with $y_S = g x_{g^{-1}S}$. We can then form the Γ_C-space,

$$E_C(A) = \coprod \{ |sd_{|C|} E_{\bullet}([m], \underline{k})| \colon A = ([m], \rho), \underline{k} \in \mathrm{Hom}(A, \mathbb{N}_0)\} .$$

Its corresponding spectrum is the Eilenberg-MacLane spectrum $H\mathbb{Z}$ with trivial C-action. In order to get more complicated Γ (and Γ_C-spaces) we involve $THH_{\bullet}$ and the map $\theta_{\bullet}$ from (4.5). Let $\underline{k}\colon P_0[m] \to \mathbb{N}_0$ be an additive function, and F a functor with smash product. We define

$$\Sigma_{\underline{k}} = \prod_{i=1}^{m} \Sigma_{k_i}, \; F_{\underline{k}} = \prod_{i=1}^{m} F_{k_i} .$$

The simplicial map $\theta_\bullet$ extends to a simplicial map

$$\theta_\bullet\colon E_\bullet \Sigma_{\underline{k}} \times THH_\bullet(F_{\underline{k}}) \to THH_\bullet(F_{\underline{k}})$$

upon using the old $\theta_\bullet$ on each of the coordinates, and we still have the relations (4.4) satisfied.

Given $\phi\colon [m] \to [n]$ there is a cyclic map

$$THH(\phi)\colon THH_\bullet(F_{\underline{k}}) \to THH_\bullet(F_{\phi^*(\underline{k})}) \,. \tag{4.10}$$

Indeed, if $\underline{l} = \phi_\sharp(\underline{k})$ then $l_j = \sum_{i \in \phi^{-1}(j)} k_i$.

The ordering of $\phi^{-1}(j)$ as a subset of $[m]$ and the wedge sum

$$\mu\colon F_a \times F_b \to F_{a+b}$$

defines a map from $\prod_{j=1}^n \prod_{i \in \phi^\sharp\{0,j\}} F_{k_i}$ to $F_{\underline{l}}$. Finally we have a projection from $F_{\underline{k}}$ onto $\prod_{j=1}^n \prod_{i \in \phi^\sharp\{0,j\}} F_{k_i}$, and we can apply the functor THH to get $THH(\phi)$. Similarly, using the sum (concatenation) map from $\Sigma_a \times \Sigma_b$ to Σ_{a+b} there is a cyclic map

$$\mu(\phi)\colon E_\bullet \Sigma_{\underline{k}} \to E_\bullet \Sigma_{\underline{l}} \,. \tag{4.11}$$

We must examine functoriality. In the special case where $\phi\colon [m] \to [n]$ and $\psi\colon [n] \to [r]$ are *order preserving*, then it is easily seen that

$$\begin{gathered} THH(\psi) \circ THH(\phi) = THH(\psi \circ \phi) \\ \mu(\psi) \circ \mu(\phi) = \mu(\psi \circ \phi) \,. \end{gathered} \tag{4.12}$$

Moreover, for order preserving ϕ, $\theta_\bullet$ is functorial in the sense that we have a commutative diagram

$$\begin{array}{ccc} E_\bullet \Sigma_{\underline{k}} \times THH_\bullet(F_{\underline{k}}) & \xrightarrow{\theta_\bullet(\underline{k})} & THH_\bullet(F_{\underline{k}}) \\ \downarrow \mu(\phi) \times THH(\phi) & & \downarrow THH(\phi) \\ E_\bullet \Sigma_{\underline{l}} \times THH_\bullet(F_{\underline{l}}) & \xrightarrow{\theta_\bullet(\underline{l})} & THH_\bullet(F_{\underline{l}}) \,. \end{array} \tag{4.13}$$

In order to handle more general set maps we introduce the cyclic set

$$F_\bullet(\phi, \underline{k}) \subseteq \prod_{j=1}^n E_\bullet \Bigg(\coprod_{i \in \phi^\sharp\{0,j\}} [k_i], [l_j] \Bigg), \quad \underline{l} = \phi_\sharp(\underline{k}) \,,$$

defined by the condition

$$f_j = 0 \text{ if card } \{ i \mid \phi(i) = j,\ k_i \neq 0 \} = 1 \,. \tag{**}$$

Here $\phi\colon [m] \to [n]$ is arbitrary. There is a natural map of cyclic sets

$$\lambda(\phi, \underline{k})\colon E_\bullet([m], \underline{k}) \to F_\bullet(\phi, \underline{k})$$

defined by projection onto the relevant components.

As for functoriality, if ϕ and ψ are composable there is a product map induced by composition

$$\mu_F\colon F_\bullet(\psi, \phi_\sharp(\underline{k})) \times F_\bullet(\phi, \underline{k}) \to F_\bullet(\psi\phi, \underline{k}) \,.$$

To be precise, if $\underline{l} = \phi_\sharp(\underline{k})$, $\underline{p} = \psi_\sharp(\underline{l})$ then the component h_v is defined to make the following diagram commutative

$$\begin{array}{ccc} \coprod_{i \in (\psi\phi)^\sharp\{0,v\}} [k_i] & \xrightarrow{h_v} & [p_v] \\ \uparrow \sigma_{\phi, \psi^\sharp\{0,v\}} & & \uparrow g_v \\ \coprod_{j \in \psi^\sharp\{0,v\}} \coprod_{i \in \phi^\sharp\{0,j\}} [k_i] & \xrightarrow{\amalg f_j} & \coprod_{j \in \psi^\sharp\{0,v\}} [l_j] \end{array}$$

It is direct from the definitions involved that the following diagram is commutative

$$(4.14) \qquad \begin{array}{ccc} E_\bullet([m], k) & \xrightarrow{(\lambda, E(\phi))} & E_\bullet([n], l) \times F_\bullet(\phi, \underline{k}) \\ \downarrow (E(\psi\phi), \lambda) & & \downarrow (E(\psi), \lambda, \mathrm{id}) \\ E_\bullet([r], p) \times F_\bullet(\psi\phi, \underline{k}) & \xrightarrow{(id, \mu_F)} & E_\bullet([r], p) \times F_\bullet(\psi, \underline{l}) \times F_\bullet(\phi, \underline{k}). \end{array}$$

We have $\coprod_{i \in \phi^\sharp\{0,j\}} [k_i] = [l_j]$ so that

$$E_\bullet\left(\coprod_{i \in \phi^\sharp\{0,j\}} [k_i], [l_j] \right) = E_\bullet([l_j], [l_j]) = E_\bullet \Sigma_{l_j}$$

and there is a corresponding cyclic map

$$\phi_1 \colon F_\bullet(\phi, \underline{k}) \to E_\bullet \Sigma_{\underline{l}}$$

which is equivariant with respect to the right action over

$$\mu_1(\phi) \colon \Sigma_{\underline{k}} \to \Sigma_{\underline{l}}$$

(compare (4.11): $\mu(\phi) = E_\bullet(\mu_1(\phi))$. We can now define the cyclic map

$$(4.15) \qquad \pi(\phi) \colon F_\bullet(\phi, \underline{k}) \times THH_\bullet(F_{\underline{k}}) \xrightarrow{(\phi_1, THH(\phi))} E_\bullet \Sigma_{\underline{l}} \times THH_\bullet(F_{\underline{l}}) \xrightarrow{\theta(\underline{l})} THH_\bullet(F_{\underline{l}})$$

where $\theta_\bullet(\underline{l}) = (\theta_\bullet(l_1), \ldots, \theta_\bullet(l_n))$, and $\theta_\bullet(l_j)$ is the map from (4.5). It follows from (4.4) that $\pi(\phi)$ has the following equivariance property

$$(4.16) \qquad \pi(\phi)(g_1 f g_2, t) = g_2^{-1} \pi(\phi)(x, g_1^{-1} t g_1) g_2$$

for $g_1 \in \Sigma_{\underline{l}}$, $g_2 \in \Sigma_{\underline{k}}$ and where the action of g_2 on the right hand side of (4.15) is via $\mu_1(\phi) \colon \Sigma_{\underline{k}} \to \Sigma_{\underline{l}}$.

We finally let

$$X(\phi) \colon E_\bullet([m], \underline{k}) \times THH_\bullet(F_{\underline{k}}) \to E_\bullet([n], \underline{l}) \times THH_\bullet(F_{\underline{l}})$$

be the cyclic map defined by

$$X(\phi)(e, t) = (E(\phi)(e), \pi(\phi)(\lambda(\phi, \underline{k})(e), t))$$

Lemma 4.17 *For based maps $\phi \colon [m] \to [n]$, $\psi \colon [n] \to [r]$, $X(\psi\phi) = X(\psi) \circ X(\phi)$.*

Proof. We have already showed that $E(\psi \circ \phi) = E(\psi) \circ E(\phi)$ so we have left to examine the functoriality of $\pi(\phi)$.

Let $\underline{l} = \phi_*(\underline{k})$, $p = \psi_\#(\underline{l})$. Assuming first that ϕ and ψ are order preserving. We can then combine (4.13), applied to ψ, with the diagram

$$\begin{array}{ccc} E_\bullet\Sigma_{\underline{p}} \times E_\bullet\Sigma_{\underline{p}} \times THH_\bullet(F_{\underline{p}}) & \xrightarrow{\mu \times id} & E_\bullet\Sigma_{\underline{p}} \times THH_\bullet(F_{\underline{p}}) \\ \downarrow (\mathrm{id}, \theta(\underline{p})) & & \downarrow \theta(\underline{p}) \\ E_\bullet\Sigma_{\underline{p}} \times THH_\bullet(F_{\underline{p}}) & \xrightarrow{\theta(\underline{p})} & THH_\bullet(F_{\underline{p}}) \end{array}$$

which is commutative by the second formula in (4.4), to show that

$$(4.18) \quad \begin{array}{ccc} F_\bullet(\psi, \underline{l}) \times F_\bullet(\phi, \underline{k}) \times THH_\bullet(F_{\underline{k}}) & \xrightarrow{(\mu_F, \mathrm{id})} & F_\bullet(\psi\phi, \underline{k}) \times THH_\bullet(F_{\underline{k}}) \\ \downarrow (\mathrm{id}, \pi(\phi)) & & \downarrow \pi(\psi\phi) \\ F_\bullet(\psi, \underline{l}) \times THH_\bullet(F_{\underline{l}}) & \xrightarrow{\pi(\psi)} & THH_\bullet(F_{\underline{p}}) \end{array}$$

is commutative.

We claim that (4.18) commutes for all based set maps. Since a set map is the composition of an order preserving map and a permutation (of the non-zero elements) there is really only two cases to consider, namely the cases where either ϕ or ψ is a based permutation. Then $F(\phi, \underline{k})$ or $F(\psi, \underline{l})$ is a one-point space, and the commutativity of (4.18) follows from the equivariance property (4.16). Indeed, if ψ is a permutation then the two maps around in (4.18) are

$$(e, t) \to \pi(\phi)(e\psi, t), \quad (e, t) \to \psi^{-1}\pi(\phi)(e, t)\psi$$

and if ϕ is a permutation the two compositions are

$$(e, t) \to \pi(\phi e, t), \quad (e, t) \to \pi(e, \phi^{-1} t \phi)$$

In either case the two compositions are identical by (4.16). □

We can define the wanted equivariant Γ-space structure on $THH(F)$. Let C be a cyclic group of order r, $A = ([m], \rho)$ with $\rho\colon C \to \mathrm{Aut}_0([m])$, and $\phi\colon ([m], \rho) \to ([n], \rho)$ a morphism in Γ_C^{op}. We define

$$T_C F(A) = \coprod \{|sd_r(E_\bullet([m], \underline{k}) \times THH_\bullet(F_{\underline{k}}))| \colon \underline{k} \in Hom(P_0[m], \mathbb{N}_0)\}\,,$$

$$(4.19) \qquad T_C F(\phi) = |sd_C X(\phi)|\,.$$

Here the C-action on $T_C F(A)$ is the conjugation action when we view (4.19) as the space of mappings with domain $\mathrm{Hom}(P_0[n], \mathbb{N}_0)$ (with its C-action induced from ρ) and the range as the C-space induced from the simplicial C-action on the subdivision. It is clear from (4.17) that we have defined a functor

$$T_C(F)\colon \Gamma_C^{op} \to \{C - \text{spaces}\}\,.$$

We have left to prove that $T_C F$ is a special Γ_C-space, i.e. that it satisfies (4.1).

Proposition 4.20 *For any two functors with smash product, the product of projections define a C-equivariant homotopy equivalence*

$$|\phi_\bullet| \colon |sd_r THH_\bullet(F' \times F'')| \to |sd_r THH_\bullet(F')| \times |sd_r THH_\bullet(F'')|\,.$$

Proof. A map between equivariant CW-complexes is an equivariant homotopy equivalence if it induces ordinary homotopy equivalences on all fixed sets, [A, Sect. 2]. It thus suffices to show that

$$|\phi_\bullet|: |sd_r THH_\bullet(F' \times F'')^{C_r}| \to |sd_r THH_\bullet(F')^{C_r}| \times |sd_r THH_\bullet(F'')^{C_r}|$$

is a homotopy equivalence.

The domain and range for $|\phi_\bullet|$ are H-spaces, hence simple, so we can check on integral homology whether $|\phi_\bullet|$ is a homotopy equivalence or not. Moreover, for each simplicial space the skeleton filtration of $|X_\bullet|$ defines a spectral sequence with abutment $H_*(|X_\bullet|)$ and with $E^1_{p,q} = H_q(X_p)$.

Given any FSP, define a simplicial group $H^r_q(F)_\bullet$ by

$$[p] \mapsto H_q(sd_r THH_p(F)^{C_r}) .$$

We show the projections define a homotopy equivalence of simplicial groups

$$H^r_q(\phi)_\bullet : H^r_q(F' \times F'')_\bullet \to H^r_q(F')_\bullet \times H^r_q(F'')_\bullet$$

This in turn will imply that the $|\phi_\bullet|$ induces an isomorphism on the E^1-term of the spectral sequences, and hence is an integral homology isomorphism.

There are (not unit preserving) inclusions ψ_i: $F^{(i)} \to F' \times F''$ of FSP's, giving simplicial maps $H^r_q(\psi_i)_\bullet$ for $i = 1, 2$. We consider the sum

$$\psi_\bullet = H^r_q(\psi_1)_\bullet + H^r_q(\psi_2)_\bullet .$$

One composition is the identity, $H^r_q(\phi)_\bullet \circ \psi_\bullet = \mathrm{id}$. We shall construct a simplicial homotopy between the identity and the other composition.

$$K: \Delta[1]_\bullet \times H_q(sd_r THH_\bullet(F' \times F'')^{C_r}) \to H_q(sd_r THH_\bullet(\mathrm{F}' \times \mathrm{F}'')^{C_r}) .$$

A p-simplex of $\Delta[1]_\bullet$ is a weakly increasing map σ: $[p] \to [1]$, so is determined by the number k for which $\sigma(k-1) = 0$ and $\sigma(k) = 1$. Let $F = F' \times F''$ and consider

$$f: S^{|i|R} \to F(S^{i_0})^{(r)} \wedge \ldots \wedge F(S^{i_p})^{(r)} ,$$

representing an element of $sd_r THH_p(F)$.

Suppose first $r = 1$. Write $\pi_{\sigma,1}$ and $\pi_{\sigma,2}$ for the following compositions where $v = (')$ or $('')$

$$F(S^{i_0}) \wedge \ldots \wedge F(S^{i_p}) \to F^{(v)}(S^{i_0}) \wedge \ldots \wedge F^{(v)}(S^{i_{k-1}}) \wedge F(S^{i_k}) \wedge \ldots \wedge F(S^{i_p})$$
$$\to F(S^{i_0}) \wedge \ldots \wedge F(S^{i_p}) .$$

Here the first map is projection onto $F^{(v)}(S^{i_t})$, $0 \leqq t \leqq k-1$, and the second is induced from the inclusion of $F^{(v)}$ into $F = F' \times F''$.

Assigning $(\pi_{\sigma,1} \circ f, \pi_{\sigma,2} \circ f)$ to f induces two maps

$$\Delta[1]_p \times H_q(THH_p(F)) \to H_q(THH_p(F))$$

which we can add to get K_p. In the special case where $\sigma(i) = 0$ for all i, we let $K_p = \mathrm{id}$. If $\sigma(i) = 1$ for all i then $K_p = \psi_p \circ H^r_q(\phi_p)$. One can easily check that $K_\bullet$ is a simplicial map, thus defines the required simplicial homotopy.

Finally, if $r > 1$, we use $\pi^{(r)}_{\sigma,1}$ and $\pi^{(r)}_{\sigma,2}$ to get equivariant maps, and define $K^{(r)}_p$ using the sum of these two maps to obtain the required homotopy. □

Remark 4.21 The above map K_p can be defined directly on the space level,

$$\Delta[1]_p \times sd_r THH_p(F)^{C_r} \to sd_r THH_p(F)^{C_r}.$$

However, this will not be a simplicial map; one will only have the simplicial identities satisfied up to homotopy. This is the reason that we apply the integral homology functor.

Corollary 4.22 *The functor $T_{C_r}F$ defined in* (4.19) *is a special Γ_G-space.*

In [Sh], Shimakawa constructs to each special Γ_G-space T_G a G-equivariant spectrum $\mathbb{B}T_G$. It is an "almost" $G - \Omega$ spectrum in the sense that the structure maps

$$S^W \wedge \mathbb{B}_V T_G \to \mathbb{B}_{W \oplus V} TG$$

adjoin to become G-homotopy equivalences for each pair of $\mathbb{R}G$-modules with $V^G \neq 0$. In particular, $V \mapsto \Omega\mathbb{B}_{V \oplus \mathbb{R}} T_G$ is a $G - \Omega$ spectrum. Moreover, the natural map (adjoined to the inclusion of $T_G([1])$ in the 1-skeleton)

$$T_G([1]) \to \Omega\mathbb{B}_{\mathbb{R}} T_G$$

is an equivariant group completion.

Recall the terminology that a G-infinite loop space is the zero'th space in a $G - \Omega$ spectrum, and that a G-infinite loop map is the zero'th level of map between $G - \Omega$ spectra.

The above applies to the Γ_C-space $T_C F$ defined in (4.19).

Proposition 4.23 *For each finite cyclic group C and each functor with smash product $THH(F)$ is a C-infinite loop space in such a way that the product $H(\mathbb{Z}) \times THH(F)$ is the C-infinite loop space associated with the Γ_C-space of* (4.19).

Proof. Since the cyclic space $|E\Sigma_\bullet(\underline{k})|$ is equivariantly contractible and since THH is a Morita-invariant by (3.9),

$$|sd_r(E\Sigma_\bullet(\underline{k}) \times THH_\bullet(F_k))| \simeq_{C_r} |sd_r THH_\bullet(F)|.$$

The resulting map

$$T_C F([1]) \to \mathbb{Z} \times THH(F),$$

which maps the k'th term into $\{k\} \times THH(F)$, is an equivariant group completion. We have left to see that $\mathbb{Z}$, with its standard infinite loop space structure arising from the C-trivial Eilenberg-Maclane spectrum, splits off. To this end we can project $T_C F$ to the Γ_C-space E_C given by

$$[n] \to \coprod |sd_r E\Sigma_\bullet(\underline{k})|, \quad \underline{k} \in \operatorname{Hom}(P_0[n], \mathbb{N}_0)$$

and observe that the projection is split (as Γ_C-spaces) by the inclusion of the base point in $THH(F_{\underline{k}})$ (FSP's take value in pointed spaces).

Since $|sd_r E\Sigma_\bullet(\underline{k})|$ is C-contractible, the equivariant Γ-space E_C is homotopy equivalent to

$$([n], \rho) \to \operatorname{Hom}(P_0[n], \mathbb{N}_0).$$

The spectrum of E_C is the Eilenberg-Maclane spectrum $H(\mathbb{Z})$, with trivial action of C. □

In (4.19) one may replace $THH_\bullet(F_k)$ with the (diagonal of the) bi-simplicial space $X_{\bullet,\bullet}(r)$ from the proof of Proposition 3.9, and one gets

Corollary 4.24 *Morita-invariance is a C-infinite loop map*

$$|THH_\bullet(F)| \simeq_C |THH_\bullet(M_k(F))| .$$

For the FSP of Example 3.2(i), $\underline{\underline{\Gamma}}$, the C-equivariant homotopy type of $THH(\underline{\underline{\Gamma}})$ was determined in (3.7). Since $Q_C(\Lambda)$ is the universal C-equivariant infinite loop space generated by Λ we have a C-infinite loop map

$$Q_C(\Lambda B\Gamma) \to THH(\underline{\underline{\Gamma}})$$

which is seen to be a C-homotopy equivalence of C-infinite loop spaces. Thus we also have for each k,

$$THH(M_k(\underline{\underline{\Gamma}})) \simeq_C Q_C(\Lambda B\Gamma) .$$

The maps constructed above for the cyclic groups of order p^n fit together for varying n, essentially by 3.14, to give the following conclusion which is what we will use in the paragraphs below.

Proposition 4.25 *There is a stable homotopy equivalence*

$$\underset{\substack{\longleftarrow \\ n}}{\operatorname{holim}} |sd_{p^n} THH_\bullet(M_k(\underline{\underline{\Gamma}}))|^{C_{p^n}} \simeq \underset{\substack{\longleftarrow \\ n}}{\operatorname{holim}}\, Q_{C_{p^n}}(\Lambda B\Gamma)^{C_{p^n}}$$

with the limit varying over the integers.

There is an analogue of (4.25) where we vary over all cyclic groups C_r and take limits over r ordered by division.

5 The cyclotomic trace

For a ring R, consider the following string of maps

$$|N_\bullet(\mathrm{GL}_n(R))| \xrightarrow{i} |N_\bullet^{cy}(\mathrm{GL}_n(R))| \xrightarrow{s} |N^{cy}_{\otimes,\bullet}(M_n(R))| \simeq |N^{cy}_{\otimes,\bullet}(R)|$$

with the notations of Sect. 2 and Sect. 3. The first map is from (2.4), the second embeds $\mathrm{GL}_n(R)^k \subset M_n(R)^{\otimes k}$, and the third is Morita-invariance. After suitable stabilization one gets a map

$$\mathrm{Tr}\colon \mathrm{BGL}(R)^+ \to |N^{cy}_{\otimes,\bullet}(R)|$$

which on homotopy groups induces the trace map, due to K. Dennis, from algebraic K-theory of R to Hochschild homology of R, cf. [W2].

It follows from (2.12) that Dennis' trace map lifts to a map

$$\mathrm{BGL}(R)^+ \to \underset{\longleftarrow}{\operatorname{holim}} |sd_{p^n} N^{cy}_{\otimes,\bullet}(R)^{C_{p^n}}|^{\Phi_p} . \tag{5.1}$$

We generalize in this section the above to the 'rings up to homotopy' associated with FSP's. The range in this situation becomes the (fixed set) of the topological

Hochschild spectrum. The domain $K(F)$ is essentially Waldhausen's generalization of algebraic K-theory.

More precisely, let F be any FSP. Its ring up to homotopy was defined in (3.3), and its homotopy units, denoted $(QF)^\times$ or $\mathrm{GL}_1(F)$, is the limit of maps $f \in \Omega^n F(S^n)$ for which there exists $g \in \Omega^m F(S^m)$ with

$$S^n \wedge S^m \xrightarrow{f \wedge g} F(S^n) \wedge F(S^m) \xrightarrow{\mu} F(S^n \wedge S^m)$$

homotopic to $\mathbb{1}_{S^{n+m}}$, i.e. the union of the invertible components in QF. More generally, let

(5.2) $$\mathrm{GL}_k(F) = (QM_k(F))^\times \ .$$

Observe for the FSP $\underline{\underline{\Gamma}}$ of Example 3.2 associated with a monoid Γ that

$$\mathrm{GL}_1(\underline{\underline{\Gamma}}) = \underrightarrow{\mathrm{holim}}\, H(S^n \wedge \Gamma_+)(\simeq \underrightarrow{\lim}\, H(S^n \wedge \Gamma_+)) \ ,$$

the limit of the monoid of homotopy equivalences of $S^n \wedge \Gamma_+$.

In general, $\mathrm{GL}_k(F)$ is an associative monoid with classifying space $\mathrm{BGL}_k(F)$. The wedge multiplication of 3.2 (ii)

(5.3) $$\alpha\colon \mathrm{GL}_k(F) \times \mathrm{GL}_l(F) \to \mathrm{GL}_{k+l}(F)$$

induces a topological monoid $\coprod_k \mathrm{BGL}_k(F)$. Its group-completion defines the algebraic K-theory of F:

Definition 5.4 [B] The algebraic K-theory space of F is the group-completion

$$K(F) \times \mathbb{Z} = \Omega B\left(\coprod_k \mathrm{BGL}_k(F)\right) .$$

Alternatively, as in the case of algebraic K-theory of rings,

$$K(F) \times \mathbb{Z} \simeq \mathrm{BGL}_\infty(F)^+ \times \mathbb{Z} \ ,$$

Quillen's plus construction on $\mathrm{BGL}_\infty(F) = \underrightarrow{\mathrm{holim}}\, \mathrm{BGL}_k(F)$.

The infinite loop space structure on $K(F)$ can be specified via a Γ-structure similar to the one defined on $THH(F)$ in the previous section. Indeed, Σ_k acts on $\mathrm{GL}_k(F)$ by conjugation and there is a simplicial map

(5.5) $$\theta_\bullet\colon E_\bullet\Sigma_k \times N_\bullet(\mathrm{GL}_k(F)) \to N_\bullet(\mathrm{GL}_k(F))$$

given by

$$\theta_\bullet(\sigma_0, \ldots, \sigma_p; [g_1|\ldots|g_p]) = [\sigma_0^{-1} g_1 \sigma_1 | \sigma_1^{-1} g_2 \sigma_2 | \ldots | \sigma_{p-1}^{-1} g_p \sigma_p] \ .$$

This has the equivariance property (4.4), and we can define the Γ-space

(5.6) $$K[m] = \coprod \{ |E_\bullet([m], \underline{k}) \times N_\bullet(\mathrm{GL}_{\underline{k}}(F))| \colon \underline{k} \in \mathrm{Hom}(P_0[m], \mathbb{N}_0) \}$$

where $\mathrm{GL}_{\underline{k}}(F) = \mathrm{GL}_{k_1}(F) \times \ldots \times \mathrm{GL}_{k_m}(F)$, and k_i is the value of $\underline{k}$ on $\{0, i\}$. The group completion of $K[1]$ is homotopy equivalent to $K(F)$ in (5.4).

We similarly define an infinite loop structure on

(5.7) $$K^{cy}(F) = \Omega B\left(\coprod_k |N_\bullet^{cy}(\mathrm{GL}_k(F))|\right) .$$

Indeed the construction of (4.4) also gives a Σ_k-invariant map

$$\Theta_\bullet\colon \delta(E_\bullet(\Sigma_k)\times N_\bullet^{cy}(\mathrm{GL}_k(F)))\to N_\bullet^{cy}(\mathrm{GL}_k(F))$$

and hence as in Sect. 4 a Γ_{C_r}-structure on

(5.8) $$N^{cy}(r)=\coprod_k |sd_r(E_\bullet(\Sigma_k)\times N_\bullet^{cy}(\mathrm{GL}_k(F)))|\,.$$

The maps D and $\Phi_p=\Delta_p^{-1}$ introduced in Sect. 1 and Sect. 2 give Γ-maps from $N^{cy}(p^n)^{C_{p^n}}$ to $N^{cy}(p^{n-1})^{C_{p^{n-1}}}$, hence a Γ-structure on $[\operatorname{holim}_{\leftarrow} N^{cy}(p^n)^{C_{p^n}}]^{h\Phi_p}$ and this exhibits an infinite loop space structure on

$$[\operatorname{holim}_{\leftarrow} K^{cy}(F)^{C_{p^n}}]^{h\Phi_p}\,.$$

The map from (2.13) induces an infinite loop map

(5.9) $$I\colon K(F)\to\left(\operatorname{holim}_{\leftarrow} K^{cy}(F)^{C_{p^n}}\right)^{\Phi_p}$$

cf. (1.11), (2.13).

Consider the simplicial map

$$S_\bullet\colon N_\bullet^{cy}(GL_k(F))\to THH_\bullet(M_k(F))$$

which sends a p-simplex $(f_0,\ldots,f_p)$ with $f_i\in \operatorname{holim}_{\rightarrow}\Omega^n M_k(F)(S^n)^\times$ into the smash product $f_0\wedge\ldots\wedge f_p$. By definition, $\Theta_\bullet(id_\bullet\times S_\bullet)=S_\bullet\Theta_\bullet$, so $S=|S_\bullet|$ induces a map of equivariant Γ-structures, hence a C_r—infinite loop map

$$K^{cy}(F)\to THH(F)\times\mathbb{Z}\,.$$

We next define a map Φ_p which makes the diagram below commute

(5.10) $$\begin{array}{ccc} |sd_{p^{n-1}}N_\bullet^{cy}(GL_k(F))|^{C_{p^{n-1}}} & \xrightarrow{S} & |sd_{p^{n-1}}THH_\bullet(F_k)|^{C_{p^{n-1}}} \\ \downarrow \Delta_p & & \uparrow \Phi_p \\ |sd_{p^n}N_\bullet^{cy}(GL_k(F))|^{C_{p^n}} & \xrightarrow{S} & |sd_{p^n}THH_\bullet(F_k)|^{C_{p^n}}\,. \end{array}$$

The outcome is then an infinite loop map

(5.11) $$S\colon [\operatorname{holim}_{\leftarrow} K^{cy}(F)^{C_{p^n}}]^{h\Phi_p}\to(\operatorname{holim}_{\leftarrow} THH(F)^{C_{p^n}})^{h\Phi_p}\times\mathbb{Z}\,.$$

To define Φ_p let $R=\mathbb{R}C_{p^n}$, $\bar R=\mathbb{R}C_{p^{n-1}}$ be the regular representations of C_{p^n} and $C_{p^{n-1}}$ respectively, so that $\bar R=R^{C_p}$, when we identify $C_{p^{n-1}}=C_{p^n}/C_p$. Consider the map

$$\begin{aligned}\mathrm{Fix}_p\colon \mathrm{Map}_{C_{p^n}}(S^{i_0R}\wedge\ldots\wedge S^{i_kR}, F(S^{i_0})^{(p^n)}\wedge\ldots\wedge F(S^{i_k})^{(p^n)})\to\\ \mathrm{Map}_{C_{p^{n-1}}}(S^{i_0\bar R}\wedge\ldots\wedge S^{i_k\bar R}, F(S^{i_0})^{(p^{n-1})}\wedge\ldots\wedge F(S^{i_k})^{(p^{n-1})})\end{aligned}$$

which takes f to the induced map f^{C_p} on C_p fixed sets. It induces a simplicial map

$$\Phi_{p,\bullet}\colon sd_{p^n}THH_\bullet(F)^{C_{p^n}}\to sd_{p^{n-1}}THH_\bullet(F)^{C_{p^{n-1}}}$$

whose realization is the map Φ_p which makes (5.10) commutative. It is easy to see, and left for the reader that $D\Phi_p=\Phi_p D$.

Definition 5.12 (i) Let F be any FSP. Define its cyclotomic trace functor at p to be

$$TC(F, p) = [\underset{\longleftarrow}{\text{holim}}\, THH(F)^{C_{p^n}}]^{h\Phi_p}$$

with the infinite loop space structure from above.

(ii) The cyclotomic trace at p is the infinite loop map $\text{Trc} = \text{proj} \circ S \circ I$,

$$\text{Trc}: K(F) \to TC(F, p)$$

with I from (5.9), S from (5.11) and proj the projection away from $\mathbb{Z}$ (cf. 4.13).

We could in all the above have taken homotopy inverse limits over all natural numbers rather than just the powers of a single prime p to get a functor $TC(F)$. This functor however would not really be stronger than the products of the $TC(F, p)$. In particular for the profinite completions one would have the equivalence

$$TC(F)^\wedge \simeq \prod TC(F, p)^\wedge \,.$$

Let

$$\beta: TC(F, p) \to THH(F)$$

be the map induced by projecting the homotopy inverse limit to its zero'th term. The composition $\beta \circ \text{Trc}$ is (for any p) the topological Dennis trace map considered in [B].

Remark 5.13 T. Goodwillie has pointed out to us that it is sometimes advisable to interchange the role of Φ and D in the definition of $TC(F, p)$, i.e. that there is a homotopy equivalence

$$TC(F, p) \simeq \left[\underset{\Phi}{\underset{\longleftarrow}{\text{holim}}}\, THH(F)^{C_{p^n}}\right]^{hD} .$$

This amounts in our case to the fact that for a double string $\Phi, D: S_n \rightrightarrows S_{n-1}, n \geqq 0$

$$\left(\underset{D}{\underset{\longleftarrow}{\text{holim}}}\, S_n\right)^{h\Phi} \simeq \left(\underset{\Phi}{\underset{\longleftarrow}{\text{holim}}}\, S_n\right)^{hD} .$$

To see this one can replace (S_*, Φ, D) with a double string $(f(S_*), f(\Phi), f(D))$ where the maps are fibrations. For example, $f(S_0) = S_0$ and $f(S_1)$ is the subset of $S_1 \times S_0^I \times S_0^I$ of points (x_1, σ, τ) with $\Phi(x_1) = \sigma(0)$ and $D(x_1) = \tau(0)$ Then use that the double homotopy fibre can be calculated in two ways in the diagram

$$\begin{array}{ccc} \Pi f(S_n) & \xrightarrow{f(\Phi)-1} & \Pi f(S_n) \\ \downarrow{\scriptstyle f(D)-1} & & \downarrow{\scriptstyle f(D)-1} \\ \Pi f(S_n) & \xrightarrow{f(\Phi)-1} & \Pi f(S_n)\,. \end{array}$$

We shall now calculate (the completion of) the functor $TC(F; p)$ when $F = \underline{\underline{\Gamma}}$, the FSP associated with a group-like monoid. We have

$$THH_\bullet(\underline{\underline{\Gamma}})^{C_{p^n}} \simeq Q_{C_{p^n}}(\Lambda B\Gamma)^{C_{p^n}}$$

and using the proof of (2.6) it is easy to see that Φ_p is homotopic to the composition.

$$\Phi_p\colon Q_{C_{p^n}}(\Lambda B\Gamma)^{C_{p^n}} \to Q_{C_{p^{n-1}}}(\Lambda B\Gamma^{C_p})^{C_{p^{n-1}}} \stackrel{\cong}{\leftarrow} Q_{C_{p^{n-1}}}(\Lambda B\Gamma)^{C_{p^{n-1}}} .$$

Here the first map takes $f\colon S^V \to S^V \wedge \Lambda B\Gamma_+$ into its induced map f^{C_p} on C_p fixed sets, and the second map is induced from the power map

$$\Delta_p\colon \Lambda B\Gamma \stackrel{\cong}{\to} \Lambda B\Gamma^{C_p},\ \Delta_p(\sigma)(z) = \sigma(z^p) .$$

We have left to determine the homotopy fibre of

$$\Phi_p - 1\colon \operatorname{holim}_{\longleftarrow} Q_{C_{p^n}}(\Lambda B\Gamma)^{C_{p^n}} \to \operatorname{holim}_{\longleftarrow} Q_{C_{p^{n-1}}}(\Lambda B\Gamma)^{C_{p^{n-1}}}$$

where the homotopy limit is over the inclusions of fixed sets.

Consider for $m < n$ the covering space

$$EC_{p^m} \times_{C_{p^m}} \Lambda X \to EC_{p^n} \times_{C_{p^n}} \Lambda X$$

of order p^{n-m}. The associated stable transfer maps are denoted

$$t_n^m\colon Q(EC_{p^n} \times_{C_{p^n}} \Lambda X) \to Q(EC_{p^m} \times_{C_{p^m}} \Lambda X) .$$

Then $t_n^m \simeq t_{m+1}^m \circ \ldots \circ t_n^{n-1}$ and we form the homotopy inverse limit

(5.14) $$C(X, p) = \operatorname{holim}_{\longleftarrow} Q(EC_{p^n} \times_{C_{p^n}} \Lambda X) .$$

Lemma 5.15 *For a space X with ΛX of finite type, the completions*

$$C(X, p)_p^\wedge \simeq \tilde{Q}(\Sigma_+(ES^1 \times_{S^1} \Lambda X))_p^\wedge$$

are homotopy equivalent ($\tilde{Q} = \Omega^\infty S^\infty$).

Proof. Let $Y = \Lambda X$. The S^1-transfer defines a map

$$\tau\colon \tilde{Q}(\Sigma_+(ES^1 \times_{S^1} Y)) \to \operatorname{holim}_{\longleftarrow} Q(EC_{p^n} \times_{C_{p^n}} Y)$$

which we must show becomes an equivalence after p-adic completion.

It can be assumed that Y is a free S^1 CW-complex, by replacing Y by $Y \times ES^1$, and we can induct over the S^1-skeleton. The induction starts with $Y = S^1 \times Z$ with Z a finite set of points, which is a trivial case to check. The inductive step is to show that

$$\tilde{\tau}\colon \tilde{Q}(\Sigma(ES^1_+ \wedge_{S^1} S^1_+ \wedge Z)) \to \operatorname{holim}_{\longleftarrow} \tilde{Q}(ES^1_+ \wedge_{C_{p^n}} S^1_+ \wedge Z)) ,.$$

becomes a p-adic equivalence. We divide out the action, and $\tilde{\tau}$ becomes a map from $\tilde{Q}(\Sigma Z)$ to $\operatorname{holim}_{\longleftarrow} \tilde{Q}(S^1_+/C_{p^n} \wedge Z)$. We decompose

$$\tilde{Q}(S^1_+/C_{p^n} \wedge Z) \simeq \tilde{Q}(S^1/C_{p^n} \wedge Z) \times \tilde{Q}(Z) ,$$

and use that the homotopy inverse limit of

$$\tilde{Q}(Z) \xleftarrow{p} \tilde{Q}(Z) \xleftarrow{p} \tilde{Q}(Z) \xleftarrow{p} \ldots$$

is p-adically trivial to conclude that

$$\underset{\longleftarrow}{\operatorname{holim}}\, \tilde{Q}(S^1_+/C_{p^n} \wedge \Lambda Z)^\wedge_p \simeq \underset{\longleftarrow}{\operatorname{holim}}\, \tilde{Q}(S^1/C_{p^n} \wedge Z)^\wedge_p .$$

Finally, the diagram

$$\begin{array}{ccc} \tilde{Q}(\Sigma Z) & \xrightarrow{\tilde{\tau}_{n+1}} & \tilde{Q}(S^1/C_{p^{n+1}} \wedge Z) \\ \downarrow \mathrm{id} & & \downarrow \\ \tilde{Q}(\Sigma Z) & \xrightarrow{\tilde{\tau}_n} & \tilde{Q}(S^1/C_{p^n} \wedge Z) \end{array}$$

is homotopy commutative, and each $\tilde{\tau}_n$ is a homotopy equivalence. Hence

$$\tilde{\tau}\colon \tilde{Q}(\Sigma Z) \xrightarrow{\simeq} \underset{\longleftarrow}{\operatorname{holim}}\, \tilde{Q}(S^1/C_{p^n} \wedge Z) . \quad \square$$

We thank T. Goodwillie and R. Cohen for help with the argument above.

Lemma 5.15 applies to spaces X with finite fundamental group and with $H_*(X; \hat{\mathbb{Z}})$ finitely generated in each degree. It does not in general apply when $\pi_1 X$ is infinite.

The simplest such case, $X = S^1$ is illuminating and will now be discussed in some detail. We have the S^1-homotopy equivalence

$$\Lambda S^1 \simeq \coprod_{n \in \mathbb{Z}} S^1(n) .$$

Here $S^1(n) = S^1(\mathbb{C}^{\otimes n})$ and $\mathbb{C}$ has the standard S^1-structure. Then

$$\Sigma_+(ES^1 \times_{S^1} \Lambda S^1) \simeq \bigvee_{n \in \mathbb{Z}} \Sigma_+(ES^1 \times_{S^1} S^1(n))$$

and $ES^1 \times_{S^1} S^1(n) \sim BC_n$ for $n \neq 0$ (and equivalent to $BS^1 \times S^1$ for $n = 0$). It follows that

$$\tilde{Q}(\Sigma_+(ES^1 \times_{S^1} \Lambda S^1)) \simeq \prod_{n \neq 0} \tilde{Q}(\Sigma_+ BC_n) \times \tilde{Q}(\Sigma_+(BS^1 \times S^1))$$

where $\prod$ means the weak product of infinite loop spaces, corresponding to wedge sum of suspension spectra. In particular the homotopy groups are *sums*.

We next attempt to calculate the homotopy groups (with $\mathbb{F}_p$ coefficients) of $C(S^1; p)$. Let $n = p^i k$ with $(k, p) = 1$. Then

$$EC_{p^m} \times_{C_{p^m}} S^1(n) \simeq \begin{cases} BC_{p^i} \times S^1(k)/C_{p^{m-i}}, & m \geqq i \\ BC_{p^m} \times S^1(n), & m < i . \end{cases}$$

The transfer

$$t_m^{m-1}\colon Q(EC_{p^m} \times_{C_{p^m}} S^1(n)) \to Q(EC_{p^{m-1}} \times_{C_{p^{m-1}}} S^1(n))$$

can correspondingly be calculated to be

$$t_m^{m-1} = \begin{cases} \mathrm{id}_+ \wedge T_1, & m > i \\ T_m^{m-1} \wedge \mathrm{id}_+, & m \leqq i \end{cases}$$

with $T_1\colon Q(S^1) \to Q(S^1)$ the transfer associated to the covering $t \mapsto t^p$ of S^1, and $T_m^{m-1}\colon Q(BC_{p^m}) \to Q(BC_{p^{m-1}})$ the transfer of the covering $BC_{p^{m-1}} \to BC_{p^m}$.

Let $H_*^{\mathrm{spec}}(\ ;\mathbb{F}_p)$ denote the spectrum homology with $\mathbb{F}_p$ coefficients,

$$H_*^{\mathrm{spec}}(QX;\mathbb{F}_p) = H_*(X;\mathbb{F}_p)\,.$$

It is a standard fact that the induced maps

$$(T_m^{m-1})_*\colon H_*(BC_{p^m};\mathbb{F}_p)\to H_*(BC_{p^{m-1}};\mathbb{F}_p)$$
$$(T_1)_*\colon H_*(S^1;\mathbb{F}_p)\to H_*(S^1;\mathbb{F}_p)$$

are isomorphisms in odd dimensions and zero in even dimensions. Suppose $i>0$. Then

$$H_r(EC_{p^m}\times_{C_{p^m}}S^1(n);\mathbb{F}_p) = \mathbb{F}_p\oplus\mathbb{F}_p \text{ for } r>1\,.$$

The first summand $\mathbb{F}_p$ is $H_r(BC_{p^m};\mathbb{F}_p)\otimes H_0(S^1;\mathbb{F}_p)$ when $m<i$ (and $H_r(BC_{p^i};\mathbb{F}_p)\otimes H_0(S^1;\mathbb{F}_p)$ when $m\geqq i$). The second summand is $H_{r-1}(BC_{p^m};\mathbb{F}_p)\otimes H_1(S^1(n);\mathbb{F}_p)$ for $m<i$ (and $H_{r-1}(BC_{p^i};\mathbb{F}_p)\otimes H_1(S^1;\mathbb{F}_p)$ when $m\geqq i$). It follows that $(t_m^{m-1})_*$ in dimension r can be tabulated as

$$(t_m^{m-1})_r = \begin{cases} 0\oplus 1 & \text{for } m>i\\ 0\oplus 1 & \text{for } m\leqq i,\ r \text{ even}\\ 1\oplus 0 & \text{for } m\leqq i,\ r \text{ odd}\end{cases}$$

We can now calculate (for fixed k) that

$$\varprojlim_m \sum_{i=1}^{\infty}{}^{\oplus}\, H_r(EC_{p^m}\times_{C_{p^m}}S^1(p^ik);\mathbb{F}_p) = \begin{cases}\mathbb{F}_p[t], & r \text{ even}\\ \mathbb{F}_p[[t]], & r \text{ odd}\end{cases}$$

i.e. an infinite sum of $\mathbb{F}_p$'s when r is even and an infinite product of $\mathbb{F}_p$'s when r is odd. Indeed for even r, the inverse system is constant equal to $\mathbb{F}_p[t]$ whereas for odd r it is the system $\mathbb{F}_p[t]/\langle t^m\rangle$ with limit $\mathbb{F}_p[[t]]$.

When $r<2p-3$, the Hurewicz map

$$\pi_r(QX;\mathbb{F}_p)\to H_r^{\mathrm{spec}}(QX;\mathbb{F}_p)$$

is an isomorphism, so for each $(k,p)=1$,

$$\varprojlim_m \pi_r Q\left(EC_{p^m}\times_{C_{p^m}}\bigvee_{i=0}^{\infty} S^1(p^ik);\mathbb{F}_p\right) = \begin{cases}\mathbb{F}_p[t], & r \text{ even}\\ \mathbb{F}_p[[t]], & r \text{ odd}\end{cases}$$

in the same range. For odd r less than $2p-3$, $\pi_r(C(S^1;p);\mathbb{F}_p)$ is therefore not countable and hence we see that

$$\pi_r(C(S^1,p);\mathbb{F}_p)\ncong \pi_r(Q\Sigma_+(ES^1\times_{S^1}\Lambda S^1);\mathbb{F}_p)\,.$$

The p-completion preserves (co)fibrations of spectra by [BK, p. 62] so

$$(X\wedge S^0/p)_p^\wedge \simeq X_p^\wedge\wedge S^0/p$$

and hence by [BK, p. 183]

$$\begin{aligned}\pi_*(X_p^\wedge;\mathbb{F}_p) &\cong \operatorname{Ext}(\mathbb{Z}/p^\infty;\pi_*(X;\mathbb{F}_p))\\ &\quad\oplus \operatorname{Hom}(\mathbb{Z}/p^\infty;\pi_*(X;\mathbb{F}_p))\\ &\cong \pi_*(X;\mathbb{F}_p)\,.\end{aligned}$$

In conclusion, (5.15) is false for $X=S^1$.

Let $C(X,p) \xrightarrow{B} Q(\Lambda X)$ be the projection onto the first factor in the inverse system, or after the identification in Lemma 5.15, the S^1-transfer. Consider also the map

$$1 - \Delta_p\colon Q(\Lambda X) \to Q(\Lambda X)$$

where Δ_p is the p'th power map.

Definition 5.16 Let $TC(X,p)$ be the homotopy inverse limit of B and $1 - \Delta_p$, such that there is a homotopy Cartesian diagram

$$\begin{array}{ccc} TC(X,p) & \xrightarrow{\alpha} & C(X,p) \\ \downarrow \beta & & \downarrow B \\ Q(\Lambda X) & \xrightarrow{1-\Delta_p} & Q(\Lambda X)\,. \end{array}$$

Theorem 5.17 *For every group-like monoid, $TC(\underline{\Gamma}, p) \simeq TC(B\Gamma, p)$.*

Proof. Let us write $X = B\Gamma$. There is a well-known decomposition of $Q_{C_{p^n}}(\Lambda X)^{C_{p^n}}$

$$(*) \qquad Q_{C_{p^n}}(\Lambda X)^{C_{p^n}} \simeq \prod_{i=0}^{n} Q(EC_{p^{n-i}} \times_{C_{p^{n-i}}} \Lambda X)$$

basically due to [tD]. We recall the proof of $(*)$.

Consider the cofibration

$$(EC_{p^n})_+ \to S^0 \to \Sigma(EC_{p^n})\,.$$

We take smash product with ΛX_+ and obtain a homotopy fibration of fixed point sets

$$Q_{C_{p^n}}(EC_{p^n} \times \Lambda X)^{C_{p^n}} \to Q_{C_{p^n}}(\Lambda X)^{C_{p^n}} \to \tilde{Q}_{C_{p^n}}(\Lambda X_+ \wedge \Sigma EC_{p^n})^{C_{p^n}}\,.$$

Now, $f \to f^{C_p}$ induces a homotopy equivalence

$$\tilde{Q}_{C_{p^n}}(\Lambda X_+ \wedge \Sigma EC_{p^n})^{C_{p^n}} \simeq Q_{C_{p^{n-1}}}(\Lambda X^{C_p})^{C_{p^{n-1}}}$$

where $C_{p^{n-1}} = C_{p^n}/C_p$. This is clear from equivariant obstruction theory. Also, the equivariant transfer induces a homotopy equivalence

$$Q(EC_{p^n} \times_{C_{p^n}} \Lambda X) \xrightarrow{\simeq} Q_{C_{p^n}}(EC_{p^n} \times \Lambda X)^{C_{p^n}}$$

cf. [A, Theorem 5.3]. Hence we obtain a homotopy fibration

$$(**) \qquad Q(EC_{p^n} \times_{C_{p^n}} \Lambda X) \xrightarrow{a_n} Q_{C_{p^n}}(\Lambda X)^{C_{p^n}} \xrightarrow{b_n} Q_{C_{p^{n-1}}}(\Lambda X^{C_p})^{C_{p^{n-1}}}\,.$$

The mapping b_n is split by the inclusion

$$\beta_n\colon Q_{C_{p^{n-1}}}(\Lambda X^{C_p})^{C_{p^{n-1}}} \to Q_{C_{p^n}}(\Lambda X)^{C_{p^n}}$$

which includes ΛX^{C_p} into ΛX, and views an $\mathbb{R}C_{p^{n-1}}$-module as an $\mathbb{R}C_{p^n}$-module via the projection $C_{p^n} \to C_{p^{n-1}}$. Let

$$\alpha_n\colon Q_{C_{p^n}}(\Lambda X)^{C_{p^n}} \to Q(EC_{p^n} \times_{C_{p^n}} \Lambda X)$$

be the induced splitting of a_n in $(**)$, well-defined up to homotopy.

Let $\varphi_n = \Delta_p^{-1} \circ b_n$ and $\psi_n = \beta_n \circ \Delta_p$ so we have split fibrations

$$Q(EC_{p^n} \times_{C_{p^n}} \Lambda X) \underset{\alpha_n}{\overset{a_n}{\leftrightarrows}} Q_{C_{p^n}}(\Lambda X)^{C_{p^n}} \underset{\psi_n}{\overset{\varphi_n}{\leftrightarrows}} Q_{C_{p^{n-1}}}(\Lambda X)^{C_{p^{n-1}}}$$

with $a_n \alpha_n \simeq 1 - \psi_n \varphi_n$. Let $d_n\colon Q_{C_{p^n}}(\Lambda X)^{C_{p^n}} \to Q_{C_{p^{n-1}}}(\Lambda X)^{C_{p^{n-1}}}$ be the inclusion of fixed sets. We have the relations (up to homotopy)

$$d_{n-1} \circ \varphi_n \simeq \varphi_{n-1} \circ d_n, \quad n > 1$$

$$a_{n-1} \circ t_n^{n-1} \simeq d_n \circ a_n, \quad n \geqq 1$$

$$d_n \circ \psi_n \simeq \psi_{n-1} \circ d_{n-1}, \quad n > 1$$

$$d_1 \circ \psi_1 \simeq \Delta_p .$$

From this we easily see that

$$t_n^{n-1} \circ \alpha_n \simeq \alpha_{n-1} \circ d_n, \quad n > 1$$

$$t_1^0 \circ \alpha_1 \simeq d_1 - \Delta_p \circ \varphi_1 .$$

The maps α_n induce a homotopy class

$$\alpha\colon \underset{\alpha_n}{\operatorname{holim}_{\leftarrow}} Q_{C_{p^n}}(\Lambda X)^{C_{p^n}} \to C(X, p) .$$

(Its p-completion is unique by (2.10)). There is a homotopy commutative diagram

$$\begin{array}{ccc} TC(\underline{\Gamma}, p) & \xrightarrow{\alpha} & C(X, p) \\ \downarrow \beta & & \downarrow \beta \\ Q(\Lambda X) & \xrightarrow{1 - \Delta_p} & Q(\Lambda X) \end{array}$$

which we must show to be homotopy Cartesian. We show that the homotopy fibres of α and $1 - \Delta_p$ are homotopy equivalent via the map induced by β. It follows from the proof of (*) above, that the maps

$$\Phi, D\colon Q_{C_{p^n}}(\Lambda X)^{C_{p^n}} \to Q_{C_{p^{n-1}}}(\Lambda X)^{C_{p^{n-1}}}$$

become homotopy equivalent to

(5.18)
$$D(x_0, \ldots, x_n) = (t_1^0 x_1 + \Delta_p x_0, t_2^1 x_2, \ldots, t_n^{n-1} x_n)$$
$$\Phi(x_0, \ldots, x_n) = (x_0, \ldots, x_{n-1})$$

We use (5.13) and see from (5.18) that

$$\underset{\Phi}{\operatorname{holim}_{\leftarrow}} Q_{C_{p^n}}(\Lambda X)^{C_{p^n}} \simeq \prod_{n=0}^{\infty} Q(EC_{p^n} \times_{C_{p^n}} \Lambda X)$$

with D given by

$$D(x_0, x_1, \ldots) = (t_1^0 x_1 + \Delta_p x_0, t_2^1 x_2, \ldots) .$$

We thus have a diagram of homotopy fibrations (with $Y_n = Q(EC_{p^n} \times_{C_{p^n}} \Lambda X)$),

$$\begin{array}{ccccc} hF(\Delta_p - 1) & \longrightarrow & Y_0 & \xrightarrow{\Delta_p - 1} & Y_0 \\ \downarrow & & \downarrow & & \downarrow \\ TC(X, p) & \longrightarrow & \prod_{n=0}^{\infty} Y_n & \xrightarrow{D-1} & \prod_{n=0}^{\infty} Y_n \\ \alpha \downarrow & & \downarrow & & \downarrow \\ C(X, p) & \longrightarrow & \prod_{n=1}^{\infty} Y_n & \xrightarrow{D-1} & \prod_{n=1}^{\infty} Y_n \, . \end{array} \tag{5.19}$$ □

Taken together Lemma 5.15 and Theorem 5.17 give a calculation of the p-adic completion of $TC(\underline{\underline{\Gamma}}, p)$ in terms of functors which have been extensively examined in algebraic topology. It seems unlikely however that the completion

$$TC(\underline{\underline{\Gamma}}, p) \to TC(\underline{\underline{\Gamma}}, p)^{\wedge}$$

in general induces an injection on homotopy since there can be $\lim^{(1)}$-terms in the homotopy groups of $C(X, p)$.

We end the section with some remarks to clarify the relationship between the range of the cyclotomic map and Connes' cyclic homology, [Co, J].

Let Γ be a discrete group. For the corresponding topological Hochschild homology space $THH(B\Gamma) = Q(\Lambda B\Gamma)$ we have

$$\pi_i(THH(\underline{\underline{\Gamma}})) \otimes \mathbb{Q} = HH_i(\mathbb{Q}\Gamma) \, ,$$

and similarly

$$\pi_i(Q(ES^1 \times_{S^1} \Lambda B\Gamma)) \otimes \mathbb{Q} = HC_i(\mathbb{Q}\Gamma) \, ,$$

by [J]. Thus $C(B\Gamma, p)$ and $THH(B\Gamma)$ can be thought of as topological versions of cyclic homology and Hochschild homology, respectively. One might wonder about the topological analogue of Connes' exact sequence.

Let λ be the canonical line bundle over BS^1 and λ_X its pull-back to $ES^1 \times_{S^1} \Lambda X$. Consider the Thom spectrum $\mathrm{Th}(-\lambda_X)$, defined as the direct limit of the spectra

$$\mathrm{Th}(-\lambda_{n,x}) = \Sigma^{-2N(n)} \mathrm{Th}(\mu_{n,X}) \, .$$

Here λ_n is the restriction of λ to $\mathbb{C}P^n$, $\lambda_n \oplus \mu_n$ is trivial of complex dimension $N(n)$, and $\mu_{n,X}$ is the pull-back to $S^{2n+1} \times_{S^1} \Lambda X$.

Proposition 5.20 *There is a homotopy fibration*

$$\Omega^\infty \Sigma \mathrm{Th}(-\lambda_X) \to \tilde{Q}\Sigma_+(ES^1 \times_{S^1} \Lambda X))_p^{\wedge} \xrightarrow{\mathrm{Trf}} Q(\Lambda X)_p^{\wedge}$$

where Trf *denotes the* S^1*-transfer.*

The argument for $X = pt$ is given in [R]; the general case is similar and left to the reader. We note that the rational homotopy groups of $\mathrm{Th}(-\lambda_X)$ can be calculated as

$$\pi_i \mathrm{Th}(-\lambda_X) \otimes \mathbb{Q} \cong \pi_{i+2}(Q(ES^1 \times_{S^1} \Lambda X)) \otimes \mathbb{Q}$$

by the Thom isomorphism for rational homology. In particular, the rational homotopy groups of the fibration in Proposition 5.20 give the exact sequence of Connes:

$$\ldots \to HC_{i+1}(\hat{\mathbb{Q}}_p\Gamma) \xrightarrow{S} HC_{i-1}(\hat{\mathbb{Q}}_p\Gamma) \xrightarrow{B} HH_i(\hat{\mathbb{Q}}_p\Gamma) \to \ldots$$

for every discrete group Γ.

Note also from (5.15) and Theorem 5.17 the exact sequence

$$\ldots \to \pi_i TC_p(B\Gamma) \otimes \mathbb{Q} \to HH_i(\hat{\mathbb{Q}}_p\Gamma) \oplus HC_{i-1}(\hat{\mathbb{Q}}_p\Gamma) \to HH_i(\hat{\mathbb{Q}}_p\Gamma) \to \ldots$$

6 Assembly maps and Soulé's embedding

A pairing of rings $R_1 \otimes R_2 \to R_3$ gives a pairing of spectra

$$K(R_1) \wedge K(R_2) \to K(R_3)$$

induced from tensor product of matrices. The inclusion of $\mathrm{BGL}_1(R_1)$ into $\coprod_{n \geq 0} \mathrm{BGL}_n(R_1)$ induces a (based) map

$$\mathrm{BGL}(R_1)_+ \to K(R_1)$$

Let $R_1 = R\Gamma$ with R commutative. Then $\Gamma \subset \mathrm{GL}(R\Gamma)$, and one gets a map $B\Gamma_+ \to \mathrm{BGL}(R\Gamma)_+$. We can further take $R_2 = R$ and use the product to get a map of spectra (cf. [L])

$$\mu\colon B\Gamma_+ \wedge K(R) \to K(R\Gamma).$$

This is often called the assembly map.

Similarly, if $\mu\colon F_1 \wedge F_2 \to F_3$ is a pairing of FSP's we get an induced map

$$\mu\colon \mathrm{GL}_m(F_1) \times \mathrm{GL}_n(F_2) \to \mathrm{GL}_{mn}(F_3)$$

It associates to $f \in \Omega^i \mathrm{Map}([m], [m] \wedge F_1(S^i))$ and $g \in \Omega^j \mathrm{Map}([n], [n] \wedge F_2(S^j))$ the composition

$$[m] \wedge [n] \wedge S^i \wedge S^j \xrightarrow{f \wedge g} F_1(S^i) \wedge F_2(S^j) \xrightarrow{\mu} F_3(S^i \wedge S^j)$$

and induces

(6.1) $$\mu\colon N_\bullet((\mathrm{GL}_m(F_1)) \times N_\bullet(\mathrm{GL}_n(F_2)) \to N_\bullet(\mathrm{GL}_{nm}(F_3)).$$

This leads to the pairing of spectra

$$K(F_1) \wedge K(F_2) \to K(F_3).$$

There are completely analogous pairings for $K^{cy}(F)$, $THH(F)$ and $TC(F, p)$. Moreover the cyclotomic trace preserves the pairings up to homotopy. As for rings we can restrict one factor to 1×1 matrices. This gives the homotopy commutative diagram

(6.2) $$\begin{array}{ccc} \mathrm{BGL}_1(F_1)_+ \wedge K(F_2) & \xrightarrow{\mu} & K(F_3) \\ \downarrow 1 \wedge \mathrm{Trc} & & \downarrow \mathrm{Trc} \\ \mathrm{BGL}_1(F_1)_+ \wedge TC(F_2, p) & \xrightarrow{\mu} & TC(F_3, p) \end{array}$$

where the bottom line uses the composition

$$\mathrm{BGL}_1(F_1) \xrightarrow{I} \underset{\longleftarrow}{\mathrm{holim}}\,(|sd_{p^n} N_\bullet^{cy}(\mathrm{GL}(F_1))|^{C_{p^n}})^{\Phi_p} \xrightarrow{S} TC(F_1, p)$$

on the first factor (cf. (5.9), (5.11)), before applying the pairing for $TC(-, p)$. If F is a commutative FSP so that in addition to (3.1) the diagram

$$\begin{array}{ccc} F(X) \wedge F(Y) & \xrightarrow{\mu} & F(X \wedge Y) \\ \downarrow T & & \downarrow F(T) \\ F(Y) \wedge F(X) & \xrightarrow{\mu} & F(Y \wedge X) \end{array}$$

is commutative then μ gives a pairing from $F \wedge F$ to F and we can take $F_i = F$ for $i = 1, 2, 3$ in (6.2). Another specialization which will be important to us is where $F_1 = F_3$ and $F_2 = \mathrm{Id}$.

Let Γ be a topological group-like monoid and consider the FSP $\underline{\underline{\Gamma}}$ from (3.2). We write $A(B\Gamma)$ instead of $K(\underline{\underline{\Gamma}})$ since (by defintion) $K(\underline{\underline{\Gamma}})$ is the version of Waldhausen's A-functor with $\pi_0 A = \mathbb{Z}$, cf. [B], [W2]. With this notation we can specialize (6.2) to

(6.3)
$$\begin{array}{ccc} B\Gamma_{1+} \wedge A(B\Gamma_2) & \to & A(B(\Gamma_1 \times \Gamma_2)) \\ \downarrow 1 \wedge \mathrm{Trc} & & \downarrow \mathrm{Trc} \\ B\Gamma_{1+} \wedge TC(\underline{\underline{\Gamma}}_2, p) & \to & TC(\underline{\underline{\Gamma}}_1 \times \underline{\underline{\Gamma}}_2, p)\,. \end{array}$$

In (6.2) and (6.3) the smash products are to be taken in the category of infinite loop spaces (spectra), i.e.

$$X \wedge E = \underset{\longrightarrow}{\lim}\, \Omega^n(X \wedge E_n)$$

where E_n is the n'th deloop of E.

The functor $TC(\underline{\underline{\Gamma}}, p)$ was calculated in (5.15) and (5.17). The corresponding calculation of the lower horizontal map in (6.3) is as follows. There is an obvious pairing

$$\begin{aligned} B\Gamma_{1+} \wedge Q(EC_{p^n} \times_{C_{p^n}} \Lambda B\Gamma_2) &\to Q(EC_{p^n} \times_{C_{p^n}} B\Gamma_1 \times \Lambda B\Gamma_2) \\ &\to Q(EC_{p^n} \times_{C_{p^n}} \Lambda B(\Gamma_1 \times \Gamma_2))\,. \end{aligned}$$

It commutes with the transfer map, so induces a pairing

$$\mu_C\colon B\Gamma_{1+} \wedge C(B\Gamma_2, p) \to C(B(\Gamma_1 \times \Gamma_2), p)$$

compatible with the pairing

$$\mu_H\colon B\Gamma_{1+} \wedge Q(\Lambda B\Gamma_2) \to Q(\Lambda B(\Gamma_1 \times \Gamma_2))$$

Since $\mu_H \circ (1 \wedge \Delta_p) = \Delta_p \circ \mu_H$ there is an induced pairing

$$\mu_{TC}\colon B\Gamma_{1+} \wedge TC(B\Gamma_2, p) \to TC(B(\Gamma_1 \times \Gamma_2), p)\,.$$

As an addendum to the proof of (5.17) we have

Lemma 6.4 *The bottom map in* (6.3) *is homotopic to the map* μ_{TC} *under the identification in* (5.17). □

We shall in particular make use of (6.3) (and (6.4)) in two special cases, namely $\Gamma_2 = 1$ where the horizontal maps are the assembly maps, and in the case where $\Gamma_1 = \Gamma_2 = \Gamma$ is a commutative group. In this situation we can compose the diagram with the maps induced by multiplication to obtain $\mu_\Gamma\colon B\Gamma_+ \wedge A(B\Gamma) \to A(B\Gamma)$ and correspondingly when $A(B\Gamma)$ is replaced by $TC(B\Gamma, p)$.

A transformation $f\colon F_1 \to F_2$ of FSP's induces a map

$$f_*\colon K(F_1) \times \mathbb{Z} \to K(F_2) \times \mathbb{Z}\ .$$

For a group G and a subgroup Γ of finite index we also have a map in the other direction

$$i^*\colon K(\underline{\underline{G}}) \to K(\underline{\underline{\Gamma}})$$

which we call 'Restriction' and sometimes denote Res or Res_G^Γ. To define it, note that the forgetful map

$$\mathrm{Map}_G(G, G) \to \mathrm{Map}_\Gamma(G, G)$$

together with the identification

$$\mathrm{Map}_\Gamma(G, G) \cong \mathrm{Map}(G/\Gamma, G/\Gamma \times \Gamma)$$

(which depends on a choice of transversal $G/\Gamma \subset G$ to Γ) defines a map

$$i^\#\colon \underline{\underline{G}} \to M_k(\underline{\underline{\Gamma}}),\ k = |G:\Gamma|\ .$$

There is an induced map

$$i^*\colon \mathrm{GL}_n(\underline{\underline{G}}) \to \mathrm{GL}_{nk}(\underline{\underline{\Gamma}})$$

which defines

$$i^*\colon A(BG) \to A(B\Gamma)\ .$$

It is direct from the definitions to check the following lemma whose proof is left to the reader.

Lemma 6.5 (Frobenius reciprocity) *Let $\Gamma \subset G$ be a pair of groups with $|G:\Gamma| < \infty$. Then the diagram below is homotopy commutative in the category of infinite loop spaces*

$$\begin{array}{ccc}
B\Gamma_+ \wedge A(BG) & \xrightarrow{B(i) \wedge \mathrm{id}} & BG_+ \wedge A(BG) \\
\downarrow{\scriptstyle \mathrm{id} \wedge i^*} & & \downarrow{\scriptstyle \mu_G} \\
B\Gamma_+ \wedge A(B\Gamma) & & A(BG) \\
& \searrow^{\mu_\Gamma} \quad \swarrow^{i^*} & \\
& A(B\Gamma)\ . &
\end{array}$$

Similarly, $i^\#$ induces a cyclic map

$$THH_\bullet(\underline{\underline{G}}) \to THH_\bullet(M_k(\underline{\underline{\Gamma}}))$$

and in turn a map of r-fold subdivisions. Composing with Morita invariance, Sect. 3, we get a C_r-equivariant mapping

$$i_r^*: |sd_r THH_\bullet(\underline{\underline{G}})| \to |sd_r THH_\bullet(\underline{\underline{\Gamma}})|$$

for each r, and $D_r \circ i_{rs}^* = i_s^* \circ D_r$ with D from (1.12). In particular the $i_{p^n}^*$ induces

$$i^*: \operatorname{holim}_{\longleftarrow} |sd_{p^n} THH_\bullet(\underline{\underline{G}})|^{C_{p^n}} \to \operatorname{holim}_{\longleftarrow} |sd_{p^n} THH_\bullet(\underline{\underline{\Gamma}})^{C_{p^n}}$$

compatible with the operation of Φ_p from (5.12), and hence

$$i^*: TC(\underline{\underline{G}}, p) \to TC(\underline{\underline{\Gamma}}, p) .$$

The analogue of Lemma 6.5 is valid for $TC(-, p)$ but we shall have no use of this fact in the paper. More important for our purpose is the homotopy commutative diagram of infinite loop maps

$$(6.6) \qquad \begin{array}{ccc} A(BG) & \xrightarrow{i^*} & A(B\underline{\underline{\Gamma}}) \\ \downarrow \mathrm{Trc} & & \downarrow \mathrm{Trc} \\ TC(\underline{\underline{G}}, p) & \xrightarrow{i^*} & TC(\underline{\underline{\Gamma}}, p) . \end{array}$$

We shall often write Res_G^Γ instead of i^*.

Let us next define a map from $\tilde{Q}(S^1 \wedge \mathbb{C}P^\infty)_p^\wedge$ into $A(*)_p^\wedge$ which we shall call Soulé's embedding as it generalizes a construction from [S2]. Stable homotopy classes of $(S^1 \wedge \mathbb{C}P^\infty)_p^\wedge$ then give potential elements of $\pi_* A(*) \otimes \hat{\mathbb{Z}}_p$, but the assignement need not be injective of course.

The usual norm maps

$$\mathrm{Norm}: \mathbb{Z}[C_{p^n}]^\times \to \mathbb{Z}[C_{p^{n-1}}]^\times$$

give rise to an inverse system. We consider an element in its limit,

$$u = (u_n) \in \lim_{\longleftarrow} \mathbb{Z}[C_{p^n}]^\times .$$

The inclusion of (1×1)-matrices induces a map $\mathrm{BGL}_1(\underline{\Gamma}) \to A(B\Gamma)$, and since

$$\pi_1 \mathrm{BGL}_1(\underline{\Gamma}) = \pi_0 \mathrm{GL}_1(\underline{\Gamma}) = \mathbb{Z}\Gamma^\times ,$$

any unit $u_n \in \mathbb{Z}C_{p^n}^\times$ produces an element $u_n \in \pi_1 A(BC_{p^n})$.

It follows from (6.5) that the compositions

$$u_n^\#: BC_{p^n+} \wedge S^1 \xrightarrow{1 \wedge u_n} BC_{p^n+} \wedge A(BC_{p^n}) \xrightarrow{\mu} A(BC_{p^n}) \xrightarrow{i^*} A(*)$$

are compatible up to homotopy when n varies. Thus we have a homotopy class

$$u^\#: \operatorname{holim}_{\longleftarrow} BC_{p^n+} \wedge S^1 \to A(*)$$

which restricts to the $u_n^\#$ for each n. In fact $u^\#$ is uniquely determined by this property. This follows from Milnor's exact sequence [Mi], provided

$$\lim_{\longleftarrow}{}^{(1)} [BC_{p^n+} \wedge S^2, A(*)] = 0 .$$

But the $\lim\limits_{\longleftarrow}{}^{(1)}$ vanishes because the group in question decomposes as a product of the finite group $\pi_2 A(*)$ and a compact group, cf. Sect. 2.

Lemma 6.7 *The p-adic completions of* $\underrightarrow{\operatorname{holim}}\, BC_{p^n}$ *and* $\mathbb{C}P^\infty$ *are homotopy equivalent (via the universal Bockstein).*

Proof. It is clear that $\underrightarrow{\operatorname{holim}}\, BC_{p^n} \simeq B\mu_{p^\infty}$, where μ_{p^∞} are the p-power roots of 1. There is a fibration

$$H(\mathbb{Q}, 1) \to B\mu_{p^\infty} \to H(\mathbb{Z}_{(p)}, 2)$$

and since the completion of $H(\mathbb{Q}, 1)$ is trivial, this gives the equivalence $\beta\colon (B\mu_{p^\infty})_p^\wedge \to (\mathbb{C}P^\infty)_p^\wedge$. □

After completion we then get

$$u^\#\colon (\mathbb{C}P_+^\infty)_p^\wedge \wedge S^1 \to A(*)_p^\wedge$$

which since $A(*)$ is an infinite loop space extends to the map (Soulé embedding)

$$u^\#\colon \tilde{Q}(\Sigma_+(\mathbb{C}(P^\infty))_p^\wedge \to A(*)_p^\wedge\ . \tag{6.8}$$

If we substitute for C_{p^n} the dihedral groups D_{2p^n} of order $2p^n$ in the above we can associate, to a compatible system of units

$$v = (v_n) \in \underleftarrow{\lim}\, K_1(\mathbb{Z}D_{2p^n}) \otimes \hat{\mathbb{Z}}_p\,,$$

a map

$$v^\#\colon \tilde{Q}(\Sigma_+ BO(2))_p^\wedge \to A(*)_p^\wedge\ . \tag{6.9}$$

Indeed, we just have to notice the homotopy equivalence

$$(\underrightarrow{\operatorname{holim}}\, BD_{2p^n})_p^\wedge \simeq BO(2)_p^\wedge\ .$$

Let us next relate the two embeddings (6.8) and (6.9). The inclusion $i\colon C_{p^n} \to D_{2p^n}$ has index 2 and it induces a homomorphism

$$i^*\colon K_1(\mathbb{Z}D_{2p^n}) \to K_1(\mathbb{Z}C_{p^n})^{\mathbb{Z}/2} \tag{6.10}$$

for each n.

Lemma 6.11 *Suppose* $v \in \underleftarrow{\lim}\, K_1(\mathbb{Z}D_{2p^n}) \otimes \hat{\mathbb{Z}}_p$ *maps to* $u \in \underleftarrow{\lim}\, K_1(\mathbb{Z}C_{p^n}) \otimes \hat{\mathbb{Z}}_p$ *under* i^*. *Then there is a homotopy commutative diagram*

$$\begin{array}{ccc} \tilde{Q}(\Sigma_+ BO(2))_p^\wedge & \xrightarrow{v^\#} & A(*)_p^\wedge \\ & \nwarrow \qquad \nearrow{\scriptstyle u^\#} & \\ & \tilde{Q}(\Sigma_+ \mathbb{C}P^\infty)_p^\wedge & \end{array}\ .$$

Proof. It follows from (6.5) with $G = D_{2p^n}$ and $\Gamma = C_{p^n}$ that there is a homotopy commutative diagram

$$\begin{array}{ccc} BD_{2p^n_+} \wedge S^1 & \xrightarrow{v_n^\#} & A(*)_{(p)} \\ & \nwarrow \qquad \nearrow{\scriptstyle u_n^\#} & \\ & BC_{p^n+} \wedge S^1 & \end{array}\ .$$

The lemma now follows by letting n tend to infinity and completing at p. □

Remark. The p-localization $i^* \otimes \mathbb{Z}_{(p)}$ of the map in (6.10) is a isomorphism for $n \geqq 2$. This is clear for p odd by general considerations, and it follows for $p = 2$ from [OT, Proposition 4.2], combined with the fact that $SK_1(\mathbb{Z}C_{2^n}) = 0$ and $SK_1(\mathbb{Z}D_{2^{n+1}}) = 0$.

Next, let us make a few remarks about the space $\tilde{Q}(\Sigma_+(BO(2)))$. First,

$$\tilde{Q}(\Sigma_+(BO(2))) \simeq \tilde{Q}S^1 \times \tilde{Q}(\Sigma BO(2))$$

and since $\pi_1 A(*) = \mathbb{Z}/2$ the factor $\tilde{Q}S^1 \to A(*)$, induced from $S^1 \to A(*)$ needs not concern us very much. Secondly, the usual embedding $BO(2) \to BO$ gives a map from $\Sigma BO(2)$ to BBO, which extends over $\tilde{Q}(\Sigma BO(2))$ upon using that BBO is an infinite loop space. By Bott periodicity, $BBO \cong SU/SO$ so we have

$$(6.12) \qquad h\colon \tilde{Q}(\Sigma_+ BO(2)) \to QS^1 \times SU/SO\,,$$

which is rationally an equivalence. We denote its fibre BCO_2. The loop map Ωh is split by a standard argument using the Becker-Gottlieb transfer of $B(\Sigma_n \int O(2)) \to BO(2^n)$, so

$$Q(BO(2)) \simeq Q(*) \times BO \times CO_2$$

It is not so clear, however, if the similar splitting holds on the delooped level.

Let us recall from [BM] how to construct elements in $\varprojlim \mathbb{Z}[C_{p^n}]^\times$. Choose an integer g which generates the units modulo p^2. Let $g_n = g^{p^{n-1}}$; it has order $p-1$ in $(\mathbb{Z}/p^n)^\times$. Consider the element $u_n \in \mathbb{Z}C_{p^n}$ given by the formula

$$(6.13) \qquad u_n = \left(T^{(1-g_n)/2}\,\frac{T^{g_n-1}}{T-1}\right)^{p-1} - \frac{g_n^{p-1}-1}{p^n}\sum_{i=0}^{p^n-1} T_n^i$$

where $T \in C_p^n$ is the generator. This is a unit, and moreover

$$u = (u_n) \in \varprojlim \mathbb{Z}C_{p^n}^\times$$

by Lemma 4.8 of [BM]. The element u_n and its conjugates under the Galois substitutions $T \to T^i$, $(i, p) = 1$ are the analogues in $\mathbb{Z}[C_{p^n}]$ of the usual cyclotomic units in $\mathbb{Z}[\zeta_{p^n}]$.

It turns out that the Soulé embedding $u^\#$ associated to (6.13) is not sufficient for our purpose: composed with the cyclotomic trace it is trivial on rational homotopy groups in dimensions $\equiv 1 \pmod{2p-2}$.

To remedy this we work instead with A-theory based on localized spheres, or based on completed spheres. We write $A(X; R)$ for this form of A-theory with $R = \mathbb{Z}[1/g]$, $\mathbb{Z}_{(p)}$ or $\hat{\mathbb{Z}}_p$. The local case is treated in [W3]. For a discrete group G there is a linearization map

$$L\colon A(BG; R) \to K(RG)$$

which is a rational homotopy equivalence. Moreover, $\pi_1(L)$ is an isomorphism.

Consider the elements in the localized group rings

(6.14) $$v_n = 1/g T^{(1-g)/2} \frac{T^g - 1}{T - 1} \in RC_{p^n}$$

where $R = \mathbb{Z}[1/g]$. If $p = 2$, take $g = 3$.

These elements are compatible under norms, cf. [BM, Proposition 3.1], and as above they imply an embedding

(6.15) $$v^{\#}\colon \tilde{Q}(\Sigma_+(\mathbb{C}P^\infty))_p^\wedge \to A(*; \mathbb{Z}[1/g])_p^\wedge \ .$$

Remark 6.16 With the use of techniques from [OT] one may prove that

$$i^*\colon K_1(RD_{2p^n}) \otimes \mathbb{Z}_{(p)} \to K_1(RC_{p^n})^{\mathbb{Z}/2} \otimes \mathbb{Z}_{(p)}$$

is surjective also for the ring $R = \mathbb{Z}[1/g]$. Of course, only the prime $p = 2$ presents any problem.

It follows from compactness that $\varprojlim i^* \otimes \hat{\mathbb{Z}}_p$ is onto, and hence that the map $v^{\#}$ in (6.15) factors as

(6.17) $$\begin{array}{ccc} \tilde{Q}(\Sigma_+(\mathbb{C}P^\infty))_p^\wedge & \xrightarrow{v^{\#}} & A(*; \mathbb{Z}[1/g])_p^\wedge \\ & \searrow \quad \nearrow_{v^{\#}} & \\ & \tilde{Q}(\Sigma_+(BO(2)))_p^\wedge \ . & \end{array}$$

The difference between $A(*; \mathbb{Z}[1/g])$ and $A(*)$ has been studied in [W3]. First recall from [Gr] the homotopy fibration

(6.18) $$\prod_{l \mid g} \mathbb{K}(\mathbb{F}_l) \to \mathbb{K}(\mathbb{Z}) \to \mathbb{K}(\mathbb{Z}[1/g])$$

where $\mathbb{K}(R)$ is Quillen's space (from the Q-construction) with

$$\pi_i \mathbb{K}(R) = K_i(R) \quad \text{for } i \geqq 0 \ .$$

The homotopy exact sequence of (6.18) breaks up into short exact sequences as follows.

(6.19) $$0 \to K_i(\mathbb{Z}) \to K_i(\mathbb{Z}[1/g]) \to \sum_{l \mid g}^{\oplus} K_{i-1}(\mathbb{F}_l) \to 0 \ .$$

This follows from [S1, Theorem 3]. Let $\mathbb{A}(X)$ denote Waldhausen's A-functor constructed from the category of finitely dominated spaces with weak equivalences and cofibrations. It has $\pi_0 \mathbb{A}(X) = K_0(\mathbb{Z}\pi_1 X)$, and the connected covers of $\mathbb{A}(X)$ and $A(X)$ are homotopy equivalent.

In [W3] it is proved that

(6.20) $$\begin{array}{ccc} \mathbb{A}(*) & \to & \mathbb{A}(*; \mathbb{Z}[1/g]) \\ \downarrow & & \downarrow \\ \mathbb{K}(\mathbb{Z}) & \to & \mathbb{K}(\mathbb{Z}[1/g]) \end{array}$$

is homotopy Cartesian, at least after outside completion at p. This gives an exact sequence similar to (6.19), and since $K_{4i}(\mathbb{F}_l) = 0$ for $i > 0$.

(6.21) $$\begin{array}{c} A_{4i+1}(*) \otimes \mathbb{Z}_{(p)} \cong A_{4i+1}(*; \mathbb{Z}[1/g]) \otimes \mathbb{Z}_{(p)}, \ i > 0 \\ A_1(*; \mathbb{Z}[1/g]) \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)} \oplus (A_1(*) \otimes \mathbb{Z}_{(p)}) \end{array}$$

with $A_i = \pi_i A$.

7 Induction and the cyclotomic trace

This section and the next is concerned with the evaluation of the composition

$$(7.1) \qquad \tilde{Q}(\Sigma_+ \mathbb{C}P^\infty)_p^\wedge \xrightarrow{\varepsilon^\#} A(*)_p^\wedge \xrightarrow{\mathrm{Trc}} TC(*, p)_p^\wedge$$

in spectrum homology, where $\varepsilon^\#$ is some Soulé embedding to be explicated later. By (5.15) and (5.17),

$$TC(*, p)_p^\wedge \simeq Q(*)_p^\wedge \times \mathrm{hofiber}(\hat{Q}(\Sigma_+ \mathbb{C}P^\infty)_p^\wedge \xrightarrow{\mathrm{Trf}} Q(*)_p^\wedge)$$

and since the first component of $\mathrm{Trc} \circ \varepsilon^\#$ turns out to be trivial, we are left essentially with a spectrum endomorphism of $\hat{Q}(\Sigma_+ \mathbb{C}P^\infty)_p^\wedge$ which has a single $\hat{\mathbb{Z}}_p$ in each odd dimensional homology group.

We take the opportunity here to describe the general plan of our calculation. By construction the elements in the image of $\varepsilon^\#$ are in the image of

$$\mathrm{Res}\colon A(BC_{p^m})_p^\wedge \to A(*)_p^\wedge \quad (\mathrm{Res} = \mathrm{Res}^{\{1\}}_{C_{p^m}})$$

for all $m \geqq 1$. Since Trc commutes with Res by (6.6), it suffices to evaluate

$$(7.2) \quad \tilde{Q}(\Sigma_+ BC_{p^m})_p^\wedge \xrightarrow{\varepsilon_m^*} A(BC_{p^m})_p^\wedge \xrightarrow{\mathrm{Trc}} TC(BC_{p^m}, p)_p^\wedge \xrightarrow{\mathrm{Res}} TC(*, p)_p^\wedge$$

for each m, where $\varepsilon_m^* = \mu \circ (1 \wedge \varepsilon_m)$, and $\varepsilon_m\colon S^1 \to A(BC_{p^m})$ is to be specified in Sect. 8.

Let j_m be the natural map

$$j_m\colon (\underset{\overleftarrow{D}}{\mathrm{holim}}\, THH(\underline{\underline{\Gamma}})^{C_{p^n}})^{h\Phi} \to \underset{\overleftarrow{D}}{\mathrm{holim}}\, THH(\underline{\underline{\Gamma}})^{C_{p^n}} \to THH(\underline{\underline{\Gamma}})^{C_{p^m}}$$

where we have written $THH(\underline{\underline{G}})^{C_{p^m}}$ instead of $|sd_{p^m} THH_\bullet(G)|^{C_{p^m}}$. We use the notation $\mathrm{Trc}^{(m)} = j_m \circ \mathrm{Trc}$, and have by (3.7)

$$THH(\underline{\underline{\Gamma}})^{C_{p^m}} \simeq Q_{C_{p^m}}(\Lambda B\Gamma)^{C_{p^m}}$$

so that

$$\mathrm{Trc}^{(m)}\colon A(B\Gamma) \to Q_{C_{p^m}}(\Lambda B\Gamma)^{C_{p^m}}\,.$$

We must evaluate

$$\mathrm{Res}\colon THH(\underline{\underline{\Gamma}})^{C_{p^m}} \to THH(*)^{C_{p^m}}\,.$$

Let $\Lambda_{[1]}X$ be the subspace of ΛX of homotopically trivial loops, i.e. loops which extend over the disc. It is a component of ΛX when $X = B\Gamma$ for a discrete Γ, and there is a projection

$$\mathrm{pr}_{[1]}\colon Q(\Lambda X) \to Q(\Lambda_{[1]}X)\,.$$

In (7.22) below we show that there is a factorization

$$(7.3) \qquad \begin{array}{ccc} THH(\underline{\underline{C}}_{p^m})^{C_{p^m}} & \xrightarrow{\mathrm{Res}} & THH(*)^{C_{p^m}} \\ \downarrow \mathrm{pr}_{[1]} & & \downarrow \simeq \\ Q_{C_{p^m}}(\Lambda_{[1]} BC_{p^m})^{C_{p^m}} & \xrightarrow{\mathrm{Res}_{[1]}} & Q_{C_{p^m}}(*)^{C_{p^m}} \end{array}$$

and in (7.18) we identify $\mathrm{Res}_{[1]}$ in terms of a more computable (and well-known) map in homotopy theory, namely an equivariant transfer mapping.

Consider the map

$$\mathrm{Fix}\colon Q_{C_{p^m}}(\Lambda_{[1]}B\Gamma)^{C_{p^m}} \to Q((\Lambda_{[1]}B\Gamma)^{C_{p^m}})$$

which sends a point represented by a C_{p^m}-equivariant map $f\colon S^V \to S^V \wedge \Lambda B\Gamma_+$ to its induced map on the C_{p^m}-fixed set. It is a section to the inclusion which embeds $(\Lambda_{[1]}B\Gamma)^{C_{p^m}}$ into $\Lambda_{[1]}B\Gamma$ and $Q(-)$ into $Q_{C_{p^m}}(-)$,

(7.4) $$i_m\colon Q((\Lambda_{[1]}B\Gamma)^{C_{p^m}}) \to Q_{C_{p^m}}(\Lambda_{[1]}B\Gamma)^{C_{p^m}}\,.$$

Special properties of the unit ε is used in (8.14) below to show that $pr_{[1]}\circ \mathrm{Trc}^{(m)}\circ \varepsilon^{\#}$ essentially factors over i_m. Thus in the splitting (cf. [tD] or (5.17) above)

(*) $$Q_{C_{p^m}}(\Lambda_{[1]}BC_{p^m})^{C_{p^m}} \xrightarrow{\simeq} \prod_{n=0}^{m} Q(EC_{p^n}\times_{C_{p^n}}(\Lambda_{[1]}BC_{p^m})^{C_{p^{m-n}}})$$

$pr_{[1]}\circ \mathrm{Trc}^{(m)}\circ \varepsilon^{\#}$ only has non-trivial component corresponding to the factor with $n=0$.

Remark 7.5 It is not true that $\mathrm{Trc}^{(m)}\circ\varepsilon^{\#}$ is concentrated in one component of $Q_{C_{p^m}}(\Lambda BC_{p^m})^{C_{p^m}}$ under the splitting analogous to (*): it first happens after we apply the projection $pr_{[1]}$.

The relation

$$\Phi_p\circ \mathrm{Trc}^{(m)} \simeq Q(\Delta_p)\circ \mathrm{Trc}^{(m-1)}$$

used in Sect. 5 in connection with the definition of Trc may be iterated to give the homotopy commutative diagram

(7.6) $$\begin{array}{ccc} A(BC_{p^m}) & \xrightarrow{pr_{[1]}\circ \mathrm{Trc}^{(m)}} & Q_{C_{p^m}}(\Lambda_{[1]}BC_{p^m})^{C_{p^m}} \\ \downarrow \mathrm{Tr} & & \downarrow \mathrm{Fix} \\ Q(\Lambda BC_{p^m}) & \xrightarrow{Q(\Delta_p^m)} & Q((\Lambda_{[1]}BC_{p^m})^{C_{p^m}})\,. \end{array}$$

Notice here that $(\Lambda BC_{p^m})^{C_{p^m}} = (\Lambda_{[1]}BC_{p^m})^{C_{p^m}}$. There is a homotopy commutative diagram of assembly maps

(7.7) $$\begin{array}{ccc} BC_{p^m+}\wedge A(BC_{p^m}) & \xrightarrow{\mu_A} & A(BC_{p^m}) \\ \downarrow 1\wedge \mathrm{Trc} & & \downarrow \mathrm{Trc} \\ BC_{p^m+}\wedge TC(BC_{p^m},p) & \xrightarrow{\mu_{TC}} & TC(BC_{p^m},p) \\ \downarrow 1\wedge\beta & & \downarrow\beta \\ BC_{p^m+}\wedge Q(\Lambda BC_{p^m}) & \xrightarrow{\mu_H} & Q(\Lambda BC_{p^m}) \end{array}$$

where the bottom horizontal map is the obvious one, cf. Sect. 6, and 5.16. In order now to complete the calculation of (7.2) in spectrum homology (with $\mathbb{Z}/p^m$ coefficients) we just need two calculations, namely the calculation of

(7.8) $$S^1 \xrightarrow{\varepsilon_m^*} A(BC_{p^m}) \xrightarrow{\mathrm{Tr}} Q(\Lambda BC_{p^m})$$

and the calculation (in homology) of

(7.9) $$Q((\Lambda_{[1]}BC_{p^m})^{C_{p^m}}) \xrightarrow{\mathrm{Res}_{[1]}} Q_{C_{p^m}}(*)^{C_{p^m}} \xrightarrow{\alpha_m} Q(BC_{p^m})$$

where $\alpha_m: Q_{C_{p^m}}(*)^{C_{p^m}} \to Q(BC_{p^m})$ is a special case of the splitting used in the proof of (5.17).

The rest of this section studies the restriction map Res_G^Γ in TC-theory in some special cases, sufficient for our calculations outlined above. The arguments are very round about, based in part on the affirmed Segal conjecture. One would like to have a more general understanding of the restriction map.

Let us fix a cyclic p-group C of order r. We have

$$THH(\underline{\underline{G}}) = |sd_r THH_\bullet(\underline{\underline{G}})| \simeq_C Q_C(\Lambda BG)$$

by (3.7). For $|G:\Gamma| < \infty$ we shall study the equivariant C-map

(7.10) $$\mathrm{Res}_G^\Gamma(C): Q_C(\Lambda BG) \to Q_C(\Lambda B\Gamma) .$$

We are mostly interested in the induced map on fixed sets, which contain the necessary information about Res_G^Γ on the TC-functor, but it is better to work equivariantly for some of the arguments.

We begin by examining the components of ΛBG; each component is preserved by the C-action since it extends to a circle action. We assume G to be a discrete group. We have

$$\pi_0 \Lambda BG = [S^1_+, BG] ,$$

the free homotopy classes. Thus the components of ΛBG are indexed by the conjugacy classes $[g]$, $g \in G$.

Lemma 7.11 *For a discrete group G*

$$\Lambda BG = \coprod \Lambda_{[g]} BG \text{ and } \Lambda_{[g]} BG \cong BC_G[g] ,$$

where $C_G[g]$ denotes the centralizer of g.

Proof. The map

$$\phi: N_\bullet^{cy}(G) \to N_\bullet(G; AdG) \cong E_\bullet G \times_G \mathrm{Ad}G ,$$

given by the formula

$$\phi(g_0, \ldots, g_k) = [g_0|\ldots|g_k]g$$

with $g = \prod g_i$, identifies $|N_\bullet^{cy}(G)|$ with the Borel construction of G, considered as a G-space by the adjoint representation. But

$$\mathrm{Ad}_G = \coprod [g]$$

is the union of the conjugacy classes, so

$$EG \times_G \mathrm{Ad}G = \coprod EG \times_G [g] = \coprod EG \times_G G/C_G[g]$$

and the result follows from (2.6). □

For later use it is important to us to understand the power map Δ_p in terms of the decomposition of ΛBG into its connected components.

If $[g]$ is a conjugacy class which is left fixed by Δ_p then $\gamma g \gamma^{-1} = g^p$ for some γ, and it is easily checked that γ normalizes $C_G[g]$. If we identify

$$\Lambda BG = EG \times_G \operatorname{Ad} G$$

as in the proof above, then Δ_p corresponds to the p-power map on Ad G. It follows that Δ_p on $BC_G[g] = EG \times_G G/C_G[g]$ is induced from conjugation with γ on $C_G[g]$. We list this in

Corollary 7.12 *Let $g \in G$ be an element whose p'th power is conjugate to g, $\gamma g \gamma^{-1} = g^p$. Then Δ_p: $BC_G[g] \to BC_G[g]$ is induced by conjugation with γ^{-1}.*

It follows from the proof of Lemma 7.11 that

$$\Lambda_{[g]} BG = |N_\bullet^{cy}(G)_{[g]}| \;,$$

$$N_k^{cy}(G)_{[g]} = \{(g_0, \ldots, g_k) | \prod g_i \in [g]\} \;.$$

Lemma 7.13 *The component $\Lambda_{[1]}BG$ is a model for the C-equivariant classifying space $B_C(G)$.*

Proof. We have the G-covering

$$\Lambda EG \to \Lambda_{[1]} BG$$

and it suffices to show that $\Lambda EG = E_C G$, the terminal object in the category of $(C \times G)$-spaces which are free as G-space. This object in turn is characterized by the properties

$$E_C(G)^A = \begin{cases} * & \text{if } A \cap G = \{1\} \\ \phi & \text{if } A \cap G \neq \{1\} \end{cases}$$

for $A \subseteq C \times G$.

Given a A which intersects G only in $\{1\}$, there is a homomorphism ρ: $C \to G$ with graph A. Then

$$\operatorname{Map}(S^1, EG)^A = \operatorname{Map}_C(S^1, EG)$$

with C acting through ρ on EG. A C-equivariant map from S^1 is determined by its restriction to $\{e^{2\pi i t/|C|} | 0 \leqq t \leqq 1\}$, so

$$\operatorname{Map}_C(S^1, EG) = \{f: I \to EG | f(1) = f(0) \cdot g\}$$

where $g = \rho(T)$ and $T \cdot 1 = e^{2\pi i/|C|}$. Let $p_0(f) = f(0)$, giving a fibration

$$\operatorname{Map}_*^g(S^1, EG) \to \operatorname{Map}_C(S^1, EG) \xrightarrow{p_0} EG$$

with $\operatorname{Map}_*^g(S^1, EG) = \Omega_g EG \simeq *$. It follows that $\operatorname{Map}_C(S^1, EG) \simeq *$.

If $A \cap G \neq \{1\}$ then $\operatorname{Map}(S^1, EG)^A = \varnothing$, since G acts freely on EG. □

Suppose $\Gamma \subset G$ has finite index k. Then

$$\Lambda_{[1]} B\Gamma \to \Lambda_{[1]} BG$$

is a k fold covering space (use the model EG/Γ for $B\Gamma$). The finite subgroup $C \subset S^1$ acts on the covering space, and there is a C-equivariant stable map (its transfer, cf. [LMS, Chap. IV])

$$\mathrm{trf}(C)\colon \Sigma_C^\infty(\Lambda_{[1]}BG_+) \to \Sigma_C^\infty(\Lambda_{[1]}B\Gamma_+)$$

inducing the equivariant infinite loop map

$$\mathrm{Trf}_C^\Gamma(C)\colon Q_C(\Lambda_{[1]}BG_+) \to Q_C(\Lambda_{[1]}B\Gamma_+)\,.$$

We want to compare this with the restriction map from (7.10), and make the

Conjecture 7.14 *On* $Q_C(\Lambda_{[1]}BG)$, $\mathrm{Trf}_G^\Gamma(C) \simeq \mathrm{Res}_G^\Gamma(C)$ *as* C*-infinite loop maps.*

Proposition 7.15 *The conjecture* 7.14 *is true for* $C = 1$.

Proof. There is a commutative diagram (cf. [W4])

$$\begin{array}{ccccc} & Q(BG) & \xrightarrow{\mathrm{Trf}} & Q(B\Gamma) & \\ {}^{j}\swarrow & \downarrow v & & \downarrow v & \searrow^{j} \\ Q(\Lambda_{[1]}BG) & A(BG) & \xrightarrow{\mathrm{Res}} & A(B\Gamma) & Q(\Lambda_{[1]}B\Gamma) \\ \searrow & \downarrow \mathrm{Tr} & & \downarrow \mathrm{Tr} & \swarrow \\ & Q(\Lambda BG) & \xrightarrow{\mathrm{Res}} & Q(\Lambda B\Gamma) & . \end{array}$$

Here v is induced from the inclusion of G or Γ in $GL_1(\underline{\underline{G}})$ or $GL_1(\underline{\underline{\Gamma}})$, j embeds BG into the constant loops and Trf is the usual transfer of the covering $B\Gamma \to BG$.

Given the diagram, the lemma follows because $j\colon BG \to \Lambda_{[1]}BG$ is a homotopy equivalence. □

For two subgroups Γ and Ω of finite index in G, we choose double coset representatives $g_\nu \in G$, $\nu = 1, \dots, r$

$$G = \coprod_{\nu=1}^{r} \Gamma g_\nu \Omega\,.$$

There are inclusions

$$i\colon \Gamma \to G, \quad i(g)\colon \Gamma \cap g\Omega g^{-1} \to \Gamma$$

$$j\colon \Omega \to G, \quad j(g)\colon \Gamma \cap g\Omega g^{-1} \to \Omega$$

($j(g)$ conjugates with g^{-1}).

Lemma 7.16 (*Double coset formula*). *The* C*-equivariant stable maps* $j^* \circ i_*$ *and* $\sum_{\nu=1}^r j(g_\nu)_* \circ i(g_\nu)^*$ *from* $THH(\underline{\underline{\Gamma}})$ *to* $THH(\underline{\underline{G}})$ *are equivariantly homotopic.*

Proof. The proof is the standard one, based on the commutative diagram of FSP's

$$\begin{array}{ccl} \underline{\underline{G}} & \xrightarrow{\ j^*\ } & M_k(\underline{\underline{\Omega}}) \\ \uparrow & & \uparrow \Phi \\ \underline{\underline{\Gamma}} & \xrightarrow{\prod i(g_\nu)^*} & \prod M_{k(\nu)}(\underline{\underline{\Gamma}}_\nu), \quad \Gamma_\nu = \Gamma \cap g_\nu \Omega g_\nu^{-1} \end{array}$$

with Φ being the wedge product (direct sum) of the $j(g_\nu)$. This gives a corresponding diagram upon applying $THH(\ \ )$. Use Morita invariance to complete the proof. □

We specialize to the symmetric groups Σ_n. Define the C-equivariant stable map

$$\chi_n\colon THH(\underline{\underline{\Sigma}}_n) \to Q_C(*)$$

to be the composition.

$$\chi_n\colon THH(\underline{\underline{\Sigma}}_n) \xrightarrow{i^*} THH(\underline{\underline{\Sigma}}_{n-1}) \xrightarrow{p_*} THH(\underline{\underline{1}})$$

where $i\colon \Sigma_{n-1} \subset \Sigma_n$ and $p\colon \Sigma_{n-1} \to \{1\}$. With the identification of $THH(\underline{\underline{G}})$ we get a C-stable map

$$\chi_n\colon Q_C(\Lambda B\Sigma_n) \to Q_C(*)\,.$$

Lemma 7.17 $\chi_{n+1} \simeq_C \chi_n * [1]$, *where* $*$ *indicates loop sum, and* $[1] \in Q_C(*)$ *is represented by the identity.*

Proof. By (7.16) the composition

$$THH(\underline{\underline{\Sigma}}_n) \xrightarrow{i_*} THH(\underline{\underline{\Sigma}}_{n+1}) \xrightarrow{i^*} THH(\underline{\underline{\Sigma}}_n)$$

is the wedge sum of the identity and the composition

$$THH(\underline{\underline{\Sigma}}_n) \to THH(\underline{\underline{\Sigma}}_{n-1}) \xrightarrow{C_T} THH(\underline{\underline{\Sigma}}_n)\,.$$

Here C_T is conjugation with the n-cycle, so is homotopic to the identity. Compose with p_*. □

The C-equivariant transfer of the n-fold covering

$$\Lambda E\Sigma_n \times_{\Sigma_n} [n] \to \Lambda_{[1]} B\Sigma_n$$

gives us an equivariant mapping

$$\Lambda_{[1]} B\Sigma_n \to Q_C(\Lambda E\Sigma_n \times_{\Sigma_n} [n])$$

which we can compose with the projection of $\Lambda E\Sigma_n \times_{\Sigma_n} [n]$ into a point to get

$$\chi_n^t\colon \Lambda_{[1]} B\Sigma_n \to Q_C(*)$$

Standard properties of the transfer show that $\coprod_n \chi_n^t$ extends over the group completion

$$\Lambda_{[1]} B\Sigma_\infty^+ \times \mathbb{Z} = \Omega B\left(\coprod_{n=0}^{\infty} \Lambda_{[1]} B\Sigma_n\right)$$

to induce a C-equivariant mapping

$$\chi_\infty^t\colon \Omega B\left(\coprod_{n=0}^{\infty} \Lambda_{[1]} B\Sigma_n\right) \to Q_C(*)\,.$$

This is a C-homotopy equivalence by the equivariant analogue of the Barratt-Priddy-Quillen theorem. Here is a quick argument for this result when C is a p-group and when we complete at p: the left hand side is known to be a model for

$Q_C(*)$, see e.g. [Sh], so both source and target for χ^t_∞ has homotopy fixed sets equal to fixed sets (after p-completion) by the affirmed Segal conjecture, [C]. That is,

$$\Omega B\left(\coprod_n \Lambda_{[1]} B\Sigma_n\right)^{hD} \simeq_p \Omega B\left(\coprod_n \Lambda_{[1]} B\Sigma_n\right)^D, \quad D \subseteq C$$

where $X^{hD} = \mathrm{Map}_D(ED, X)$. But χ^t_∞ is a non-equivariant homotopy equivalence, so induces an equivalence of the homotopy fixed points. Hence

$$(\chi^t_\infty)^D\colon \Omega B\left(\coprod_n \Lambda_{[1]} B\Sigma_n\right)^D \to Q_C(*)^D$$

is a p-complete homotopy equivalence, and the equivariant Whitehead theorem ([A], (2.7)) implies that χ^t_∞ is an equivariant homotopy equivalence

Proposition 7.18 *Let C be a cyclic p-group. For finite G and $\Gamma = 1$, the p-complete version of* (7.14) *is true.*

Proof. Since both $\mathrm{Res}(C)$ and $\mathrm{Trf}(C)$ are stable maps they are determined by their restrictions to $\Lambda_{[1]}BG$. Let $\rho\colon G \to \Sigma_n$, $n = |G|$ be the regular representation. It displays G as a subgroup of Σ_n such that $\rho(G)\cdot\Sigma_{n-1} = \Sigma_n$. The double coset formula gives the diagram

$$\begin{array}{ccc} THH(\underline{\underline{G}}) & \xrightarrow{\ \rho\ } & THH(\underline{\underline{\Sigma}}_n) \\ \downarrow \mathrm{Res}(C) & & \downarrow i^* \\ THH(*) & \longrightarrow & THH(\underline{\underline{\Sigma}}_{n-1}) \end{array} .$$

It follows that

$$\chi_n \circ \rho \simeq_C \mathrm{Res}(C) .$$

By (7.17) the C-maps χ_n extend to the group completion,

$$\chi_\infty\colon \Lambda_{[1]} B\Sigma^+_\infty \times \mathbb{Z} \to Q_C(*) .$$

This is a C-map, and a non-equivariant homotopy equivalence. Indeed, forgetting the action of C, it follows from (7.15) that the above χ_n is homotopic to χ^t_n, the map constructed from the transfer above. It follows as for χ^t_∞ that

$$\chi^{hD}_\infty\colon \Omega B\left(\coprod_n \Lambda_{[1]} B\Sigma_n\right)^{hD} \to Q_C(*)^{hD}$$

is an equivalence, and so χ_∞ is a p-complete equivariant homotopy equivalence.

Finally $(\chi^t_\infty)^{-1} \circ \chi_\infty$ is a stable C-homotopy equivalence of $Q_C(S^0)$. The homotopy classes of such are the units in the Burnside ring $A(C)$, cf. [tD]. Since C is cyclic $A(C)^\times = \{\pm 1\}$, and as $(\chi^t_\infty)^{-1} \circ \chi_\infty = 1$ in $A(1)^*$, it is the identity for C also. Finally $\chi^t_\infty \circ \rho\colon \Lambda_{[1]}BG \to Q_C(*)$ is equal to $\mathrm{Trf}(C)$. □

At last we examine the restriction map on the other components of ΛBG. This requires a more concrete Morita invariance than the one used in Sect. 3. As motivation, consider first the linear analogue in the case $C = 1$.

Recall that a group ring is self-dual via the isomorphism

$$\delta\colon \mathbb{Z}G \to \operatorname{Hom}(\mathbb{Z}G, \mathbb{Z})$$

which maps group elements $g \in G$ to the characteristic map δ_g ($\delta_g(h) = 1$ if $g = h$, $\delta_g(h) = 0$ if $h \in G - \{g\}$). Thus for $\alpha \in \mathbb{Z}G$, $\delta_\alpha(\gamma) = \delta_1(\alpha\bar{\gamma})$ where $\bar{\gamma}$ is the usual conjugate ($\bar{\gamma} = \Sigma n_g g^{-1}$ if $\gamma = \sum n_g g$), and δ_1 picks out the coefficient of $1 \in G$.

It follows that

$$\Delta\colon \mathbb{Z}G \otimes \mathbb{Z}G \to \operatorname{Hom}(\mathbb{Z}G, \mathbb{Z}G);\ \Delta(\alpha \otimes \beta) = \alpha\delta_\beta$$

is an isomorphism. Composition of maps on the right hand side corresponds under Δ to the product

$$(\mathbb{Z}G \otimes \mathbb{Z}G) \otimes (\mathbb{Z}G \otimes \mathbb{Z}G) \xrightarrow{\mu} \mathbb{Z}G \otimes \mathbb{Z}G$$

$$(*) \qquad \mu(\alpha \otimes \beta \otimes \alpha_1 \otimes \beta_1) = \alpha \otimes \delta_\beta(\alpha_1)\beta_1\,,$$

and the trace homomorphism corresponds to the map

$$(7.19) \qquad \operatorname{Tr}_0\colon \mathbb{Z}G \otimes \mathbb{Z}G \to \mathbb{Z},\ \operatorname{Tr}_0(\alpha \otimes \beta) = \delta_\beta(\alpha)\,.$$

The cyclic object $N^{cy}_{\otimes,\bullet}(\operatorname{Hom}(\mathbb{Z}G, \mathbb{Z}G))$ translates under Δ into the cyclic object $N^{cy}_{\otimes,\bullet}(\mathbb{Z}G \otimes \mathbb{Z}G; \mu)$ and (7.19) induces a map into the constant cyclic object $\mathbb{Z}$ $(= N_{\otimes,\bullet}(\mathbb{Z}))$. Indeed, on k-simplices

$$(7.20) \qquad \operatorname{Tr}_k\colon N^{cy}_{\otimes,\bullet}(\mathbb{Z}G \otimes \mathbb{Z}G,\ \mu) \to \mathbb{Z}$$

is the composition $\operatorname{Tr}_k = \operatorname{Tr}_0 \circ d_1 \circ \ldots \circ d_k$.

The product in $(*)$, and hence the face operators in $N^{cy}_{\otimes,\bullet}(\mathbb{Z}G \otimes \mathbb{Z}G, \mu)$, are "monomial" in the sense of mapping group elements into group elements. The same is not the case for the degeneracy operators, because the identity in $\operatorname{Hom}(\mathbb{Z}G, \mathbb{Z}G)$ does not correspond to a monomial in $\mathbb{Z}G \otimes \mathbb{Z}G = \mathbb{Z}[G \times G]$, but to the sum $\sum (g \otimes g)$ of the diagonal elements.

Only "monomial" operations generalize (in a straightaway manner) to the topological setting of FSP's. Thus in a topological version of $N^{cy}_{\otimes,\bullet}(\mathbb{Z}G \otimes \mathbb{Z}G, \mu)$ we need to abandon degeneracy operators. Equivalently, we must work with FSP's without units (but equipped with a stabilization map and with a product, cf. [B]). We shall only consider a special example, namely the functor

$$\mathscr{G}(X) = G_+ \wedge G_+ \wedge X$$

of based spaces. There is an obvious stabilization

$$\mathscr{G}(X) \wedge Y \to \mathscr{G}(X \wedge Y)$$

and a product

$$\mu\colon \mathscr{G}(X) \wedge \mathscr{G}(Y) \to \mathscr{G}(X \wedge Y)$$

induced from $(*)$. Specifically, μ is given as the composition

$$\mu\colon (G_+ \wedge G_+ \wedge X) \wedge (G_+ \wedge G_+ \wedge Y) \to G_+ \wedge G_+ \wedge G_+ \wedge G_+ \wedge X \wedge Y$$

$$\xrightarrow{\mu_0 \wedge 1 \wedge 1} G_+ \wedge G_+ \wedge X \wedge Y$$

with

$$\mu_0(g_1, g_2, g_3, g_4) = \begin{cases} (g_1, g_4) & \text{if } g_2 = g_3 \\ * & \text{if } g_2 \neq g_3 \end{cases} .$$

Let $THH_\bullet(\mathscr{G})$ be the Δ-space (=simplicial space without degeneracy operators, [RS]) with k-simplices defined in (3.4). It is a cyclic Δ-space. There are Δ-space homotopy equivalences

$$\Delta\colon THH_\bullet(\mathscr{G}) \to THH_\bullet(M_{|G|}(\underline{1})) .$$

$$\mathrm{Tr}\colon THH_\bullet(\mathscr{G}) \to Q(*)$$

The first one is induced from $\delta\colon \mathscr{G}(X) \to \mathrm{Map}(G_+, G_+ \wedge X)$ with $\delta(g, h, x) = (g, x)\delta_h$. The second is given by (7.20) with

$$\mathrm{Tr}_0\colon \varinjlim \Omega^n(G_+ \wedge G_+ \wedge S^n) \to \varinjlim \Omega^n S^n$$

induced from the map $\bar{\varepsilon}_1\colon G_+ \wedge G_+ \to 1_+$ defined as

$$(g, h) \to \delta_h(g) = \begin{cases} 1 & \text{if } h = g \\ * & \text{if } h \neq g \end{cases} .$$

The realizations (as Δ-spaces) are denoted $THH(\quad)$ as before. For FSP's (with units), the two notions of $THH(\quad)$ are homotopy equivalent by [Se1, Appendix].

Morita-invariance (cf. Sect. 3) can be viewed as the composite homotopy equivalence

$$THH(M_{|G|}(\underline{1})) \xleftarrow[\simeq]{\Delta} THH(\mathscr{G}) \xrightarrow[\simeq]{\mathrm{Tr}} Q(*) .$$

We finally can combine with the subdivisions. To define a C-equivariant trace map (C cyclic of order r) we can use the formula (7.20) to specify

$$\mathrm{Tr}_k\colon sd_r THH_k(\mathscr{G}) \to Q_C(*)$$

Hence it suffices to give the C-equivariant map

$$\mathrm{Tr}_0\colon sd_r THH_0(\mathscr{G}) \to Q_C(*) ,$$

or equivalently the map

$$\mathrm{Tr}_0\colon \varinjlim \Omega^{mR}(S^{mR} \wedge (G_+ \wedge G_+)^{(r)}) \to \varinjlim \ \Omega^{mR} S^{mR} ,$$

with $R = \mathbb{R}C$. It is induced from $\bar{\varepsilon}_1^{(r)}\colon (G_+ \wedge G_+)^{(r)} \to 1_+$ given by

$$((g_1, h_1), \dots, (g_r, h_r)) \to \begin{cases} 1 & \text{if } h_i = g_{i+1} \text{ for all } i \\ * & \text{if not} \end{cases} .$$

We obtain a C-equivariant homotopy equivalence

(7.21) $$|sd_r THH_\bullet(M_{|G|}(\underline{1}))| \xleftarrow[\simeq]{|sd_r\Delta|} |sd_r THH_\bullet(\mathscr{G})| \xrightarrow{\mathrm{Tr}} Q_C(*) .$$

This is a description of Morita-invariance which is more convenient for our purpose than the one from Sect. 3, but, of course, it is less general.

Proposition 7.22 *For G finite and $g \neq 1$ the restriction map*

$$\mathrm{Res}^1_G\colon Q_C(\Lambda_{[g]}BG) \to Q_C(*)$$

is equivariantly null-homotopic.

Proof. Under the equivalence

$$\Lambda BG \simeq_C |sd_r N^{cy}_\bullet(G)|$$

the complement $\Lambda_{[1]}BG^\perp$ of $\Lambda_{[1]}BG$ corresponds to the subspace of $sd_r N^{cy}_\bullet(G)$ whose $(k-1)$-simplices are the kr-tuples $(g_0, \ldots, g_{kr-1})$ with $\prod g_i \neq 1$. Let

$$\delta_1^{(r)}\colon G_+^{(r)} \to 1_+$$

be the map with

$$\delta_1^{(r)}(g_1, \ldots, g_r) = \begin{cases} 1 & \text{if } \prod g_i = 1\,. \\ * & \text{otherwise} \end{cases}$$

It induces a map

$$\delta_1^{(r)}\colon \underset{\substack{\longrightarrow \\ m}}{\mathrm{holim}}\ \Omega^{mR}(S^{mR} \wedge G_+^{(r)}) \to \underset{\substack{\longrightarrow \\ m}}{\mathrm{holim}}\ \Omega^{mR}S^{mR}\,,$$

or in other words a map

$$\delta_1^{(r)}\colon sd_r THH_0(\underline{\underline{G}}) \to Q_C(*)\,.$$

Consider the simplicial subspace $sd_r THH_\bullet(\underline{\underline{G}})^\perp_{[1]}$ of $sd_r THH_\bullet(\underline{\underline{G}})$ whose k-simplices are annihilated by $\delta_1^{(r)} \circ d_1 \circ \ldots \circ d_k$ (i.e. map to the base point $* \in Q_G(*)$, represented by the constant map). Its realization $THH(\underline{\underline{G}})^\perp_{[1]}$ is homotopy equivalent to $Q_C(\Lambda_{[1]}BG^\perp)$ via (3.7).

We need to show that the composition

$$|sd_r THH_\bullet(\underline{\underline{G}})| \xrightarrow{\mathrm{Res}} |sd_r THH_\bullet(M_{|G|}(\underline{\underline{1}}))| \xrightarrow{(7.21)} Q_C(*)$$

is equivariantly null-homotopic on the subspace $THH(\underline{\underline{G}})^\perp_{[1]}$. For this we define corresponding subspaces $THH(M_{|G|}(\underline{\underline{1}}))^\perp_{[1]}$ and $THH(\mathscr{G})^\perp_{[1]}$.

There is a C-equivariant diagram

$$\begin{array}{ccccc}
\mathrm{Map}(G_+, G_+ \wedge S^m)^{(r)} & \xrightarrow{\mu^{(r)}} & \mathrm{Map}(G_+, G_+ \wedge S^{mR}) & \xrightarrow{\varepsilon_1} & \mathrm{Map}(G_+, S^{mR}) \\
\uparrow \Delta & & \uparrow \Delta & & \uparrow \Delta \\
(G_+ \wedge G_+ \wedge S^m)^{(r)} & \xrightarrow{\mu^{(r)}} & G_+ \wedge G_+ \wedge S^{mR} & \xrightarrow{\bar{\varepsilon}_1} & G_+ \wedge S^{mR}
\end{array}$$

with $\mu^{(r)}$ being the iterated product, and ε_1, $\bar{\varepsilon}_1$ the following maps

$$\varepsilon_1(f)(g) = (\delta_1 \wedge \mathrm{id})(g^{-1}f(g))$$

$$\bar{\varepsilon}_1(g, h, x) = (\delta_1 \wedge \mathrm{id})(g^{-1}h, x)\,.$$

In the induced diagram,

$$\begin{array}{ccc} \underrightarrow{\operatorname{holim}}\, \Omega^{mR}\operatorname{Map}(G_+, G_+ \wedge S^m)^{(r)} & \xrightarrow{\varepsilon_1^{(r)}} & \underrightarrow{\operatorname{holim}}\, \Omega^{mR}\operatorname{Map}(G_+, S^{mR}) \\ \uparrow \Delta & & \uparrow \Delta \\ \underrightarrow{\operatorname{holim}}\, \Omega^{mR}\mathscr{G}(S^m)^{(r)} & \xrightarrow{\bar{\varepsilon}_1^{(r)}} & \underrightarrow{\operatorname{holim}}\, \Omega^{mR}(G_+ \wedge S^{mR}) \end{array}$$

the vertical maps are equivariant homotopy equivalences.

Define the simplicial subspaces $sd_r THH_\bullet(M_{|G|}(\underline{1}))^{\perp}_{[1]}$ and $sd_r THH_\bullet(\mathscr{G})^{\perp}_{[1]}$ to have k-simplices which are annihilated by $\varepsilon_1^{(r)} \circ d_1 \circ \ldots \circ d_k$ and $\bar{\varepsilon}_1^{(r)} \circ d_1 \circ \ldots \circ d_k$, respectively. (Here d_i is the i'th face operator in the relevant subdivision).

It is clear that

$$\Delta\colon sd_r THH_\bullet(\mathscr{G})^{\perp}_{[1]} \to sd_r(THH_\bullet(M_{[G]}(\underline{1}))^{\perp}_{[1]}$$

is a simplicial homotopy equivalence, and that we have

$$\operatorname{Res}\colon sd_r THH_\bullet(\underline{G})^{\perp}_{[1]} \to sd_r THH_\bullet(M_{|G|}(\underline{1}))^{\perp}_{[1]}$$

Hence to complete the argument it suffices to note that

$$\operatorname{Tr}\colon |sd_r THH_\bullet(\mathscr{G})| \to Q_C(*)$$

is constant on $|sd_r THH_\bullet(\mathscr{G})^{\perp}_{[1]}|$ by definitions. □

8 Homological calculations of the cyclotomic trace

This section completes the evaluation of (7.1) and (7.2) along the lines explained in the beginning of Sect. 7. Let R be one of the rings $\mathbb{Z}$, $\mathbb{Z}[1/g]$ or $\hat{\mathbb{Z}}_p$ where g is a generator of $(\mathbb{Z}/p^2)^\times$, and let $A(X, R)$ denote A-theory based on R-local or R-complete spheres. More precisely, let $S^n_R = S^n$, $S^n[1/g]$ or $(S^n)^\wedge_p$ in the three cases, and consider the "FSP"

(8.1) $$\underline{\Gamma}_R(S^n) = S^n_R \wedge \Gamma_+,\ \Gamma = \Omega X \quad (X \text{ connected and based})$$

In the definition of $K(F)$ one only uses the values of F (and its matrix FSP $M_k(F)$) on spheres. Thus (8.1) is enough to define $K(\underline{\Gamma}_R)$; we write

$$A(X; R) = K(\underline{\Gamma}_R)$$

and remind the reader that $A_1(X; R) \approx K_1(R[\pi_1 X]$. Consider an element $\varepsilon \in \underleftarrow{\lim}\,(RC_{p^m})^\times$. Its m'th component defines an element $[\varepsilon_m] \in A_1(BC_{p^m}; R)$. We are interested in

(8.2) $$(\operatorname{pr}_{[1]} \circ \operatorname{Trc}^{(m)})_* [\varepsilon_{m+k}] \in \pi_1(Q_{C_{p^m}}(\Lambda_{[1]} BC_{p^{m+k}})^{C_{p^m}}) \otimes R$$

for $m \geqq 0$ and $k \geqq 0$, cf. Sect. 7 for notation. Let us first however calculate

$$\operatorname{Tr}_*[\varepsilon_m] \in \pi_1 Q(\Lambda BC_{p^m}) \otimes R$$

We have

(8.3) $$\pi_1 Q(\Lambda BC_{p^m}) = HH_1(\mathbb{Z}C_{p^m}) \oplus (\pi_1 Q(*))^{\oplus p^m}$$

where $HH_1(\mathbb{Z}C_{p^m})$ is the first Hochschild homology group, i.e. the first homology of the Hochschild complex $N^{cy}_{\otimes}(\mathbb{Z}C_{p^m})$. The first component in (8.3) is the spectrum homology,

$$H_1^{\text{spec}}(Q(\Lambda BC_{p^m})) = H_1(\Lambda BC_{p^m}) = HH_1(\mathbb{Z}C_{p^m}) .$$

The second factor is one $\mathbb{Z}/2$ for each component of ΛBC_{p^m}. We shall disregard the second summand in (8.3). In fact we are mainly interested in the cyclotomic trace for odd primes, and then

$$\pi_1 Q(\Lambda BC_{p^m}) \otimes \hat{\mathbb{Z}}_p = HH_1(\hat{\mathbb{Z}}_p C_{p^m}) \approx HH_1(\mathbb{Z}C_{p^m}) .$$

In any case we can always project from homotopy to spectrum homology.

Instead of using the cyclic complex $N^{cy}_{\otimes}(RG)$ to calculate $HH_1(RG)$ it is often more convenient to use the isomorphic bar construction $B_*(RG^c; G)$ of G with coefficients in the group ring RG viewed as a G module via conjugation. In our case G is abelian so RG^c has trivial G action. Concretely the isomorphism is given by

$$N^{cy}_{\otimes}(RG) \xrightarrow{\cong} B_*(RG^c; G)$$

(8.4) $$g_0 \otimes \ldots \otimes g_k \mapsto g[g_1 | \ldots | g_k];\ g = \prod g_i .$$

Let $T \in G$ be a generator (of order p^m) and define elements in $H_1(G; RG^c)$ by

$$\beta_{m,k} = T^k[T] .$$

Under the inverse of (8.4) they become $\beta_{m,k} = 1/k[1 \otimes T^k]$ if $(k, p) = 1$ and $\beta_{m,k} = 1/(k-1)[T \otimes T^{k-1}]$ if p divides k.

Lemma 8.5 *Let $\varepsilon_m \in RG^\times$, and write $\varepsilon_m = \sum a_i T^i$, $\varepsilon_m^{-1} = \sum b_j T^j$. Then*

$$\mathrm{Tr}(\varepsilon_m) = \sum_{k=0}^{p^m-1} c_k(\varepsilon_m)\beta_{m,k}, \quad c_k(\varepsilon_m) = \sum_{i+j \equiv k(p^m)} i a_i b_j .$$

Proof. By definition, $\mathrm{Tr}(\varepsilon_m) = [\varepsilon_m^{-1} \otimes \varepsilon_m]$ in $N^{cy}_{\otimes}(RG)$, and using (8.4) we get in the bar construction

$$\mathrm{Tr}(\varepsilon_m) = \sum a_i b_j T^{i+j}[T^i],\ 0 \leqq i, j < p^m$$

This is homologous to $\sum c_k(\varepsilon_m)\beta_{m,k}$. □

Let us decompose $HH_1(RC_{p^m})$ as

$$HH_1(RC_{p^m}) = K_0 \oplus K_1$$

where K_0 is the part generated by $\beta_{m,k}$ with $(k, p) = 1$ and K_1 is the rest. Then K_0 is $\phi(p^m)$ copies of $\mathbb{Z}/p^m$. We shall use below the following result from [BM], Proposition 3.7.:

(8.6) $$\mathrm{proj}_{K_1}(1 - \Delta_p)\mathrm{Tr}(\varepsilon_m) = 0 \text{ in } HH_1(RC_{p^m}) \otimes \mathbb{Z}/p^{m-1} .$$

We return to (8.2). The target is naturally decomposed (as in (5.17)) into

(8.7) $$Q_{C_{p^m}}(\Lambda_{[1]} BC_{p^{m+k}})^{C_{p^m}} \simeq \prod_{n=0}^{m} Q(EC_{p^n} \times_{C_{p^n}} (\Lambda_{[1]} BC_{p^{m+k}})^{C_{p^{m-n}}}) .$$

Lemma 8.8 *Let $S \in C_{p^m}$ have order p^i and let $\Lambda_{[S]}BC_{p^m}$ be the associated component of ΛBC_{p^m}. Then*

$$\text{(i)}\ EC_{p^n} \times_{C_{p^n}} \Lambda_{[S]}BC_{p^m} \simeq BC_{p^n} \times \Lambda_{[S]}BC_{p^m} \quad \text{for } i \leqq m-n$$

$$\text{(ii)}\ EC_{p^n} \times_{C_{p^n}} \Lambda_{[S]}BC_{p^m} \simeq BC_{p^{m-i}} \times BC_{p^{n+i}} \quad \text{for } i > m-n\,.$$

Proof. Let $f\colon S^1 \to \Lambda_{[S]}BC_{p^m}$ be the parametrization of an S^1-oribt; it induces on fundamental groups a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}/p^m$ with cokernel $\mathbb{Z}/p^{m-i}$. Set $g = EC_{p^n} \times_{C_{p^n}} f$. There is the diagram of homotopy fibrations

$$\begin{array}{ccccc} \Lambda_{[S]}BC_{p^m} & \xrightarrow{j} & EC_{p^n} \times_{C_{p^n}} \Lambda_{[S]}BC_{p^m} & \to & BC_{p^n} \\ \uparrow f & & \uparrow g & & \| \\ S^1 & \xrightarrow{p^n} & S^1 & \to & BC_{p^n}\,. \end{array}$$

On fundamental groups it induces the exact diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}/p^m & \xrightarrow{j_*} & \pi_1 & \to & \mathbb{Z}/p^n & \to & 0 \\ & & \uparrow f_* & & \uparrow g_* & & \| & & \\ 0 & \to & \mathbb{Z} & \xrightarrow{p^n} & \mathbb{Z} & \to & \mathbb{Z}/p^n & \to & 0\,. \end{array}$$

Thus $\operatorname{cok} f_* \cong \operatorname{cok} g_* = \mathbb{Z}/p^{m-i}$ and $\ker g_* = p^{i+n}\mathbb{Z}$ so that π_1 is an extension of $\mathbb{Z}/p^{n+i}$ by $\mathbb{Z}/p^{m-i}$. Using that $\mathbb{Z}/p^m \xrightarrow{j_*} \pi_1 \to \operatorname{cok} g_*$ is surjective it follows that the extension is split. □

The $(m-n)$'th iterate of Δ_p defines a homeomorphism

$$\hat{\Delta}_p^{m-n}\colon \coprod \Lambda_{[S]}BC_{p^{m+k}} \to (\Lambda_{[1]}BC_{p^{m+k}})^{C_{p^{m-n}}}$$

where the disjoint union runs over the elements $S \in C_{p^{m+k}}$ of order dividing p^{m-n}. Thus in (8.7)

$$EC_{p^n} \times_{C_{p^n}} (\Lambda_{[1]}BC_{p^{m+k}})^{C_{p^{m-n}}} \simeq BC_{p^n} \times (\Lambda_{[1]}BC_{p^{m+k}})^{C_{p^{m-n}}}\,.$$

Recall from (5.18) that the inclusion of fixed sets

$$D\colon Q_{C_{p^m}}(\Lambda_{[1]}BC_{p^{m+k}})^{C_{p^m}} \to Q_{C_{p^{m-1}}}(\Lambda_{[1]}BC_{p^{m+k}})^{C_{p^{m-1}}}$$

under (8.7) corresponds to

(8.9) $$D(x_0, x_1, \ldots, x_m) = (i_m^{m-1}(x_0) + t_1^0(x_1), t_2^1(x_2), \ldots, t_m^{m-1}((x_m)))$$

where i_m^{m-1} is the inclusion (since $\Delta_p \simeq \mathrm{id}$ on $\Lambda_{[1]}$) and t_n^{n-1} is the transfer associated to the C_p-covering

$$BC_{p^{n-1}} \times (\Lambda_{[1]}BC_{p^{m+k}})^{C_{p^{m-n}}} \to BC_{p^n} \times (\Lambda_{[1]}BC_{p^{m+k}})^{C_{p^{m-n}}}\,.$$

Write $j^{(m)}$ for the inclusion

$$j^{(m)}\colon Q(\Lambda X^{C_{p^m}}) \to Q_{C_{p^m}}(\Lambda X)^{C_{p^m}}$$

and

$$\mathrm{Fix}^{(m)}\colon Q_{C_{p^m}}(\Lambda X)^{C_{p^m}} \to Q(\Lambda X^{C_{p^m}})$$

for its one-sided inverse which maps a C_{p^m}-equivariant element $f: S^V \to S^V \wedge \Lambda X_+$ into its induced map of C_{p^m} fixed sets.

From (8.7), (8.8) we get the decomposition

$$Q_{C_{p^n}}(\Lambda_{[1]} BC_{p^{m+k}})^{C_{p^m}} \xrightarrow{\simeq} \prod_{n=0}^{m} Q(BC_{p^n} \times (\Lambda_{[1]} BC_{p^{m+k}})^{C_{p^{m-n}}}) .$$

There are p^{m-n} components in $(\Lambda_{[1]} BC_{p^{m+k}})^{C_{p^{m-n}}}$, and since the functor $Q(-)$ converts disjoint unions into direct products we get an embedding

$$\delta \colon \prod_{n=0}^{m-1} Q(BC_{p^n})^{p^{m-n}} \to \prod_{n=0}^{m} Q(BC_{p^n} \times (\Lambda_{[1]} BC_{p^{m+k}})^{C_{p^{m-n}}})$$

(into the first m factors).

Proposition 8.10 *If p is odd, then the p-primary component of the element $(pr_{[1]}\mathrm{Trc}^{(m)})_*[\varepsilon_{m+k}]$ belongs to the image of $j_*^{(m)}$, modulo* $\mathrm{Im}\,\delta_*$

Proof. Let $\mathrm{pr}_{[1]}\mathrm{Trc}_n^{(m)}$ be the component in $Q(BC_{p^n} \times (\Lambda_{[1]} BC_{p^{m+k}})^{p^{m-n}})$. We have

$$(*) \qquad \pi_1 Q(BC_{p^n} \times (\Lambda_{[1]} BC_{p^{m+k}})^{C_{p^{m-n}}})_{(p)} = (\mathbb{Z}/p^n \oplus \mathbb{Z}/p^{m+k})^{\oplus p^{m-n}}$$

The transfer map t_n^{n-1} induces a surjection of $\mathbb{Z}/p^n$ onto $\mathbb{Z}/p^{n-1}$ and multiplication by p on each of the p^{m-n} summands $\mathbb{Z}/p^{m+k}$. There is no component of $\mathrm{pr}_{[1]}\mathrm{Trc}_m^{(m)}$ in the first summand $\mathbb{Z}/p^n$ in $(*)$. This follows from the homotopy commutative diagram

$$\begin{array}{ccc} A(BC_{p^{m+k}}) & \to & Q(BC_{p^m} \times \Lambda_{[1]} BC_{p^{m+k}}) \\ \downarrow f_* & & \downarrow \text{proj} \\ A(*) & \to & Q(BC_{p^m}) \end{array}$$

where f_* is the induced map in A-theory associated to $BC_{p^{m+k}} \to B\{1\}$. Indeed, $f_*[\varepsilon_{m+k}] = 0$ since the augmentation of $\varepsilon_{m+k} \in RC_{p^{m+k}}$ is equal to $1 \in R$. We have left to show that the projection of $\mathrm{pr}_{[1]}\mathrm{Trc}_n^{(m)}[\varepsilon_{m+k}]$ into the summands $(\mathbb{Z}/p^{m+k})$ is trivial. Let us write $T_n^{(m)}[\varepsilon_{m+k}]$ for this projection. Since t_n^{n-1} is multiplication by p on these summands and since $D \circ \mathrm{Trc}^{(m)} = \mathrm{Trc}^{(m-1)}$ we get from (8.9) that

$$(8.11) \qquad T_{n-1}^{(m-1)}[\varepsilon_{m+k}] = p \cdot T_n^{(m)}[\varepsilon_{m+k}] \quad \text{for } n > 1 .$$

We need a similar relation for $n = 1$. Let

$$P_1^{(m-1)} \colon Q(\Lambda BC_{p^{m+k}}) \to Q\left(\coprod_{S^{p^{m-1}}=1} \Lambda_{[S]} BC_{p^{m+k}} \right)$$

be the projection and let

$$\hat{\Delta}_p^{m-1} \colon \coprod_{S^{p^{m-1}}=1} \Lambda_{[S]} BC_{p^{m+k}} \to (\Lambda_{[1]} BC_{p^{m+k}})^{C_{p^{m-1}}}$$

be the homeomorphism induced from the $(m-1)$'st iterate of Δ_p. Then

$$(8.12) \qquad t_1^0 T_1^{(m)} = \hat{\Delta}_p^{m-1} P_1^{(m-1)} (1 - \Delta_p)\mathrm{Tr} .$$

Indeed by (7.6), $\mathrm{Fix}^{(m)}\mathrm{Trc}^{(m)} = \hat{\Delta}_p^m \mathrm{Tr}$, and hence

$$(**) \qquad T_0^{(m)} = \mathrm{Fix}^m T^{(m)} = \hat{\Delta}_p^m P_1^{(m)} \mathrm{Tr} .$$

From (8.9),

$$T_0^{(m-1)} = i_m^{m-1} T_0^{(m)} + t_1^0 T_1^{(m)}$$

and with one more application of (**),

$$\begin{aligned} t_1^0 T_1^{(m)} &= \hat{\Delta}_p^{m-1} P_1^{(m-1)} \mathrm{Tr} - i_m^{m-1} \hat{\Delta}_p^m P_1^{(m)} \mathrm{Tr} \\ &= \hat{\Delta}_p^{m-1} P_1^{(m-1)} (1 - \Delta_p) \mathrm{Tr} . \end{aligned}$$

It follows from (8.6) that

$$p T_1^{(m)} [\varepsilon_{m+k}] \equiv 0 \ (\mathrm{mod}\, p^{m+k-1})$$

and hence upon using (8.11) that

$$(8.13) \qquad p^{i+1} T_i^{(m)} [\varepsilon_{m+k}] = 0, \quad 1 \leqq i \leqq m .$$

We finally make use of the norm compatibility of the ε_{m+k}, or equivalently that the restriction in A-theory of $[\varepsilon_{m+k+1}]$ is $[\varepsilon_{m+k}]$. If we write $\mathrm{Res} = \mathrm{Res}_{C_{p^{m+k+1}}}^{C_{p^{m+k}}}$ then

$$\mathrm{Res}\, \mathrm{T}_i^{(m)} [\varepsilon_{m+k+1}] = T_i^{(m)} [\varepsilon_{m+k}]$$

By (8.13), $p^{i+1} T_i^{(m)} [\varepsilon_{m+k+i+1}] = 0$ in $\mathbb{Z}/p^{m+k+i+1}$. Any homomorphism from $\mathbb{Z}/p^{m+k+i+1}$ to $\mathbb{Z}/p^{m+k}$ has $\mathbb{Z}/p^{i+1}$ contained in its kernel, and we can conclude that $T_i^{(m)} [\varepsilon_{m+k}] = 0$ for $i > 0$. □

Consider the composition $\varepsilon_m^* = \mu \circ (\varepsilon_m \wedge 1)$,

$$\varepsilon_m^* \colon \tilde{Q}(S^1 \wedge (BC_{p^m})_+) \to A(BC_p^{m+k}, R)$$

and let as above

$$\hat{\Delta}_p^m \colon Q(\Lambda BC_{p^m}) \to Q((\Lambda_{[1]} BC_{p^m})^{C_{p^m}})$$

be the homeomorphism induced from iterating Δ_p.

We have from (8.10) with $k = 0$ that

$$\mathrm{pr}_{[1]} \mathrm{Trc}^{(m)} [\varepsilon_m] = \mathrm{pr}_{[1]} \mathrm{Trc}_0^{(m)} [\varepsilon_m] + \delta(\bar{\varepsilon}_m)$$

with

$$\mathrm{pr}_{[1]} \mathrm{Trc}_0^{(m)} [\varepsilon_m] \in \pi_1 Q((\Lambda_{[1]} BC_{p^m})^{C_{p^m}})_{(p)} ,$$

$$\bar{\varepsilon}_m \in \pi_1 \left(\prod_{n=1}^{m-1} Q(BC_{p^n})^{p^{m-n}} \right) .$$

The element $\bar{\varepsilon}_m$ induces a homotopy class

$$\bar{\varepsilon}_m^* \colon \tilde{Q}(S^1 \wedge (BC_{p^m})_+) \to \prod_{n=1}^{m-1} Q(BC_{p^n} \times (\Lambda_{[1]} BC_{p^m})^{C_{p^{m-n}}}) \subset Q_{C_{p^m}} (\Lambda_{[1]} BC_{p^m})^{C_{p^{m-n}}}$$

where the first map uses the multiplication

$$BC_{p^m} \times (\Lambda_{[1]} BC_{p^m})^{C_{p^{m-n}}} \to (\Lambda_{[1]} BC_{p^m})^{C_{p^{m-n}}}$$

induced from the inclusion of BC_{p^m} in $(\Lambda_{[1]}BC_{p^m})^{C_{p^m}}$ and the product in $\Lambda_{[1]}BC_{p^m}$. It seems likely that in fact $\bar{\varepsilon}_m = 0$.

Proposition 8.14 *Let p be odd. After p-completion*

$$\mathrm{pr}_{[1]} \circ \mathrm{Trc}^{(m)} \circ \varepsilon_m^* \simeq j^{(m)} \circ \hat{\Delta}_p^m \circ \mathrm{Tr} \circ \varepsilon_m^* + \bar{\varepsilon}_m^*$$

Proof. We write $C = C_{p^m}$ and do not indicate p-completion in the notation. There is the homotopy commutative diagram

$$\begin{array}{ccccc} \tilde{Q}(BC_+ \wedge S^1) & \xrightarrow{1 \wedge \varepsilon_m} & BC_+ \wedge A(BC) & \xrightarrow{\mu_A} & A(BC) \\ & & \downarrow 1 \wedge \mathrm{pr}_{[1]}\mathrm{Trc}^{(m)} & & \downarrow \mathrm{pr}_{[1]}\mathrm{Trc}^{(m)} \\ & & BC_+ \wedge Q_C(\Lambda_{[1]}BC)^C & \xrightarrow{\mu_H} & Q_C(\Lambda_{[1]}BC)^C \end{array}$$

where μ_H is the composition

$$\begin{aligned} BC_+ \wedge Q_C(\Lambda_{[1]}BC)^C &\to (\Lambda_{[1]}BC)^C \wedge Q_C(\Lambda_{[1]}BC)^C \\ &\to Q_C(\Lambda_{[1]}BC \times \Lambda_{[1]}BC)^C \\ &\to Q_C(\Lambda_{[1]}BC)^C . \end{aligned}$$

The commutativity of the square expresses that the cyclotomic trace commutes with assembly maps, cf. Sect. 6. From (8.10) we have the factorization

$$\begin{array}{ccc} S^1 & \xrightarrow{\varepsilon_m} & A(BC) \\ \downarrow t^{(m)} & & \downarrow T^{(m)} \\ Q((\Lambda_{[1]}BC)^C) & \xrightarrow{j^{(m)}} & Q_C(\Lambda_{[1]}BC)^C \end{array}$$

where $T^{(m)}$ is the composition of $\mathrm{pr}_{[1]} \circ \mathrm{Trc}^{(m)}$ and the projection onto

$$Q(BC_{p^m} \times \Lambda_{[1]}BC_{p^m}) \times \prod_{n=0}^{m-1} Q((\Lambda_{[1]}BC_{p^m})^{p^{m-n}}$$

It follows that $T^{(m)} \circ \mu_A \circ (1 \wedge \varepsilon_m)$ is homotopic to the composition

$$(*) \qquad \tilde{Q}(BC_+ \wedge S^1) \xrightarrow{1 \wedge t^{(m)}} BC_+ \wedge Q((\Lambda_{[1]}BC)^C) \xrightarrow{\mu} Q((\Lambda_{[1]}BC)^C)$$

$$\xrightarrow{j^{(m)}} Q_C(\Lambda_{[1]}BC)^C$$

Together (7.6) and (8.10) show that

$$t^{(m)} \simeq \hat{\Delta}_p^m \circ \mathrm{Tr} \circ \varepsilon_m$$

and if one further uses (7.7) and the commutative diagram

$$\begin{array}{ccc} BC_+ \wedge Q((\Lambda_{[1]}BC)^C) & \longrightarrow & Q((\Lambda_{[1]}BC)^C) \\ \uparrow 1 \wedge \hat{\Delta}_p^m & & \uparrow \hat{\Delta}_p^m \\ BC_+ \wedge Q(\Lambda BC) & \xrightarrow{\mu_H} & Q(\Lambda BC) \end{array}$$

the composition $(*)$ becomes homotopic to $j^{(m)} \circ \hat{\Delta}_p^m \circ \mathrm{Tr} \circ \varepsilon^*$ □

Remark 8.15 If $p = 2$, then p-completion does not kill the extra p^m summands $\mathbb{Z}/2$ in (8.3), and

$$\pi_1(Q(\Lambda BC)) \to HH_1(\hat{\mathbb{Z}}_p C)$$

is not injective. Thus it is not obvious, although very likely, that (8.10) and hence (8.14) are correct as stated. However, the weaker forms where homotopy classes (of infinite loop maps) are replaced by their induced homomorphisms hold true. This is all what is really needed for our key calculational result to be presented below in (8.20).

Before we can continue our calculations it is convenient to examine the fixed point structure of the discrete model $\Omega B(\coprod \Lambda_{[1]} B\Sigma_n)$ for $Q_{C_{p^m}}(*)$. By (7.11),

$$(\Lambda_{[1]} B\Sigma_n)^{C_{p^m}} = \coprod \Lambda_{[S]} B\Sigma_n = \coprod BC[S]$$

where S runs over the conjugacy classes of elements in Σ_n with $S^{p^m} = 1$. Conjugacy classes in symmetric groups are determined by types (=cycle decompositions). If S has type $(1)^{a_0} \ldots (p^m)^{a_m}$ then

$$C[S] = \prod_{i=0}^{m} \Sigma_{a_i} \int C_{p^i}$$

so

$$(\Lambda_{[1]} B\Sigma_n)^{C_{p^m}} = \coprod_S \prod_{i=0}^{m} E\Sigma_{a_i} \times_{\Sigma_{a_i}} (BC_{p^i})^{a_i}$$

and therefore

$$\text{(8.16)} \qquad \Omega B\left(\coprod_{n=1}^{\infty} \Lambda_{[1]} B\Sigma_n\right)^{C_{p^m}} = \prod_{i=0}^{m} \Omega B\left(\coprod_k E\Sigma_k \times_{\Sigma_k} (BC_{p^i})^k\right).$$

The i'th factor on the right-hand side is a model for $Q(BC_{p^i})$, cf. [May] or [Se1] and (8.16) is just the decomposition of $Q_C(*)^C$. One possible identification is via the transfer (compare the discussion preceeding (7.18)): the space

$$E\Sigma_k \times X^k \times_{\Sigma_k} [k] = E\Sigma_k \times_{\Sigma_{k-1}} X^k$$

projects onto X, and we may compose the (non-equivariant) transfer of

$$E\Sigma_k \times_{\Sigma_{k-1}} X^k \to E\Sigma_k \times_{\Sigma_k} X^k$$

with the projection to obtain

$$\chi_k^X \colon E\Sigma_k \times_{\Sigma_k} X^k \to Q(X).$$

These maps induce the equivalence

$$\text{(8.17)} \qquad \chi_\infty^X \colon \Omega B(\coprod E\Sigma_k \times_{\Sigma_k} X^k) \xrightarrow{\simeq} Q(X).$$

(One may show that $(\chi_\infty^t)^{C_{p^m}} \simeq \prod_{i=0}^m \chi_\infty^{BC_{p^i}}$, where χ_∞^t is the equivariant transfer used in (7.18); this gives another proof that $\Omega B(\coprod \Lambda_{[1]} B\Sigma_n)$ is a model for $Q_C(*)$). Let

$T \in \Sigma_{p^m}$ be the cycle of order p^m, and let $(\Lambda'_{[1]} B\Sigma_{p^m})^{C_{p^m}}$ be the image of the natural map from $(\Lambda_{[1]} BC_{p^m})^{C_{p^m}}$ to $(\Lambda_{[1]} B\Sigma_{p^m})^{C_{p^m}}$, i.e.

$$(\Lambda'_{[1]} B\Sigma_{p^m})^{C_{p^m}} = \coprod_{i=0}^{m} \Lambda_{[T^{p^i}]} B\Sigma_{p^m}$$

The centralizers of T^{p^i} are $\Sigma_{p^i} \int C_{p^{m-i}}$, so

$$(\Lambda'_{[1]} B\Sigma_{p^m})^{C_{p^m}} = \coprod_{i=0}^{m} E\Sigma_{p^i} \times_{\Sigma_{p^i}} (BC_{p^{m-i}})^{p^i}$$

and

$$Q((\Lambda'_{[1]} B\Sigma_{p^m})^{C_{p^m}}) = \prod_{i=0}^{m} Q(E\Sigma_{p^i} \times_{\Sigma_{p^i}} (BC_{p^{m-i}})^{p^i})$$

With this identification

$$\chi^t_{p^m}\colon Q((\Lambda_{[1]} B\Sigma_{p^m})^{C_{p^m}}) \to \prod_{i=0}^{m} Q(BC_{p^{m-i}}) \tag{8.18}$$

is equal to the product of the compositions

$$Q(E\Sigma_{p^i} \times_{\Sigma_{p^i}} (BC_{p^{m-i}})^{p^i}) \to Q(Q(BC_{p^{m-i}})) \to Q(BC_{p^{m-i}})$$

of $\chi_{p^i}^{BC_{p^{m-i}}}$ and the action $QQ(-) \to Q(-)$. For $i = 0$ in particular

$$\chi^t_{p^m}\colon Q(\Lambda_{[T]} B\Sigma_{p^m}) \to Q(BC_{p^m}) \tag{8.19}$$

is just the identity.

Let $u_{2n} \in H_{2n}(\mathbb{C}P^\infty; \mathbb{Z})$ and $e_n \in H_n(BC_{p^m}; \mathbb{Z}/p^m)$ be generators, and let

$$\alpha_m\colon Q_{C_{p^m}}(*)^{C_{p^m}} \to Q(BC_{p^m})$$

be the splitting onto the first factor, used in the definition of $TC(*, p)$ in Sect. 5. Here is the main calculational result of the entire paper.

Theorem 8.20 *In spectrum homology with $\mathbb{Z}/p^m$ coefficients, the composition*

$$\tilde{Q}(S^1 \wedge \mathbb{C}P^\infty_+) \xrightarrow{\varepsilon^\#} A(*, R) \xrightarrow{\mathrm{Trc}_p^{(m)}} (Q_{C_{p^m}}(*)^{C_{p^m}})^\wedge_p \xrightarrow{\alpha_m} Q(BC_{p^m})^\wedge_p$$

maps $e_1 \otimes u_{2n}$ into $(\Sigma\, (1/k)^{n+1} c_k(\varepsilon_m)) e_{1+2n}$. Here k varies over the integers $1 \leqq k \leqq p^m - 1$ which are prime to p and $c_k(\varepsilon_m)$ is the integer from (8.5), and $R = \mathbb{Z}[1/2]$, $\mathbb{Z}_{(p)}$ or $\hat{\mathbb{Z}}_p$.

Proof. Let us drop the notation for p-completion. We consider the commutative squares

$$\begin{array}{ccccc}
\tilde{Q}(S^1 \wedge \mathbb{C}P^\infty_+) & \xrightarrow{\varepsilon^\#} & A(*; R) & \xrightarrow{\mathrm{Trc}^{(m)}} & Q_{C_{p^m}}(*)^{C_{p^m}} \\
\uparrow 1 \wedge e_2^\vee & & \uparrow \mathrm{Res} & & \uparrow \mathrm{Res} \\
\tilde{Q}(S^1 \wedge BC_{p^m+}) & \xrightarrow{\varepsilon_m^*} & A(BC_{p^m}; R) & \xrightarrow{\mathrm{Trc}^{(m)}} & Q_{C_{p^m}}(\Lambda BC_{p^m})^{C_{p^m}}
\end{array}$$

with $e_2^\vee : BC_{p^m} \to \mathbb{C}P^\infty$ the generator of $H^2(BC_{p^m}; \mathbb{Z})$. Since $1 \wedge e_2^\vee$ is an isomorphism in odd dimensional spectrum homology with $\mathbb{Z}/p^m$ coefficients, it suffices to calculate the lower horizontal composition followed by Res $= \mathrm{Res}_{C_{p^m}}^{\{1\}}$. By (7.22),

$$\mathrm{Res} \circ \mathrm{Trc}^{(m)} \simeq \mathrm{Res} \circ \mathrm{pr}_{[1]} \circ \mathrm{Trc}^{(m)} \simeq \mathrm{Trf} \circ pr_{[1]} \circ \mathrm{Trc}^{(m)}$$

The composition

$$\alpha_m \circ \mathrm{trf}\colon Q_{C_{p^m}}(\Lambda_{[1]} BC_{p^m})^{C_{p^m}} \to Q_{C_{p^m}}(*)^{C_{p^m}} \xrightarrow{\alpha_m} Q(BC_{p^m})$$

annihilates the subset

$$\prod_{n=1}^{m-1} Q(BC_{p^n})^{p^{m-1}} \subset \prod_{n=0}^{m} Q(BC_{p^n} \times (\Lambda_{[1]} BC_{p^m})^{C_{p^{m-n}}})$$

and (8.14) shows that

$$\alpha_m \circ \mathrm{Res} \circ \mathrm{Trc}^{(m)} \circ \varepsilon_m^* \simeq \alpha_m \circ j^{(m)} \circ \hat{\Delta}_m^p \circ \mathrm{Tr} \circ \varepsilon_m^* .$$

We know from Sect. 6 that Tr commutes with the pairings μ, so have

$$\begin{array}{ccc} \tilde{Q}(S^1 \wedge BC_{p^m+}) & \xrightarrow{\varepsilon_m^*} & A(BC_{p^m}; R) \\ \downarrow \mathrm{Tr}(\varepsilon_m) \wedge 1 & & \downarrow \mathrm{Tr} \\ Q(\Lambda BC_{p^m}) \wedge BC_{p^m+} & \xrightarrow{\mu} & Q(\Lambda BC_{p^m}) \end{array}$$

Let $\rho\colon \Lambda_{[1]} BC_{p^m} \to \Lambda_{[1]} B\Sigma_{p^m}$ be induced from the usual inclusion of C_{p^m} into Σ_{p^m}. Since $\mathrm{Res} \simeq \mathrm{Trf} = \chi_\infty^t \circ \rho$ by (7.18),

$$\mathrm{Res} \circ j^{(m)} \simeq \chi_{p^m}^t \circ \rho \circ j^{(m)}$$

on $Q((\Lambda_{[1]} BC_{p^m})^{C_{p^m}})$. We have left to calculate the composition

$$\begin{aligned} \tilde{Q}(S^1 \wedge BC_{p^m+}) &\xrightarrow{\mu(\mathrm{Tr}(\varepsilon_m) \wedge 1)} Q(\Lambda BC_{p^m}) \xrightarrow{e} Q(\Lambda' B\Sigma_{p^m}) \\ &\xrightarrow{\hat{\Delta}_p^m} Q((\Lambda'_{[1]} B\Sigma_{p^m})^{C_{p^m}}) \\ &\xrightarrow{\chi_{p^m}^t} \prod Q(BC_{p^{m-i}}) \\ &\xrightarrow{\alpha_m} Q(BC_{p^m}) \end{aligned}$$

where

$$\Lambda' B\Sigma_{p^m} = \coprod_{i=0}^{m} \Lambda_{[T^{p^i}]} B\Sigma_{p^m} .$$

The infinite loop map $\alpha_m \chi_{p^m}^t \hat{\Delta}_p^m$ is determined by its value on $\Lambda' B\Sigma_{p^m}$, and is trivial on all components $\Lambda_{[T^{p^i}]} B\Sigma_{p^m}$ with $i > 0$. Moreover by (8.19) the

$$\chi_{p^m}^t \circ \rho \circ \hat{\Delta}_p^m\colon Q(\Lambda_{[T]} B\Sigma_{p^m}) \to Q(BC_{p^m})$$

becomes the identity when we identify $\Lambda_{[T]}B\Sigma_{p^m} = BC_{p^m}$. This reduces us to calculate the image of $e_1 \otimes e_{2n}$ under

$$\tilde{Q}(S^1 \wedge BC_{p^m+}) \xrightarrow{\mathrm{Tr}(\varepsilon_m)\wedge 1} Q(\Lambda BC_{p^m}) \wedge BC_{p^m+}$$

$$\xrightarrow{\mu\circ pr} Q\Big(\coprod_{(k,p)=1} \Lambda_{[T^k]}BC_{p^m}\Big)$$

$$\xrightarrow{pr\circ\rho} Q(\Lambda_{[T]}B\Sigma_{p^m}).$$

According to (8.5), $e_1 \otimes e_{2n}$ is mapped to the class $(\Sigma c_k(\varepsilon_m)\cdot \beta_{m,k}) \otimes e_{2n}$.

Let $\psi^k\colon C_{p^m} \to C_{p^m}$ be the homomorphism which takes T to T^k. It induces homomorphisms of BC_{p^m} and ΛBC_{p^m} making the diagram

$$\begin{array}{ccc} \Lambda_{[T]}BC_{p^m} & \xrightarrow{\psi^k} & \Lambda_{[T^k]}BC_{p^m} \\ \downarrow eval & & \downarrow eval \\ BC_{p^m} & \xrightarrow{\psi^k} & BC_{p^m} \end{array}$$

commutative. Since ψ^k is inner in Σ_{p^m}, the inclusion $BC_{p^m} \xrightarrow{\rho} B\Sigma_{p^m}$ and the composition $\rho\circ\psi^k$ are (freely) homotopic. In particular they induce identical homomorphisms on homology.

Since $\Lambda_{[T]}BC_{p^m} \to \Lambda_{[T]}B\Sigma_{p^m}$ is a homotopy equivalence, and since $\beta_{m,k}$ all go to e_1 under the evaluation map, $e_1 \otimes e_{2n}$ is mapped to

$$\Sigma c_k(\varepsilon_m)(\psi^k_*)^{-1}(e_{1+2n}) = (\Sigma(1/k)^{1+n}c_k)e_{1+2n}$$

This is the claimed formula. □

The result above can be formulated entirely in the framework of group homology. This is carried out in [BM] where the numbers $\Sigma(1/k)^{n+1}c_k$ are also examined for various choices of $\varepsilon \in \varprojlim RC_{p^n}^\times$. This will be used in the next section.

For small primes the reader can easily carry out the calculation. Consider for example the case $C_{p^m} = C_5 (p = 5, m = 1)$. Take $g = 2$; it generates the multiplicative group of $\mathbb{Z}/25$. In the local case $R = \mathbb{Z}/[1/2]$, a possible unit is $\varepsilon_1(T) = 1/2(T^2 + T^{-2})$. Its inverse is

$$\varepsilon_1(T)^{-1} = 1 - T - T^{-1} + T^2 + T^{-2}$$

and c_k is the coefficient of T^k in

$$T\frac{d\varepsilon_1}{dT}\varepsilon_1(T)^{-1} = -2T + 2T^2 - 2T^3 + 2T^4$$

Hence in $\mathbb{Z}/5$ we have

$$(8.21)\qquad \sum_{k=1}^{4} (1/k)^{1+n}c_k = \begin{cases} 3 \text{ if } n \equiv 0 \pmod 4 \\ 4 \text{ if } n \equiv 2 \pmod 4 \\ 0 \text{ if } n \equiv 1 \pmod 2 \end{cases}$$

Corollary 8.22 *In spectrum homology, the composition*

$$\tilde{Q}(\Sigma_+ \mathbb{C}P^\infty)^\wedge_p \xrightarrow{\varepsilon^\#} A(*; R)^\wedge_p \xrightarrow{\text{Trc}} TC(*, p)^\wedge_p \xrightarrow{\alpha} \tilde{Q}(\Sigma_+ \mathbb{C}P^\infty)^\wedge_p$$

multiplies with the p-adic number $\varprojlim_m (\Sigma(1/k)^{1+n} c_k(\varepsilon_m))$ *in dimension* $1 + 2n$, *where the sum runs over integers* $1 \leqq k \leqq p^m - 1$ *which are prime to p.*

Proof. This follows from the proof of (5.15), from (8.20) and from the commutative diagram

$$\begin{array}{ccc} \tilde{Q}(S^1 \wedge \mathbb{C}P^\infty_+)^\wedge_p & \xrightarrow{\alpha \circ \text{Trc} \circ \varepsilon^\#} & \tilde{Q}(S^1 \wedge \mathbb{C}P^\infty_+)^\wedge_p \\ \uparrow 1 \wedge e_2^\vee & & \downarrow \text{Trf} \\ \tilde{Q}(S^1 \wedge (BC_{p^m})_+)^\wedge_p & \xrightarrow{\alpha_m \text{Trc}^{(m)} \varepsilon_m^*} & \tilde{Q}(S^1 \wedge (BC_{p^m})_+) \end{array}$$

together with the fact that the S^1-transfer

$$\text{Trf}_*\colon H_*(S^1 \wedge \mathbb{C}P^\infty_+, \mathbb{Z}/p^m) \to H_*(S^1 \wedge BC_{p^m+}; \mathbb{Z}/p^m)$$

is an isomorphism in odd degrees, cf. [MMM]. □

9 The main theorems

The cyclotomic trace functor of a point was calculated in Sect. 5. After completion at p we have

(9.1) $$TC(*, p)^\wedge_p \simeq Q(*)^\wedge_p \times \text{hofibre}(\tilde{Q}(\Sigma_+ \mathbb{C}P^\infty)^\wedge_p \xrightarrow{\text{Trf}} Q(*)^\wedge_p)\,.$$

Let $R = \mathbb{Z}$, $\mathbb{Z}[1/g]$ or $\hat{\mathbb{Z}}_p$ and let $\varepsilon \in \varprojlim (RC_{p^m})^\times$. There is an induced map $\varepsilon^\#\colon \tilde{Q}(\Sigma_+(\mathbb{C}P^\infty))^\wedge_p \to A(*; R)$ and we are interested in the composition

(9.2) $$\tilde{Q}(\Sigma_+(\mathbb{C}P^\infty))^\wedge_p \xrightarrow{\varepsilon^\#} A(*; R)^\wedge_p \xrightarrow{\text{Trc}} TC(*, p)^\wedge_p$$

for various choices of ε.

The first component of Trc under the splitting (9.1) is the topological Dennis trace and we first evaluate $\text{Tr} \circ \varepsilon^\#$.

We must recall the relevant choices of ε in the three cases $R = \mathbb{Z}$, $\mathbb{Z}[1/g]$ and $\hat{\mathbb{Z}}_p$.

$$\text{(i)}\ R = \mathbb{Z}\colon \varepsilon_n = T_n^{(1-g_n)/2} \left(\frac{T_n^{g_n} - 1}{T_n - 1}\right)^{p-1} - \frac{g_n^{p-1} - 1}{p^n} \sum_{i=0}^{p^n-1} T_n^i$$

(9.3) $$\text{(ii)}\ R = \mathbb{Z}[1/g]\colon \varepsilon_n = 1/g T_n^{(1-g)/2} \frac{T_n^g - 1}{T_n - 1}$$

$$\text{(iii)}\ R = \hat{\mathbb{Z}}_p;\ \varepsilon_n = \frac{\lambda - T_n}{\omega(1 - \lambda)}\,.$$

Here $T_n \in C_{p^n}$ is the generator, g is an integer which generates $(\mathbb{Z}/p^2)^\times$, $g_n = g^{p^{n-1}}$, $\lambda \in \hat{\mathbb{Z}}_p^\times$ is a certain non-trivial $(p-1)$'st root of 1 which we shall not further specify at present, and ω: $\mathbb{F}_p^\times \to \hat{\mathbb{Z}}_p^\times$ is the Teichmuller character.

Lemma 9.4 *Let p be an odd prime, and let $\varepsilon \in \varprojlim RC_{p^n}$ be as in* (9.3). *In case* (i) *and case* (ii), $\mathrm{Tr} \circ \varepsilon^\#$ *is homotopically trivial. In case* (iii), $\mathrm{Tr} \circ \varepsilon^\#$ *is homotopic to the S^1-transfer composed with an automorphism of $\tilde{Q}(\Sigma_+ \mathbb{C}P^\infty)$.*

Proof. The composition is induced from the direct limit (over m) of the compositions

$$\begin{aligned} S^1 \times BC_{p^m} &\xrightarrow{\mathrm{Tr}(\varepsilon_m)\times 1} Q(\Lambda BC_{p^m}) \times BC_{p^m} \\ &\xrightarrow{\mathrm{pr}_{[1]}} Q(\Lambda_{[1]} BC_{p^n}) \times BC_{p^m} \\ &\xrightarrow{\mu} Q(\Lambda_{[1]} BC_{p^m}) \\ &\xrightarrow{\chi'_{p^m}} Q(*) \end{aligned}$$

where we have not indicated p-adic completion in the notation. Thus to prove the lemma it suffices to show that

$$[\mathrm{pr}_{[1]}\mathrm{Tr}(\varepsilon_m)] = 0 \text{ in } \pi_1 Q(\Lambda_{[1]} BC_{p^m})_{(p)} .$$

We have (cf. [BM]):

$$\pi_1 Q(\Lambda BC_{p^m})_{(p)} \simeq HH_1(RC_{p^m}) \simeq H_1(C_{p^m}; RC_{p^m})$$

and the projection $pr_{[1]}$ corresponds to the homomorphism

$$H_1(C_{p^n}; RC_{p^m}) \to H_1(C_{p^m}; R)$$

induced from the homomorphism δ_1: $RC_{p^m} \to R$ which picks out the coefficient of $1 \in C_{p^m}$. With the notation from Lemma 8.5,

$$[\mathrm{pr}_{[1]}\mathrm{Tr}(\varepsilon_m)] = c_0(\varepsilon_m)\beta_{m,0} .$$

In case (i) and (ii) of (9.3), $\varepsilon_m \in (RC_{p^m})^\times$ is visibly symmetric in that $a_i = a_{-i}$ in the expression $\varepsilon_m = \Sigma a_i T^i$, $i \in \mathbb{Z}/p^n$. The same is then true for ε_m^{-1} and hence

$$c_0(\varepsilon_m) = \sum_{i=1}^{p^m-1} i a_i b_{-i} = 0 .$$

In case (iii), $c_0(\varepsilon_m) = \dfrac{1}{1-\lambda}$. Indeed,

$$a_0 = \frac{\lambda}{\omega(1-\lambda)}, \; a_1 = \frac{-1}{\omega(1-\lambda)}$$

which gives $\lambda b_0 - b_{p^n-1} = \omega(\lambda - 1)$ and $\lambda b_i = b_{i-1}$ for $i \neq 0$, and hence $c_0(\varepsilon_m) = 1/1-\lambda$ as claimed. Consequently

$$\mu \circ \mathrm{pr}_{[1]} \circ (\mathrm{Tr}(\varepsilon_m) \times 1)\colon S^1 \wedge (BC_{p^m})_+ \to Q(\Lambda_{[1]} BC_{p^m})$$

sends the homology class $e_1 \otimes e_{2m} \in H_{2m+1}(S^1_+ \wedge BC_{p^m};\ \mathbb{Z}/p^m)$ non-trivially to $H^{\mathrm{spec}}_{2m+1}(Q(\Lambda_{[1]}BC_{p^m});\ \mathbb{Z}/p^m)$.

Using Frobenius reciprocity, Lemma 6.5, we get

$$\begin{array}{ccc} S^1 \wedge (BC_{p^m})_+ & \to & Q(\Lambda_{[1]}BC_{p^m}) \\ \downarrow 1\wedge i_m & & \uparrow \mathrm{Res} \\ S^1 \wedge (BC_{p^{m+1}})_+ & \to & Q(\Lambda_{[1]}BC_{p^{m+1}}) \end{array}$$

and hence a mapping

$$\underrightarrow{\mathrm{holim}}\, S^1 \wedge (BC_{p^m})_+ \to \underleftarrow{\mathrm{holim}}\, Q(\Lambda_{[1]}BC_{p^m})\,.$$

Both spaces are understood to be completed at p. The left hand side is $\Sigma_+(\mathbb{C}P^\infty)$ by (6.7), and the right hand side is $\tilde{Q}(\Sigma_+\mathbb{C}P^\infty)$ according to Proposition 5.15 and Proposition 7.15. The S^1-transfer

$$\mathrm{Trf}\colon \tilde{Q}(\Sigma_+\mathbb{C}P^\infty) \to Q(BC_{p^m})\,,$$

induces a surjective homomorphism

$$H_{1+2n}(\Sigma_+\mathbb{C}P^\infty) \to H_{1+2n}(BC_{p^m})$$

for all $m \geqq 1$, [MMM]. Thus to evaluate in spectrum homology the map induced from

$$(*) \qquad \tilde{Q}(\Sigma_+\mathbb{C}P^\infty)^\wedge_p \to \tilde{Q}(\Sigma_+\mathbb{C}P^\infty)^\wedge_p$$

we can compose with the S^1-transfer in the range and with the inclusion of $\tilde{Q}(\Sigma_+BC_{p^m})$ into $\tilde{Q}(\Sigma_+\mathbb{C}P^\infty)^\wedge_p$ in the domain.

The above homology calculation shows that (*) induces an isomorphism in homology and so defines a homotopy equivalence of spectra. Its composition with the S^1-transfer $Q(\Sigma_+\mathbb{C}P^\infty) \to Q(*)$ is equal to $\mathrm{Tr} \circ \varepsilon^\#$. □

Recall for any space X that $Wh(X) = Wh^{Diff}(X)$ is the homotopy fibre of

$$\mathrm{Tr}\colon A(X) \to Q(X)$$

and that

$$A(X) = Wh(X) \times Q(X)$$

(cf. [W4]). Similarly for $R = \mathbb{Z}[1/g]$ or $\hat{\mathbb{Z}}_p$ we can define $Wh(X;R)^\wedge_p$ as the fibre of

$$\mathrm{Tr}\colon A(X;R)^\wedge_p \to Q(X)^\wedge_p$$

and obtain a splitting

$$A(X;R)^\wedge_p \simeq Wh(X;R)^\wedge_p \times Q(X)^\wedge_p\,.$$

Comparing with (9.1) we see that the cyclotomic trace gives a map

$$\mathrm{Trc}\colon Wh(*;R)^\wedge_p \to \mathrm{hofibre}(\tilde{Q}(\Sigma_+\mathbb{C}P^\infty)^\wedge_p \xrightarrow{\mathrm{Trf}} Q(*)^\wedge_p)\,.$$

If $R = \mathbb{Z}$ or $\mathbb{Z}[1/g]$, Lemma 9.4 shows that the Soulé embedding $\varepsilon^{\#}$ lands entirely in $Wh(*; R)_p^{\wedge}$ whereas for $R = \hat{\mathbb{Z}}_p$, we naturally have

$$\varepsilon^{\#}\colon \text{hofibre}(\tilde{Q}(\Sigma_+ \mathbb{C}P^{\infty})_p^{\wedge} \to Q(*)_p^{\wedge}) \to Wh(*; \hat{\mathbb{Z}}_p)_p^{\wedge} .$$

In all three cases, the composition (see (5.16) for the definition of α)

$$Q(\Sigma_+ \mathbb{C}P^{\infty})_p^{\wedge} \xrightarrow{\varepsilon^{\#}} A(*; R)_p^{\wedge} \xrightarrow{\alpha \circ \mathrm{Trc}} Q(\Sigma_+ \mathbb{C}P^{\infty})_p^{\wedge}$$

was calculated in spectrum homology in Corollary 8.22; it multiplies by the p-adic number

$$\theta_n(\varepsilon) = \varprojlim \sum_{(k,p)=1} (1/k)^{1+n} c_k(\varepsilon_m)$$

in dimension $1 + 2n$. These numbers were expressed in terms of more common number theoretic functions in [BM, Theorem 4.9] and in [Be]. We quote the result. Let $L_p(\ \ , \omega^{-n})$ be the p-adic L-function with respect to the $(-n)$'th power of the Teichmuller character.

Theorem 9.5 *Let p be an odd prime. Then* (i) $R = \mathbb{Z}$: *The element* $\varepsilon = (\varepsilon_n)$ *listed* 9.3(i) *has* $\theta_n(\varepsilon) = (\omega(g)^{-n} - 1)L_p(1 + n, \omega^{-n})$
(ii) $R = \mathbb{Z}[1/g]$: *The element* ε *from* 9.3(ii) *has* $\theta_n(\varepsilon) = (1 - g^{-n})L_p(1 + n, \omega^{-n})$.
(iii) $R = \hat{\mathbb{Z}}_p$: *There exists a non trivial* $\lambda \in \hat{\mathbb{Z}}_p^{\times}$ *with* $\lambda^{p-1} = 1$ *such that the unit* ε *in* 9.3 (iii) *has* $\theta_n(\varepsilon) \in \hat{\mathbb{Z}}_p^{\times}$ *for* $n \not\equiv -1 \pmod{p-1}$. *Moreover, if* $p > 3$ *and* $n \equiv -1 \pmod{p-1}$ *then the p-adic valuation on* $\theta_n(\varepsilon)$ *is* $v_p(\theta_n(\varepsilon)) = 1 + v_p(1 + n)$.

Actually the results of [BM] and [Be] are a little stronger than indicated in Theorem 9.5 in that [BM] shows that we cannot choose "better" units than the ones specified in (9.3).

Remark 9.6 (i) For $p = 2$ and $R = \mathbb{Z}[1/3]$ we have the compatible units $\varepsilon_n = \frac{1}{3}(1 + T_n + T_n^{-1})$, and in this case $\alpha \circ \mathrm{Trc} \circ \varepsilon^{\#}$ multiplies with 2 in $H_{1+4n}(-; \hat{\mathbb{Z}}_2)$ and is zero in $H_{3+4n}(-; \hat{\mathbb{Z}}_2)$.

(ii) For p odd, let B_k be the k'th Bernoulli number, that is,

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} (B_k/k!)t^k$$

The value of $L_p(1 + n, \omega^{-n}) \in \hat{Q}_p$ can be given modulo p^m as follows: let $k = p^{m-1}(p - 1) - n$. Then

$$B_k/k \equiv (p^{k-1} - 1)L_p(1 + n, \omega^{-n}) \pmod{p^m} .$$

In particular if $n \equiv 0 \pmod{p-1}$ then

$$(g^{-n} - 1)L_p(1 + n, \omega^{-n})$$

is a p-adic unit by von Staudt's theorem. For n odd, $L_p(1 + n, \omega^{-n}) = 0$. For n even and $n \not\equiv 0 \pmod{p-1}$, $(g^{-n} - 1)$ is a p-adic unit and $L_p(1 + n, \omega^{-n})$ is a unit if and only if p does not divide $B_{p-1-n}/(p - 1 - n)$. This happens for all even n precisely if p is a regular prime.

Since $L_p(1+n, \omega^{-n}) = 0$ for n odd, the composition $\mathrm{Trc} \circ \varepsilon^{\#}$ is only non-zero in dimensions congruent to 1 (mod 4) when $R = \mathbb{Z}$ or $\mathbb{Z}[1/g]$. Let us therefore project from $\mathbb{C}P^{\infty} = BS^1$ to $BO(2)$, and consider

$$\mathrm{Trc}\colon Wh(*; \mathbb{Z}[1/g])_p^{\wedge} \to \tilde{Q}(\Sigma_+ BO(2))_p^{\wedge}$$

By (6.10) and (6.11), the Soulé embedding $\varepsilon^{\#}\colon \tilde{Q}(\Sigma_+ \mathbb{C}P^{\infty})_p^{\wedge} \to Wh(*; \mathbb{Z}[1/g])_p^{\wedge}$ factors over $\tilde{Q}(\Sigma_+ BO(2))_p^{\wedge}$, so we have the composition

(9.7) $$\tilde{Q}(\Sigma_+ BO(2))_p^{\wedge} \xrightarrow{\varepsilon^{\#}} Wh(*; \mathbb{Z}[1/g])_p^{\wedge} \xrightarrow{\alpha \circ \mathrm{Trc}} \tilde{Q}(\Sigma_+ BO(2))_p^{\wedge} .$$

Corollary 9.8 *The composition in* (9.7) *is a homotopy equivalence at odd regular primes.*

For odd primes p,

(9.9) $$\tilde{Q}(\Sigma_+ \mathbb{C}P^{\infty})_p^{\wedge} = \prod_{i=0}^{p-2} \tilde{Q}(X_i)_p^{\wedge}$$

cf. [McG], where the i'th factor has homology concentrated in dimensions $1+2n$ with $n \equiv i(p-1)$. The even components together give a splitting of $\tilde{Q}(\Sigma_+ BO(2))_p^{\wedge}$. The S^1-transfer from $\tilde{Q}(\Sigma_+ \mathbb{C}P^{\infty})$ is known to be non-trivial precisely on the factor $\tilde{Q}(X_{p-2})_p^{\wedge}$.

Corollary 9.10 *For $R = \hat{\mathbb{Z}}_p$ and with the units of* 9.3 (iii) *the composition*

$$\tilde{Q}(\Sigma_+ \mathbb{C}P^{\infty})_p^{\wedge} \xrightarrow{\varepsilon^{\#}} Wh(*; \hat{\mathbb{Z}}_p)_p^{\wedge} \xrightarrow{\mathrm{Trc}} \tilde{Q}(\Sigma_+ \mathbb{C}P^{\infty})_p^{\wedge}$$

is a homotopy equivalence of the factors of (9.9) *corresponding to $0 \leq i \leq p-3$. For $p > 3$ it induces multiplication by $(1+n)$ in $H_{1+2n}^{\mathrm{spec}}(\tilde{Q}(X_{p-2}))$ when $1 + 2n \equiv -1 (\mathrm{mod}\, 2(p-1))$.*

The natural map from $A(*)_p^{\wedge}$ to $A(*, \mathbb{Z}[1/g])_p^{\wedge}$ has fibre $K(\mathbb{F}_g)_p^{\wedge} = (\mathrm{Im}\, J \times \mathbb{Z})_p^{\wedge}$. By elementary obstruction theory there are no non-trivial maps from $\Sigma\mathbb{C}P^{\infty}$ into $BK(\mathbb{F}_g)_p^{\wedge}$. Consequently the Soulé embedding

$$\Sigma_+ \mathbb{C}P^{\infty} \xrightarrow{\varepsilon^{\#}} A(*; \mathbb{Z}[1/g])_p^{\wedge}$$

restricted to $\Sigma\mathbb{C}P^{\infty}$ lifts to $A(*)_p^{\wedge}$. The composition

(9.11) $$\Sigma\mathbb{C}P^{\infty} \xrightarrow{\varepsilon^{\#}} A(*)_p^{\wedge} \xrightarrow{\alpha \circ \mathrm{Trc}} \tilde{Q}(\Sigma_+ \mathbb{C}P^{\infty})_p^{\wedge}$$

was evaluated in homology above, and we get

Corollary 9.12 (i) *The cyclotomic trace map*

$$\mathrm{Trc}\colon Wh(*)_p^{\wedge} \to Q(\Sigma BO(2))_p^{\wedge}$$

is split surjective if p is an (odd) regular prime.

(ii) *It is a rational equivalence if $L_p(1+2n, \omega^{-2n}) \neq 0$ for all n.*

The rest of the section is devoted a proof of the K-theory analogue of Novikov's conjecture. Let Γ be a *discrete* group. We introduce the following

Condition (C) There exists a prime p such that

$$\text{(i) } L_p(1 + 2n, \omega^{-2n}) \neq 0 \text{ for all } n$$

$$\text{(ii) } H_i(B\Gamma) \otimes \mathbb{Q} \to [\varprojlim H_i(B\Gamma; \mathbb{Z}/p^n)] \otimes \mathbb{Q}$$

is injective for each i.

No prime number is known where (i) is not satisfied, and it may well be true in general. However, as with regular primes it seems one cannot prove that there are even an infinite set of primes where (i) is valid. The second condition is satisfied for all primes if the integral group homology $H_i(\Gamma) = H_i(B\Gamma)$ is finitely generated.

Theorem 9.13 *Suppose Γ satisfies Condition* (C). *Then the assembly map $B\Gamma_+ \wedge A(*) \to A(B\Gamma)$ is rationally split injective; it induces a split injection on homotopy groups of spectra.*

Proof. Let us first note that we may use Waldhausen's decomposition of $A(X)$ as a product of $Q(X)$ and $Wh(X) = Wh^{\mathrm{Diff}}(X)$ to divide the problem into two. We need to show that the two maps

$$B\Gamma_+ \wedge Q(*) \to Q(B\Gamma)$$

$$B\Gamma_+ \wedge Wh(*) \to Wh(B\Gamma)$$

are rationally injective. The first is obvious. For the second we use that the cyclotomic trace commutes with the assembly maps, cf. (6.2), so that we have the homotopy commutative diagram

$$\begin{array}{ccc} B\Gamma_+ \wedge Wh(*) & \xrightarrow{\mu_A} & Wh(B\Gamma) \\ \downarrow 1 \wedge \mathrm{Trc} & & \downarrow \mathrm{Trc} \\ B\Gamma_+ \wedge TC(*, p) & \xrightarrow{\mu_{TC}} & TC(B\Gamma, p)\,. \end{array}$$

We further compose with

$$\alpha\colon TC(X, p) \to C(X, p)\,,$$

complete at p, and use (5.15) to get the diagram

$$(*) \qquad \begin{array}{ccc} B\Gamma_+ \wedge Wh(*) & \to & Wh(B\Gamma) \\ \downarrow & & \downarrow \\ B\Gamma_+ \wedge \tilde{Q}(\Sigma_+ \mathbb{C}P^\infty)_p^\wedge & \to & \operatorname{holim}_{\leftarrow} Q(EC_{p^n} \times_{C_{p^n}} \Lambda B\Gamma)_p^\wedge\,. \end{array}$$

The bottom horizontal map μ_{TC} is the composition

$$\begin{aligned} B\Gamma_+ \wedge Q(\Sigma_+ \mathbb{C}P^\infty)_p^\wedge &\xrightarrow{1 \wedge t_n} B\Gamma_+ \wedge Q(BC_{p^n}) \\ &\longrightarrow Q(EC_{p^n} \times_{C_{p^n}} B\Gamma) \\ &\xrightarrow{1 \times i} Q(EC_{p^n} \times_{C_{p^n}} \Lambda B\Gamma) \end{aligned}$$

with t_n: $Q(\Sigma_+ \mathbb{C}P^\infty) \to Q(BC_{p^n})$ the relevant S^1-transfer, and i: $B\Gamma \to \Lambda B\Gamma$ the inclusion in the constant loops, cf. the discussion following (6.3).

The reader is reminded that the smash products in (*) take place in the category of infinite loop spaces (or spectra); so by definition

$$B\Gamma_+ \wedge \tilde{Q}(\Sigma_+ \mathbb{C}P^\infty) \simeq \tilde{Q}(\Sigma_+(B\Gamma \times \mathbb{C}P^\infty))$$

and the assembly map μ_{TC} in (*) is induced from

$$1 \times i\colon \mathbb{C}P^\infty \times B\Gamma \to \mathbb{C}P^\infty \times \Lambda B\Gamma$$

where i is the inclusion into the constant loops. There are S^1-equivariant maps

$$\Lambda B\Gamma_+ \xrightarrow{\mathrm{pr}_{[1]}} \Lambda_{[1]} B\Gamma_+ \xleftarrow{\ i\ } B\Gamma_+$$

where as before $\Lambda_{[1]}B\Gamma$ is the component of homotopically trivial loops. The inclusion i is non-equivariantly a homotopy equivalence, so induces an equivalence

$$Q(BC_{p^n} \times B\Gamma) \xrightarrow{\simeq} Q(EC_{p^n} \times_{C_{p^n}} \Lambda_{[1]} B\Gamma)\,.$$

Its inverse composed with the map induced from $pr_{[1]}$ gives

$$P_{[1]}\colon Q(EC_{p^n} \times_{C_{p^n}} \Lambda B\Gamma) \to Q(BC_{p^n} \times B\Gamma)\,.$$

From (*) we then get the homotopy commutative diagram

$$(**)\qquad \begin{array}{ccc} B\Gamma_+ \wedge Wh(*) & \longrightarrow & Wh(B\Gamma) \\ \downarrow 1 \wedge \alpha \circ \mathrm{Trc} & & \downarrow P_{[1]} \circ \alpha \circ \mathrm{Trc} \\ \tilde{Q}(B\Gamma_+ \wedge \Sigma_+ \mathbb{C}P^\infty)_p^\wedge & \xrightarrow{T} & \operatorname{holim}_{\leftarrow} Q(BC_{p^n} \times B\Gamma)_p^\wedge\,. \end{array}$$

We show that the bottom horizontal map induces an injection on rational homotopy groups. For any spectrum the rationalized Hurewicz map is an isomorphism, so

$$\begin{aligned} \pi_m(\tilde{Q}(B\Gamma_+ \wedge \Sigma_+ \mathbb{C}P^\infty)) \otimes \mathbb{Q} &\xrightarrow{\cong} H_m^{\mathrm{spec}}(\tilde{Q}(B\Gamma_+ \wedge \Sigma_+ \mathbb{C}P^\infty)) \otimes \mathbb{Q} \\ &= \tilde{H}(B\Gamma_+ \wedge \Sigma_+ \mathbb{C}P^\infty) \otimes \mathbb{Q} \\ &\cong \Sigma^\oplus H_{m-2i-1}(B\Gamma) \otimes \mathbb{Q} \end{aligned}$$

is an isomorphism. Consider the homomorphism

$$\begin{aligned} \pi_m(\operatorname{holim}_{\leftarrow} Q(BC_{p^n} \times B\Gamma)_p^\wedge) &\to \lim_{\leftarrow} \pi_m(Q(BC_{p^n} \times B\Gamma)_p^\wedge) \\ &\to \lim_{\leftarrow} H_m(BC_{p^n} \times B\Gamma; \mathbb{Z}/p^n) \\ &\to \lim_{\leftarrow} \Sigma^\oplus H_{m-2i-1}(B\Gamma; \mathbb{Z}/p^n)\,. \end{aligned}$$

Where the last arrow is from the Künneth theorem. The composition of this homomorphism with T_* from (**),

$$\Sigma^{\oplus} H_{m-2i-1}(B\Gamma)\otimes \mathbb{Q} \to \Sigma^{\oplus}\operatorname{ho}\varprojlim_n H_{m-2i-1}(B\Gamma; \mathbb{Z}/p^n)\otimes \mathbb{Q}$$

can be identified with the (direct sum of the) natural homomorphism induced from reducing from $\mathbb{Z}$ coefficients to $\mathbb{Z}/p^n$ coefficient. This follows because the S^1-transfer $\tilde{Q}(\Sigma_+ \mathbb{C}P^\infty)\to Q(BC_{p^n})$ in odd dimensional spectrum homology surjects $\mathbb{Z}$ to $\mathbb{Z}/p^n$, cf. [MMM]. Condition C(ii) now tells us that $T_*\otimes\mathbb{Q}$ is injective.

We next use the Soulé embedding from (9.7)

$$\varepsilon^{\#}\colon \tilde{Q}(\Sigma_+ BO(2)) \to Wh(*; \mathbb{Z}[1/g])_p^{\wedge}$$

whose composition with the cyclotomic trace is multiplication with $(1-g^{-2k})L_p(1+2k, \omega^{-2k})$ is spectrum homology in degree $4k+1$ by (9.5.ii). From (6.20) we know that the 1-connected cover of $Wh(*; \mathbb{Z}[1/g])_p^{\wedge}$ is rationally equivalent to $Wh(*)_p^{\wedge}$, so

$$\pi_m((B\Gamma_+ \wedge Wh(*))_p^{\wedge})\otimes\mathbb{Q} \cong \pi_m((B\Gamma_+ \wedge Wh(*; \mathbb{Z}[1/g]))_p^{\wedge})\otimes\mathbb{Q}$$

for $m>1$. With this and the above, we are reduced to showing that $(1\wedge \alpha\circ \mathrm{Trc}\circ\varepsilon^{\#})$ induces an isomorphism from the rational vector space

$$\varprojlim_s H_n(\Sigma_+(B\Gamma\times BO(2)); \mathbb{Z}/p^s)\otimes\mathbb{Q} = \sum_{k\geqq 1}^{\oplus}\varprojlim_s H_{n-4k-1}(B\Gamma; \mathbb{Z}/p^s)\otimes\mathbb{Q}$$

to itself. The induced map on the k'th summand is multiplication with $(1-g^{-2k})L_p(1+2k, \omega^{-2k})$ which we assume in Condition C(i) to be non-trivial. □

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