# The *d*-Majorization Polytope

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## Abstract

We investigate geometric and topological properties of d-majorization – a generalization of classical majorization to positive weight vectors  $d \in \mathbb{R}^n$ . In particular, we derive a new, simplified characterization of d-majorization which allows us to work out a halfspace description of the corresponding d-majorization polytopes. That is, we write the set of all vectors which are d-majorized by some given vector  $y \in \mathbb{R}^n$  as an intersection of finitely many half spaces, i.e. as solutions to an inequality of the type  $Mx \leq b$ . Here b depends on y while M can be chosen independently of y. This description lets us prove continuity of the d-majorization polytope (jointly with respect to d and y) and, furthermore, lets us fully characterize its extreme points. Interestingly, for  $y \geq 0$  one of these extreme points classically majorizes every other element of the d-majorization polytope.

Moreover, we show that the induced preorder structure on  $\mathbb{R}^n$  admits minimal and maximal elements. While the former are always unique the latter are unique if and only if they correspond to the unique minimal entry of the *d*-vector.

Keywords: majorization relative to d, d-majorization polytope, convex polytopes, extreme points

2020 MSC: 15B51, 26A51, 52B11, 52B12

### 1. Introduction

The concept of *d*-majorization is a natural generalization of the classical notion of majorization as we will see in a moment. But let us first describe a deep relation between *d*majorization, *d*-stochastic matrices, and quantum physics. Readers who are not familiar with the relevant quantum-mechanical terminology can skip the following two paragraphs as although we will draw some connections to the physics literature throughout this article—we will not use any of these results.

Over the last few years, sparked by Brandão, Horodecki, Oppenheim [5, 26], and further pursued by others [19, 21, 35, 46, 40, 1], thermo-majorization and in particular its resource theory approach has been a widely discussed and researched topic in quantum physics. Here the central question is: Given a fixed "background temperature" T > 0 as well as initial and target states (density operators) of a quantum system, can the former be mapped to the latter by means of a thermal operation? Here the set of all thermal operations which constitutes a compact, convex semigroup within the set of all quantum channels [37, App. C] consists of all linear maps which can be approximated arbitrarily well by quantum channels of the form

$$\Phi(\cdot) = \operatorname{tr}_B\left(U\left((\cdot) \otimes \frac{e^{-H_B/T}}{\operatorname{tr}(e^{-H_B/T})}\right)U^*\right),$$

where  $H_B$  is an arbitrary "bath" Hamiltonian,  $H_S$  the system's Hamiltonian, U any unitary operator satisfying the stabilizer condition  $U(H_S \otimes \mathbb{1}_B + \mathbb{1}_S \otimes H_B)U^* = H_S \otimes \mathbb{1}_B + \mathbb{1}_S \otimes H_B$ , and tr<sub>B</sub> denotes the partial trace operation with respect to the "bath", cf. [27, 6].

Two key properties of thermal operations are that  $e^{-H_S/T}$  is a fixed point of all thermal operations—that is, the Gibbs state of the system is preserved—and that they commute with the system's "natural" dynamics at all times, i.e.

$$\Phi(e^{-iH_S t}(\cdot)e^{iH_S t}) = e^{-iH_S t}\Phi(\cdot)e^{iH_S t},\tag{1}$$

for all  $t \ge 0$  which is equivalent to  $[\Phi, \mathrm{ad}_{H_S}] = 0$  [36]. However, these two features do *not* fully characterize thermal operations, meaning there exist quantum channels which satisfy the above properties but lie outside the set of thermal operations [13]. For a comprehensive introduction to this topic we refer to the review article [34].

For finite-dimensional systems, the above commutation relation (1) allows for a partial answer to the state-conversion problem posed above because it reveals that  $\Phi$  and  $\mathrm{ad}_{H_S}$  share common invariant subspaces. In particular, if  $H_S \in \mathbb{C}^{n \times n}$  is diagonal with non-degenerate spectrum the set of diagonal density matrices is a common invariant and the action of thermal operations on it coincides with the action of d-stochastic<sup>1</sup> matrices on  $\mathbb{R}^n$ , cf. [34, Thm. 1]. The vector d then consists of the diagonal<sup>2</sup> entries of the matrix  $e^{-H_S/T}$  (up to a global constant which can be disregarded). Thus, if the initial state and the final state are both "diagonal" then the d-stochastic matrices fully characterize all possible state transitions [26, 30].

From a mathematical point of view this puts us in the realm of majorization relative to a positive vector  $d \in \mathbb{R}^n$  as introduced by Veinott [51] and, in the quantum regime, by Ruch, Schranner, and Seligman [45]. For positive d, some vector x is said to be d-majorized by y, denoted by  $x \prec_d y$ , if there exists a d-stochastic matrix A such that x = Ay. A variety of characterizations of  $\prec_d$  and d-stochastic matrices can be found in the work of Joe [28] and in Prop. 1 below.

Certainly, the concept of classical majorization as first introduced by Muirhead [42] and more widely spread by Hardy, Littlewood, and Pólya [23] is a special case of *d*-majorization. More precisely, one says that a vector  $x \in \mathbb{R}^n$  is classically majorized by  $y \in \mathbb{R}^n$ , denoted by  $x \prec y$ , if  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$  and  $\sum_{i=1}^j x_{[i]} \leq \sum_{i=1}^j y_{[i]}$  for all  $j = 1, \ldots, n-1$ , where  $x_{[i]}, y_{[i]}$ are the components of x, y in decreasing order. This is well known to be equivalent to the existence of a doubly-stochastic matrix A, that is, a *d*-stochastic matrix with  $d = (1, \ldots, 1)^{\top}$ , such that x = Ay [23, Thm. 46]. A comprehensive survey on classical majorization can be found in [39]. For numerous applications in various fields of science we also refer to [33, 12, 43, 4, 48, 17].

<sup>&</sup>lt;sup>1</sup>A matrix  $A \in \mathbb{R}^{n \times n}$  is said to be *d*-stochastic if it is column-stochastic with Ad = d, where column-stochastic means that all its entries are non-negative and each column sums up to one.

<sup>&</sup>lt;sup>2</sup>Certainly, it suffices to require that the density matrices are represented in an eigenbasis of  $H_S$  in which case *d* consists of the eigenvalues of  $e^{-H_S/T}$ .

Interestingly, for  $x, y \in \mathbb{R}^n$  one has  $x \prec y$  if and only if x lies in the convex hull of all permutations of y (as shown in, e.g., [44], or as a direct consequence of Birkhoff's theorem [39, Ch. 2, Thm. A.2]). Therefore the set  $\{x \in \mathbb{R}^n : x \prec y\}$  is a convex polytope with at most n! corners. Hence it has a half-space description, that is, it can be written as the intersection of finitely many half-spaces or equivalently as the solution to finitely many linear (in-)equalities. The precise result as first stated in [11, Thm. 1] reads as follows: Given  $y \in \mathbb{R}^n$  one has

$$\{x \in \mathbb{R}^n : x \prec y\} = \left\{x \in \mathbb{R}^n : \left(\sum_{j=1}^n x_i = \sum_{j=1}^n y_i\right) \land \left(\forall_{m \in \{0,1\}^n} \ m^\top x \le \sum_{i=1}^{m_1 + \dots + m_n} y_{[i]}\right)\right\}$$

This result motivated us to work out a half-space description of "the" *d*-majorization polytope to study its extreme points as well as further topological properties.

In the main, this manuscript is concerned with analyzing the preorder structure of d-majorization and its geometry. It is organized as follows: First, in Section 2 we extend the list of existing characterizations of d-majorization by a novel one, which allows to check d-majorization in finitely many steps (Prop. 1 (vi)), and we identify the minimal and maximal elements of this preorder (Thm. 3). In Section 3 we briefly revisit convex polytopes and their equivalent descriptions before studying d-majorization from this perspective in Section 4. In particular, the novel characterization of Prop. 1 allows us to work out a half-space description of "the" corresponding d-majorization polytope (Thm. 10) and to prove its continuity with respect to d and y (Thm. 12). Moreover, we identify its extreme points (Thm. 14) and – as for classical majorization – we show that the number of extreme points is always upper bounded by n! (Coro. 15). Finally we conclude that if the "initial" vector y is non-negative, then one of the extreme points majorizes every element of the polytope classically (Thm. 16).

#### 2. Characterizations and Preorder Properties of *d*-Majorization

For the purpose of this paper be aware of the following notions and notations:

- In accordance with Marshall and Olkin [39],  $\mathbb{R}^n_+$  ( $\mathbb{R}^n_{++}$ ) denotes the set of all real vectors with non-negative (strictly positive) entries. Whenever it is clear that x is a real vector of length n we occasionally write  $x \ge 0$  (x > 0) to express non-negativity (strict positivity) of its entries.
- e shall denote the column vector of ones, i.e.  $\mathbf{e} = (1, \dots, 1)^{\mathsf{T}}$ .
- $\|\cdot\|_1$  is the usual 1-norm on  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ ).
- $S_n$  is the symmetric group (the group of all permutations of  $\{1, \ldots, n\}$ ).
- The standard simplex  $\Delta^{n-1} \subseteq \mathbb{R}^n$  is given by the convex hull of all standard basis vectors  $e_1, \ldots, e_n$  and precisely contains all probability vectors, that is, all vectors  $x \in \mathbb{R}^n_+$  with  $e^{\top}x = 1$ .

Having reviewed classical vector majorization as well as its polytope properties in the introduction, let us now dive into the "non-symmetric" case of majorization, that is, the case where doubly stochastic matrices (having e as "left" and "right" fixed point) are replaced by d-stochastic matrices (having e and  $d \in \mathbb{R}^n_{++}$  as "left" and "right" fixed point, respectively). As also explained in the introduction this concept is closely related to thermo-majorization and quantum thermodynamics in general, and we will occasionally point out some connections to the physics literature if appropriate. For the following definition we mostly<sup>3</sup> follow [39, p. 585].

**Definition 1.** Let  $d \in \mathbb{R}^n_{++}$  and  $x, y \in \mathbb{R}^n$  be given. A square matrix  $A \in \mathbb{R}^{n \times n}$  is said to be *column-stochastic* if  $A_{ij} \geq 0$  for all i, j = 1, ..., n (i.e.  $A \in \mathbb{R}^{n \times n}_+$ ) and  $\mathbb{e}^{\top} A = \mathbb{e}^{\top}$ . If, additionally, Ad = d then A is said to be *d-stochastic*. The set of all *d*-stochastic  $n \times n$  matrices is denoted by  $s_d(n)$ . Moreover, x is said to be *d-majorized* by y, denoted by  $x \prec_d y$ , if there exists  $A \in s_d(n)$  such that x = Ay.

In particular,  $x \prec_d y$  implies  $e^{\top}x = e^{\top}Ay = e^{\top}y$ . Also note that this definition of  $\prec_d$  naturally generalizes to complex vectors, cf. also [20].

**Remark 1.** (i) For any  $d \in \mathbb{R}^n_{++}$ , the set  $s_d(n)$  constitutes a convex, compact subsemigroup of  $\mathbb{C}^{n \times n}$  with identity element  $I_n$ . In particular it acts contractively in the 1-norm: for all  $z \in \mathbb{R}^n$  and  $A \in \mathbb{R}^{n \times n}_+$  with  $e^{\top}A = e^{\top}$  one has the estimate

$$||Az||_1 = \sum_{i=1}^n \left| \sum_{j=1}^n A_{ij} z_j \right| \le \sum_{i,j=1}^n A_{ij} |z_j| = \sum_{j=1}^n \left( \sum_{i=1}^n A_{ij} \right) |z_j| = \sum_{j=1}^n |z_j| = ||z||_1.$$
(2)

- (ii) Note that d-majorization is a special case of so-called matrix majorization: Given matrices A ∈ ℝ<sup>m×p</sup>, B ∈ ℝ<sup>n×p</sup> keeping in mind footnote 3 one says A majorizes B (denoted by A ≺ B) if there exists X ∈ ℝ<sup>m×n</sup><sub>+</sub> with e<sup>T</sup>X = e<sup>T</sup> such that XB = A [10, 9]. With this one recovers d-majorization by setting B = (d y), A = (d x) because then A ≺ B holds iff x ≺<sub>d</sub> y as is readily verified.
- (iii) By Minkowski's theorem [7, Thm. 5.10], the previous point implies that  $s_d(n)$  can be written as the convex hull of its extreme points. However—unless d = e—this does not prove to be all too helpful as stating said extreme points (for n > 2) becomes quite delicate<sup>4</sup>. To substantiate this the extreme points for n = 3 and non-degenerate  $d \in \mathbb{R}^3_{++}$  can be found in Lemma 18 (Appendix A).
- (iv) If some entries of the *d*-vector coincide, then  $\prec_d$  is known to be a preordering but not a partial ordering. Contrary to what is written in [28, Rem. 4.2] this in general does not change if the entries of *d* are pairwise distinct: To see this, consider  $d = (3, 2, 1)^{\top}$ ,  $x = (1, 0, 0)^{\top}$ ,  $y = (0, \frac{2}{3}, \frac{1}{3})^{\top}$ , and

$$A = \begin{pmatrix} 0 & 1 & 1 \\ \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{pmatrix} \in s_d(3) \,.$$

<sup>&</sup>lt;sup>3</sup>Usually *d*-stochastic matrices are defined via  $d^{\top}A = d^{\top}$  and A e = e which is equivalent to the definition below as it only differs by transposing once. This is because we consider d, x, y to be usual column vectors whereas [28, 39] consider row vectors.

<sup>&</sup>lt;sup>4</sup>The number of extreme points of  $s_d(n)$  is lower bounded by n! and upper bounded by  $\binom{n^2}{2n-1}$ , cf. [28, Rem. 4.5].

Then Ax = y and Ay = x so  $x \prec_d y \prec_d x$ , but obviously  $x \neq y$ .

Now let us summarize the known characterizations of  $\prec_d$ : While the equivalences of (i) through (v) in the following proposition are due to Joe [28, Thm. 2.2], number (vi) will be a new result of ours. Moreover, (vii) is related to the definition most prominent among the physics literature, called "thermo-majorization curves" [26]. Indeed the criterion (vii) we present here is a more explicit version of [2, Thm. 4]; for more on this, cf. Ch. 4.3.

**Proposition 1.** Let  $d \in \mathbb{R}^n_{++}$  and  $x, y \in \mathbb{R}^n$  be given. The following are equivalent.

- (i)  $x \prec_d y$
- (ii)  $\sum_{j=1}^{n} d_j \psi(\frac{x_j}{d_j}) \leq \sum_{j=1}^{n} d_j \psi(\frac{y_j}{d_j})$  for all continuous, convex functions  $\psi : D(\psi) \subseteq \mathbb{R} \to \mathbb{R}$  such that  $\{\frac{x_j}{d_j} : j = 1, \dots, n\}, \{\frac{y_j}{d_j} : j = 1, \dots, n\} \subseteq D(\psi).$
- (*iii*)  $\sum_{j=1}^{n} (x_j td_j)_+ \leq \sum_{j=1}^{n} (y_j td_j)_+$  for all  $t \in \mathbb{R}$  where  $(\cdot)_+ := \max\{\cdot, 0\}$ .
- (*iv*)  $\sum_{j=1}^{n} (x_j td_j)_+ \le \sum_{j=1}^{n} (y_j td_j)_+$  for all  $t \in \{\frac{x_i}{d_i}, \frac{y_i}{d_i} : i = 1, \dots, n\}.$
- (v)  $||x td||_1 \le ||y td||_1$  (i.e.  $\sum_{j=1}^n |x_j td_j| \le \sum_{j=1}^n |y_j td_j|$ ) for all  $t \in \mathbb{R}$ .
- (vi)  $e^{\top}x = e^{\top}y$  and  $||x \frac{y_i}{d_i}d||_1 \le ||y \frac{y_i}{d_i}d||_1$  for all i = 1, ..., n.

(vii)  $\mathbf{e}^{\top} x = \mathbf{e}^{\top} y$  and for all  $j = 1, \dots, n-1$ 

$$\sum_{i=1}^{j} x_{\sigma(i)} \leq \min_{i=1,\dots,n} \left( \mathbb{e}^{\top} \left( y - \frac{y_i}{d_i} d \right)_+ + \frac{y_i}{d_i} \left( \sum_{k=1}^{j} d_{\sigma(k)} \right) \right)$$

where  $\sigma \in S_n$  is any permutation such that  $\frac{x_{\sigma(1)}}{d_{\sigma(1)}} \ge \ldots \ge \frac{x_{\sigma(n)}}{d_{\sigma(n)}}$ .

*Proof.* (v)  $\Rightarrow$  (vi): For t large enough all entries of x - td, y - td are non-positive so

$$-e^{\top}(x-td) = ||x-td||_1 \le ||y-td||_1 = -e^{\top}(y-td)$$

and thus  $e^{\top}x \ge e^{\top}y$ . Doing the same for -t large enough gives  $e^{\top}x \le e^{\top}y$  so together  $e^{\top}x = e^{\top}y$ .

(vi)  $\Rightarrow$  (v): Define  $P := \{\frac{x_i}{d_i}, \frac{y_i}{d_i} : i = 1, \dots, n\}$ ; w.l.o.g. |P| > 1. As argued before,  $\mathbb{e}^\top x = \mathbb{e}^\top y$  implies  $||x - td||_1 = ||y - td||_1$  on  $t \in (-\infty, \min P] \cup [\max P, \infty)$ . Now define

$$g_x : [\min P, \max P] \to \mathbb{R}_+$$
$$t \mapsto \|x - td\|_1 = \sum_{i=1}^n d_i \left| \frac{x_i}{d_i} - t \right|$$

and  $g_y$  analogously. Thus all that is left to show is  $g_x(t) \leq g_y(t)$  for all  $t \in [\min P, \max P]$ .

First note that  $g_x(\min P) = g_y(\min P)$ ,  $g_x(\max P) = g_y(\max P)$ , and  $g_x(\frac{y_i}{d_i}) \le g_y(\frac{y_i}{d_i})$  for all  $i = 1, \ldots, n$  by assumption, meaning we can choose  $t \in (\min P, \max P) \setminus \{\frac{y_1}{d_1}, \ldots, \frac{y_n}{d_n}\}$ .

Hence there exists  $i = 0, \ldots, n$  such that  $\frac{y_i}{d_i} < t < \frac{y_{i+1}}{d_{i+1}}$  where  $\frac{y_0}{d_0} := \min P$ ,  $\frac{y_{n+1}}{d_{n+1}} := \max P$ . Defining  $\lambda := (\frac{y_{i+1}}{d_{i+1}} - t)/(\frac{y_{i+1}}{d_{i+1}} - \frac{y_i}{d_i}) \in (0, 1)$  and using convexity of  $g_x$  we compute

$$g_x(t) = g_x \left( \lambda \frac{y_i}{d_i} + (1-\lambda) \frac{y_{i+1}}{d_{i+1}} \right) \le \lambda g_x \left( \frac{y_i}{d_i} \right) + (1-\lambda) g_x \left( \frac{y_{i+1}}{d_{i+1}} \right)$$
$$\le \lambda g_y \left( \frac{y_i}{d_i} \right) + (1-\lambda) g_y \left( \frac{y_{i+1}}{d_{i+1}} \right)$$
$$= g_y \left( \lambda \frac{y_i}{d_i} + (1-\lambda) \frac{y_{i+1}}{d_{i+1}} \right) = g_y(t) \,.$$

In the last line we used that  $g_y$  is affine linear on each interval  $\left[\frac{y_i}{d_i}, \frac{y_{i+1}}{d_{i+1}}\right]$ .

As stated before, the equivalence of (i) through (v) is due to [28, Thm. 2.2]. However for the sake of this work being self-contained (and possibly filling some gaps in the literature) let us show a proof, or at least sketch the ideas. First of all (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) is obvious.

(i)  $\Rightarrow$  (v): There exists  $A \in s_d(n)$  which maps y to x so A(y - td) = Ay - tAd = x - td for all  $t \in \mathbb{R}$ , hence (v) is a direct consequence of (2).

 $(v) \Rightarrow (iv)$ : Because  $e^{\top}x = e^{\top}y$ , just as in the proof of Lemma 9 trace equality and trace norm inequality implies the inequality for the positive part of the vectors.

(iv)  $\Rightarrow$  (ii): Let a continuous convex function  $\psi : D(\psi) \subseteq \mathbb{R} \to \mathbb{R}$  be given such that  $P := \{\frac{x_j}{d_j}, \frac{y_j}{d_j} : j = 1, \dots, n\} \subseteq D(\psi)$ . In particular one can construct a continuous function  $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$  such that

- $\tilde{\psi}(\frac{x_j}{d_j}) = \psi(\frac{x_j}{d_j})$  and  $\tilde{\psi}(\frac{y_j}{d_j}) = \psi(\frac{y_j}{d_j})$  for all  $j = 1, \dots, n$
- $\tilde{\psi}$  is piecewise linear with change in slope only at the elements of P.
- $\tilde{\psi}$  is convex (evident because  $\psi$  is convex).

In other words  $\tilde{\psi}$  is the "piecewise linearization" of  $\psi$  (with respect to P). Thus it suffices to prove (ii) for all such  $\tilde{\psi}$  because then

$$\sum_{j=1}^{n} d_j \psi\left(\frac{x_j}{d_j}\right) = \sum_{j=1}^{n} d_j \tilde{\psi}\left(\frac{x_j}{d_j}\right) \le \sum_{j=1}^{n} d_j \tilde{\psi}\left(\frac{y_j}{d_j}\right) = \sum_{j=1}^{n} d_j \psi\left(\frac{y_j}{d_j}\right)$$

Now let  $\phi : \mathbb{R} \to \mathbb{R}$  continuous, convex and piecewise linear (with respect to P) be given. Then  $\phi$  can be written as a (non-negative) linear combination of the maps<sup>5</sup> { $\phi_p : p \in P$ } where  $\phi_p(t) := (p - t)_+$ . But all  $\phi_p$  satisfy (ii) by assumption, hence  $\phi$  does as well.

(ii)  $\Rightarrow$  (i): The idea here is much in the spirit of Kemperman [29, Thm. 2]. Finding  $A \in \mathbb{R}^{n \times n}_+$  with  $e^{\top}A = e^{\top}$ , Ad = d and Ay = x is equivalent (by vectorization, cf. [38, Ch. 2.4]) to finding a solution  $z \in \mathbb{R}^{n^2}_+$  to

$$\begin{pmatrix} y^{\top} \otimes I_n \\ d^{\top} \otimes I_n \\ I_n \otimes e^{\top} \end{pmatrix} z = \begin{pmatrix} x \\ d \\ e \end{pmatrix}$$

<sup>&</sup>lt;sup>5</sup>This is true up to an affine linear map which due to  $e^{\top}x = e^{\top}y$  yields equality in (ii), thus can be disregarded.

where  $\otimes$  is the usual Kronecker product [38, Ch. 2.2] and z = vec A. By Farkas' lemma<sup>6</sup> such a solution exists if (and only if) for all  $w \in \mathbb{R}^{3n}$  which satisfy

$$\left(\begin{pmatrix} y^{\top} \otimes I_n \\ d^{\top} \otimes I_n \\ I_n \otimes e^{\top} \end{pmatrix}^{\top} \begin{pmatrix} \vec{w_1} \\ \vec{w_2} \\ \vec{w_3} \end{pmatrix}\right)_{n(j-1)+k} = \left(\begin{pmatrix} (y \otimes I_n)\vec{w_1} & (d \otimes I_n)\vec{w_2} & (I_n \otimes e)\vec{w_3} \end{pmatrix}\right)_{n(j-1)+k} = y_j w_k + d_j w_{n+k} + w_{2n+j} \le 0 \tag{3}$$

for all  $j, k = 1, \ldots, n$  one has

$$\sum_{j=1}^{n} (x_j w_j + d_j w_{n+j} + w_{2n+j}) \le 0.$$

Consider the convex (because affine linear) functions  $\psi_j : \mathbb{R} \to \mathbb{R}, t \mapsto w_j t + w_{n+j}$  for all  $j = 1, \ldots, n$ . Then

$$\psi : \mathbb{R} \to \mathbb{R} \qquad t \mapsto \max_{j=1,\dots,n} \psi_j(t)$$

is convex and continuous as well so by assumption and because d > 0

$$\sum_{j=1}^{n} (x_j w_j + d_j w_{n+j} + w_{2n+j}) = \sum_{j=1}^{n} d_j \psi_j(\frac{x_j}{d_j}) + w_{2n+j}$$
  
$$\leq \sum_{j=1}^{n} d_j \psi(\frac{x_j}{d_j}) + w_{2n+j}$$
  
$$\leq \sum_{j=1}^{n} d_j \psi(\frac{y_j}{d_j}) + w_{2n+j}.$$

But now for every j = 1, ..., n exists k = k(j) such that  $\psi(\frac{y_j}{d_j}) = \psi_{k(j)}(\frac{y_j}{d_j})$  by definition of  $\psi$  (the maximum has to be attained by at least one of the  $\psi_k$ ). Hence

$$\sum_{j=1}^{n} (x_j w_j + d_j w_{n+j} + w_{2n+j}) \leq \sum_{j=1}^{n} d_j \psi(\frac{y_j}{d_j}) + w_{2n+j}$$
$$= \sum_{j=1}^{n} d_j \psi_{k(j)}(\frac{y_j}{d_j}) + w_{2n+j}$$
$$= \sum_{j=1}^{n} \underbrace{y_j w_{k(j)} + d_j w_{n+k(j)} + w_{2n+j}}_{\leq 0 \text{ by } (3)} \leq 0$$

so we are done. Note that we needed access to not all, but only to the piecewise linear convex functions—this is the same effect as in the proof of  $(iv) \Rightarrow (ii)$ .

(i)  $\Leftrightarrow$  (vii): This will be a direct consequence of our considerations in Section 4 so we will postpone this part of the proof to Section 4.3. Note that we will not use this result anywhere in the paper, meaning we are not at risk to run into a circular argument.

**Remark 2.** (i) In terms of numerics, Prop. 1 (vi) is the most efficient one for checking d-majorization as it encapsulates at most n+1 constraints one has to verify. In contrast,

<sup>&</sup>lt;sup>6</sup>Farkas' lemma states that for  $m, n \in \mathbb{N}$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ , the system of linear equations Ax = b has a solution in  $\mathbb{R}^n_+$  if and only if for all  $y \in \mathbb{R}^m$  which satisfy  $A^\top y \leq 0$  one has  $b^\top y \leq 0$ , refer to [47, Coro. 7.1.d] (when replacing A, b by -A, -b).

the definition of *d*-majorization, i.e. finding a *d*-stochastic matrix which maps y to x boils down to a linear programming problem which is by far not as easy to check as condition (vi). The other conditions from Prop. 1 consist of either uncountably many constraints (conditions (ii), (iii), and (v)), or at most 2n (condition (iv)) or  $n^2$  steps (condition (vii), because each of the n-1 constraints features a minimum over n numbers which one has to compute before checking the actual constraint).

(ii) In the physics lecture, also connected to the notion of thermo-majorization, a freshly published result [50] gives a direct proof of (vii)  $\Rightarrow$  (i) from Prop. 1. Given two vectors  $x, y \in \mathbb{R}^n$  of the same "trace" such that y thermo-majorizes x, Shiraishi gave a constructive algorithm for a d-stochastic matrix which sends y to x.

Recently Alhambra et al. [1] were able to find conditions under which classical majorization implies d-majorization for d from some parameter range. As their result was obtained in the context of dephasing thermalization let us reformulate it by casting it into our notation:

**Proposition 2.** The following statements hold.

- (i) Let  $x, y \in \mathbb{R}^n$  and  $d \in \mathbb{R}^n_{++}$  be given. If x, y are similarly d-ordered, i.e. there exists a permutation  $\sigma \in S_n$  such that  $\frac{x_{\sigma(1)}}{d_{\sigma(1)}} \ge \ldots \ge \frac{x_{\sigma(n)}}{d_{\sigma(n)}}$  and  $\frac{y_{\sigma(1)}}{d_{\sigma(1)}} \ge \ldots \ge \frac{y_{\sigma(n)}}{d_{\sigma(n)}}$ , then  $x \prec_d y$  holds if and only if  $x \prec y$ .
- (ii) Let  $y \in \mathbb{R}^n_+$  and  $d \in \mathbb{R}^n_{++}$ . If  $y_1 \leq \ldots \leq y_n$  and  $d_1 \geq \ldots \geq d_n$ , then  $\underline{\sigma}y \prec_d y$  for all  $\sigma \in S_n$ .
- (iii) Let  $y \in \mathbb{R}^n_+$  with  $y_1 \leq \ldots \leq y_n$  be given. Then for all  $x \in \mathbb{R}^n$  one has  $x \prec y$  if and only if  $x \prec_d y$  for all  $d \in \mathbb{R}^n_{++}$  with  $d_1 \geq \ldots \geq d_n$ .

*Proof.* (i): [28, Coro. 2.5]. Note that  $x \prec_d y$  if and only if  $\underline{\sigma}x \prec_{\underline{\sigma}d} \underline{\sigma}y$  for all  $\sigma \in S_n$  as is readily verified, so it suffices to have x, y similarly *d*-ordered. (ii): One can explicitly write down generalized T-transforms which first shift  $y_1$  to  $y_{\sigma(1)}$ , then  $y_2$  to  $y_{\sigma(2)}$ , and so on. The details are carried out in [1, p. 13 & 14]. (iii),  $\Leftarrow$ : Obvious. (iii),  $\Rightarrow$ : Let  $x \in \mathbb{R}^n$  with  $x \prec y$  be given and let  $\tau \in S_n$  be any permutation such that  $\underline{\tau}x_1 \leq \ldots \leq \underline{\tau}x_n$ . Then (ii) implies  $\underline{\sigma} \underline{\tau}x \prec_d \underline{\tau}x$  for all  $\sigma \in S_n$  and all  $d \in \mathbb{R}^n_{++}$  with  $d_1 \geq \ldots \geq d_n$ , so choosing  $\sigma = \tau^{-1}$  yields  $x \prec_d \underline{\tau}x$ . On the other hand  $\underline{\tau}x$  and y are similarly *d*-ordered for all such *d*—because  $(\underline{\tau}x)_1 \leq \ldots \leq \underline{(\tau x)_n} \atop d_n$  and  $\frac{y_1}{d_1} \leq \ldots \leq \frac{y_n}{d_n}$ —so we have  $\underline{\tau}x \prec_d y$  by (i). Using that  $\prec_d$  is a preorder this yields  $x \prec_d \underline{\tau}x \prec_d y$ , that is,  $x \prec_d y$  as claimed.

To conclude this section we make some statements about minimal and maximal elements of the preorder  $\prec_d$ .

**Theorem 3.** Let  $d \in \mathbb{R}^n_{++}$  be given. The following statements hold.

- (i) d is the unique minimal element within  $\{x \in \mathbb{R}^n : \mathbb{e}^\top x = \mathbb{e}^\top d\}$  with respect to  $\prec_d$ .
- (ii) Let  $j, k \in \{1, \ldots, n\}$ ,  $j \neq k$  be given. Then  $e_j \prec_d e_k$  if and only if  $d_j \geq d_k$ .
- (iii) If k is chosen such that  $d_k$  is minimal in d, then  $(\mathbb{e}^{\top}d)e_k$  is maximal in  $(\mathbb{e}^{\top}d)\Delta^{n-1} = \{x \in \mathbb{R}^n_+ : \mathbb{e}^{\top}x = \mathbb{e}^{\top}d\}$  with respect to  $\prec_d$ . It is the unique maximal element in  $(\mathbb{e}^{\top}d)\Delta^{n-1}$  with respect to  $\prec_d$  if and only if  $d_k$  is the unique minimal element of d.

*Proof.* (i) Consider  $d e^{\top}/(e^{\top}d) \in s_d(n)$  which maps any  $x \in \mathbb{R}^n$  with  $e^{\top}x = e^{\top}d$  to d so  $d \prec_d x$ . Uniqueness is obvious as d is a fixed point of every d-stochastic matrix.

(ii): Applying Prop. 1 (vi),  $e_j \prec_d e_k$  is equivalent to  $||e_j - \frac{1}{d_k}d||_1 \leq ||e_k - \frac{1}{d_k}d||_1$  (as the n-1 other inequalities read  $1 \leq 1$  and thus are redundant). But this is satisfied iff

$$\left|1 - \frac{d_j}{d_k}\right| + \frac{d_k}{d_k} \le \frac{d_j}{d_k} + \left|1 - \frac{d_k}{d_k}\right| \quad \Leftrightarrow \quad \left|\frac{d_j}{d_k} - 1\right| \le \frac{d_j}{d_k} - 1$$

which obviously holds iff  $\frac{d_j}{d_k} - 1 \ge 0$ , that is,  $d_j \ge d_k$ .

(iii): W.l.o.g.  $e^{\top}d = 1$ . Because  $d_k \leq d_j$  for all  $j = 1, \ldots, n$ , (ii) implies  $e_j \prec_d e_k$  which, using convexity of  $\prec_d$ , shows maximality of  $e_k$ . Moreover, if  $d_k$  is not the *unique* minimal element—but there exists  $i \neq k$  such that  $d_i = d_k$ —then by the same argument  $e_i$  is maximal in  $\Delta^{n-1}$  w.r.t.  $\prec_d$ , so there exist at least two maximal elements.

Thus all that is left to show is that if  $d_k$  is the unique minimal element of d, then  $e_k \prec_d x$  for any  $x \in \Delta^{n-1}$  implies  $x = e_k$  meaning there cannot be any maximal element other than  $e_k$ . Indeed  $e_k \prec_d x$  by Prop. 1 (vi) is equivalent to  $||e_k - \frac{x_i}{d_i}d||_1 \leq ||x - \frac{x_i}{d_i}d||_1$  for all  $i = 1, \ldots, n$  which after a straightforward computation reads

$$1 + \frac{x_i}{d_i} (1 - d_i - d_k) \le 1 - 2 \sum_{\{\alpha: \frac{x_\alpha}{d_\alpha} < \frac{x_i}{d_i}\}} x_\alpha + \frac{x_i}{d_i} \left( 1 - 2d_i - 2 \sum_{\{\alpha: \frac{x_\alpha}{d_\alpha} > \frac{x_i}{d_i}\}} d_\alpha \right)$$
$$\le 1 + \frac{x_i}{d_i} (1 - 2d_i) \le 1 + \frac{x_i}{d_i} (1 - d_i - d_k).$$

Hence all these inequalities are actually equalities; in particular the last step then implies  $x_i = 0$  for all  $i \neq k$  because  $d_i > d_k$  by assumption.

**Remark 3.** The fact that every  $e_1, \ldots, e_n$  is maximal in the standard simplex  $\Delta^{n-1}$  for d = e is lost in the general setting (consider the example from Remark 1 (iv)).

However, for strictly positive vectors  $z \in \mathbb{R}^{n}_{++}$  one still has  $(e^{\top}z)e_k \not\prec_d z$  for all  $k = 1, \ldots, n$ . More generally, if  $y \prec_d z$  then y has to be strictly positive as well; otherwise the corresponding transformation matrix (non-negative entries) would contain a row of zeros which—due to d > 0—contradicts d being one of its fixed points. This is a special case of strict positivity of matrix-D-majorization [16, Coro. 4.7]

#### 3. Descriptions & Properties of Convex Polytopes

Before we investigate what Proposition 1 tells us about sets of the form  $\{x \in \mathbb{R}^n : x \prec_d y\}$ for some  $y \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^n_{++}$  we need some basic knowledge of convex polytopes. Usually, convex polytopes are introduced as subsets of  $\mathbb{R}^n$  which can be written as the convex hull of finitely many vectors from  $\mathbb{R}^n$ , cf. [47, Ch. 7.2], [22, Ch. 3]. Now it is well known that such polytopes can be characterized via finitely many affine half-spaces: More precisely a set  $P \subset \mathbb{R}^n$  is a convex polytope if and only if P is bounded and there exist  $m \in \mathbb{N}$ ,  $A \in \mathbb{R}^{m \times n}$ , and  $b \in \mathbb{R}^m$  such that  $P = \{x \in \mathbb{R}^n : Ax \leq b\}$  [47, Coro. 7.1c]. These characterizations of convex polytopes are also known as vertex- and halfspace-description  $(\mathcal{V}\text{-} \text{ and } \mathcal{H}\text{-description})$ , respectively [22, Ch. 3.6]. **Remark 4.** Let any  $A \in \mathbb{R}^{m \times n}$ ,  $b, b' \in \mathbb{R}^m$ , and  $p \in \mathbb{R}^n$  be given. The following observations are readily verified:

$$\{x \in \mathbb{R}^n : Ax \le b\} \cap \{x \in \mathbb{R}^n : Ax \le b'\} = \{x \in \mathbb{R}^n : Ax \le \min\{b, b'\}\}$$

$$\{x \in \mathbb{R}^n : Ax \le b\} + p = \{x \in \mathbb{R}^n : Ax \le b + Ap\}$$

$$b \le b' \quad \Leftrightarrow \quad \{x \in \mathbb{R}^n : Ax \le b\} \subseteq \{x \in \mathbb{R}^n : Ax \le b'\}.$$

Here and henceforth, we for simplicity use the convention that min and max operate entrywise on vectors, e.g.,  $\min\{b, b'\} = (\min\{b_j, b'_j\})_{j=1}^m$  for  $b, b' \in \mathbb{R}^m$ . The above results are not too surprising as A in some sense describes the geometry of the polytope which intuitively should not change under the above operations.

In the following we are primarily interested in the case where A and b have a very particular structure. After all, in the introduction we have seen that the orientation of the half-spaces which describe classical majorization are precisely given by the collection of all binary vectors  $\{0,1\}^n$ . Thus to introduce this special structure we define

$$M := \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ e^{\top} \\ -e^{\top} \end{pmatrix} \in \mathbb{R}^{2^n \times n},$$

$$(4)$$

where the rows of  $M_j \in \mathbb{R}^{\binom{n}{j} \times n}$  consist of all elements of  $\{0, 1\}^n$  which sum up to j. The order of these rows can be chosen arbitrarily but shall be fixed henceforth.

Moreover,  $b \in \mathbb{R}^{2^n}$  will be partitioned in the same way and our focus will be on the case where  $b'_n + b'_{n+1} = 0$  or, equivalently,

$$b = \begin{pmatrix} b'_1 \\ \vdots \\ b'_{n-1} \\ b'_n \\ -b'_n \end{pmatrix} \in \mathbb{R}^{2^n}.$$

with  $b'_j \in \mathbb{R}^{\binom{n}{j}}$  for all j = 1, ..., n; in particular  $b'_n \in \mathbb{R}$ . The last two conditions on M and b obviously allow us to guarantee that any solution of  $Mx \leq b$  satisfies the "trace condition"  $e^{\top}x = \sum_{j=1}^{n} x_j = b'_n$ .

**Lemma 4.** Let M be the matrix (4) and let  $b \in \mathbb{R}^{2^n}$  with  $b'_n + b'_{n+1} = 0$  be given. If  $\{x \in \mathbb{R}^n : Mx \leq b\}$  is non-empty then it is a convex polytope of dimension at most n-1.

Proof. Because  $b'_n + b'_{n+1} = 0$  by assumption, all solutions to  $Mx \leq b$  have to satisfy  $e^{\top}x = b'_n$  which reduces the dimension of  $\{x \in \mathbb{R}^n : Mx \leq b\}$  by at least 1. Now if we can show that  $\{x \in \mathbb{R}^n : Mx \leq 0\} = \{0\}$  then the set in question is bounded (cf. [47, Ch. 8.2]) so by the above characterization of convex polytopes we are done. Indeed let  $x \in \mathbb{R}^n$  satisfy  $Mx \leq 0$ . Then  $M_1x \leq 0$ , implying  $x_i \leq 0$  for all  $i = 1, \ldots, n$ . But this together with the "trace condition"  $e^{\top}x = 0$  shows  $x_i = 0$  for all i, as desired.

Now an immediate question is the one concerning the vertices (extreme points) of convex polytopes of the above form. This will be the topic of the remainder of this section. For this we need a characterization of the extreme points, given a  $\mathscr{H}$ -description of a convex polytope.

To characterize these extreme points in a convenient way we need a formalism to pick rows from the matrix M and the corresponding entry of the vector b. Since we do not want to fix any "unnatural" order of the rows of M, we define an abstract mapping which does this job for us. While this procedure would not be necessary for classical majorization – because there the vector b is of convenient structure – for studying general d-majorization this map will be indispensable.

**Definition 2.** Let  $a \in \{0,1\}^n$ ,  $a \neq 0$  be given. Then the row vector a corresponds to a (unique) row of  $M \in \mathbb{R}^{2^n \times n}$ , hence there exist unique  $m_1 \in \{1, \ldots, n\}, m_2 \in \{1, \ldots, \binom{n}{m_1}\}$  such that a is the  $m_2$ -th row of the submatrix  $M_{m_1}$  of M in (4). This lets us define

$$\mathfrak{m}: \{0,1\}^n \to \mathbb{N}_0 \times \mathbb{N}_0$$
$$a \mapsto \begin{cases} (0,0) & \text{if } a = 0\\ (m_1,m_2) & \text{else} \end{cases}$$

Now given  $b \in \mathbb{R}^{2^n}$  with  $b'_n + b'_{n+1} = 0$  we can define an analogous mapping

$$\mathfrak{b}: \{0,1\}^n \to \mathbb{R}$$
$$a \mapsto \begin{cases} 0 & \text{if } a = 0\\ (b'_{m_1})_{m_2} & \text{else} \end{cases}$$

where  $(m_1, m_2) = \mathfrak{m}(a)$ .

In other words  $Mx \leq b$  then is equivalent to  $a^{\top}x \leq \mathfrak{b}(a^{\top})$  for all  $a \in \{0,1\}^n$  together with  $e^{\top}x = b'_n$ . Certainly the maps  $\mathfrak{m}, \mathfrak{b}$  generalize to matrices  $A \in \{0,1\}^{m \times n}, b \in \mathbb{R}^m \ (m \in \mathbb{N})$  by applying them to every row of A individually<sup>7</sup>.

**Lemma 5.** Let  $b \in \mathbb{R}^{2^n}$  with  $b'_n + b'_{n+1} = 0$  as well as  $p \in \mathbb{R}^n$  be given such that  $Mp \leq b$ . The following statements are equivalent.

- (i) p is an extreme point of  $\{x \in \mathbb{R}^n : Mx \leq b\}$ .
- (ii) There exists a submatrix  $M' \in \{0,1\}^{n \times n}$  of M—one row of M' being equal to  $e^{\top}$ —such that  $M'p = \mathfrak{b}(M') =: b'$  and rank M' = n.

*Proof.* "(ii)  $\Rightarrow$  (i)": Assume there exist  $x_1, x_2 \in \{x \in \mathbb{R}^n : Mx \leq b\}$  and  $\lambda \in (0, 1)$  such that  $p = \lambda x_1 + (1 - \lambda) x_2$ . Then  $M'x_1 \leq b'$ ,  $M'x_2 \leq b'$  by assumption and thus

$$b' = M'p = \lambda M'x_1 + (1-\lambda)M'x_2 \le \lambda b' + (1-\lambda)b' = b' \quad \Rightarrow \quad M'x_1 = M'x_2 = b'.$$

<sup>&</sup>lt;sup>7</sup>For this map to be well-defined we need the assumption that no row of A appears twice. But this is rather natural because one of the two inequalities  $a^{\top}x \leq b$ ,  $a^{\top}x \leq b'$  is redundant and can be disregarded.

But  $M' \in \mathbb{R}^{n \times n}$  is of full rank so the system of linear equations M'y = b has a unique solution in  $\mathbb{R}^n$ , hence  $x_1 = x_2 = p$  is in fact an extreme point of  $\{x \in \mathbb{R}^n : Mx \leq b\}$ .

"(i)  $\Rightarrow$  (ii)": Each extreme point p of  $\{x \in \mathbb{R}^n : Mx \leq b\}$  is determined by n linearly independent equations from Mx = b so there exists a submatrix  $\hat{M} \in \mathbb{R}^{n \times n}$  of M of full rank such that  $\hat{M}p = \mathfrak{b}(\hat{M}) =: \hat{b}$ , cf. [47, Thm. 8.4 ff.]. If one row of  $\hat{M}$  equals  $e^{\top}$  then we are done. If  $\hat{M}$  features  $-e^{\top}$  replace it by  $e^{\top}$  and flip the sign of the corresponding entry in  $\mathfrak{b}(\hat{M})$ . Otherwise define

$$\tilde{M} := \begin{pmatrix} \hat{M} \\ \mathbb{e}^{\top} \end{pmatrix} \in \mathbb{R}^{(n+1) \times n} \quad \text{and} \quad \tilde{b} := \mathfrak{b}(\tilde{M}) = \begin{pmatrix} \hat{b} \\ b'_n \end{pmatrix} \in \mathbb{R}^{n+1}$$

so  $\tilde{M}p = \tilde{b}$  because p satisfies the "trace condition"  $e^{\top}p = b'_n$ . But this system of linear equations is now overdetermined so there exists a row of  $\hat{M}$  which can be replaced by  $e^{\top}$  such that the resulting matrix  $M' \in \{0, 1\}^{n \times n}$ —which still satisfies  $M'p = \mathfrak{b}(M')$ —has again full rank.

This enables—in some special cases—an explicit description of the extreme points of the polytope induced by M and b. Henceforth the symbol  $\Lambda$  will denote the lower triangular matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & \cdots & \cdots & 1 \end{pmatrix}.$$
 (5)

**Definition 3.** Let  $b \in \mathbb{R}^{2^n}$  with  $b'_n + b'_{n+1} = 0$  and arbitrary  $\sigma \in S_n$  be given. Denote by  $\underline{\sigma}$  the permutation matrix<sup>8</sup> induced by  $\sigma$ . Then the unique solution to

$$\Lambda \,\underline{\sigma} x = \mathfrak{b} \left( \Lambda \,\underline{\sigma} \right) =: b'_{\sigma} \tag{6}$$

(with  $\Lambda$  from (5)) shall be denoted by  $x = E_b(\sigma)$ .

Now  $E_b(\sigma)$  is of the following simple form.

**Lemma 6.** Let  $b \in \mathbb{R}^{2^n}$  with  $b'_n + b'_{n+1} = 0$ , arbitrary  $\sigma \in S_n$ , as well as  $p \in \mathbb{R}^n$  be given. Then for all j = 1, ..., n and for all  $\sigma \in S_n$ 

$$(E_{b}(\sigma))_{\sigma(j)} = \mathfrak{b}\Big(\sum_{i=1}^{j} e_{\sigma(i)}^{\top}\Big) - \mathfrak{b}\Big(\sum_{i=1}^{j-1} e_{\sigma(i)}^{\top}\Big) = (b'_{\sigma})_{j} - (b'_{\sigma})_{j-1}$$
(7)

(with  $b'_{\sigma}$  from (6)), as well as  $E_{b+Mp}(\sigma) = E_b(\sigma) + p$ .

*Proof.* The *j*-th row of (6) for  $x = E_b(\sigma), j = 1, ..., n$  reads

$$\sum_{i=1}^{j} (E_b(\sigma))_{\sigma(i)} = \left(\sum_{i=1}^{j} e_{\sigma(i)}^{\top}\right) E_b(\sigma) = \mathfrak{b}\left(\sum_{i=1}^{j} e_{\sigma(i)}^{\top}\right)$$

<sup>&</sup>lt;sup>8</sup>Given some permutation  $\sigma \in S_n$  the corresponding permutation matrix is given by  $\sum_{i=1}^n e_i e_{\sigma(i)}^\top$ . In particular the identities  $(\underline{\sigma}x)_j = x_{\sigma(j)}$  and  $\underline{\sigma} \circ \underline{\tau} = \underline{\tau} \cdot \underline{\sigma}$  hold.

which implies (7). Also one readily verifies

$$(b + Mp)'_{\sigma} = b'_{\sigma} + \underbrace{\left(\sum_{i=1}^{j} p_{\sigma(i)}\right)_{j=1}^{n}}_{\text{corresponding entry in } Mp} = b'_{\sigma} + \Lambda \,\underline{\sigma}p$$

for any  $\sigma \in S_n$ , so  $E_{b+Mp}(\sigma) = E_b(\sigma) + p$  by uniqueness of the solution of (6).

Clearly if  $E_b(\sigma) \in \{x \in \mathbb{R}^n : Mx \leq b\}$  for some  $\sigma \in S_n$ , then it is an extreme point by Lemma 5 although, in general, not every  $E_b(\sigma)$  needs to be in  $\{x \in \mathbb{R}^n : Mx \leq b\}$  for arbitrary  $b \in \mathbb{R}^{2^n}$  with  $b'_n + b'_{n+1} = 0$  (cf. Example 1, Appendix D). However for the wellstructured polytopes we will deal with later on all of these  $E_b(\sigma)$  lie within the polytope, in which case they are the only extreme points:

**Theorem 7.** Let  $b \in \mathbb{R}^{2^n}$  with  $b'_n + b'_{n+1} = 0$  be given such that  $\{E_b(\sigma) : \sigma \in S_n\} \subset \{x \in \mathbb{R}^n : Mx \leq b\}$ . Then every extreme point of  $\{x \in \mathbb{R}^n : Mx \leq b\}$  is of the form  $E_b(\sigma)$  for some  $\sigma \in S_n$ , and therefore  $\{x \in \mathbb{R}^n : Mx \leq b\} = \operatorname{conv}\{E_b(\sigma) : \sigma \in S_n\}$ .

Proof. Let  $p \in \{x \in \mathbb{R}^n : Mx \leq b\}$  be extremal so by Lemma 5 there exists a submatrix  $M' \in \{0,1\}^{n \times n}$  of M of full rank, one row of M' being equal to  $e^{\top}$ , such that  $M'p = \mathfrak{b}(M')$ . By Minkowski's theorem [7, Thm. 5.10] if we can show that  $p = E_b(\sigma)$  for some  $\sigma \in S_n$  this would conclude the proof.

Indeed consider any two rows  $m_1, m_2$  of M'. The idea will be to show that  $m_{\min}p = \mathfrak{b}(m_{\min})$  and  $m_{\max}p = \mathfrak{b}(m_{\max})$  where  $m_{\min} := \min\{m_1, m_2\}, m_{\max} := \max\{m_1, m_2\}$ . Thus M' can be extended by these rows while keeping the equality  $M'p = \mathfrak{b}(M')$ . This would conclude the proof by means of an abstract result which, roughly speaking, states that any matrix of full rank which has  $e^{\top}$  as a row and satisfies this min-max-property necessarily features a submatrix of the form  $\Lambda \underline{\sigma}$  for some  $\sigma \in S_n$  (Lemma 19 (iii), Appendix B), so  $p = E_b(\sigma)$ .

Now note that  $m_{\min} \leq m_{\max}$ , meaning one finds  $\tau \in S_n$  such that  $m_{\min}$ ,  $m_{\max}$  are rows of  $\Lambda \underline{\tau}^{9}$ , as well as  $m_{\min} + m_{\max} = m_1 + m_2$ . This has two immediate consequences: firstly,  $M'p = \mathfrak{b}(M')$  and  $Mp \leq b$  imply

$$\mathfrak{b}(m_1) + \mathfrak{b}(m_2) = (m_1 + m_2)p = (m_{\min} + m_{\max})p \le \mathfrak{b}(m_{\min}) + \mathfrak{b}(m_{\max})$$

and secondly,  $\Lambda \underline{\tau} E_b(\tau) = b'_{\tau}$  and  $M E_b(\tau) \leq b$  (because  $E_b(\tau) \in \{x \in \mathbb{R}^n : Mx \leq b\}$  by assumption) yield

$$\mathfrak{b}(m_{\min}) + \mathfrak{b}(m_{\max}) = (m_{\min} + m_{\max})E_b(\tau) = (m_1 + m_2)E_b(\tau) \le \mathfrak{b}(m_1) + \mathfrak{b}(m_2)$$

Combining these two we get

$$\mathfrak{b}(m_1) + \mathfrak{b}(m_2) = (m_{\min} + m_{\max})p \le \mathfrak{b}(m_{\min}) + \mathfrak{b}(m_{\max}) \le \mathfrak{b}(m_1) + \mathfrak{b}(m_2),$$

that is,  $(m_{\min} + m_{\max})p = \mathfrak{b}(m_{\min}) + \mathfrak{b}(m_{\max})$ . But  $m_{\min}p \leq \mathfrak{b}(m_{\min})$ ,  $m_{\max}p \leq \mathfrak{b}(m_{\max})$  (due to  $Mp \leq b$ ), hence  $m_{\min}p = \mathfrak{b}(m_{\min})$  and  $m_{\max}p = \mathfrak{b}(m_{\max})$ . As stated before, this means

<sup>&</sup>lt;sup>9</sup>If  $m_{\min} = 0$  then one can trivially find  $\tau \in S_n$  such that  $m_{\max}$  is a row of  $\Lambda \underline{\tau}$ .

we can extend M' by  $m_{\min}, m_{\max}$ —assuming these were not part of M' in the first place—to a matrix M'' which still satisfies  $M''p = \mathfrak{b}(M'')$ . Repeating this enlargement process over and over will terminate eventually as  $\{0,1\}^n$  is finite, yielding  $\tilde{M}$  of full rank, containing  $e^{\top}$ , and, most importantly, for any two rows  $m_1, m_2$  of  $\tilde{M}, \min\{m_1, m_2\}$  and  $\max\{m_1, m_2\}$  are rows of  $\tilde{M}$  as well. Therefore Lemma 19 (iii) (Appendix B) yields a permutation  $\sigma \in S_n$ such that every row of  $\Lambda \underline{\sigma}$  is a row of  $\tilde{M}$ , so  $\tilde{M}p = \mathfrak{b}(\tilde{M})$  implies  $\Lambda \underline{\sigma}p = \mathfrak{b}(\Lambda \underline{\sigma})$ , that is,  $p = E_b(\sigma)$ .

**Remark 5.** This result is remarkable because it shows that under certain requirements on b the special structure of M allows to simplify the procedure of finding the extreme points of the induced polytope significantly. Usually one would have to determine all invertible submatrices of M, solve the corresponding linear equations (cf. [47, Thm. 8.4 ff.]), and finally check whether these solutions satisfy the remaining inequalities. However, M has way more invertible submatrices than actual extreme points (i.e. n!) in this case.

Following Remark 10 (Appendix B) one can even improve Thm. 7: If the matrix M' corresponding to p contains two incomparable rows  $m_1, m_2$ , that is,  $m_1 \geq m_2 \geq m_1$ , then there exist at least two permutations  $\sigma_1, \sigma_2 \in S_n$  such that  $E_b(\sigma_1) = p = E_b(\sigma_2)$ . Notably in such a situation the map  $\sigma \mapsto E_b(\sigma)$  is not injective.

### 4. The *d*-Majorization Polytope

#### 4.1. Characterizing the d-Majorization Polytope

With the tools surrounding convex polytopes developed in Section 3 we are finally ready to explore the "geometry" of *d*-majorization. For this let us consider the set of all vectors which are *d*-majorized by some  $y \in \mathbb{R}^n$ ; more generally we introduce the map

$$M_d: \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$$
$$S \mapsto \bigcup_{y \in S} \{ x \in \mathbb{R}^n : x \prec_d y \}$$

where  $\mathcal{P}$  denotes the power set. For convenience  $M_d(y) := M_d(\{y\})$  for any  $y \in \mathbb{R}^n$ , which then equals the set of all vectors which are *d*-majorized by y. Note that the idea here is close to—but should not be confused with—the (*d*-)majorization polytope of two vectors [9] which, given two real vectors, is the set of all (*d*-)stochastic matrices which map one vector to the other.

**Lemma 8.** Let  $d \in \mathbb{R}^n_{++}$ . Then  $M_d$  is a closure operator<sup>10</sup>. In particular, for any  $x, y \in \mathbb{R}^n$  one has  $x \prec_d y$  if and only if  $M_d(x) \subseteq M_d(y)$ .

*Proof.* The first statement is a simple consequence of the *d*-stochastic matrices  $s_d(n)$  forming a semigroup with identity. For the second statement note that  $x \prec_d y$ , that is,  $x \in M_d(y)$  implies  $M_d(x) \subseteq M_d(M_d(y)) = M_d(y)$ .

<sup>&</sup>lt;sup>10</sup>Recall that an operator J on the power set  $\mathcal{P}(S)$  of a set S is called *closure operator* or *hull operator* if it is extensive  $(X \subseteq J(X))$ , increasing  $(X \subseteq Y \Rightarrow J(X) \subseteq J(Y))$  and idempotent (J(J(X)) = J(X)) for all  $X, Y \in \mathcal{P}(S)$ , cf., e.g., [8, p. 42].

Now Prop. 1 (vi) directly implies

$$M_{d}(y) = \bigcap_{i=1}^{n} \left\{ x \in \mathbb{R}^{n} : e^{\top}x = e^{\top}y \land \left\| x - \frac{y_{i}}{d_{i}}d \right\|_{1} \leq \left\| y - \frac{y_{i}}{d_{i}}d \right\|_{1} \right\}$$
$$= \bigcap_{i=1}^{n} \left\{ x \in \mathbb{R}^{n} : e^{\top}\left(x - \frac{y_{i}}{d_{i}}d\right) = e^{\top}\left(y - \frac{y_{i}}{d_{i}}d\right) \land \left\| x - \frac{y_{i}}{d_{i}}d \right\|_{1} \leq \left\| y - \frac{y_{i}}{d_{i}}d \right\|_{1} \right\}$$
$$= \bigcap_{i=1}^{n} \left\{ \left\{ \tilde{x} \in \mathbb{R}^{n} : e^{\top}\tilde{x} = e^{\top}\left(y - \frac{y_{i}}{d_{i}}d\right) \land \left\| \tilde{x} \right\|_{1} \leq \left\| y - \frac{y_{i}}{d_{i}}d \right\|_{1} \right\} + \frac{y_{i}}{d_{i}}d \right\}$$
(8)

for all  $y \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^n_{++}$ .

**Lemma 9.** Let  $z \in \mathbb{R}^n$ . Then

$$\{x \in \mathbb{R}^n : e^{\top}x = e^{\top}z \land \|x\|_1 \le \|z\|_1\} = \{x \in \mathbb{R}^n \mid x \prec (e^{\top}z_+, -e^{\top}z_-, 0, \dots, 0)^{\top}\}$$

where  $z = z_+ - z_-$  is the unique decomposition of z into positive and negative part, i.e.  $z_+ = (\max\{z_j, 0\})_{j=1}^n, z_- = (-z)_+ = (\max\{-z_j, 0\})_{j=1}^n) \in \mathbb{R}^n_+.$ 

*Proof.* For what follows let  $\hat{z} := (e^{\top}z_+, -e^{\top}z_-, 0, \dots, 0).$ 

" $\supseteq$ ": Majorization by definition forces  $e^{\top}x = e^{\top}z_{+} - e^{\top}z_{-} = e^{\top}(z_{+} - z_{-}) = e^{\top}z$ . Also if  $x \prec \hat{z}$  then there exists a doubly stochastic matrix A which maps  $\hat{z}$  to x so using (2) we compute

$$||x||_1 = ||A\hat{z}||_1 \le ||\hat{z}||_1 = e^\top z_+ + e^\top z_- = \sum_{j=1}^n |z_j| = ||z||_1.$$

"  $\subseteq$  ": Decompose  $x = x_+ - x_-$  with  $x_+, x_- \in \mathbb{R}^n_+$  as above. By assumption

$$e^{\top}x = e^{\top}x_{+} - e^{\top}x_{-} = e^{\top}z_{+} - e^{\top}z_{-} = e^{\top}z_{+}$$
  
 $\|x\|_{1} = e^{\top}x_{+} + e^{\top}x_{-} \le e^{\top}z_{+} + e^{\top}z_{-} = \|z\|_{1}$ 

so taking the sum of these conditions gives  $e^{\top}x_+ \leq e^{\top}z_+$ . Thus for all  $k = 1, \ldots, n-1$ 

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} (x_{[i]})_{+} \le e^{\top} x_{+} \le e^{\top} z_{+} = e^{\top} z_{+} + \underbrace{0 + \ldots + 0}_{k-1 \text{ zeros}} = \sum_{i=1}^{k} \hat{z}_{[i]}$$

which—together with  $e^{\top}x = e^{\top}z$ —shows  $x \prec \hat{z}$ .

The previous lemma is the key to transferring the  $\mathcal H\text{-}\mathrm{description}$  of classical majorization over to  $d\text{-}\mathrm{majorization}$ 

**Theorem 10.** Let  $y \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^n_{++}$ . Then  $M_d(y) = \{x \in \mathbb{R}^n : Mx \leq b\}$  with M being the matrix (4) and

$$b = \min_{i=1,\dots,n} \begin{pmatrix} e^{\top} (y - \frac{y_i}{d_i} d)_+ e + \frac{y_i}{d_i} M_1 d \\ \vdots \\ e^{\top} (y - \frac{y_i}{d_i} d)_+ e + \frac{y_i}{d_i} M_{n-1} d \\ e^{\top} y \\ -e^{\top} y \end{pmatrix} \in \mathbb{R}^{2^n}.$$
(9)

*Proof.* Using [11, Thm. 1] as well as (8), Lemma 9, and Remark 4 we find

$$\begin{split} M_{d}(y) &= \bigcap_{i=1}^{n} \left( \left\{ \tilde{x} \in \mathbb{R}^{n} : e^{\top} \tilde{x} = e^{\top} \left( y - \frac{y_{i}}{d_{i}} d \right) \land \|\tilde{x}\|_{1} \leq \left\| y - \frac{y_{i}}{d_{i}} d \right\|_{1} \right\} + \frac{y_{i}}{d_{i}} d \right) \\ &= \bigcap_{i=1}^{n} \left( \left\{ x \in \mathbb{R}^{n} : x \prec \begin{pmatrix} e^{\top} (y - \frac{y_{i}}{d_{i}} d)_{+} \\ -e^{\top} (y - \frac{y_{i}}{d_{i}} d)_{-} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\} + \frac{y_{i}}{d_{i}} d ) \\ &= \bigcap_{i=1}^{n} \left\{ x \in \mathbb{R}^{n} : Mx \leq \begin{pmatrix} e^{\top} (y - \frac{y_{i}}{d_{i}} d)_{+} \\ e^{\top} (y - \frac{y_{i}}{d_{i}} d)_{+} \\ e^{\top} (y - \frac{y_{i}}{d_{i}} d)_{+} \\ -e^{\top} (y - \frac{y_{i}}{d_{i}} d)_{+} \end{pmatrix} + \frac{y_{i}}{d_{i}} M d \right\} \\ &= \left\{ x \in \mathbb{R}^{n} : Mx \leq \min_{i=1,\dots,n} \begin{pmatrix} e^{\top} (y - \frac{y_{i}}{d_{i}} d)_{+} e + \frac{y_{i}}{d_{i}} M_{1} d \\ e^{\top} (y - \frac{y_{i}}{d_{i}} d)_{+} e + \frac{y_{i}}{d_{i}} M_{1} d \\ e^{\top} y &= -e^{\top} y \end{pmatrix} \right\}. \end{split}$$

Setting d = e in Thm. 10—together with Lemma 20 (iii) (Appendix C)—recovers the known  $\mathscr{H}$ -description of classical majorization [11, Thm. 1] as expected.

The previous theorem shows that, roughly speaking,  $\prec_d$  and  $\prec$  share the same geometry, i.e. the faces of  $M_d(y)$  for arbitrary  $y \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^n_{++}$  are all parallel to some face of a classical majorization polytope, but the precise location of the halfspaces (respectively faces) may differ.

**Remark 6.** The description of *d*-majorization via halfspaces is not only conceptionally interesting, it also enables an algorithmic computation of the extreme points of  $M_d(y)$ . In general, the problem of converting an  $\mathscr{H}$ -description to a  $\mathscr{V}$ -description is known as vertex enumeration problem and well-studied in the field of convex polytopes and computational geometry, see [3] for an overview. For arbitrary polytopes this is a hard problem but in our case—due to the particular structure of  $M_d(y)$ —one can achieve an explicit (even analytic) solution, cf. Section 4.3.

# 4.2. Geometric and Topological Properties of the d-Majorization Polytope

If  $M_d$  acts on a set consisting of more than one vector we can state further geometric and topological results. This will be of use when treating continuity questions of the map  $(d, P) \mapsto M_d(P)$  afterwards.

**Theorem 11.** Let  $d \in \mathbb{R}^n_{++}$  and an arbitrary subset  $P \subseteq \mathbb{R}^n$  be given. Then the following statements hold.

- (i) If P lies within a trace hyperplane, i.e. there exists  $c \in \mathbb{R}$  such that  $e^{\top}x = c$  for all  $x \in P$ , then  $M_d(P)$  is star-shaped with respect to  $\frac{c}{e^{\top}d}d$ .
- (ii) If P is convex, then  $M_d(P)$  is path-connected.

(iii) If P is compact, then  $M_d(P)$  is compact.

*Proof.* (i): Every  $x \in P$  is directly connected to  $\frac{e^{\top}x}{e^{\top}d}d = \frac{c}{e^{\top}d}d$  within  $M_d(P)$  (Thm. 3 & convexity of  $\prec_d$ ). (ii): Let  $y, z \in P, \lambda \in [0, 1]$  be arbitrary. Then  $\lambda y + (1 - \lambda)z \in P$  and

$$\lambda \frac{\mathrm{e}^{\top} y}{\mathrm{e}^{\top} d} d + (1-\lambda) \frac{\mathrm{e}^{\top} z}{\mathrm{e}^{\top} d} d = \frac{\mathrm{e}^{\top} (\lambda y + (1-\lambda)z)}{\mathrm{e}^{\top} d} d \in M_d(\lambda y + (1-\lambda)z) \subseteq M_d(P).$$

Thus  $\frac{e^{\top}y}{e^{\top}d}d$  and  $\frac{e^{\top}z}{e^{\top}d}d$  are path-connected in  $M_d(P)$ , hence (ii) follows from (i).

(iii): As matrix multiplication is continuous,  $M_d(P) = \{Az : A \in s_d(n), z \in P\}$  is compact as the image of the compact set  $s_d(n) \times P$  under a continuous function.  $\Box$ 

The previous theorem still holds when extending  $\prec_d$  to complex vectors. Also one might hope that Thm. 11 (ii) is not optimal in the sense that convexity of general P implies convexity of  $M_d(P)$ . Example 2 (Appendix D), however, gives a negative answer.

Now the description of  $M_d(y)$  as a convex polytope is powerful enough to answer continuity questions regarding the map  $M_d$ .

**Theorem 12.** Let  $\mathcal{P}_c(\mathbb{R}^n)$  denote the collection of all compact subsets of  $\mathbb{R}^n$  and let  $\delta$  be the Hausdorff metric<sup>11</sup> on  $\mathcal{P}_c(\mathbb{R}^n)$  with respect to  $\|\cdot\|_1$ . Then the following statements hold.

- (i) For all  $d \in \mathbb{R}^n_{++}$ ,  $M_d$  is non-expansive under  $\delta$ , that is, for all  $P, P' \in \mathcal{P}_c(\mathbb{R}^n)$  one has  $\delta(M_d(P), M_d(P')) \leq \delta(P, P')$ .
- (ii) The following map is continuous:

$$M : \mathbb{R}^n_{++} \times (\mathcal{P}_c(\mathbb{R}^n), \delta) \to (\mathcal{P}_c(\mathbb{R}^n), \delta)$$
$$(d, P) \mapsto M_d(P)$$

*Proof.* Note that the image of a compact set under  $M_d$  remains compact by Thm. 11 (iii) so this guarantees that the image of the map M is contained in  $\mathcal{P}_c(\mathbb{R}^n)$  and that  $\delta(M_d(P), M_d(P'))$  is well-defined. (i): As a direct consequence of (2) one has

$$\max_{z \in M_d(P)} \min_{w \in M_d(P')} \|z - w\|_1 = \max_{\substack{A \in s_d(n) \\ z_1 \in P}} \min_{\substack{B \in s_d(n) \\ z_2 \in P'}} \|Az_1 - Bz_2\|_1 \le \max_{\substack{A \in s_d(n) \\ z_1 \in P}} \min_{z_1 \in P} \|A(z_1 - z_2)\|_1$$
$$\le \max_{\substack{A \in s_d(n) \\ z_1 \in P}} \min_{z_2 \in P'} \|z_1 - z_2\|_1 = \max_{z \in P} \min_{w \in P'} \|z - w\|_1.$$

Interchanging P and P' yields the desired estimate.

(ii): Our proof can be divided into the following five steps.

$$\delta(A,B) := \max\left\{\max_{z \in A} \min_{w \in B} d(z,w), \max_{z \in B} \min_{w \in A} d(z,w)\right\}$$

is a metric on the space of all non-empty compact subsets of X, cf. [31, §21.VII].

<sup>&</sup>lt;sup>11</sup>Given a metric space (X, d) and  $A, B \subseteq X$  non-empty and compact, the Hausdorff distance

Step 1: For all  $y \in \mathbb{R}^n$  the vector b from (9) continuously depends on  $d \in \mathbb{R}^n_{++}$  as a composition and a finite sum of continuous functions, using that the minimum over finitely many continuous functions remains continuous. Here we use that d > 0 so  $d \mapsto \frac{1}{d}$  is continuous.

Step 2: If a sequence  $(b^{(m)})_{m\in\mathbb{N}} \subset \mathbb{R}^{2^n}$  with  $b_{2^n-1}^{(m)} + b_{2^n}^{(m)} = 0$  for all  $m \in \mathbb{N}$  converges to  $b \in \mathbb{R}^{2^n}$  in norm and all the induced convex polytopes  $\{x \in \mathbb{R}^n : Mx \leq b^{(m)}\}, \{x \in \mathbb{R}^n : Mx \leq b\}$  are non-empty, then  $\lim_{m\to\infty} \delta(\{x \in \mathbb{R}^n : Mx \leq b^{(m)}\}, \{x \in \mathbb{R}^n : Mx \leq b\}) = 0$ .

This follows directly from [32, Thm. 2.4]—or, originally, [24]—which yields a constant  $c_M > 0$  (only depending on M) such that

$$\delta(\{x \in \mathbb{R}^n : Mx \le b^{(m)}\}, \{x \in \mathbb{R}^n : Mx \le b\}) \le c_M \|b^{(m)} - b\|_1 \stackrel{m \to \infty}{\to} 0$$

Step 3:  $d \mapsto M_d(y)$  is continuous on  $\mathbb{R}^n_{++}$  for all  $y \in \mathbb{R}^n$ .

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Let  $d^{(m)} \subset \mathbb{R}^n_{++}$  be a sequence with limit  $d \in \mathbb{R}^n_{++}$ . As shown in Step 1 this implies that  $b^{(m)} = b(d^{(m)}) \subset \mathbb{R}^{2^n}_{++}$  converges to  $b(d) \in \mathbb{R}^{2^n}_{++}$  so Step 2 together with Thm. 10 yields

$$\lim_{n \to \infty} M_{d^{(m)}}(y) = \lim_{m \to \infty} \{ x \in \mathbb{R}^n : Mx \le b^{(m)} \} = \{ x \in \mathbb{R}^n : Mx \le b \} = M_d(y) .$$

Step 4:  $d \mapsto M_d(P)$  is continuous on  $\mathbb{R}^n_{++}$  for all  $P \in \mathcal{P}_c(\mathbb{R}^n)$ .

As before let  $d^{(m)} \subset \mathbb{R}_{++}^n$  be a sequence which converges to  $d \in \mathbb{R}_{++}^n$  and let  $\varepsilon > 0$  be given. Because P is compact one finds  $y_1, \ldots, y_k \in P$ ,  $k \in \mathbb{N}$  with  $P \subseteq \bigcup_{i=1}^k B_{\varepsilon/2}(y_i)$ . On the other hand (by Step 2) for every  $i = 1, \ldots, k$  one finds  $N_i \in \mathbb{N}$  such that  $\delta(M_{d^{(m)}}(y_i), M_d(y_i)) < \frac{\varepsilon}{2}$ for all  $m \ge N_i$ . We want to show  $\delta(M_{d^{(m)}}(P), M_d(P)) < \varepsilon$  for all  $m \ge N := \max\{N_1, \ldots, N_k\}$ which would imply the claim.

Let  $m \geq N$  and  $x \in M_{d^{(m)}}(P)$  so one finds  $A \in s_{d^{(m)}}(n)$  and  $y \in P$  such that x = Ay. First compactness of P yields  $y_i \in P$  such that  $||y - y_i|| < \frac{\varepsilon}{2}$ . Then  $Ay_i$  is in  $M_{d^{(m)}}(y_i)$  which lets us pick  $\tilde{x} \in M_d(y_i) \subset M_d(P)$  with  $||Ay_i - \tilde{x}||_1 < \frac{\varepsilon}{2}$  (because  $\delta(M_{d^{(m)}}(y_i), M_d(y_i)) < \frac{\varepsilon}{2}$ ). Using (2) we compute

$$\|x - \tilde{x}\|_{1} \le \|Ay - Ay_{i}\|_{1} + \|Ay_{i} - \tilde{x}\|_{1} \le \|y - y_{i}\|_{1} + \|Ay_{i} - \tilde{x}\|_{1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Analogously for every  $\tilde{x} \in M_d(P)$  one finds  $x \in M_{d^{(m)}}(P)$  such that  $||x - \tilde{x}||_1 < \varepsilon$  which by definition of  $\delta$  implies  $\delta(M_{d^{(m)}}(P), M_d(P)) < \varepsilon$  for all  $m \ge N$ .

Step 5: M is continuous (in the product topology).

Let  $(d^{(m)}, P_m)_{m \in \mathbb{N}} \subset \mathbb{R}^n_{++} \times \mathcal{P}_c(\mathbb{R}^n)$  converge to  $(d, P) \in \mathbb{R}^n_{++} \times \mathcal{P}_c(\mathbb{R}^n)$  in the product topology, i.e.  $\lim_{m\to\infty} \|d^{(m)} - d\|_1 = 0$  and  $\lim_{m\to\infty} \delta(P^{(m)}, P) = 0$ . By Step 4 the former implies  $\lim_{m\to\infty} \delta(M_{d^{(m)}}(P), M_d(P)) = 0$  so using that  $M_{d^{(m)}}$  is non-expansive we find

$$\begin{split} \delta(M_{d^{(m)}}(P^{(m)}), M_d(P)) &\leq \delta(M_{d^{(m)}}(P^{(m)}), M_{d^{(m)}}(P)) + \delta(M_{d^{(m)}}(P), M_d(P)) \\ &\leq \delta(P^{(m)}, P) + \delta(M_{d^{(m)}}(P), M_d(P)) \stackrel{m \to \infty}{\to} 0 \,. \end{split}$$

- **Remark 7.** (i) Continuity of the map M is supported by the fact that the half-spaces limiting  $M_d(y)$  are independent of d, y. An example of a discontinuous relation between  $A \in \mathbb{R}^{m \times n}$  and the induced polytope  $\{x \in \mathbb{R}^n : Ax \leq b\}$  can be found in Example 6.
  - (ii) While  $M_d$  is, in principle, defined for arbitrary  $d \in \mathbb{R}^n$  the continuity statement from Thm. 16 (ii) fails if the domain is extended to  $\mathbb{R}^n_+ \times P_c(\mathbb{R}^n)$ . A counterexample is given in Example 3 (Appendix D).

#### 4.3. Analyzing the d-Majorization Polytope

So far we learned that the majorization polytope  $M_d(y)$  induced by a single vector  $y \in \mathbb{R}^n$ with respect to some  $d \in \mathbb{R}^n_{++}$  differs from the classical majorization polytope not in the orientation of the faces but only in their precise location. By Thm. 10 this difference is fully captured by the following map:

$$f: [0, \mathbf{e}^{\top} d] \to \mathbb{R}$$
$$c \mapsto \min_{i=1,\dots,n} \left( \mathbf{e}^{\top} \left( y - \frac{y_i}{d_i} d \right)_+ + \frac{y_i}{d_i} c \right)$$
(10)

Thus if we want to learn more about the d-majorization polytope we are well-advised to study the properties of (10).

**Lemma 13.** Let  $y \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^n_{++}$  be given and let  $\sigma \in S_n$  be any permutation which orders  $(\frac{y_i}{d_i})_{i=1}^n$  decreasingly, i.e.  $\frac{y_{\sigma(1)}}{d_{\sigma(1)}} \geq \ldots \geq \frac{y_{\sigma(n)}}{d_{\sigma(n)}}$ . Then the map (10) has the following properties.

- (i) f is continuous, piecewise linear, and concave.
- (ii) For arbitrary  $j = 1, \ldots, n$  and  $c \in (\sum_{i=1}^{j-1} d_{\sigma(i)}, \sum_{i=1}^{j} d_{\sigma(i)})$

$$f(c) = \sum_{i=1}^{j-1} y_{\sigma(i)} + \frac{y_{\sigma(j)}}{d_{\sigma(j)}} \left( c - \sum_{i=1}^{j-1} d_{\sigma(i)} \right)$$

as well as  $f'(c) = \frac{y_{\sigma(j)}}{d_{\sigma(j)}}$  so the (weak) derivative of f is monotonically decreasing.

- (iii) For all j = 0, ..., n one has  $f(\sum_{i=1}^{j} d_{\sigma(i)}) = \sum_{i=1}^{j} y_{\sigma(i)}$  so in particular f(0) = 0 and  $f(e^{\top}d) = e^{\top}y$ .
- (iv) Let k = 1, ..., n 1,  $\tau \in S_n$ , and pairwise different  $\alpha_1, ..., \alpha_k \in \{1, ..., n\}$  be given. Then

$$\sum_{j=1}^{k} f\left(\sum_{i=1}^{\alpha_{j}-1} d_{\tau(i)}\right) + f\left(\sum_{i=1}^{k} d_{\tau(\alpha_{i})}\right) \ge \sum_{j=1}^{k} f\left(\sum_{i=1}^{\alpha_{j}} d_{\tau(i)}\right).$$

(v) For all j = 1, ..., n - 1

$$\frac{f\left(\sum_{i=1}^{j} d_{[i]}\right) - f\left(\sum_{i=1}^{j-1} d_{[i]}\right)}{d_{[j]}} \ge \frac{f\left(\sum_{i=1}^{j+1} d_{[i]}\right) - f\left(\sum_{i=1}^{j} d_{[i]}\right)}{d_{[j+1]}}$$

*Proof.* (i): The minimum over finitely many affine linear functions (in particular these functions are continuous & concave) is piecewise linear, continuous, and concave. (ii): Direct consequence of Lemma 20 (Appendix C). (iii): Follows from (ii) together with continuity of f. (iv): Because f is continuous & concave (-f is continuous & convex) this is a direct consequence of Prop. 1 (ii) (for d = e) together with Lemma 21 (Appendix C) and f(0) = 0from (iii). (v): Define  $\tilde{d} := (d_{[j]}, d_{[j+1]})^{\top} \in \mathbb{R}^2_{++}$ . Evidently

$$\left(\sum_{i=1}^{j} d_{[i]}\right)\tilde{d} = \begin{pmatrix} d_{[j]}\sum_{i=1}^{j} d_{[i]} \\ d_{[j+1]}\sum_{i=1}^{j} d_{[i]} \end{pmatrix} \prec_{\tilde{d}} \begin{pmatrix} d_{[j]}\sum_{i=1}^{j+1} d_{[i]} \\ d_{[j+1]}\sum_{i=1}^{j-1} d_{[i]} \end{pmatrix}$$

due to minimality of  $\tilde{d}$  w.r.t.  $\prec_{\tilde{d}}$  (Thm. 3 (i)) and because the entries of the two vectors sum up to the same. Again Prop. 1 (ii) for  $d \to \tilde{d}$  yields

$$\begin{aligned} d_{[j]}f\Big(\sum_{i=1}^{j} d_{[i]}\Big) + d_{[j+1]}f\Big(\sum_{i=1}^{j} d_{[i]}\Big) &= d_{[j]}f\Big(\frac{d_{[j]}\sum_{i=1}^{j} d_{[i]}}{d_{[j]}}\Big) + d_{[j+1]}f\Big(\frac{d_{[j+1]}\sum_{i=1}^{j} d_{[i]}}{d_{[j+1]}}\Big) \\ &\geq d_{[j]}f\Big(\frac{d_{[j]}\sum_{i=1}^{j+1} d_{[i]}}{d_{[j]}}\Big) + d_{[j+1]}f\Big(\frac{d_{[j+1]}\sum_{i=1}^{j-1} d_{[i]}}{d_{[j+1]}}\Big) \\ &= d_{[j]}f\Big(\sum_{i=1}^{j+1} d_{[i]}\Big) + d_{[j+1]}f\Big(\sum_{i=1}^{j-1} d_{[i]}\Big) \end{aligned}$$

because -f is convex, which readily implies (v).

For all rows  $m \in \{0,1\}^n$  of M, the *b*-vector of the *d*-majorization polytope satisfies  $\mathfrak{b}(m) = f(md)$  so, in slight abuse of notation,  $M_d(y) = \{x \in \mathbb{R}^n : Mx \leq f(Md)\}.$ 

**Remark 8.** Recall that in the physics literature, thermo-majorization is usually defined via curves of the following form: Given any vector  $z \in \mathbb{R}^n$  and  $d \in \mathbb{R}^n_{++}$  consider the piecewise linear, continuous curve fully characterized by the elbow points  $\{\left(\sum_{i=1}^j d_{\sigma(i)}, \sum_{i=1}^j z_{\sigma(j)}\right)\}_{j=0}^n$ , where  $\sigma$  is any permutation such that  $\frac{z_{\sigma(1)}}{d_{\sigma(1)}} \geq \ldots \geq \frac{z_{\sigma(n)}}{d_{\sigma(n)}}$ . Then a vector y is said to thermomajorize x if  $e^{\top}x = e^{\top}y$  and if the curve induced by y is never below the curve induced by x [26]. But by the previous lemma this thermo-majorization curve is precisely the function f which characterizes the polytope meaning  $x \prec_d y$  is equivalent to  $f_x(c) \leq f_y(c)$  for all  $c \in [0, e^{\top}d]$ ; more on this in a bit.

While this confirms the (well-known) equivalence of d-majorization and thermo-majorization, we can reduce the comparison of the two curves to just the "elbow points" of the lower curve—as already observed in [2, Thm. 4]—by means of the following elegant proof:

Proof of Prop. 1 (i)  $\Leftrightarrow$  (vii). By Lemma 8,  $x \prec_d y$  is equivalent to  $M_d(x) \subseteq M_d(y)$  which by Thm. 10 and Remark 4 holds if and only if  $e^{\top}x = e^{\top}y$  and  $f_x((Md)_i) \leq f_y((Md)_i)$  for all  $i = 1, \ldots, 2^n - 2$ . Now we may apply Lemma 13 and the "elbow point principle" (i.e. only check the elbow points of the *lower concave* curve, similar to the proof of Prop. 1) to arrive at the equivalent condition:  $e^{\top}x = e^{\top}y$  and  $\sum_{i=1}^{j} x_{\sigma(i)} = f_x(\sum_{i=1}^{j} d_{\sigma(i)}) \leq f_y(\sum_{i=1}^{j} d_{\sigma(i)})$  for all  $j = 1, \ldots, n - 1$ , which concludes the proof.  $\Box$ 

Another advantage of introducing and studying the function f is that its properties transfer to  $M_d(y)$  which suffices to fully characterize the extreme points of the *d*-majorization polytope, thus generalizing [44]:

**Theorem 14.** Let  $y \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^n_{++}$ . Then the extreme points of  $M_d(y)$  are precisely the  $E_b(\sigma)$ ,  $\sigma \in S_n$  from Definition 3. In particular  $M_d(y) = \operatorname{conv} \{E_b(\sigma) : \sigma \in S_n\}$ .

*Proof.* If we can show  $\{E_b(\sigma) : \sigma \in S_n\} \subseteq M_d(y)$  then by Thm. 7 we are done. Let arbitrary  $\sigma \in S_n$  be given. Showing  $E_b(\sigma) \in M_d(y)$  by Thm. 10 is equivalent to showing  $ME_b(\sigma) \leq b$ , i.e.

$$\sum_{i=1}^{j} (E_b(\sigma))_{\sigma(\alpha_i)} = \left(\sum_{i=1}^{j} e_{\sigma(\alpha_i)}\right)^\top E_b(\sigma) \le \mathfrak{b}\left(\sum_{i=1}^{j} e_{\sigma(\alpha_i)}^\top\right) = f\left(\sum_{i=1}^{j} d_{\sigma(\alpha_i)}\right)$$

for all j = 1, ..., n - 1 and all pairwise different  $\alpha_1, ..., \alpha_j \in \{1, ..., n\}$  where f is the map from (10). Be aware that writing  $\sigma(\alpha_i)$  instead of  $\alpha_i$  in above inequality yields an equivalent problem as the  $\alpha_i$  can be chosen arbitrarily anyway; this will be advantageous because the expression  $(E_b(\sigma))_{\sigma(\alpha_i)}$  is easier to handle than  $(E_b(\sigma))_{\alpha_i}$ . Indeed by Lemma 6

$$(E_b(\sigma))_{\sigma(\alpha_j)} = \mathfrak{b}\Big(\sum_{i=1}^{\alpha_j} e_{\sigma(i)}^{\top}\Big) - \mathfrak{b}\Big(\sum_{i=1}^{\alpha_j-1} e_{\sigma(i)}^{\top}\Big) = f\Big(\sum_{i=1}^{\alpha_j} d_{\sigma(i)}\Big) - f\Big(\sum_{i=1}^{\alpha_j-1} d_{\sigma(i)}\Big)$$

for all j = 1, ..., k. Hence showing  $E_b(\sigma) \in M_d(y)$  is equivalent to

$$\sum_{j=1}^{k} \left( f\left(\sum_{i=1}^{\alpha_j} d_{\sigma(i)}\right) - f\left(\sum_{i=1}^{\alpha_j-1} d_{\sigma(i)}\right) \right) \le f\left(\sum_{i=1}^{k} d_{\sigma(\alpha_i)}\right)$$

which holds due to Lemma 13 (iv).

We immediately obtain the following result.

**Corollary 15.** Let  $y \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^n_{++}$ . Then  $M_d(y)$  is a non-empty convex polytope of dimension at most n-1 and, moreover, has at most n! extreme points.

Proof. For non-emptiness note that  $y \in M_d(y)$  because  $I_n \in s_d(n)$ . By Thm. 10 there exists  $b \in \mathbb{R}^{2^n}$  such that  $M_d(y) = \{x \in \mathbb{R}^n : Mx \leq b\}$  so  $M_d(y)$  is a convex polytope of at most n-1 dimensions (Lemma 4). Finally the extreme points of  $M_d(y)$  are given by  $\{E_b(\sigma) : \sigma \in S_n\}$  (Thm. 14) which due to  $|S_n| = n!$  concludes the proof.  $\Box$ 

- **Remark 9.** (i) These results (Thm. 14 & Coro. 15) recently appeared in the physics literature for the special case  $y \ge 0$  [1, Sec. 2.2] but with an entirely different proof strategy: Alhambra et al. explicitly constructed a family  $\{P^{(\pi,\alpha)}\}_{\alpha}$  of *d*-stochastic matrices called " $\beta$ -permutations" with the property that  $\{P^{(\pi,\alpha)}y\}_{\alpha}$  contains all extreme points of  $M_d(y)$ , which—in our language—necessarily have to be of the form  $E_b(\sigma)$  [35, Lemma 12].
  - (ii) Analyzing and structuring the situations when  $M_d(y)$  has less than n! corners, i.e. when d, y are chosen such that the map  $E_b$  is not injective reveals further connections between  $M_d(y)$  and the map f that determines the entries of the vector b. We will treat this question in a forthcoming paper [18].

Now one of these extreme points has the property of classically majorizing every other element inside the d-majorization polytope. The result, which is of particular interest, e.g., to tackle reachability questions in quantum control theory [49, 15], reads as follows:

**Theorem 16.** Let  $y \in \mathbb{R}^n_+$ ,  $d \in \mathbb{R}^n_{++}$ . Then there exists  $z \in M_d(y)$  such that  $x \prec z$  for all  $x \in M_d(y)$ , i.e.  $M_d(y) \subseteq M_e(z)$ , and this z is unique up to permutation.

More precisely if  $\sigma \in S_n$  orders d decreasingly, that is,  $d_{\sigma(1)} \geq \ldots \geq d_{\sigma(n)}$ , then one has  $M_d(y) \subseteq M_e(E_b(\sigma))$ . Thus z can be chosen to be the extreme point  $E_b(\sigma)$ , i.e. the solution to

$$\Lambda \underline{\sigma} z = \min_{i=1,...,n} \begin{pmatrix} \mathbb{e}^+ (y - \frac{y_i}{d_i} d)_+ + \frac{y_i}{d_i} d_{\sigma(1)} \\ \vdots \\ \mathbb{e}^\top (y - \frac{y_i}{d_i} d)_+ + \frac{y_i}{d_i} \sum_{j=1}^{n-1} d_{\sigma(j)} \\ \mathbb{e}^\top y \end{pmatrix} \stackrel{(10)}{=} \left( f \left( \sum_{i=1}^j d_{\sigma(i)} \right) \right)_{j=1}^n.$$
(11)

Moreover  $M_d(z) \subseteq M_e(z)$ , and  $\frac{z}{d}$  and d are similarly ordered, i.e.  $\frac{z_{\sigma(1)}}{d_{\sigma(1)}} \ge \ldots \ge \frac{z_{\sigma(n)}}{d_{\sigma(n)}}$ .

Proof. Uniqueness of such z (up to permutation) is the easiest to show: If there exist  $z_1, z_2 \in M_d(y)$  such that  $M_d(y) \subseteq M_{e}(z_i)$  for i = 1, 2, then in particular  $z_{2-i} \in M_d(y) \subseteq M_{e}(z_i)$ . Hence  $z_1 \prec z_2 \prec z_1$  and one finds a permutation  $\tau \in S_n$  such that  $z_1 = \underline{\tau} z_2$ .

For existence let  $\sigma \in S_n$  be any permutation which orders d decreasingly. From Thm. 14 we know that  $z := E_b(\sigma) \in M_d(y)$ . By (11) for all  $j = 1, \ldots, n-1$  one finds

$$z_{\sigma(j)} = f\left(\sum_{i=1}^{j} d_{\sigma(i)}\right) - f\left(\sum_{i=1}^{j-1} d_{\sigma(i)}\right) = f\left(\sum_{i=1}^{j} d_{[i]}\right) - f\left(\sum_{i=1}^{j-1} d_{[i]}\right).$$
(12)

Therefore  $\frac{z_{\sigma(j)}}{d_{\sigma(j)}} \ge \frac{z_{\sigma(j+1)}}{d_{\sigma(j+1)}}$  is equivalent to

$$\frac{f\left(\sum_{i=1}^{j} d_{[i]}\right) - f\left(\sum_{i=1}^{j-1} d_{[i]}\right)}{d_{[j]}} \ge \frac{f\left(\sum_{i=1}^{j+1} d_{[i]}\right) - f\left(\sum_{i=1}^{j} d_{[i]}\right)}{d_{[j+1]}}$$

which holds due to Lemma 13 (v), meaning  $\frac{z}{d}$  and d are indeed similarly ordered, as claimed. More importantly because  $z \in \mathbb{R}^n_+$  (stochastic matrices preserve non-negativity of y) one even has  $z_{\sigma(j)} = z_{[j]}$  for all j = 1, ..., n because  $z_{\sigma(j)} \ge \frac{d_{\sigma(j)}}{d_{\sigma(j+1)}} z_{\sigma(j+1)} = \frac{d_{[j]}}{d_{[j+1]}} z_{\sigma(j+1)} \ge z_{\sigma(j+1)}$ . Now recall that  $M_{\mathbb{e}}(z) = \{x \in \mathbb{R}^n : Mx \le b'_z\}$  [11, Thm. 1] where  $b'_z$  is of the following

Now recall that  $M_{\mathbb{e}}(z) = \{x \in \mathbb{R}^n : Mx \leq b'_z\}$  [11, Thm. 1] where  $b'_z$  is of the following form: the first  $\binom{n}{1}$  entries equal  $z_{[1]} = z_{\sigma(1)}$ , the next  $\binom{n}{2}$  entries equal  $z_{[1]} + z_{[2]} = z_{\sigma(1)} + z_{\sigma(2)}$ and so forth until  $\binom{n}{n-1}$  entries equaling  $\sum_{i=1}^{n-1} z_{[i]} = \sum_{i=1}^{n-1} z_{\sigma(i)}$ . Writing  $M_d(y) = \{x \in \mathbb{R}^n : Mx \leq b\}$  (Thm. 10), if we can show that  $b \leq b'_z$  then we get  $M_d(y) \subseteq M_{\mathbb{e}}(z)$  (Remark 4) as desired.

For all k = 1, ..., n - 1 and all  $\tau \in S_n$  by Lemma 22 (Appendix C)—which we may apply because  $\frac{y_i}{d_i} \ge 0$  for all *i*—we compute

$$\begin{split} \mathfrak{b}\Big(\sum_{j=1}^{k} e_{\tau(j)}^{\top}\Big) &= \min_{i=1,\dots,n} e^{\top} \Big(y - \frac{y_i}{d_i}d\Big)_+ + \frac{y_i}{d_i} \sum_{j=1}^{k} d_{\tau(j)} \\ &\leq \max_{\tau \in S_n} \min_{i=1,\dots,n} e^{\top} \Big(y - \frac{y_i}{d_i}d\Big)_+ + \frac{y_i}{d_i} \sum_{j=1}^{k} d_{\tau(j)} \\ &= \min_{i=1,\dots,n} e^{\top} \Big(y - \frac{y_i}{d_i}d\Big)_+ + \frac{y_i}{d_i} \Big(\max_{\tau \in S_n} \sum_{j=1}^{k} d_{\tau(j)}\Big) \\ &= \min_{i=1,\dots,n} e^{\top} \Big(y - \frac{y_i}{d_i}d\Big)_+ + \frac{y_i}{d_i} \Big(\sum_{j=1}^{k} d_{[j]}\Big) \stackrel{(12)}{=} \sum_{i=1}^{k} z_{\sigma(i)} = \mathfrak{b}_z' \Big(\sum_{j=1}^{k} e_{\tau(j)}^{\top}\Big) \end{split}$$

so  $b \leq b'_z$  as claimed. To conclude the proof note that by Lemma 8  $z \in M_d(y)$  implies  $M_d(z) \subseteq (M_d \circ M_d)(y) = M_d(y) \subseteq M_e(z)$ .

Non-negativity of y in Thm. 16 is actually necessary as Example 4 (Appendix D) shows.

Given our knowledge of this maximal point (w.r.t. classical majorization) in the d-majorization polytope, one can now give a necessary condition for when the initial vector y itself is this maximal element.

**Corollary 17.** Let  $y \in \mathbb{R}^n_+$ ,  $d \in \mathbb{R}^n_{++}$ . If  $\frac{y}{d}$  and d are similarly ordered, that is, there exists a permutation  $\sigma \in S_n$  such that  $d_{\sigma(1)} \geq \ldots \geq d_{\sigma(n)}$  and  $\frac{y_{\sigma(1)}}{d_{\sigma(1)}} \geq \ldots \geq \frac{y_{\sigma(n)}}{d_{\sigma(n)}}$ , then  $M_d(y) \subseteq M_e(y)$ .

Proof. If d and  $\frac{y}{d}$  are similarly ordered (by means of  $\sigma$ ) then, by Thm. 16,  $M_d(y) \subseteq M_{\mathfrak{e}}(E_b(\sigma))$ with  $\Lambda \underline{\sigma} E_b(\sigma) = (f(\sum_{i=1}^j d_{\sigma(i)}))_{j=1}^n = (\sum_{i=1}^j y_{\sigma(i)})_{j=1}^n = \Lambda \underline{\sigma} y$  by Lemma 13 (iii), hence  $E_b(\sigma) = y$ .

Be aware that the converse to Coro. 17 does not hold, refer to Example 5, Appendix D. To see how the *d*-majorization polytope behaves (aside from continuity) when changing only *d* while leaving the initial vector *y* untouched we refer to Example 7. This example also illustrates Coro. 17 because the whole trajectory  $\{d(\lambda) : \lambda \in [0,1]\}$  taken by the *d*-vector satisfies  $\frac{y_1}{(d(\lambda))_1} \geq \ldots \geq \frac{y_n}{(d(\lambda))_n}$  so maximality of *y* (w.r.t. classical majorization) is preserved throughout.

# Appendix A. Extreme Points of $s_d(3)$

**Lemma 18.** Let  $d \in \mathbb{R}^3_{++}$  with  $d_1 > d_2 > d_3$ .

(i) If  $d_1 \ge d_2 + d_3$ , then the 10 extreme points of  $s_d(3)$  are given by

$$\begin{array}{c} I_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{d_3}{d_2} & 1 \\ 0 & \frac{d_3}{d_2} & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_3}{d_1} & 0 & 1 \\ 0 & 1 & 0 \\ \frac{d_3}{d_1} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_2}{d_1} & 1 & 0 \\ \frac{d_2 - d_3}{d_1} & 0 & 1 \\ \frac{d_3}{d_1} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 - \frac{d_3}{d_1} & \frac{d_3}{d_2} & 0 \\ 0 & 1 - \frac{d_3}{d_1} & 1 & 0 \\ \frac{d_2}{d_1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_2}{d_1} & 1 & 0 \\ \frac{d_2}{d_1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_3}{d_1} & 0 & 1 \\ \frac{d_3}{d_1} & 1 - \frac{d_3}{d_2} & 0 \\ 0 & \frac{d_3}{d_2} & 0 \end{pmatrix} \\ \begin{pmatrix} 1 - \frac{d_2 - d_3}{d_1} & 1 - \frac{d_3}{d_2} & 0 \\ 0 & \frac{d_2 - d_3}{d_1} & 1 - \frac{d_3}{d_2} & 0 \\ 0 & \frac{d_2 - d_3}{d_1} & 0 & 1 \\ 0 & \frac{d_3}{d_2} & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_2}{d_1} & 1 - \frac{d_3}{d_2} & 1 \\ \frac{d_2}{d_1} & 0 & 0 \\ 0 & \frac{d_3}{d_2} & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_2 + d_3}{d_1} & 1 & 1 \\ \frac{d_2}{d_1} & 0 & 0 \\ 0 & \frac{d_3}{d_2} & 0 \end{pmatrix}$$

(ii) If  $d_1 < d_2 + d_3$ , then the 13 extreme points of  $s_d(3)$  are given by

$$\begin{array}{c} I_{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{d_{3}}{d_{2}} & 1 \\ 0 & \frac{d_{3}}{d_{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_{3}}{d_{1}} & 0 & 1 \\ 0 & 1 & 0 \\ \frac{d_{3}}{d_{1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_{2}}{d_{1}} & 1 & 0 \\ \frac{d_{2} - d_{3}}{d_{1}} & 0 & 1 \\ \frac{d_{3}}{d_{1}} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 - \frac{d_{3}}{d_{1}} & \frac{d_{3}}{d_{2}} & 0 \\ 0 & 1 - \frac{d_{3}}{d_{2}} & 1 \\ \frac{d_{3}}{d_{1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_{2}}{d_{1}} & 1 & 0 \\ \frac{d_{2} - d_{3}}{d_{1}} & 0 & 0 \\ 0 & \frac{d_{3}}{d_{2}} & 0 \end{pmatrix} \\ \begin{pmatrix} 1 - \frac{d_{2} - d_{3}}{d_{1}} & 1 - \frac{d_{3}}{d_{2}} & 0 \\ 0 & \frac{d_{3}}{d_{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_{2}}{d_{1}} & 1 - \frac{d_{3}}{d_{2}} & 1 \\ \frac{d_{2} - d_{3}}{d_{1}} & 1 - \frac{d_{3}}{d_{2}} & 0 \\ 0 & \frac{d_{3}}{d_{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_{2}}{d_{1}} & 1 - \frac{d_{3}}{d_{2}} & 1 \\ \frac{d_{2}}{d_{1}} & 0 & 0 \\ 0 & \frac{d_{3}}{d_{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{d_{1} - d_{3}}{d_{2}} & 1 \\ \frac{d_{2}}{d_{1}} & 0 & 0 \\ 0 & \frac{d_{3}}{d_{2}} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{d_{1} - d_{3}}{d_{2}} & 1 \\ 1 - \frac{d_{3}}{d_{1}} & 1 - \frac{d_{1} - d_{3}}{d_{2}} & 1 \\ \frac{d_{2}}{d_{1}} & 0 & 0 \\ 1 - \frac{d_{2}}{d_{1}} & 1 - \frac{d_{1} - d_{2}}{d_{3}} \end{pmatrix} \begin{pmatrix} 0 & \frac{d_{1} - d_{3}}{d_{2}} & 1 \\ \frac{d_{2}}{d_{1}} & 0 & 0 \\ 1 - \frac{d_{2}}{d_{1}} & 0 & 1 - \frac{d_{1} - d_{2}}{d_{3}} \\ \frac{d_{3}}{d_{1}} & 0 & 0 \end{pmatrix} \end{pmatrix}$$

*Proof.* The respective number of extreme points is stated in [28, Remark 4.5] or, more recently, [41, Ch. IV]. Then one only has to verify that the above matrices (under the given assumptions) are in fact extremal in  $s_d(3)$ .

Once we allow components of d to coincide, the above extreme points simplify slightly (as already observed in [28, Remark 4.5]). Within the setting of (i) if  $d_2 = d_3$  then one is left with 7 extreme points. For (ii) if either  $d_1 = d_2$  or  $d_2 = d_3$  then one has 10 extreme points and if  $d_1 = d_2 = d_3$  then there are 6 extreme points—namely the 3 × 3 permutation matrices—which recovers Birkhoff's theorem, cf. [39, Thm. 2.A.2].

### Appendix B. Appendix to Section 3

In order to keep the main part of this paper sufficiently structured we outsourced some of the lengthy and technical lemmata.

**Lemma 19.** Let  $A \in \{0,1\}^{m \times n}$  be a matrix such that  $\operatorname{rank}(A) = n$ ,  $e^{\top}$  is a row of A, and for any two rows  $a_1$ ,  $a_2$  of A their entrywise minimum  $\min\{a_1, a_2\}$  and maximum  $\max\{a_1, a_2\}$  are rows of A, as well. Then the following statements hold.

- (i) There exists a row a of A such that ae = n 1.
- (ii) For every row a of A with ae > 1 one finds a row  $\tilde{a}$  of A such that  $\tilde{a}e = ae 1$  and  $\tilde{a} \leq a$ .
- (iii) There exist rows  $a_1, \ldots, a_{n-1}$  of A and a permutation  $\sigma \in S_n$  such that

$$\Lambda \underline{\sigma} = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ e^{\top} \end{pmatrix}.$$
(B.1)

*Proof.* (i): For all j = 1, ..., n define  $S_j := \{a : a \text{ is a row of } A \text{ and } ae_j = 0\} \setminus \{0\}$  as the collection of all (non-zero) rows of A the j-th entry of which vanishes<sup>12</sup>. Defining  $a_j := \max S_j$  this is a row of A (due to the maximum property) with  $a_j e \leq n - 1$ . It is obvious that any non-zero row a of A is in  $S_j$  if and only if  $a \leq a_j$ —hence  $a_j = a_k$  for any two j, k implies  $S_j = S_k$ . Now there exists  $k \in \{1, \ldots, n\}$  such that

$$a_k \mathbf{e} = \max_{j=1,\dots,n} a_j \mathbf{e} \quad (\ge 1) \,. \tag{B.2}$$

If  $a_k e = n - 1$  then we are done. If  $a_k e < n - 1$  then one finds an index  $i \neq k$  such that  $a_k e_i = 0$ . Therefore  $a_k \in S_i$  so  $a_k \leq a_i$  but  $a_k e \geq a_i e$  by (B.2); this shows  $a_k = a_i$  and thus  $S_k = S_i$ . We claim that this forces all rows a of A to satisfy  $a(e_i + e_k) \in \{0, 2\}$ ; but then the

<sup>&</sup>lt;sup>12</sup>It may happen that A contains (at most, due to rank condition) one column of ones so (at most) one of the  $S_j$  might be empty, but one can still guarantee the existence of some j such that  $S_j \neq \emptyset$  (because  $n \ge 2$ , the case n = 1 is trivial).

linear span of all rows of A has this property as well so it cannot contain  $e_k$ , contradicting rank(A) = n.

Indeed let a be any row of A. If  $a \in S_k = S_i$  then  $ae_i = ae_k = 0 = ae_i + ae_k$ . If  $a \notin S_k = S_i$  then  $ae_i = ae_k = 1$  so  $ae_i + ae_k = 2$ .

(ii): We prove this via induction. The case n = 1 is trivial. Now for  $n \to n+1$  let  $A \in \{0,1\}^{m \times (n+1)}$  with the above properties be given. Be aware of the following argument: For  $\tau = \begin{pmatrix} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n+1 \\ 2 & 3 & \cdots & j & 1 & j+1 & \cdots & n+1 \end{pmatrix}$  and all  $j = 1, \ldots, n+1$  the matrix

$$A_j := A \underline{\tau_j} \begin{pmatrix} 0 & \cdots & 0 \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathbb{R}^{m \times n}$$

is the original matrix A but without the *j*-th column. Using this is easy to see that  $\operatorname{rank}(A_j) = n$  (follows from, e.g., [25, Thm. 0.4.5.(c)]),  $e^{\top} \in A_j$ , and the min-max condition for the rows of  $A_j$  hold for all  $j = 1, \ldots, n + 1$ . Hence we may apply the induction hypothesis to any of these matrices  $A_j$ .

Now consider any row a of A with  $a \ge 1$ . There are two cases which, once verified, conclude the proof of (ii).

Case 1:  $a = e^{\top}$ . By (i) we find  $j \in \{1, \ldots, n+1\}$  such that  $e^{\top} - e_j^{\top}$  is a row of A.

Case 2:  $a \neq e^{\top}$  so there exists  $j \in \{1, \ldots, n+1\}$  such that  $ae_j = 0$ . Consider  $A_j \in \mathbb{R}^{m \times n}$  and the truncated row b of  $A_j$  corresponding to a. By induction hypothesis (be = ae > 1) we find  $\tilde{b} \in A_j$  such that  $\tilde{b}e = be - 1$  and  $\tilde{b} \leq b$ . Now there exists a row  $\hat{a}$  of A which becomes  $\tilde{b}$  when removing the j-th entry. Defining  $\tilde{a} := \min\{\hat{a}, a\}$  we know that this is a row of A (min-max-property of A) and  $\tilde{a} \leq a$  as well as  $\tilde{a}e = \tilde{b}e - 1 = ae - 1$ .

(iii): By assumption  $e^{\top} \in A$  so using (ii) A contains some  $a_{n-1}$  of row sum n-1, which in turn yields  $a_{n-2} \in A$  of row sum n-2 with  $a_{n-2} \leq a_{n-1}$ , and so forth. Eventually one ends up with rows  $a_1, \ldots, a_{n-1}$  of A which satisfy  $a_j e = j$  for all  $j = 1, \ldots, n-1$  as well as  $a_1 \leq \ldots \leq a_{n-1}$ ; hence there exists a permutation  $\tau \in S_n$  such that (B.1) holds.  $\Box$ 

**Remark 10.** With an analogous argument one can show that every such matrix A contains a standard basis vector as a row, and that for every row a with ae < n one finds  $\tilde{a} \in A$  with  $\tilde{a}e = ae + 1$  and  $\tilde{a} \ge a$ . Therefore *every* row of A can be completed to a matrix of the form (B.1).

## Appendix C. Appendix to Section 4.3

**Lemma 20.** Let  $y \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^n_{++}$  be arbitrary and let  $\sigma \in S_n$  satisfy<sup>13</sup>

$$\frac{y_{\sigma(1)}}{d_{\sigma(1)}} \ge \frac{y_{\sigma(2)}}{d_{\sigma(2)}} \ge \dots \ge \frac{y_{\sigma(n)}}{d_{\sigma(n)}}.$$
(C.1)

Then the following statements hold.

<sup>&</sup>lt;sup>13</sup>Such a permutation  $\sigma$  always exists as it is just the decreasing ordering of the vector  $\frac{y}{d} := (\frac{y_i}{d_i})_{i=1}^n$ .

(i) For all  $c \in \mathbb{R}$ ,  $k = 1, \ldots, n$ .

$$\mathbf{e}^{\top} \left( y - \frac{y_{\sigma(k)}}{d_{\sigma(k)}} d \right)_{+} + \frac{y_{\sigma(k)}}{d_{\sigma(k)}} c = \frac{y_{\sigma(1)}}{d_{\sigma(1)}} c - \sum_{i=1}^{k-1} \left( \frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(i+1)}}{d_{\sigma(i+1)}} \right) \left( c - \sum_{j=1}^{i} d_{\sigma(j)} \right).$$

(ii) Let  $c \in (0, e^{\top}d]$  and let  $k \in \{1, \ldots, n\}$  be the unique index such that  $c - \sum_{i=1}^{k-1} d_{\sigma(i)} > 0$ but  $c - \sum_{i=1}^{k} d_{\sigma(i)} \leq 0$ . Then

$$\min_{i=1,\dots,n} \left( \mathbb{e}^{\top} \left( y - \frac{y_i}{d_i} d \right)_+ + \frac{y_i}{d_i} c \right) = \left( \sum_{i=1}^{k-1} y_{\sigma(i)} \right) + \frac{y_{\sigma(k)}}{d_{\sigma(k)}} \left( c - \sum_{i=1}^{k-1} d_{\sigma(i)} \right).$$

(*iii*) If d = e, then  $\min_{i=1,...,n} e^{\top} (y - y_i e)_+ + k y_i = \sum_{i=1}^k y_{[i]}$  for all k = 1,...,n.

*Proof.* (i): This identity comes from

$$\frac{y_{\sigma(1)}}{d_{\sigma(1)}}c - \sum_{i=1}^{k-1} \left(\frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(i+1)}}{d_{\sigma(i+1)}}\right)c = \frac{y_{\sigma(1)}}{d_{\sigma(1)}}c - \frac{y_{\sigma(1)}}{d_{\sigma(1)}}c + \frac{y_{\sigma(k)}}{d_{\sigma(k)}}c = \frac{y_{\sigma(k)}}{d_{\sigma(k)}}c$$

as well as

$$e^{\top} \left( y - \frac{y_{\sigma(k)}}{d_{\sigma(k)}} d \right)_{+} = \sum_{j=1}^{k-1} \left( \frac{y_{\sigma(j)}}{d_{\sigma(j)}} - \frac{y_{\sigma(k)}}{d_{\sigma(k)}} \right) d_{\sigma(j)}$$
$$= \sum_{j=1}^{k-1} \sum_{i=j}^{k-1} \left( \frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(i+1)}}{d_{\sigma(i+1)}} \right) d_{\sigma(j)} = \sum_{i=1}^{k-1} \sum_{j=1}^{i} \left( \frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(i+1)}}{d_{\sigma(i+1)}} \right) d_{\sigma(j)}$$

where in the last step one just re-enumerates the index set  $\{(i, j) : 1 \le j \le i \le k - 1\}$ . (ii): Using (i)

$$\begin{split} \min_{i=1,\dots,n} \mathbf{e}^{\top} \Big( y - \frac{y_i}{d_i} d \Big)_+ &+ \frac{y_i}{d_i} c = \min_{\ell=1,\dots,n} \mathbf{e}^{\top} \Big( y - \frac{y_{\sigma(\ell)}}{d_{\sigma(\ell)}} d \Big)_+ &+ \frac{y_{\sigma(\ell)}}{d_{\sigma(\ell)}} c \\ &= \frac{y_{\sigma(1)}}{d_{\sigma(1)}} c - \max_{\ell=1,\dots,n} \sum_{i=1}^{\ell-1} \Big( \frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(i+1)}}{d_{\sigma(i+1)}} \Big) \Big( c - \sum_{j=1}^i d_{\sigma(j)} \Big) \,. \end{split}$$

There are two important things to notice here: the expression  $\frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(i+1)}}{d_{\sigma(i+1)}}$  is always nonnegative by (C.1) and, moreover, the map  $g : \{0, \ldots, n\} \to \mathbb{R}$ ,  $i \mapsto c - \sum_{j=1}^{i} d_{\sigma(j)}$  satisfies g(0) = c > 0,  $g(n) = c - e^{\top}d \leq 0$ , and is strictly monotonically decreasing. Thus the index k described above exists, is unique, and we get

$$\max_{\ell=1,\dots,n} \sum_{i=1}^{\ell-1} \left( \frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(i+1)}}{d_{\sigma(i+1)}} \right) \left( c - \sum_{j=1}^{i} d_{\sigma(j)} \right) = \sum_{i=1}^{k-1} \left( \frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(i+1)}}{d_{\sigma(i+1)}} \right) \left( c - \sum_{j=1}^{i} d_{\sigma(j)} \right)$$

as we simply disregard all negative summands  $(i \ge k)$ . Therefore

$$\begin{split} \min_{i=1,\dots,n} e^{\top} \left( y - \frac{y_i}{d_i} d \right)_+ &+ \frac{y_i}{d_i} c = \frac{y_{\sigma(1)}}{d_{\sigma(1)}} c - \sum_{i=1}^{k-1} \left( \frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(i+1)}}{d_{\sigma(i+1)}} \right) \left( c - \sum_{j=1}^i d_{\sigma(j)} \right) \\ & \stackrel{(i)}{=} e^{\top} \left( y - \frac{y_{\sigma(k)}}{d_{\sigma(k)}} d \right)_+ + \frac{y_{\sigma(k)}}{d_{\sigma(k)}} c \\ &= \sum_{i=1}^{k-1} d_{\sigma(i)} \left( \frac{y_{\sigma(i)}}{d_{\sigma(i)}} - \frac{y_{\sigma(k)}}{d_{\sigma(k)}} \right) + \frac{y_{\sigma(k)}}{d_{\sigma(k)}} c \\ &= \left( \sum_{i=1}^{k-1} y_{\sigma(i)} \right) + \frac{y_{\sigma(k)}}{d_{\sigma(k)}} \left( c - \sum_{i=1}^{k-1} d_{\sigma(i)} \right). \end{split}$$

(iii): Direct consequence of (ii).

**Lemma 21.** Let  $d \in \mathbb{R}^{n}_{++}$ , k = 1, ..., n - 1,  $\tau \in S_n$ , and pairwise different numbers  $\alpha_1, \ldots, \alpha_k \in \{1, \ldots, n\}$  be given. Then

$$v := \begin{pmatrix} \sum_{i=1}^{\alpha_1 - 1} d_{\tau(i)} \\ \vdots \\ \sum_{i=1}^{\alpha_k - 1} d_{\tau(i)} \\ \sum_{i=1}^{k} d_{\tau(\alpha_i)} \end{pmatrix} \prec \begin{pmatrix} \sum_{i=1}^{\alpha_1} d_{\tau(i)} \\ \vdots \\ \sum_{i=1}^{\alpha_k} d_{\tau(i)} \\ 0 \end{pmatrix} =: w \in \mathbb{R}^{k+1}.$$

*Proof.* W.l.o.g.  $\alpha_1 > \ldots > \alpha_k$ —reordering the  $\alpha_i$  amounts to reordering v, w but classical majorization is permutation invariant. By definition  $v \prec w$  holds iff  $e^{\top}v = e^{\top}w$  and together with  $\sum_{i=1}^{\ell} v_{[i]} \leq \sum_{i=1}^{\ell} w_{[i]}$  for all  $\ell = 1, \ldots, k$  (the former is readily verified). Because the  $\alpha_i$  are ordered decreasingly one finds unique  $\xi \in \{1, \ldots, k+1\}$  such that

$$\sum_{i=1}^{\alpha_{\xi}-1} d_{\tau(i)} < \sum_{i=1}^{k} d_{\tau(\alpha_{i})} \le \sum_{i=1}^{\alpha_{\xi-1}-1} d_{\tau(i)}$$

(where  $\alpha_0 := n + 1$  and  $\alpha_{k+1} := 0$ ). Thus  $v \prec w$  is equivalent to

$$[v] = \begin{pmatrix} \sum_{i=1}^{\alpha_{1}-1} d_{\tau(i)} \\ \vdots \\ \sum_{i=1}^{\alpha_{\xi-1}-1} d_{\tau(i)} \\ \sum_{i=1}^{k} d_{\tau(\alpha_{i})} \\ \sum_{i=1}^{\lambda_{\xi}-1} d_{\tau(i)} \\ \vdots \\ \sum_{i=1}^{\alpha_{k}-1-1} d_{\tau(i)} \\ \sum_{i=1}^{\alpha_{k}-1} d_{\tau(i)} \end{pmatrix} \prec \begin{pmatrix} \sum_{i=1}^{\alpha_{1}} d_{\tau(i)} \\ \vdots \\ \sum_{i=1}^{\alpha_{k}+1} d_{\tau(i)} \\ \vdots \\ \sum_{i=1}^{\alpha_{k}-1} d_{\tau(i)} \\ 0 \end{pmatrix} = [w].$$

The first  $\xi - 1$  partial sum conditions are evident. Now consider  $\ell \in \{\xi, \ldots, k\}$ . Then

$$\sum_{j=1}^{\ell} w_{[j]} - \sum_{j=1}^{\ell} v_{[j]} = \sum_{j=1}^{\ell} \left( \sum_{i=1}^{\alpha_j} d_{\tau(i)} \right) - \left( \sum_{j=1}^{\ell-1} \left( \sum_{i=1}^{\alpha_j-1} d_{\tau(i)} \right) + \sum_{i=1}^{k} d_{\tau(\alpha_i)} \right)$$
$$= \sum_{i=1}^{\alpha_\ell} d_{\tau(i)} + \sum_{j=1}^{\ell-1} d_{\tau(\alpha_j)} - \sum_{i=1}^{k} d_{\tau(\alpha_i)} = \sum_{i=1}^{\alpha_\ell} d_{\tau(i)} - \sum_{i=\ell}^{k} d_{\tau(\alpha_i)} \ge 0$$

where in the last step we used that the entries of d are non-negative and, more importantly, that  $\{\alpha_k, \alpha_{k-1}, \ldots, \alpha_{\ell+1}, \alpha_\ell\} \subseteq \{1, 2, \ldots, \alpha_\ell - 1, \alpha_\ell\}$  due to the ordering of the  $\alpha_i$   $\Box$ 

**Lemma 22.** Let  $m, n \in \mathbb{N}$  and  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^n_+$ ,  $z \in \mathbb{R}^m$  be given. Then

$$\max_{k=1,\dots,m} \min_{i=1,\dots,n} (x_i + y_i z_k) = \min_{i=1,\dots,n} \left( x_i + y_i \left( \max_{k=1,\dots,m} z_k \right) \right).$$

*Proof.* Direct computation:

$$\max_{k=1,\dots,m} \min_{i=1,\dots,n} (x_i + y_i z_k) \le \min_{i=1,\dots,n} \max_{k=1,\dots,m} (x_i + y_i z_k)$$
$$= \min_{i=1,\dots,n} (x_i + y_i (\max_{k=1,\dots,m} z_k)) \le \max_{k=1,\dots,m} \min_{i=1,\dots,n} (x_i + y_i z_k)$$

The first step is the usual max-min inequality, the second step works because  $y \ge 0$ , and in the final step we use that  $\min_{i=1,\dots,n} (x_i + y_i z_l) \le \max_{k=1,\dots,m} \min_{i=1,\dots,n} (x_i + y_i z_k)$  for all  $l = 1, \dots, m$ —which in particular holds for the index l which satisfies  $z_l = \max_{k=1,\dots,m} z_k$ .  $\Box$ 

# Appendix D. Examples and Counterexamples

**Example 1.** Let n = 4 so

By Definition 3 and Lemma 6

$$\{ E_b(\sigma) : \sigma \in S_4 \} = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ -3/8 \\ -1/2 \\ -1/8 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/4 \\ -1/2 \\ -1/4 \end{pmatrix}, \begin{pmatrix} 0 \\ -3/8 \\ -3/8 \\ -1/4 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -3/8 \\ 0 \\ -1/8 \end{pmatrix}, \begin{pmatrix} -5/8 \\ -3/8 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/4 \\ -1/4 \\ -1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/4 \\ -3/8 \\ -3/8 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

The second and the fourth vector from this list are the solutions to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} p = \begin{pmatrix} 0 \\ -1/2 \\ -5/8 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} p = \begin{pmatrix} 0 \\ -1/4 \\ -5/8 \\ -1 \end{pmatrix}$$

respectively and are not in  $\{x \in \mathbb{R}^4 : Mx \leq b\}$ —but every other point of  $\{E_b(\sigma) : \sigma \in S_4\}$ is in. On the other hand one readily verifies that  $p = -\frac{1}{8}(1,3,3,1)^{\top}$  satisfies  $Mp \leq b$  and solves

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} p = \begin{pmatrix} -1/2 \\ -1/4 \\ -5/8 \\ -1 \end{pmatrix}$$

so it is extreme in  $\{x \in \mathbb{R}^4 : Mx \leq b\}$  by Lemma 5 but  $p \notin \{E_b(\sigma) : \sigma \in S_4\}$ . Thus there exist extreme points of  $\{x \in \mathbb{R}^4 : Mx \leq b\}$  not of the form  $E_b(\sigma)$ .

**Example 2.** Let n = 3 and d = e (so  $\prec_d$  becomes  $\prec$ ). Consider the probability vectors  $x = \frac{1}{5}(2,1,2)^{\top}, y = \frac{1}{4}(1,2,1)^{\top}$ , and their joining line segment  $P := \operatorname{conv}\{x,y\}$ . Be aware that P as well as  $M_e(P) = \bigcup_{z \in P} \{v \in \mathbb{R}^n : v \prec z\}$  are subsets of  $\Delta^2$ . One readily verifies

$$M_{\mathfrak{e}}(P) = \{ v \in \mathbb{R}^n : v \prec x \lor v \prec y \} = M_{\mathfrak{e}}(x) \cup M_{\mathfrak{e}}(y) , \qquad (D.1)$$

refer also to Fig. D.1. Now although  $x, \tilde{y} := \frac{1}{4}(1, 1, 2)^{\top}$  are in  $M_{e}(P)$  one has

$$\frac{1}{2}x + \frac{1}{2}\tilde{y} = \frac{1}{40} \begin{pmatrix} 13\\9\\18 \end{pmatrix} = \begin{pmatrix} 0.325\\0.225\\0.45 \end{pmatrix} \stackrel{(\mathrm{D.1)}}{\not\in} M_{\mathrm{e}}(P)$$

as it is neither majorized by x nor by y; therefore  $M_{e}(P)$  is not convex.

**Example 3.** Consider  $y = (1, 1, 1)^{\top}$ ,  $\lambda \in [0, \frac{1}{2}]$ , and  $d(\lambda) = (1, \lambda, \lambda^2)$ . For all  $\lambda \in (0, \frac{1}{2}]$ —using Thm. 10 & 14—by direct computation one obtains

$$M_{d(\lambda)}(y) = \left\{ x \in \mathbb{R}^3 : Mx \le \left(3 - \lambda - \lambda^2, 2 - \lambda, 1, 3 - \lambda^2, 3 - \lambda, 2, 3, -3\right)^\top \right\}$$
$$= \operatorname{conv} \left\{ \begin{pmatrix} 3 - \lambda - \lambda^2 \\ \lambda \\ \lambda^2 \end{pmatrix}, \begin{pmatrix} 1 + \lambda - \lambda^2 \\ 2 - \lambda \\ \lambda^2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 - \lambda \\ \lambda \end{pmatrix}, \begin{pmatrix} 2 - \lambda \\ \lambda \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

as well  $as^{14}$ 

$$M_{d(\lambda)}(y) \stackrel{\lambda \to 0^+}{\to} \operatorname{conv}\left\{ \begin{pmatrix} 3\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\2\\0 \end{pmatrix}, \begin{pmatrix} 2\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\} \neq \operatorname{conv}\left\{ \begin{pmatrix} 3\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\2\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\2 \end{pmatrix} \right\} = M_{d(0)}(y) \,.$$

<sup>14</sup>For  $\lambda = 0$ , i.e. d = d(0) = (1, 0, 0) it is easy so see that every d-stochastic matrix is of the form

$$A = \begin{pmatrix} 1 & & \\ 0 & v & w \\ 0 & & \end{pmatrix} \quad \text{with arbitrary } v, w \in \Delta^2 \,.$$

Thus  $M_{d(0)}(y) = \{Ay : A \in s_{d(0)}(3)\} = \{(1 + v_1 + w_1, v_2 + w_2, v_3 + w_3)^\top : v, w \in \Delta^2\}$  which has extreme points  $(3, 0, 0)^\top, (1, 2, 0)^\top, (1, 0, 2)^\top$ .

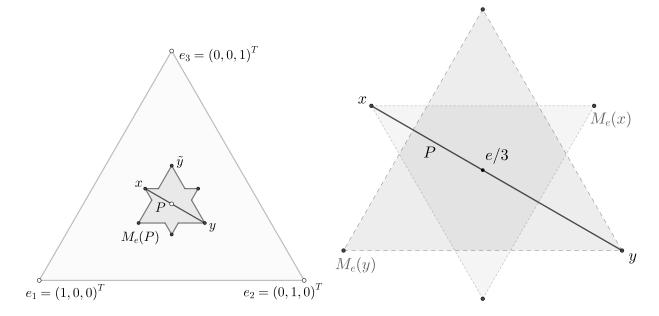


Figure D.1: Visualization of Example 2 on the 3-dimensional standard simplex. The image on the right zooms in on  $M_{\mathfrak{e}}(P)$  and shows the decomposition into  $M_{\mathfrak{e}}(x)$  and  $M_{\mathfrak{e}}(y)$ . In particular one sees that for all  $z \in P$  one has either  $z \prec x$  ( $\Leftrightarrow z \in M_{\mathfrak{e}}(x)$ ) or  $z \prec y$  ( $\Leftrightarrow z \in M_{\mathfrak{e}}(y)$ ) which implies (D.1).

**Example 4.** Let  $y = (1, 1, -1)^{\top}$ ,  $d = (1, 2, 3)^{\top}$ . Using Thm. 10 & 14, by direct computation

$$M_d(y) = \left\{ x \in \mathbb{R}^3 : Mx \le \frac{1}{6} (6, 9, 12, 12, 10, 8, 6, -6)^\top \right\}$$
$$= \operatorname{conv} \left\{ \begin{pmatrix} 1\\1\\-1 \end{pmatrix}, \begin{pmatrix} 1\\-2/3\\2/3 \end{pmatrix}, \begin{pmatrix} 1/2\\3/2\\-1 \end{pmatrix}, \begin{pmatrix} -1/3\\3/2\\-1/6 \end{pmatrix}, \begin{pmatrix} -1/3\\-2/3\\2 \end{pmatrix} \right\}$$

so the only possible candidate for the point z from Thm. 16 is the vector z = (-1/3, 2/3, 2)(because  $z_{[1]} = 2 > x_{[1]}$  for all other extreme points x). However, one has  $y \not\prec z$  because  $y_{[1]} + y_{[2]} = 1 + 1 = 2 > \frac{5}{3} = z_{[1]} + z_{[2]}$ .

**Example 5.** Let  $y = (4, 0, 1)^{\top}$ ,  $d = (4, 2, 1)^{\top}$ . Using Thm. 10 & 14, by direct computation

$$M_d(y) = \left\{ x \in \mathbb{R}^3 : Mx \le (4, 2, 1, 5, 5, 3, 5, -5)^\top \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 4\\1\\0 \end{pmatrix}, \begin{pmatrix} 4\\0\\1 \end{pmatrix}, \begin{pmatrix} 3\\2\\0 \end{pmatrix}, \begin{pmatrix} 2\\2\\1 \end{pmatrix} \right\}.$$

Therefore  $M_d(y) \subseteq M_e(y)$ , but  $\frac{y}{d} = (1, 0, 1)^{\top}$  and d are not similarly ordered. Be aware that this example generalizes to  $y = (\alpha^2, 0, 1)^{\top}$ ,  $d = (\alpha^2, \alpha, 1)^{\top}$  for all  $\alpha > 1$  so the phenomenon occurs no matter whether  $d_1 \ge d_2 + d_3$  or  $d_1 < d_2 + d_3$  (cf. Appendix A).

**Example 6.** To see discontinuity of the map

$$P(b): D(P) \to \mathcal{P}_c(\mathbb{R}^n) \qquad A \mapsto P_A(b) = \{x \in \mathbb{R}^n : Ax \le b\}$$

for arbitrary but fix  $b \in \mathbb{R}^m$  and domain<sup>15</sup> D(P) consisting of all  $A \in \mathbb{R}^{m \times n}$  such that  $P_A(0) = \{0\}$  and  $P_A(b) \neq \emptyset$ , consider the following: let

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, A_t = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ \sin(t) & \cos(t) \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

for all  $t \in (0, 1]$ . It is readily verified that

$$P_A(b) = \operatorname{conv}\left\{ \begin{pmatrix} -1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0 \end{pmatrix} \right\} \quad \text{as well as}$$
$$P_{A_t}(b) = \operatorname{conv}\left\{ \begin{pmatrix} 0\\0 \end{pmatrix}, \begin{pmatrix} -1\\0 \end{pmatrix}, \begin{pmatrix} -1\\\tan(t) \end{pmatrix} \right\} \quad \text{for all } t > 0$$

and  $(A_t)_{t\geq 0} \subset D(P)$ . Thus by definition of the Hausdorff metric

$$\delta(P_{A_t}(b), P_A(b)) \ge \max_{z \in P_A(b)} \min_{w \in P_{A_t}(b)} \|z - w\|_1 \ge \min_{w \in P_{A_t}(b)} \|\binom{1}{0} - w\|_1 = 1$$

for all t > 0 but, obviously,  $\lim_{t\to 0^+} ||A_t - A|| = 0$  so P(b) cannot be continuous.

**Example 7.** Let  $y = (3, 2, 1)^{\top}$ ,  $\lambda \in [0, 1]$  and  $d(\lambda) = (2 + \lambda, 2, 2 - \lambda)$  so

$$M_{d(0)}(y) = M_{\mathbb{P}}(y) = \operatorname{conv}\{\underline{\sigma}y : \sigma \in S_3\} \quad \text{and} \quad M_{d(1)}(y) = M_y(y) = \{y\}$$

(cf. also Example ??). Thus the parameter  $\lambda \in [0, 1]$  describes the deformation of a classical majorization polytope into a singleton. Indeed one readily computes

$$M_{d(\lambda)}(y) = \left\{ x \in \mathbb{R}^3 : Mx \le \begin{pmatrix} 3\\ \frac{6}{2+\lambda} \\ 5\\ 5-\lambda \\ 5-2\lambda \\ 6\\ -6 \end{pmatrix} \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 3\\ 2\\ 1 \end{pmatrix}, \begin{pmatrix} 3\\ 1+\lambda \\ 2-\lambda \end{pmatrix}, \frac{1}{2+\lambda} \begin{pmatrix} 4+5\lambda \\ 6\\ 2+\lambda \end{pmatrix}, \frac{1}{2+\lambda} \begin{pmatrix} 4+5\lambda \\ 6\\ 2+\lambda \end{pmatrix}, \frac{1}{2+\lambda} \begin{pmatrix} 2\lambda^2+5\lambda+2 \\ 6\\ -2\lambda^2+\lambda+4 \end{pmatrix}, \frac{1}{2+\lambda} \begin{pmatrix} -\lambda^2+6\lambda+4 \\ \lambda^2+3\lambda+2 \\ 6-3\lambda \end{pmatrix}, \frac{1}{2+\lambda} \begin{pmatrix} 2\lambda^2+5\lambda+2 \\ -2\lambda^2+4\lambda+4 \\ 6-3\lambda \end{pmatrix} \right\}.$$

Acknowledgments. The authors are grateful to Narutaka Ozawa for providing the counterexample in Remark 1 (iv) in a discussion on MathOverflow [14], as well as the anonymous referee for making them aware of reference [11] and for beneficial remarks which led to a substantially improved presentation of the material. The authors also thank Thomas Schulte-Herbrüggen for valuable and constructive comments during the preparation of this

<sup>&</sup>lt;sup>15</sup>This choice of domain ensures that the co-domain of P is  $P_c(\mathbb{R}^n)$ , i.e. that all  $P_A(b)$  are non-empty and bounded (hence compact), cf. [47, Ch. 8.2].

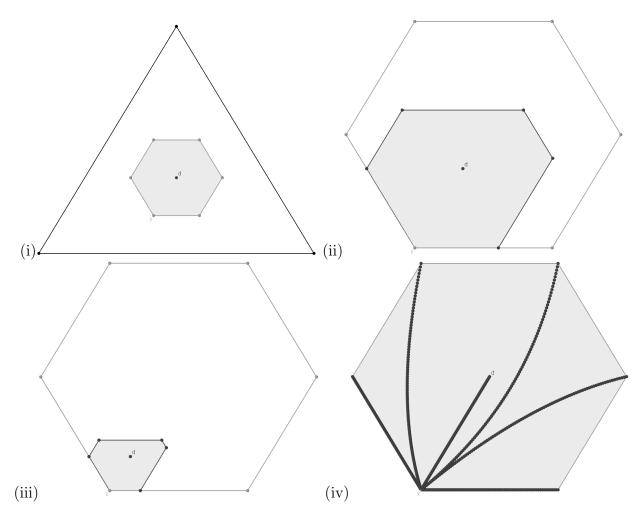


Figure D.2: Visualization of Example 7. (i): Shows  $M_{d(0)}(y) = M_{e}(y) = \operatorname{conv}\{\underline{\sigma}y : \sigma \in S_{3}\}$  inside (a multiple of) the 3-dimensional standard simplex. (ii): Zooms in on the classical majorization polytope  $M_{e}(y)$ . The shaded area is  $M_{d(\lambda)}(y)$  for  $\lambda = 0.3$ . (iii): Shows  $M_{d(\lambda)}(y)$  for  $\lambda = 0.7$ . (iv): The graph of the map  $\lambda \mapsto \operatorname{ext}(M_{d(\lambda)}(y))$ .

manuscript, and Matteo Lostaglio and Álvaro M. Alhambra for drawing their attention to some more recent publications related to thermo-majorization. This research is part of the Bavarian excellence network ENB via the International PhD Programme of Excellence *Exploring Quantum Matter* (EXQM), as well as the *Munich Quantum Valley* of the Bavarian State Government with funds from Hightech Agenda *Bayern Plus*.

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