THE d-STEP CONJECTURE FOR POLYHEDRA OF DIMENSION $d<6$

BY

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Abstract

Two functions Δ and Δ_b, of interest in combinatorial geometry and the theory of linear programming, are defined and studied. $\Delta(d, n)$ is the maximum diameter of convex polyhedra of dimension d with n faces of dimension $d-1$; similarly, $\Delta_b(d, n)$ is the maximum diameter of bounded polyhedra of dimension d with n faces of dimension $d-1$. The diameter of a polyhedron P is the smallest integer l such that any two vertices of P can be joined by a path of l or fewer edges of P. It is shown that the bounded d-step conjecture, i.e. $\Delta_b(d, 2d)=d$, is true for $d\leqslant 5$. It is also shown that the general d-step conjecture, i.e. $\Delta(d, 2d)\leqslant d$, of significance in linear programming, is false for $d\geqslant 4$. A number of other specific values and bounds for Δ and Δ_b are presented.

Introduction

In this paper two functions, Δ and Δ_b, of interest in combinatorial geometry and the theory of linear programming, are introduced and studied. For $1\leqslant d<n$, $\Delta(d, n)$ is defined as the maximum diameter of polyhedra (i.e. convex polyhedra) of dimension d with n faces of dimension $d-1$. Similarly, $\Delta_b(d, n)$ is the maximum diameter of bounded polyhedra of dimension d with n faces of dimension $d-1$. Here the *diameter* of a polyhedron P, denoted $\delta(P)$, is the smallest integer l such that any two vertices of P can be joined by a path consisting of l or fewer edges of P. The functions Δ and Δ_b provide a convenient notation for expressing a number of so-called "step" conjectures which have been circulated more or less informally in the fields of geometry and programming. One of the principal specific accomplishments of this paper is the proof of a 5-step conjecture which implies that $\Delta_b(d, n)\leqslant n-d$ whenever $n\leqslant d+5$.

The function δ can be related to the study of what might be called edge-following linear programming algorithms, which start with a vertex of the feasible region and proceed along successive edges of this region, according to some rule, until an optimum vertex is reached. The feasible region of any linear program (if not empty) is a polyhedron, and conversely, given a polyhedron P, it is always possible to construct a program with P as feasible region and specify an initial vertex so that at least $\delta(P)$ iterations are required to solve the program regardless of the edge-following algorithm employed. Thus $\Delta(d, n)$ represents, in a sense, the number of iterations required to solve the "worst" linear program of n inequalities in d variables using the "best" edge-following algorithm.

Discussing a general linear programming problem whose feasible region is determined by m linear equalities in n nonnegative variables, Dantzig writes (p. 160 of [2]): "It has been conjectured that, by proper choice of the variables to enter the basic set, it is possible to pass from any basic feasible solution to any other in m or less pivot steps, where each basic solution generated along the way must be feasible. For the case $m \leqslant 4$, the conjecture is known to be true [W. M. Hirsch, 1957, verbal communication]." Later (p. 168) he states essentially the same problem in geometric form: "In a convex region in $n-m$ dimensional space defined by n halfspaces, is m an upper bound for the minimum-length chain of vertices joining two given vertices?"

We shall call the above quoted conjecture, namely that $\Delta(d, n) \leqslant n-d$, the *general Hirsch conjecture*, and the assertion that $\Delta_b(d, n) \leqslant n-d$, the *bounded Hirsch conjecture*. The reverse inequalities, $\Delta(d, n) \geqslant n-d$, and $\Delta_b(d, n) \geqslant n-d$ when $n \leqslant 2d$, are presumably well-known and will be established incidentally in Section 2. The widely known special cases of these conjectures, namely $\Delta(d, 2d) \leqslant d$ and $\Delta_b(d, 2d) \leqslant d$, will be called respectively the *general d-step conjecture* and the *bounded d-step conjecture*.

Two major results of this paper are the evaluations $\Delta_b(5, 10)=5$ and $\Delta(4, 8)=5$, which prove the bounded 5-step conjecture alluded to in the first paragraph and disprove

d \ $n-d$	1	2	3	4	5	6	
1	1	—					
2	*	2	3	4	5	6	... $\Delta(2, n)=n-2$
3		*	3	4	5	6	... $\Delta(3, n)=n-3$
4			*	5	?	?	
5				*	?	?	
6					*	?	

Fig. 1. Values of $\Delta(d, n)$.

d \ $n-d$	1	2	3	4	5	6	7	
1	1							
2	*	2	2	3	3	4	4	... $\Delta_b(2, n)=[n/2]$
3		*	3	3	4	5	5	... $\Delta_b(3, n)=[2n/3]-1$
4			*	4	5	?	?	
5				*	5	?	?	
6					*	?	?	
7						*	?	

Fig. 2. Values of $\Delta_b(d, n)$.

the general 4-step conjecture.(1) In fact, it will be shown that $\Delta(d, 2d) \geqslant d+[d/4]$, so that excess of $\Delta(d, 2d)$ over the conjectured value tends to infinity with d. Some additional results to be developed here can be described in terms of Figs. 1 and 2, which give all the known values for $\Delta(d, n)$ and $\Delta_b(d, n)$. The asterisks indicate that each column is constant from the main diagonal downwards. Note that the d-step conjectures are concerned precisely with the values on the diagonal.

The values for $d=2$ are obvious. The values for $\Delta(3, n)$ were given in [12] and [13]. The values for $\Delta_b(3, n)$, and for $\Delta_b(d, n)$ when $n \leqslant d+4$, were established in [9]. The remaining entries are computed in this paper. In addition to these specific results, certain monotoneity properties of Figs. 1 and 2 will be proven. It will be shown that the rows in Fig. 1 and the diagonals parallel to the main diagonals in both figures are strictly increasing. It will also be shown that the rows in Fig. 2 and the columns in both figures are nondecreasing.

1. Definitions and Preliminary results

As the term is used here, a *polyhedron* is the nonempty intersection of a finite number of closed halfspaces in a finite-dimensional real vector space. A *polytope* is a bounded polyhedron. In addition to P itself, which is called the *improper face*, the *faces* of a polyhedron P are the intersections of P with its various supporting hyperplanes. We shall not consider the empty set, $\emptyset$, a face of a polyhedron. Two polyhedra are said to be *incident* if one is a face of the other. The dimension of a polyhedron P will be denoted by $\dim P$. Dimensions will also be indicated by prefixes, and the 0-faces, 1-faces, and $(d-1)$-faces

(1) Thus Dantzig's statement "For the case $m \leqslant 4$..." should be modified to read "... $m<4$...' or should be restricted to bounded regions.

of a d-polyhedron P are called respectively the *vertices*, *edges*, and *facets* of P. A *pointed* polyhedron is one which has at least one vertex. A pointed d-polyhedron is said to be *simple* if each of its vertices is incident to exactly d edges.

As we shall use the term, a (closed) *complex* is a finite set $\mathcal{K}$ whose members are polyhedra situated in some finite-dimensional space such that (a) every face of a member of $\mathcal{K}$ is a member of $\mathcal{K}$ and (b) the intersection of any two members of $\mathcal{K}$ is empty or a face of both. Any two sets of polyhedra, $\mathcal{K}_1$ and $\mathcal{K}_2$, whether complexes or not, are said to be *equivalent* if there exists a one-to-one correspondence between $\mathcal{K}_1$ and $\mathcal{K}_2$ which preserves dimension and incidence in both directions. For any polyhedron P, the *complex* of P, written $\mathcal{K}(P)$, is the complex consisting of all faces of P including P itself. Two polyhedra P and Q are said to be (*combinatorially*) *equivalent* if $\mathcal{K}(P)$ is equivalent to $\mathcal{K}(Q)$. In general the set $\mathcal{K}_F(P)=\mathcal{K}(P)\sim\mathcal{K}(F)$, where F is a face of the polyhedron P, is not a complex because certain faces of its members are not members. Nevertheless $\mathcal{K}_F(P)$ may be equivalent to the complex of a polyhedron Q, and when this happens we shall say the face F of P is *removable* and Q can be obtained from P by *removing* F. It is easy to construct examples of polyhedra with removable and nonremovable faces and to show nonequivalent polyhedra may give rise to equivalent polyhedra upon removal of a proper face from each. It is apparently a difficult task to characterize the removable faces of an arbitrary polyhedron. However, the following proposition and its immediate corollary will meet the needs of this paper.

1.1. PROPOSITION. *If the polyhedron P is contained in the strip between two parallel supporting hyperplanes H and H', then the face $H\cap P$ is removable.*

Proof. The proof consists in constructing a projective transformation of P into a polyhedron Q such that $\mathcal{K}(Q)$ is equivalent to $\mathcal{K}_{H\cap P}(P)$. From E, the vector space in which P, H, and H' are situated, construct $E\times R$, the product of E with the real numbers. Let $J=H\times R$ and $J'=H'\times R$, so that $E\times\{0\}$, J, and J' are hyperplanes in $E\times R$. Choose a point u in $J\sim(H\times\{0\})$, and for each point p in $P\sim H$, let τp denote the point at which J' is intersected by the ray issuing from u and passing through $p\times 0$. Then τ is a projective transformation, $Q=\tau(P\sim H)$ is a polyhedron in J', and the faces of Q are exactly the sets of the form $\tau(G\sim H)$, where G is a face of P not contained in H. Hence $\mathcal{K}(Q)$ is equivalent to $\mathcal{K}_{H\cap P}(P)$.

1.2. COROLLARY. *Every proper face of a polytope is removable.*

For two vertices x and y of a polyhedron P, a *path* of *length* l from x to y is a sequence of edges of P, $([x_0, x_1], [x_1, x_2], ..., [x_{l-1}, x_l])$, where $x_0=x$ and $x_l=y$. The smallest l for which such a path exists is the *distance* $\delta_P(x, y)$ between x and y in P. More generally a *facial path*, specifically a $(k_1, k_2, ..., k_r)$-*path*, from x to y in P is a sequence $(K_1, K_2, ..., K_r)$ of faces of P such that

$$x \in K_1,$$
$$y \in K_r,$$
$$\dim K_i = k_i, \quad 1 \leqslant i \leqslant r,$$
$$K_i \cap K_{i+1} \neq \emptyset, \quad 1 \leqslant i < r.$$

Note that a $(1, 1, ..., 1)$-path need not be an (ordinary) path, but for the type of problem to be considered here, this distinction will cause no difficulty.

We shall say a polyhedron belongs to *class* (d, n) if it is a pointed d-polyhedron with n facets and $1 \leqslant d < n$. Note that if a polyhedron belongs to a class, its diameter is well-defined, and conversely, if a polyhedron does not belong to any class, its diameter is undefined or zero. We also introduce the concept of a *triple*, (P, x, y), consisting of a (necessarily pointed) polyhedron P and two vertices x and y of P. We shall say (P, x, y) is bounded, simple, of class (d, n), etc. if P is. An edge or facet of P incident to a vertex v of P will be called a v-*edge* or v-*facet* of P. Thus an alternate definition for $\Delta(d, n)$ (respectively, $\Delta_b(d, n)$) is the maximum of $\delta_P(x, y)$ as (P, x, y) ranges over all triples (respectively, bounded triples) of class (d, n).

We turn now to two constructions, the product and the wedge, which will be used repeatedly in the following sections.

1.3. PROPOSITION. *If P_1 is a d_1-polyhedron and P_2 is a d_2-polyhedron, then their product $P_1 \times P_2$ is a (d_1+d_2)-polyhedron whose k-faces are precisely the sets of the form $K_1 \times K_2$, where each K_i is a k_i-face of P_i and $k = k_1 + k_2$. In particular, the vertices, edges, and facets of $P_1 \times P_2$ are respectively products of the vertices of the factors, products of a vertex of one factor by an edge of the other, and products of one factor by a facet of the other. Thus the number of facets of the product is the sum of the numbers of facets of the factors. A path α on $P_1 \times P_2$ corresponds to the locus traced out by the product of two points which trace out paths β_1 and β_2 simultaneously on their respective factors in such a way that at any time at most one of the points is not at a vertex of its factor. The paths β_1 and β_2 are the projections of α on P_1 and P_2 (it being understood that one-point projections are deleted). Thus the diameter of the product is the sum of the diameters of the factors. If both factors are bounded or simple, so is the product.*

1.4. PROPOSITION. *Suppose K is a k-face of a d-polyhedron P, $0 \leqslant k < d$. Let C denote the product of P with the half-line $L = [0, \infty)$, and let each point p in P be identified with the point $p \times 0$ of C. Let H be a hyperplane in the space spanned by C such that $H \cap P = K$ and*

H intersects the interior of C. Then C is divided by H into two $(d+1)$-polyhedra, one of which, say W, contains P. Combinatorially, W is completely determined by P and the choice of K. We shall call W the wedge over *P with* foot *K. The facets P and $H \cap C$, which will be called respectively the* lower *and* upper base *of W, are equivalent. If x is a vertex of P not in K, x^* will denote the vertex of the upper base over x, that is, the intersection of the upper base with the product of x and L. An s-face of W is either an s-face of one of the bases, or the wedge over an $(s-1)$-face T of P with foot $T \cap K$, or a face V of W intersecting both bases but not K; in the last case V is equivalent to the product of $V \cap P$ with an interval. Indeed, W may be obtained (combinatorially) from the product $P \times [0, 1]$ by collapsing $K \times [0, 1]$ into K. Thus the paths on W correspond to simultaneous paths on P and $[0, 1]$ subject to the same restriction as in 1.3 except that when the point in P is at a vertex of K the point in $[0, 1]$ may change endpoints without contributing an edge of W. The path in P is the projection parallel to L of the path on W. If P is bounded, so is W. If P is simple and K is a facet of P, then W is simple.*

Finally, we summarize some properties of simple polyhedra. We shall occasionally refer to this summary when in need of a result on simple polyhedra, even though the needed result may require some intermediate argument.

1.5. Proposition. *Suppose P is a simple pointed d-polyhedron, $d \geq 1$. Then:*

(a) *For any neighborhood N of a vertex of P, there exists a closed halfspace J such that $P \cap J$ is a d-simplex contained in N. Thus if r and s are any integers satisfying $0 \leq r \leq s \leq d$, then every r-face of P is incident to exactly $\binom{d-r}{d-s}$ s-faces of P.*

(b) *Each k-face K of P is a simple k-polyhedron. Moreover, if $k < d$, K is the intersection of $d-k$ facets of P, and conversely, the intersection of any $d-k$ facets of P is empty or a k-face of P.*

(c) *Suppose $x_0, \ldots, x_l$ are successive vertices of a path of length l on P. For each i, $0 \leq i < l$, in passing from x_i to x_{i+1} exactly one facet of P is left behind and exactly one new facet of P is met. If, in all, the vertices $x_0, \ldots, x_l$ are incident to n different facets of P, then $l \geq n-d$. If $l < d$, the path lies entirely in a face of P of dimension at most l.*

2. Some reductions

The main results of this section are "reduction theorems" in various senses. A first group of three results may be summarized as follows:

2.1. It is sufficient to consider simple polyhedra and simple polytopes when determining $\Delta(d, n)$ and $\Delta_b(d, n)$.

2.2. It is sufficient to consider $\delta_P(x, y)$ for vertices x and y not on any common facet of P when determining $\Delta(d, n)$ and $\Delta_b(d, n)$ if $n \geqslant 2d$.

2.3. It is sufficient to consider vertices x and y which are incident to unbounded edges of P when determining $\Delta(d, n)$ if $d \geqslant 2$.

From these results some recursive inequalities for Δ and Δ_b are developed, including:

2.4. Figures 1 and 2 are constant from the main diagonal downwards; that is, $\Delta(d+k, 2d+k) = \Delta(d, 2d)$ and $\Delta_b(d+k, 2d+k) = \Delta_b(d, 2d)$ for all positive integers d and k.

In view of 2.2 and 2.4 we are led naturally to consider what we shall call a *Dantzig figure*, specifically, a triple (P, x, y) of class $(d, 2d)$ with exactly d facets incident to x and exactly d other facets incident to y. We shall also recall a conjecture of Wolfe and Klee [12, 13] that any two vertices of a polytope can be joined by a so-called *W_v path*—that is, by a path which "visits" no facet more than once. Despite an apparent greater strength of this conjecture (which prompted its original formulation), we show:

2.5. THEOREM. *The following four assertions are equivalent:*

(a) *Any two vertices of a simple polytope can be joined by a W_v path.*

(b) *The bounded Hirsch conjecture is true for all d and n, $1 \leqslant d < n$.*

(c) *The bounded d-step conjecture is true for all d.*

(d) *For any simple d-dimensional bounded Dantzig figure (P, x, y), $\delta_P(x, y) = d$.*

Our first goal is to show attention may be restricted to simple polyhedra and polytopes when computing Δ and Δ_b. This was shown for Δ_b in [9] by an argument which involved passing from a polytope P to the polar polytope and applying a "pushing process" in which the vertices of the polar were pushed inward. This pushing process was carried out in such a way as to produce a new polar polytope corresponding to a simple polytope Q with diameter at least as large as the diameter of P. A similar but somewhat simpler "pulling process" was described in [4]. Had the displacements of vertices of the polar been specified in [9] as along a line through the center of the polarity, rather than in an arbitrary direction into the interior of the polar, the modified polytope Q would have been precisely the result of parallel displacements of the hyperplanes determining the facets of P. A little thought suggests that a small parallel displacement of the facets of a nonsimple polyhedron, whether bounded or unbounded, can be made to produce a simple polyhedron of no smaller diameter. This and other more-or-less plausible results concerning parallel facet displacement will be proven in a separate paper by the second author. The result we need here is the following:

2.6. LEMMA. *For any nonsimple polyhedron P of class (d, n) there exists a simple polyhedron Q of the same class such that:*

(a) *Q is bounded or unbounded according as P is.*

(b) *There exists a mapping θ of $\mathcal{K}(Q)$ onto $\mathcal{K}(P)$ such that if F and G are any faces of $\mathcal{K}(Q)$, then*

(i) *$F \supseteq G$ implies $\theta(F) \supseteq \theta(G)$;*

(ii) $\dim \theta(F) \leqslant \dim F$.

Moreover every face of $\mathcal{K}(P)$ is the image under θ of a face of $\mathcal{K}(Q)$ of the same dimension.

From this follows readily:

2.7. LEMMA. *Suppose d, n, r, and $k_1, \ldots, k_r$ are positive integers with each k_i equal to or less than d. If there exists a $(k_1, \ldots, k_r)$-path from x to y in every simple bounded triple (P, x, y) of the class (d, n), then there exists such a path in every bounded triple in that class. The foregoing statement is also true when "bounded" is replaced by "unbounded" or removed entirely.*

Proof. Suppose the "if" part of the statement is true for a particular set of values for d, n, r, and $k_1, \ldots, k_r$, and let (P, x, y) be any bounded triple of class (d, n). By 2.6 there exists a simple d-polytope Q, a mapping θ of $\mathcal{K}(Q)$ onto $\mathcal{K}(P)$ as in 2.6, and vertices x' and y' of Q mapped by θ into x and y respectively. Since we have supposed the "if" part of the statement true, there is a $(k_1, \ldots, k_r)$-path, say $(K_1', \ldots, K_r')$, from x' to y' in Q. From property (i) of 2.6 it follows $\theta(F)$ and $\theta(G)$ intersect in P whenever the faces F and G intersect in Q. Thus $(\theta(K_1'), \ldots, \theta(K_r'))$ is a path from x to y in P. In fact, by property (ii) of 2.6, it is a $(k_1^*, \ldots, k_r^*)$-path with $k_i^* \leqslant k_i$. But of course we can replace each member $\theta(K_i')$ of the path by a face K_i of P which contains it, so that $(K_1, \ldots, K_r)$ is a $(k_1, \ldots, k_r)$-path from x to y in P. This proves the lemma for polytopes. The same proof applies for unbounded or unrestricted polyhedra.

It will be seen immediately that 2.7, applied in the special case of $(1, 1, \ldots, 1)$-paths, is sufficient to prove 2.1. In the following theorem we shall show not only that 2.1, 2.2, and 2.3 hold, but that the reductions they represent may be carried out simultaneously.

2.8. THEOREM. *The value $\Delta_b(d, n)$, $1 \leqslant d < n$, can be realized as the distance between vertices x and y of a simple d-polytope P with n facets. The value $\Delta(d, n)$, $2 \leqslant d < n$, can be realized as the distance between vertices x and y of a simple d-polyhedron P with n facets in such a way that both x and y are incident to unbounded edges of P. When $n \geqslant 2d$ the requirement may be added (for Δ and Δ_b) that x and y do not lie on the same facet of P.*

Proof. The theorem is trivially true for $d \leqslant 2$; hence we shall assume in what follows that $d \geqslant 3$. The first sentence and the first part of the second have already been proven in 2.1 and 2.7.

Now suppose $n \geqslant 2d$ and consider $\Delta(d, n)$ as in the third sentence. From among the triples (P, x, y) such that $\delta_P(x, y) = \Delta(d, n)$ choose one which maximizes the dimension of the smallest face, G, of P incident to both x and y. We must show $\dim G = d$. Suppose, to the contrary, G is contained in some facet F of P. Note that

$$\delta_F(x, y) \geqslant \delta_P(x, y) = \Delta(d, n)$$

and that F is a simple $(d-1)$-polyhedron with at most $n-1$ facets of its own. We can construct from F a polyhedron F' with exactly $n-1$ facets by a suitable number of successive truncations of vertices other than x and y. (For such other vertices not to exist would lead to the conclusion $G = [x, y]$, an absurdity for $d \geqslant 3$. Each truncation, of course, increases the number of vertices by $d-2$.) It may be verified that F' is simple, that $\delta_{F'}(x, y) \geqslant \delta_F(x, y)$, that G', the intersection of G with F', is the smallest face of F' containing x and y, and that $\dim G' = \dim G$. Since F' is simple and $n-1 \geqslant 2(d-1)$, there must be at least one facet K of F' not incident to either x or y. (This is an immediate result of counting the number of facets incident to the simple vertices x and y. Similar computations will be relied upon without comment in the proofs of other results in this paper.) Let W be a wedge over F' with foot K, and let y^* be the vertex over y in the upper base of W. Then W is a simple d-polyhedron with exactly n facets, and

$$\delta_W(x, y^*) \geqslant \delta_{F'}(x, y) \geqslant \Delta(d, n),$$

where, by the definition of $\Delta(d, n)$, equality must actually hold. Now what of the smallest face S of W containing both x and y^*? Since S is not a face of either base of W, it follows from the characterization of faces of the wedge in 1.4 that $S \cap F'$ contains x and y and therefore G'. Since G' is not all of S, $\dim S > \dim G$. But then the triple (W, x, y^*) contradicts the choice of (P, x, y) so as to maximize the dimension of G. Thus $\dim G = d$, and we have verified the third sentence of 2.8 for polyhedra. The same argument applies to polytopes.

To complete the proof of 2.8, we must consider a triple (P, x, y) satisfying all the requirements of the theorem relative to $\Delta(d, n)$ except possibly for the existence of unbounded edges incident to x and y. We may as well suppose P unbounded for if P were bounded we could remove a vertex without affecting other essential properties of P. Let L be the line through x and y, let c be the midpoint of the segment $[x, y]$, and let C be the union of all rays in P which issue from c. Then C is a polyhedral cone and $C \cap L = \{c\}$. Let H

be a hyperplane which supports C at c and contains L, and let H' be a translate of H intersecting the interior of C. P is divided by H' into two polyhedra, and the one of these, P', which contains x and y is bounded. If H' is chosen sufficiently close to H, then P' is a simple d-polytope with at most $n+1$ facets such that at least one edge incident to x intersects the facet $P \cap H'$ and at least one edge incident to y intersects $P \cap H'$. Now let (P'', x'', y'') be a triple obtained from (P', x, y) by removing $P \cap H'$ and truncating enough vertices to bring the number of facets up to n. It is not difficult to see that (P'', x'', y'') satisfies all the requirements of the theorem relative to $\Delta(d, n)$.

Using the foregoing general results, we shall next establish some simple inductive inequalities for $\Delta(d, n)$ and $\Delta_b(d, n)$. These inequalities, interpreted in terms of Figs. 1 and 2, are exactly the monotoneity conditions mentioned in the introduction.

2.9. Proposition. *If $1 < d < n$, then*

$$\text{(a)}\ \Delta(d, n+1) > \Delta(d, n), \qquad \text{(b)}\ \Delta_b(d, n+1) \geqslant \Delta_b(d, n),$$
$$\text{(c)}\ \Delta(d+1, n+2) > \Delta(d, n), \qquad \text{(d)}\ \Delta_b(d+1, n+2) > \Delta_b(d, n).$$

Proof. Part (b) is easily proved by the truncation process used in the proof of 2.8. Parts (c) and (d) follow from 1.3; if P is a d-polyhedron of diameter $\Delta(d, n)$ with n facets, then $P \times [0, 1]$ is a $(d+1)$-polyhedron of diameter $\Delta(d, n)+1$ with $n+2$ facets. For part (a) let (P, x, y) be the triple of diameter $\Delta(d, n)$ described in 2.3, let g be an unbounded y-edge of P, form a new polyhedron P' from P by truncating y, and denote by y' the new endpoint of g. Then P' is a d-polyhedron with $n+1$ facets and $\delta_{P'}(x, y') = \Delta(d, n)+1$.

2.10. Proposition. *If $1 < d < n$, then*

$$\text{(a)}\ \Delta(d+1, n+1) \geqslant \Delta(d, n),$$
$$\text{(b)}\ \Delta_b(d+1, n+1) \geqslant \Delta_b(d, n),$$

with equality in both cases if $n \leqslant 2d$.

Proof. Consider (b); the proof for (a) is analogous. Let P be a d-polytope of diameter $\Delta_b(d, n)$ with n facets, let K be any facet of P, and let W be the wedge over P with foot K as constructed in 1.4. Then W is a $(d+1)$-polytope with $n+1$ facets and its diameter is at least $\Delta_b(d, n)$. This proves the inequality for (b). Now assume $n \leqslant 2d$ and let (Q, x, y) be a triple of class $(d+1, n+1)$ such that $\delta_Q(x, y) = \Delta_b(d+1, n+1)$. Since Q has fewer than $2(d+1)$ facets, it must have a facet P incident to both x and y. P is itself a d-polytope with at most n facets, and clearly $\delta_P(x, y) \geqslant \delta_Q(x, y)$. The reverse inequality and hence equality in 2.10b now follow from 2.9b.

In connection with the d-step conjectures and the values of $\Delta(d, 2d)$ and $\Delta_b(d, 2d)$, we are led to introduce some special terminology. As was mentioned earlier, a triple (P, x, y) is called a d-dimensional Dantzig figure if P is a d-polyhedron with $2d$ facets of which exactly d are incident to x and exactly d others are incident to y. The name is suggested by a problem posed by Dantzig in [3], which asks for the maximum of $\delta_P(x, y)$ as (P, x, y) ranges over all bounded d-dimensional Dantzig figures. From 2.3 it follows that the answer, whatever its value may be, is exactly $\Delta_b(d, 2d)$. In this same connection, let us also recall a notion from [12] and [13]. A path $([x_0, x_1], ..., [x_{l-1}, x_l])$ in a polyhedron P is said to be a W_v *path* if for $i<j<k$, x_j is incident to a facet whenever x_i and x_k are. Less formally, a path is a W_v path if it does not revisit any facet of P. The following proposition concerning W_v paths is easily proven:

2.11. PROPOSITION.

(a) *A path lying in a face F of a polyhedron P is a W_v path in F if and only if it is a W_v path in P.*

(b) *A path in the product of two polyhedra is a W_v path if and only if its projections in the factors are W_v paths.*

(c) *If a path in a wedge over P is a W_v path, then so is its projection in P.*

The final result of this section demonstrates the relationship among Dantzig figures, W_v paths, and the step conjectures.

2.12. LEMMA. *Let m and d be positive integers, with $d\geqslant 2$ if $m>1$. Then each of the following statements implies the next:*

(a) *The vertices x and y of any bounded simple m-dimensional Dantzig figure (P, x, y) can be joined by a path of length at most m.*

(b) *Any two vertices of a simple polytope of class $(d, d+m)$ can be joined by a W_v path.*

(c) *The bounded Hirsch conjecture is true for class $(d, d+m)$, i.e. $\Delta_b(d, d+m)\leqslant m$.*

Proof. In 2.10 part (b) it was shown, though not explicitly stated, that the bounded d-step conjecture is true for some dimension m if and only if the bounded Hirsch conjecture is true for all classes $(d, d+m)$. This is nothing more than an observation that the maximum entry in any column of Fig. 2 can be found on the diagonal. Thus, because of 2.8, statement (a) implies statement (c). In fact, from an application of 2.8 and 1.5c it follows (b) implies (c). To see that (a) implies (b), let (P, x, y) be any bounded simple triple of class $(d, d+m)$. Write $y_0=y$ and let F_0 be the smallest face of P incident to x and y. Then (F_0, x, y_0) is a bounded simple triple of class $(d', d'+m')$, where $d'\leqslant d$ and $m'\leqslant m$. Indeed, since no facet of F_0 is incident to both x and y_0, $m' = d' + k$, where k is the number of facets of F_0 incident to neither x nor y_0. If $k > 0$, let G be any facet of F_0 missing x and y, and form the wedge F_1 over F_0 with foot G. The triple (F_1, x, y_1), where y_1 is the vertex over y_0, is a bounded simple triple of class $(d'+1, d'+m'+1)$ with $k-1$ facets incident to neither x nor y_1 and none incident to both. Repeating the wedging process, we must obtain in k steps a bounded simple triple (F_k, x, y_k) of class $(d'+k, d'+m'+k) = (m', 2m')$ with no facets incident to both x and y_k. If we suppose statement (a) true, then since (F_k, x, y_k) is a simple Dantzig figure, since $m' \leqslant m$, and since 2.9d holds, we conclude there is a path of length at most m' from x to y_k in F_k. By 1.5 such a path is of length exactly m' and must be a W_v path. Now the projection of this W_v path back through the sequence of triples, (F_i, x, y_i), produces a W_v path in F from x to y. Since this is what is required in (b), the lemma is proven.

The theorem, 2.5, now follows directly from the above lemma and 2.8.

3. Facial paths in Dantzig figures

In the previous section it was shown that the study of the d-step conjecture leads naturally to the study of d-dimensional simple Dantzig figures (P, x, y). And from 1.5 it follows that $\delta_P(x, y) = d$ if and only if P admits a $(k_1, \ldots, k_r)$-path from x to y for every sequence of positive integers $(k_1, \ldots, k_r)$ whose sum is d. Thus it seems natural to study the existence of such paths for Dantzig figures, and this is the purpose of the present section. Our first results show that any simple Dantzig figure admits a $(k_1, 1, k_3)$-path and that, as a consequence of a general relationship between unbounded and bounded figures, any bounded simple Dantzig figure admits a $(1, k_2, 1, k_4)$-path. The latter result implies the existence of $(1, d-2, 1)$-paths in the bounded case, a fact which is employed in the proof of the bounded 5-step conjecture in Section 4. The rest of the present section serves primarily as an introduction and supplement to Section 5, where the general 4-step conjecture is disproved by constructing a 4-dimensional simple Dantzig figure (P, x, y) admitting no $(1, 2, 1)$-paths from x to y.

If (P, x, y) is a d-dimensional Dantzig figure, then P is the intersection of simple polyhedral cones P_x and P_y with vertices x and y respectively. Each cone contains the vertex of the other in its interior, and the faces of P incident to (say) x are subsets of faces of P_x of the same dimension cut off by the cone P_y. Employing a natural correspondence, we say an x-facet F and an x-edge f are *complementary* if f is the only x-edge not lying in F or, equivalently, F is the only x-facet not containing f. The same term applies to y-facets and y-edges.

3.1. PROPOSITION. *For any positive integers d, r, and $k_1, \ldots, k_r$ such that $k_i \leqslant d$, the following are equivalent:*

(a) *Given any d-dimensional simple Dantzig figure (P, x, y), the vertices x and y can be joined by a $(k_1, \ldots, k_r)$-path.*

(b) *Given any simple d-polytope P' with $2d+1$ facets, two vertices x and y of P' not on the same facet, and a facet F of P' not incident to either x or y, then the vertices x and y can be joined by a $(k_1, \ldots, k_r)$-path, no member of which lies entirely within F.*

(c) *Given any $(d+1)$-dimensional bounded simple Dantzig figure (Q, u, v) and any facet G of Q incident to u, the vertices u and v can be joined by a $(1, k_1, \ldots, k_r)$-path, no member of which lies entirely within G.*

Proof. The proposition is easily verified when $d=1$, hence we may assume $d \geqslant 2$ for the rest of the proof.

(b)$\to$(a). Assume (b) and consider (P, x, y) as in (a). Construct from P a simple d-polytope P' with $2d+1$ facets as follows: If P is bounded, form P' by truncating a vertex of P other than x or y (such a vertex exists if $d \geqslant 2$); if P is unbounded, "truncate the vertex at infinity" by intersecting P with a halfspace J such that the interior of J contains all the vertices of P and the boundary of J intersects all the unbounded edges of P. The polytope P' will satisfy the hypotheses of (b), with the facet added by truncation as F. Let $(K'_1, \ldots, K'_r)$ be one of the paths in P' guaranteed by (b). Each K'_i is the result of a (possibly trivial) truncation of a unique face K_i of P of the same dimension, and none of the intersections $K'_i \cap K'_{i+1}$ is empty, hence $(K_1, \ldots, K_r)$ is a path satisfying the requirements of (a).

(a)$\to$(b). Assume (a) and consider (P', x, y) and F as in (b). The figure (P, x, y) obtained from (P', x, y) by removing F satisfies the hypotheses of (a), hence there exists a $(k_1, \ldots, k_r)$-path, say $(K_1, \ldots, K_r)$, joining x to y in P. It is easily checked that $(K'_1, \ldots, K'_r)$, where each K'_i is the face of P' corresponding to the face K_i of P, is a path satisfying the requirements of (b).

(a) $\wedge$ (b)$\to$(c). Suppose (a) and (b) hold and consider (Q, u, v) and G as in (c). Let $[u, x]$ be the u-edge complementary to G, and let P' be the unique facet of Q incident to x but not u. Then P' intersects v and is itself a simple d-polytope with d facets incident to x, d other facets incident to v, and at most one additional facet, which, if it exists, is the intersection of P' with G. If this extra facet is present, then (P', x, v) satisfies the conditions of (b), and the faces of the $(k_1, \ldots, k_r)$-path from x to v guaranteed by (b), together with $[u, v]$, comprise a $(1, k_1, \ldots, k_r)$-path in Q from u to v. It is easily seen that none of the members of this path lies entirely in G, hence this path satisfies the rquirements in (c). If the extra facet is missing, the same result follows from an application of (a).

(c)$\rightarrow$(b). Assume (c) and consider (P', x, y) and F as in (b). Construct the wedge Q over P' with foot F, and let u denote the vertex of Q above x. Then (Q, u, y) and G, where G is the upper base of Q, satisfy the hypotheses of (c). Let $(K_0, K_1, ..., K_r)$ be a $(1, k_1, ..., k_r)$-path from u to v as required by (c). Since none of the K_i lies in G, it follows that $K_0 = [u, x]$, each K_i intersects P' in a nonempty face K_i' of P' which does not lie in F, and each intersection $K_i' \cap K_{i+1}'$ is nonempty. Now a face K_i' may fail to have dimension k_i, so we select faces K_i'', $1 \leqslant i \leqslant r$, so that K_i'' is a face of P' of dimension k containing k_i'. The sequence $(K_1'', ..., K_r'')$ is a $(k_1, ..., k_r)$-path of P' from x to y satisfying the requirements in (b).

A number of special cases of 3.1 are of interest; we give here two corollaries which follow immediately from 3.1 and 2.3.

3.2. Corollary. *For any positive integers d and l, the following are equivalent:*

(a) *Given any d-dimensional simple Dantzig figure (P, x, y), x and y can be joined by a path of length at most l.*

(b) *Given any simple d-polytope P' with $2d+1$ facets, two vertices x and y of P' not on the same facet, and a facet F of P' not incident to either x or y, then x and y can be joined by a path of length at most l which does not intersect F.*

(c) *Given any $(d+1)$-dimensional bounded simple Dantzig figure (Q, u, v) and any facet G of Q incident to u, then u and v can be joined by a path of length at most $l+1$ which intersects G at u only.*

3.3. Corollary. $\Delta_b(d+1, 2d+2) \leqslant \Delta(d, 2d)+1$.

3.4. Proposition. *Let (P, x, y) be a d-dimensional simple Dantzig figure, and let $k_1, ..., k_4$ be positive integers such that $k_1+1+k_3 = d = 1+k_2+1+k_4$. Then:*

(a) *P admits a $(k_1, 1, k_3)$-path from x to y.*

(b) *If P is bounded it admits a $(1, k_2, 1, k_4)$-path from x to y.*

Proof. Consider an arbitrary path in P from x to y, let x_s denote the last vertex on this path incident to k_1 or fewer y-facets, and let x_t denote the vertex following x_s. From 1.5 it will follow that $(K_1, [x_s, x_t], K_3)$ is a $(k_1, 1, k_3)$-path from x to y, where K_1 and K_3 are the intersections respectively of the x-facets incident to x_s and the y-facets incident to x_t. This proves (a). An application of 3.1 to (a) proves (b).

It is a consequence of 1.5 that if a simple figure (P, x, y) admits a $(k_1, ..., k_r)$-path from x to y then it also admits a $(k_1', ..., k_s')$-path from x to y, where the sequence $(k_1', ..., k_s')$

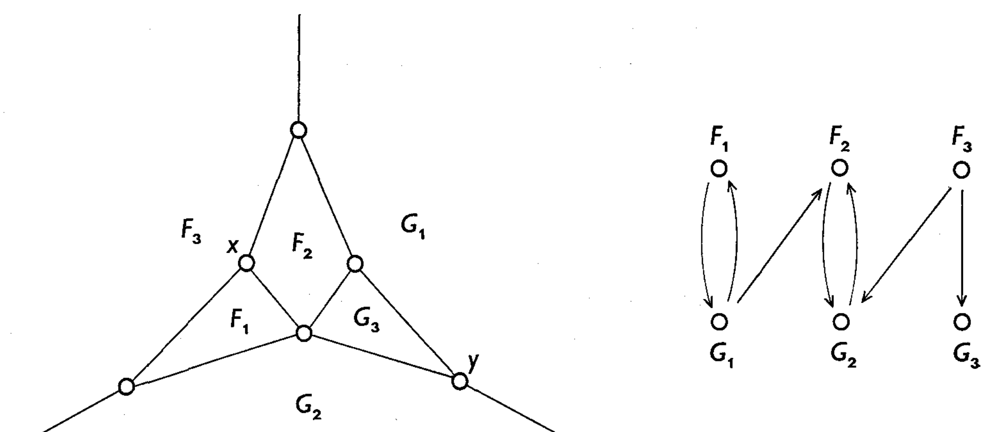


Fig. 3.

is obtained from $(k_1, ..., k_r)$ by removing adjacent terms and replacing them by a single term equal to their sum. Thus, for example, 3.5 implies that bounded simple Dantzig figures (P, x, y) of dimension d possess $(1, d-2, 1)$-paths from x to y. We do not know if such figures must have $(1, k_2, k_3, 1)$- or $(k_1, 1, 1, k_4)$-paths whenever $1+k_2+k_3+1=k_1+1+1+k_4=d$. In the rest of this section we will be concerned exclusively with $(1, d-1)$- and $(1, d-2, 1)$-paths.

As a guide in making the construction in Section 5, we have relied not only on the well-known notion of the *Schlegel diagram* of a polyhedron, but also on the notion of the *ef-diagram* of a Dantzig figure, which is described in the next paragraph. While these notions are not strictly necessary to present our results, both of them played an effective role in our investigations.

The *edge-facet diagram* (or *ef-diagram*) of a d-dimensional Dantzig figure (P, x, y) is a directed bipartite graph whose nodes are (identified with) the facets of P and whose arcs represent the $(1, d-1)$-paths from x to y and y to x. The x- and y-facets of P are referred to as x- and y-nodes of the diagram. The diagram contains a directed arc (F, G), called an x-arc, from the x-node F to the y-node G if and only if (f, G) is a $(1, d-1)$-path from x to y, where f is the x-edge complementary to F. Similarly, (G, F) is a directed arc, called a y-arc, from G to F if and only if the y-edge complementary to G and the x-facet F constitute a $(1, d-1)$-path from y to x.

In Fig. 3 above, the Schlegel diagram and the ef-diagram of a 3-dimensional non-simple unbounded Dantzig figure are shown.

A considerable amount of information about a particular Dantzig figure (or Dantzig figures in general) can be extracted from the ef-diagram. As the following proposition will show, the ef-diagram provides information about the $(1, d-2, 1)$-paths as well as the $(1, d-1)$-paths. Part (d) of the proposition will be used in analyzing the counterexample in Section 5.

3.5. PROPOSITION. *The ef-diagram of a d-dimensional Dantzig figure (P, x, y) possesses the following properties:*

(a) *There is at least one x-arc.*

(b) *For every x-arc (F, G) there is a y-arc (G', F') such that $G' \neq G$.*

(c) *If there is an x-arc (F, G) and a y-arc (G', F') such that $F' \neq F$ and $G' \neq G$, then P admits a $(1, d-1, 1)$-path (f, Q, g') from x to y, where f and g' are respectively the edges complementary to F and G'. If P is simple, the converse holds.*

(d) *If P does not admit $(1, d-2, 1)$-paths from x to y, then the ef-diagram contains exactly two x-arcs, (F, G) and (F', G'), and exactly two y-arcs, (G, F') and (G', F), and moreover $F' \neq F$ and $G' \neq G$. If P is simple, the converse holds.*

Proof. (a) P must have at least one bounded x-edge, and every bounded x-edge is the first member of a $(1, d-1)$-path from x to y.

(b) Suppose (F, G) is an x-arc. Since G intersects an x-edge, it cannot be a cone and so must contain at least one bounded y-edge g'. Since g' is bounded there exists a y-arc (G', F'), where G' is complementary to g', and since g' is not complementary to G, $G' \neq G$.

(c) It is readily checked that, if the hypotheses hold, then $(f, F' \cap G, g')$ is a path from x to y. If $F' \cap G$ has dimension less than $d-2$, the deficiency is easily removed by replacing $F' \cap G$ by a face of dimension $d-2$ containing it. Conversely, if P is simple and (f, Q, g') is a $(1, d-2, 1)$-path from x to y, then there exist (actually unique) facets F' and G such that $F' \supset f \cup Q$ and $G \supset Q \cup g'$. The facets F, G', F', and G satisfy the desired relationships.

(d) An application of (a), followed by repeated applications of (b) with care not to produce a pair of arcs as in (c), leads directly to the indicated diagram. The converse is a direct consequence of the converse to (c).

4. Proof of the bounded 5-step conjecture

The bounded d-step conjecture has been established elsewhere for $d<5$; it will be proved here for $d=5$. As we saw in 2.8, to establish the bounded d-step conjecture it suffices to show that $\delta_P(x, y) \leqslant d$ whenever (P, x, y) is a d-dimensional bounded simple Dantzig figure. By 3.4 there is always a $(1, d-2, 1)$-path (f, Q, g) from x to y in such a figure, and, of course, Q is a $(d-2)$-polytope with at least $2d-4$ and at most $2d-2$ facets. We shall first prove that if Q has a certain property defined below, then $\delta_P(x, y) \leqslant d$. We

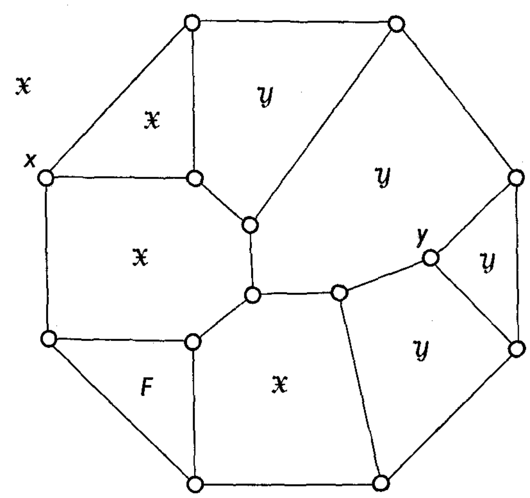


Fig. 4.

then prove the bounded 5-step conjecture by showing that this property is possessed by every simple 3-polytope with 6, 7 or 8 facets.

A d-polytope Q will be said to have *property* A provided the number of facets of Q is between $2d$ and $2d+2$, and the following condition is satisfied: If the facets of Q are divided into two disjoint classes $\mathcal{X}$ and $\mathcal{Y}$, each consisting of at most $d+1$ facets, if X (respectively Y) denotes the set of all vertices of Q which are entirely surrounded by members of $\mathcal{X}$ (respectively $\mathcal{Y}$), and if neither X nor Y is empty, then Q admits a path of length at most d joining a member of X to a member of Y.

The proposition, "Property A is possessed by every simple d-polytope with $2d$, $2d+1$, or $2d+2$ facets," would be a strengthened form of the d-step conjecture. The proposition is obviously true for $d=2$, and we show below that it is true for $d=3$. That it is false for $d=4$ may be seen by considering a wedge over the polytope shown in Fig. 4. Since the graph of Fig. 4 is planar, 3-valent, and 3-connected, Fig. 4 is at least equivalent to the Schlegel diagram of a simple 3-polytope T (cf. [8], [11], [14]). Let W be the wedge over T with foot F. Assign the lower base of W to the class $\mathcal{X}$, the upper base to the class $\mathcal{Y}$, and assign the lateral facets to $\mathcal{X}$ or $\mathcal{Y}$ according to the labeling in Fig. 4 of their intersections with the lower base. Then $X=\{x\}$, $Y=\{y^*\}$, and $\delta_W(x, y^*)=5$. Hence W lacks property A, although it is a simple polytope of class (4, 10).

There is also an example of a simple 5-polytope with 11 facets which lacks property A. Take the triple (P, x, y) of class (4, 8) and diameter 5 constructed in 5.1. Truncate the vertex at infinity to produce a triple (Q, x, y) as in 5.2 and designate the new facet F. Let u and v denote the vertices of Q where the two unbounded x-edges of P intersect F.

The segment $[u, v]$ will be an edge of Q. By intersecting Q with a halfspace J which misses u and v but contains all other vertices of Q in its interior, form a polytope Q' with facet $F' = F \cap J$, etc., and denote by G the new facet. Label F' and the x-facets of Q' as $\mathfrak{X}$ facets and G and the y-facets as $\mathfrak{Y}$ facets. Thus Q' is a simple polytope of class $(4, 10)$, $X = \{x\}$, $Y = \{y\}$, and $\delta_{Q'}(x, y) = 5$, so that Q' is an alternate to the example constructed from Fig. 4. Now let Q'' be the wedge over Q' with foot G. Assign the lower base to $\mathfrak{X}$, the upper base to $\mathfrak{Y}$ and the lateral facets according to the labeling in Q'. We have $X = \{x\}$, $Y = \{y^*\}$, and Q'' is a simple polytope of class $(5, 11)$. Now suppose, as required for property A, that α were a path from x to y^* of length 5. Consider the path β obtained by projecting α into Q'. Of course β has length at most 5. Since $\delta_{Q'}(x, y) = 5$, by the characterization of paths in wedges in 1.4, β must visit G. But G does not intersect any y-facets of Q'—only x-facets and F'. Therefore when β leaves G it either retraces to x or visits F'. Either alternative implies the length of β is 6. The contradiction shows Q'' does not have property A.

These examples of polytopes of class $(4, 10)$ and $(5, 11)$ which fail to have property A have suggested to us that there might be a simple polytope of class $(6, 12)$ lacking property A. Such an example would be nothing less than a counterexample to the 6-step conjecture.

4.1. Proposition. *If a d-dimensional bounded simple Dantzig figure (P, x, y) admits a $(1, d-2, 1)$-path $([x, x'], Q, [y', y])$ such that Q has property A, then $\delta_P(x, y) \leq d$.*

Proof. Note first that Q is the intersection of an x-facet F with a y-facet G. Each facet K of Q is the intersection with Q of a unique facet $H(K)$ of P distinct from F and G. Let the facet K be assigned to the class $\mathfrak{X}$ or $\mathfrak{Y}$ according as $H(K)$ is an x-facet or a y-facet of P, and let X and Y be defined as in the definition of property A. It is evident that $x' \in X$, $y' \in Y$, and each of the classes $\mathfrak{X}$ and $\mathfrak{Y}$ includes at most $d-1$ facets of Q. Since Q has property A, there is a path of length at most $d-2$ joining a member x'' of X to a member y'' of Y. But from the definition of the class $\mathfrak{X}$ and the fact $Q = F \cap G$, it follows x'' is the intersection of $d-1$ x-facets with G; therefore, x'' is the endpoint of an x-edge. Since also y'' is an endpoint of a y-edge, there is a path of length d from x to y.

4.2. Lemma. *Every simple 3-polytope with 6, 7, or 8 facets has property A.*

Proof. It should be clear that (if true) the lemma can be proved by exhaustion in one way or another; we give a proof by contradiction. Suppose G is a simple 3-polytope with 6, 7, or 8 facets, and suppose $\mathfrak{X}$, $\mathfrak{Y}$, X, and Y are as in the definition of property A, but suppose no members of X and Y can be joined by a path of length less than 4. Select any vertices x and y from X and Y respectively. If v, e, and f denote respectively the number of vertices, edges, and facets of G, then $v-e+f=2$ (Euler's theorem) and, because G is simple, $3v=2e$, so $v=2f-4$. Now any two vertices of G, in particular x and y, can be joined by three independent paths, that is, paths which intersect only at their common endpoints [1]. Such paths must be of length at least 4, hence $v \geq 11$. But since $v=2f-4$ we must have $f=8$ and $v=12$. Accordingly, there are two cases to be considered:

(1) For every choice of three independent paths from x to y, two of the paths are of length 4 and one is of length 5. (In this case, every vertex of G lies on one of the paths.)

(2) The vertices x and y can be joined by three independent paths of length exactly 4. (In this case exactly one vertex is not on a path.)

Before we go on with the analysis of these two cases, let us consider an arbitrary path of length 4 from x to y and note how the edges incident to the three intermediate vertices may be distributed between the two sides of the path. They cannot all be on the same side because then a facet would be incident to both x and y, contrary to the construction of $\mathfrak{X}$, $\mathfrak{Y}$, etc. They cannot alternate, as in Fig. 5a, because if $W \in \mathfrak{X}$ then $x' \in X$ and $\delta(x', y)=3$, and if $W \in \mathfrak{Y}$ then $y' \in Y$ and $\delta(x, y')=3$. Therefore, except for symmetries, including a possible interchange of $(\mathfrak{X}, X, x)$ and $(\mathfrak{Y}, Y, y)$, they must appear as in Fig. 5b, with $W \in \mathfrak{Y}$ because $W \in \mathfrak{X}$ would imply $x'' \in X$ and $\delta(x'', y)=3$.

Now consider case (1). Note that no two vertices of the same path can be connected by an edge unless that edge is already an edge of the path. Let the sets of intermediate vertices of the three paths be A, B, and C, where B is the set consisting of four vertices. Then each member of B is adjacent to a member of $A \cup C$ and, in fact, two are adjacent to A and two to C. Recalling the significance of Fig. 5b and observing the restriction $\delta(x, y) \geq 4$, we soon see that the faces of G must be disposed as in Fig. 6a except for a possible interchange of $(\mathfrak{X}, X, x)$ and $(\mathfrak{Y}, Y, y)$. Comparing the top and bottom paths from x to y in Fig. 6a with Fig. 5b, we are led to the conclusion, contrary to hypothesis, that $\mathfrak{Y}$ contains five facets. Thus case (1) cannot occur.

Finally, consider case (2). Reasoning quite analogous to that for case (1) shows that, except for symmetries, the faces of G must be disposed as in Fig. 6b. As in case (1) we

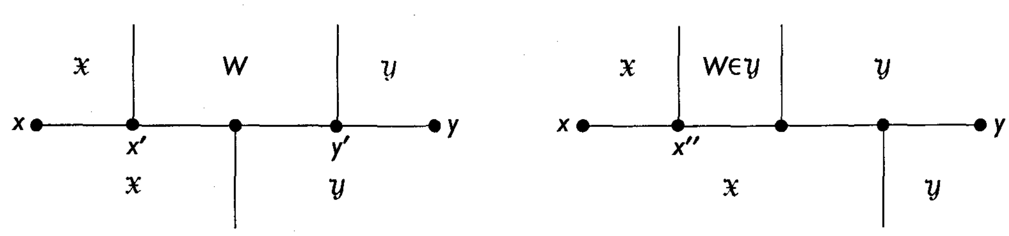


Fig. 5.

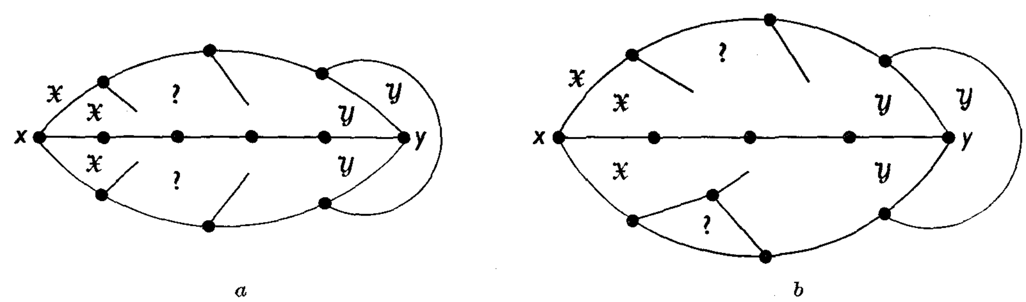


Fig. 6.

find, contrary to hypothesis, $\mathcal{Y}$ must have five members. This contradiction completes the proof of 4.2.

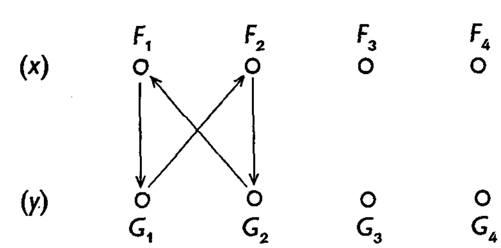
We have now proven the major result of this section:

4.3. THEOREM. $\Delta_b(5, 10) = 5$.

5. Disproof of the general 4-step conjecture

In this section our first aim is to disprove the general 4-step conjecture. We then show that $\Delta(4, 8) = 5$, $\Delta_b(4, 9) = 5$, and $\Delta(d, 2d) \geqslant d + [d/4]$.

5.1. THEOREM. *There exists a 4-dimensional unbounded simple Dantzig figure* (P, x, y) *whose edge-facet diagram is as follows:*



Thus P admits no $(1, 2, 1)$*-path from x to y, and in particular* $\delta_P(x, y) > 4$.

Proof. The second statement will follow from the first by 3.5 and the remarks following 3.4.

The 4-polyhedron P of our example is defined by means of a system of four linear equations in eight nonnegative variables. These are shown in the basic tableau S of Fig. 7 below using the detached coefficient conventions of linear programming. Thus, for example, the third equation is

$$35x_1 + 45x_2 - 6x_3 - 3x_4 + y_3 = 8.$$

	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	b
S:	6	3		−1	1				1
	3	6	−1			1			1
	35	45	−6	−3			1		8
	45	35	−3	−6				1	8

	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	b
T:	1				3	6	−1		1
		1			6	3		−1	1
			1		45	35	−3	−6	8
				1	35	45	−6	−3	8

Fig. 7.

In verifying the important properties of P we shall use the terminology and basic results of the simplex method. This will simplify the discussion, and it seems appropriate since the problems of this paper were originally raised in connection with linear programming.

If M denotes the set of all points of R^8 which satisfy the four equality constraints, then it is evident that M is a 4-dimensional flat in R^8, and since M includes an interior point of the positive orthant, namely $x_i = y_i = 1/9$, it follows that the intersection of M with the orthant is a polyhedron P of dimension 4. A second basic form of the tableau S is given in T. It is easily verified that if the matrix of y-columns in T is applied as row operations (left multiplication) on S, the tableau T results. The basic solutions of S and T, namely

$$x_1 = x_2 = x_3 = x_4 = 0, \quad y_1 = y_2 = 1, \quad y_3 = y_4 = 8,$$

and

$$y_1 = y_2 = y_3 = y_4 = 0, \quad x_1 = x_2 = 1, \quad x_3 = x_4 = 8,$$

are vertices of P which will be denoted by x and y respectively. The four x-facets F_i of P are given by $x_i = 0$, and the four y-facets G_i are given by $y_i = 0$. Utilizing the symmetries in S properly, it is a matter of only moderate labor to verify that no positive linear combination of three or fewer of the first eight columns of S is equal to the constant column. This implies that no point of P is on five or more facets, i.e. (P, x, y) is a simple Dantzig figure.

The ef-diagram of (P, x, y) can be constructed readily from S and T. For example, pivoting on y_1 in T removes y_1 from the non-basis and replaces it with x_2. This corresponds to leaving the y-facet G_1 along the y-edge determined by $y_2 = y_3 = y_4 = 0$ and terminating on the x-facet F_2. Thus the ef-diagram contains an arc (G_1, F_2). On the other hand the

y-edge complementary to G_3 is unbounded, as indicated by the nonpositivity of the column for y_3, and no arc of the ef-diagram issues from the node G_3. The completed ef-diagram is as required, and the proof of 5.1 is complete.

Of course an alternate proof that $\delta_P(x, y) > 4$ could have been obtained by considering the various feasible pivot sequences on S. It can be checked, if desired, that S can be transformed into T by successive feasible pivots on x_1, x_2, x_3, x_4, and x_1, in that order, followed by a permutation of the rows of the resulting tableau. Hence $\delta_P(x, y) = 5$.

5.2. THEOREM. $\Delta(4, 8) = \Delta_b(4, 9) = 5$.

Proof. The example (P, x, y) of 5.1 shows that $\Delta(4, 8) \geqslant 5$. To see that $\Delta_b(4, 9) \geqslant 5$, let Q be the simple 4-polytope with 9 facets obtained from P by truncating the vertex at infinity as described in the proof of 3.1. If a path in Q from x to y intersects the added facet, then by 1.5 its length must be at least 5. If the path never intersects the added facet, it is a path in P and therefore a path of length at least 5. To see that $\Delta(4, 8) \leqslant 5$ or $\Delta_b(4, 9) \leqslant 5$, consider an appropriate triple (P, x, y) as described in 2.3. By 3.4, P admits a $(1, 3)$-path (f, G) from x to y, where G is a 3-polyhedron with at most 7 facets or a 3-polytope with at most 8 facets. The desired conclusions follow from $\Delta(3, 7) = 4 = \Delta_b(3, 8)$. An alternate proof of $\Delta_b(4, 9) \leqslant 5$ can be obtained from 4.3 and 2.10b.

The example constructed in 5.1 not only disproves the d-step conjecture, it can be used to show that the excess of $\Delta(d, n)$ over the value given by the general Hirsch conjecture tends to infinity with $\min(d, n-d)$.

5.3. PROPOSITION. $\Delta(d, n) \geqslant n - d + \min\left(\left[\dfrac{d}{4}\right], \left[\dfrac{n-d}{4}\right]\right)$.

Proof. Let $k = \min([d/4], [(n-d)/4])$, so that $d = 4k + i$ and $n - d = 4k + j$. Let Q be the product of k copies of the polyhedron of 5.1. By 1.3, Q is a polytope of class $(4k, 8k)$ of diameter $5k$; hence $\Delta(4k, 8k) \geqslant 5k$. Then from j applications of 2.9a and i applications of 2.10a, it follows

$$\Delta(d, n) = \Delta(d, 8k + i + j) \geqslant 5k + j = n - d + k.$$

6. Some rough bounds for $\Delta(d, n)$ and $\Delta_b(d, n)$

In this section we collect a number of miscellaneous results on bounds for $\Delta(d, n)$ and $\Delta_b(d, n)$. It was proved in [9] through construction of specific polytopes that

$$\Delta_b(d, n) \geqslant (d-1)\left[\frac{n}{d}\right] - d + 2, \quad d > 3, \tag{6.1}$$

and

$$\Delta_b(d, n) = \left[\frac{(d-1)\,n}{d}\right] - d + 2, \quad d \leqslant 3.$$

Combining these values with the special value $\Delta_b(4, 9)=5$ from Section 5, employing the monotoneity conditions of 2.9 and 2.10, and making occasional use of 1.3, we can obtain the lower bounds for Δ_b tabulated in Fig. 8.

	$n-2d$																								
d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13	13	14
3	3	3	4	5	5	6	7	7	8	9	9	10	11	11	12	13	13	14	15	15	16	17	17	18	19
4	4	5	5	6	7	7	8	8	10	10	10	11	13	13	13	14	16	16	16	16	19	19	19	19	22
5	5	6	6	7	8	9	9	10	11	11	13	13	14	14	14	17	17	17	17	19	21	21	21	22	23
6	6	7	7	8	9	10	11	11	12	13	14	14	16	16	17	18	18	18	21	21	22	22	22	23	26
7	7	8	8	9	10	11	12	13	13	14	15	16	17	17	19	19	19	21	22	22	23	25	25	26	27
8	8	9	10	10	11	12	13	14	15	15	16	17	18	19	20	20	22	22	23	23	25	26	26	27	29
9	9	10	11	11	12	13	14	15	16	17	17	18	19	20	21	22	23	23	25	25	26	27	27	29	30
10	10	11	12	12	13	14	15	16	17	18	19	19	20	21	22	23	24	25	26	26	28	82	29	30	31
11	11	12	13	13	14	15	16	17	18	19	20	21	21	22	23	24	25	26	27	28	29	29	31	31	32
12	12	13	14	15	15	16	17	18	19	20	21	22	23	23	24	25	26	27	28	29	30	31	32	32	34

Fig. 8. Lower bounds for $\Delta_b(d, n)$.

Since the columns of Fig. 2 are constant from the main diagonal downwards, it is more efficient to tabulate Δ_b, as in Fig. 8, in terms of d and $n-2d$. For this arrangement the monotoneity conditions of 2.9 and 2.10 assert that the rows and diagonals perpendicular to the main diagonal are monotone, while the columns are strictly monotone.

A pair of results of general interest can be abstracted from Fig. 8:

6.2. PROPOSITION. $\Delta_b(d, n) \geq n-d-1 \quad$ *if* $n \leq 3d$,

$$\Delta_b(d, n) \geq n-d \qquad \text{if } n \leq 9d/4.$$

Proof. Substitution of $n=3d$ into 6.1 yields the entries $2d-1=n-d-1$ on the diagonal in Fig. 8. The monotoneity conditions, in particular the strict monotoneity of columns, complete the proof of the first inequality. Similarly the entries $5k$ at positions $d=4k$, $n-2d=k$ in Fig. 8 follow from 1.3 and the value $\Delta_b(4, 9)=5$. The second inequality then follows from the same monotoneity conditions.

Provided n is not very much larger than d, the lower bounds given in Fig. 8 can frequently be realized as the diameters of polytopes dual to the cyclic polytopes considered by Gale in [5]. For example, following Gale's presentation, we may consider a cyclic 6-polytope, Q, with 23 vertices arranged on the moment curve in R^6 in the order:

$$u\ u\ u\ u\ u\ u\ q\ q\ v\ v\ q\ q\ q\ v\ v\ q\ q\ q\ v\ v\ q\ q\ q.$$

The vertices indicated by "u" determine a facet of Q and hence a vertex x of the 6-polytope P dual to Q. Similarly, the vertices indicated by "v" determine a vertex y of P. It is a simple exercise to verify, using Gale's characterization of facets of Q, that $\delta_P(x, y)$ is 14, which is exactly the bound on $\Delta_b(6, 23)$ given in Fig. 8.(1) We have not found any polytopes of this type whose diameters exceed the bounds of Fig. 8, and we do not know of any better lower bounds for Δ than those given by 5.3.

We have no reason to believe that our lower bounds come close to representing the behavior of Δ or Δ_b. For upper bounds the situation appears to be even worse. For a limited range of values of d and n we can derive comparatively narrow limits for Δ and Δ_b using the following elementary inequalities.

6.3. PROPOSITION.

(a) $\Delta(d, 2d+k) \leqslant \Delta(d-1, 2d+k-1)+k+1$ *if* $k=0, 1$.

(b) $\Delta_b(d, 2d+k) \leqslant \Delta_b(d-1, 2d+k-1)+[k/2]+1$ *if* $k=0, 1, 2, 3$.

Proof. Consider a triple (P, x, y) of class (d, n). If $n \leqslant 2d$ or if P is bounded and $n \leqslant 2d+1$, then P admits a $(1, d-1)$-path from x to y. The inequalities, (a) for $k=0$ and (b) for $k=0, 1$, follow immediately. For the remaining inequalities it suffices to show that if $n \leqslant 2d+1$ or if P is bounded and $n \leqslant 2d+3$, then P admits a $(1, d-1, 1)$-path from x to y. Let us consider the bounded case only as the argument for the unrestricted case is similar but simpler. Note that we may assume P does not admit a $(1, d-1)$-path from x to y or from y to x since otherwise the existence of a $(1, d-1, 1)$-path is automatic. We are left with the conclusion that P has at least d facets incident to x, at least d facets incident to y, at most three facets incident to neither x nor y, and that every x-edge and every y-edge terminates on one or more of the additional facets. For all the x-edges or all the y-edges to terminate on the same one of the additional facets would lead to the clearly contradictory conclusion that P is a pyramid over that facet. Thus two of the additional facets intersect x-edges, two intersect y-edges, and hence one intersects at least one x-edge and one y-edge, i.e. P admits a $(1, d-1, 1)$-path from x to y.

6.4. COROLLARY. $\Delta(4, 9)=6$ *or* 7. $\Delta(5, 10)=6, 7,$ *or* 8.

6.5. COROLLARY. *The values of* $\Delta_b(d, n)$ *for certain* d *and* n *lie in the ranges indicated in the following table:*

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(1) This method was followed in [9] to prove, if the d-polytope P is dual to a cyclic d-polytope with n vertices, theo $\delta(P)=n-d$ for $n \leqslant 2d$. It was suggested there that $\delta(P)=[n/2]$ for $n>2d$, but this is not correct.

	$n-d$			
d	4	5	6	7
4	4	5	5–7	6–7
5	*	5	6–8	6–9
6		*	6–9	7–10
7			*	7–11

Proofs. Apply 6.3 and the known values given in Figs. 1 and 2 for the upper bounds. Use Fig. 8 and 5.3 for the lower bounds.

It has been conjectured that for polyhedra of class (d, n) the maximum number of vertices is

$$\binom{n-\left[\frac{d+1}{2}\right]}{n-d}+\binom{n-\left[\frac{d+2}{2}\right]}{n-d}$$

This conjecture has been established for $n \leqslant d+3$ in [6] and for $n \geqslant (d/2)^2-1$ in [10], hence for all n if $d \leqslant 6$. It is also known that for d-polyhedra with v vertices the maximum diameter is $v-1$, and for d-polytopes with v vertices the maximum diameter is $[(v-2)/d]+1$ [7]. These results can be combined in an obvious manner to provide some explicit but apparently extremely rough upper bounds for $\Delta(d, n)$ and $\Delta_b(d, n)$.

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