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THE DAMPED MATHIEU EQUATION

By

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Abstract. We establish an asymptotic lower bound for the minimum excitation needed to cause instability for the damped Mathieu equation. The methods used are Floquet theory and Liapunov-Schmidt, and we use a fact about the width of the instability interval for the undamped Mathieu equation. Our results are compared with published numerical data.

1. Introduction. The Mathieu equation

$$\ddot{y} + (\lambda + \varepsilon \cos 2t)y = 0 \tag{1.1}$$

gives a simple model for externally driven oscillations [2, 11] and also arises from separation of variables for elliptical regions [9]. In the engineering literature, the term $\varepsilon \cos 2t$ is often called a parametric excitation, of strength $|\varepsilon|$. For Eq. (1.1) it is known that the (λ, ε)-plane consists of regions of stability and instability bounded by curves on which there is a periodic solution. These zones of instability form tongues attached to the λ-axis at $\lambda = m^2$, $m = 0, 1, 2, \ldots$.

It is natural to consider also the damped Mathieu equation

$$\ddot{x} + c\dot{x} + (m^2 + \alpha + \varepsilon \cos 2t)x = 0, \tag{1.2}$$

where c, α, ε are small, $\lambda = m^2 + \alpha$. Many authors have observed that for $c > 0$ the tongues separate from the λ-axis. In this paper we establish an asymptotic lower bound on the minimum forcing strength ε_m needed to cause instability near $\lambda = m^2$, i.e., the distance of the mth tongue from the λ-axis, for all $m \geq 1$.

THEOREM 1.1. For small $c > 0$, $m \geq 1$,

$$\varepsilon_m \sim \hat{\varepsilon}_m := 8 \cdot \left[m!(m-1)! \cdot \frac{c}{4}\right]^{1/m}. \tag{1.3}$$

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For $c > 0$, $m \geq 1$, the vertices, i.e., the points on the mth tongue closest to the λ-axis, are given by $(\lambda, \varepsilon) \sim (m^2 + \hat{\alpha}_m, \pm\hat{\varepsilon}_m)$, where

$$\begin{aligned} \alpha_m \sim \hat{\alpha}_m &:= \frac{8}{m^2 - 1}\left[m!(m-1)! \cdot \frac{c}{4}\right]^{2/m} + \frac{c^2}{4}, \qquad m \geq 2, \\ \alpha_1 \sim \hat{\alpha}_1 &:= 0. \end{aligned} \tag{1.4}$$

For concreteness, we present a few examples of Eqs. (1.3), (1.4), which can be compared with numerical results of [7, 8] for unstable solutions of the undamped Mathieu equation.

The derivations of Eqs. (1.3), (1.4) are the principal results of this paper. There are many methods for establishing perturbation results for the Mathieu equation (1.1). We will use primarily the method of Liapunov-Schmidt, i.e., alternative problems [3, 4, 6]. For the damped Mathieu equation (1.2), we begin with Liapunov-Schmidt and then draw on [1] for a crucial fact, about the "width of the instability zones" for the undamped Mathieu equation, which is proven using the method of expansion in Fourier series [9].

The heart of the argument for $m \geq 3$ is: For $c = 0$, the bounding curves are given approximately by $(\alpha - \sum_{j=1}^{m} b_{j,m}\varepsilon^j)^2 = \rho_m^2\varepsilon^{2m}$ for a known constant ρ_m and other constants $b_{j,m}$. For small $c > 0$, it turns out that the bounding curves are given approximately by $(\alpha - \sum_{j=1}^{m} b_{j,m}\varepsilon^j)^2 = \rho_m^2\varepsilon^{2m} - (mc)^2$, so the curves are defined for approximately $|\varepsilon| \geq \hat{\varepsilon}_m = (mc/\rho_m)^{1/m}$.

Approximate bounding curves for $m = 1, 2, 3, 4$ are

$$\begin{aligned} \lambda &= 1 \pm ((\varepsilon/2)^2 - c^2)^{1/2}, \\ \lambda &= 4 + \frac{\varepsilon^2}{24} \pm \left(\frac{(\varepsilon/2)^2}{4} - (2c)^2\right)^{1/2} + \frac{c^2}{4}, \\ \lambda &= 9 + \frac{\varepsilon^2}{64} \pm \left(\frac{(\varepsilon/2)^3}{64} - (3c)^2\right)^{1/2} + \frac{c^2}{4}, \\ \lambda &= 16 + \frac{\varepsilon^2}{120} + \frac{58(\varepsilon/2)^4}{864,000} \pm \left(\frac{375(\varepsilon/2)^4}{864,000} - (4c)^2\right)^{1/2} + \frac{c^2}{4}. \end{aligned}$$

The latter two, as well as those for $m = 5, 6, 7$, are obtained using results for $c = 0$ [9]. Approximate curves for $m = 1, 2, 3$, $c = 0, .2, .4, .8$ are depicted in Fig. 1. Exact curves for different values of m should not cross, so we have restricted the domain of the curves, artificially.

2. Instability intervals and periodic solutions. First we review a basic result [10] for the undamped equation (1.1), i.e., Eq. (1.2) when $c = 0$.

THEOREM 2.1. For every ε, Eq. (1.1) has two monotonically increasing infinite sequences of real numbers $\lambda_0, \lambda_1, \lambda_2, \ldots$ and $\lambda_1', \lambda_2', \lambda_3', \ldots$ such that Eq. (1.1) has a solution of period π if and only if $\lambda = \lambda_n$, $n = 0, 1, \ldots$, and a solution of period 2π if and only if $\lambda = \lambda_n'$, $n = 1, 2, \ldots$. The λ_n and λ_n' satisfy

$$\lambda_0 < \lambda_1' \leq \lambda_2' < \lambda_1 \leq \lambda_2 < \lambda_3' \leq \lambda_4' < \lambda_3 \leq \lambda_4 < \cdots \to \infty.$$

The solutions of Eq. (1.1) are stable in the intervals (λ_0, λ_1'), (λ_2', λ_1), (λ_2, λ_3'), $(\lambda_4', \lambda_3), \dots$.

The proof of this result is based on properties of Hill's discriminant and its relationship with the characteristic multipliers of Floquet theory. We now discuss these things for the more general case of Eq. (1.2).

The damped Mathieu equation (1.2), where $\lambda = m^2 + \alpha$, can be rewritten as a linear system, which is periodic with period π, so Floquet theory [6] applies. Define two solutions $x_1(t; \lambda, \varepsilon, c)$, $x_2(t; \lambda, \varepsilon, c)$ by the initial conditions

$$x_1(0) = 1, \qquad \dot{x}_1(0) = 0 = x_2(0), \qquad \dot{x}_2(0) = 1.$$

Sometimes we will suppress dependence on λ, ε, c if the meaning is clear. A monodromy matrix

$$X(\pi) = \begin{pmatrix} x_1(\pi) & x_2(\pi) \\ \dot{x}_1(\pi) & \dot{x}_2(\pi) \end{pmatrix}$$

has trace $\Delta = x_1(\pi) + \dot{x}_2(\pi)$ and characteristic multipliers

$$\mu_\pm = \frac{\Delta \pm \sqrt{\Delta^2 - 4e^{-c\pi}}}{2}.$$

Here we have used the result that $\det X(\pi) = e^{-c\pi}$ [6, Lemma III.7.3]. Note that $\mu_+ \cdot \mu_- = e^{-c\pi}$.

Equation (1.2) has a π (or 2π)-periodic solution if and only if one of the characteristic multipliers is $+1$ (or ± 1, respectively; if a multiplier is -1 there is a 2π-periodic solution, which is not π-periodic), if and only if $\Delta = 1+e^{-c\pi}$ (or $-1-e^{-c\pi}$, respectively). Stability of $x \equiv 0$ is guaranteed if $|\mu_\pm| < 1$, i.e., $|\Delta| < 1 + e^{-c\pi}$, and instability of $x \equiv 0$ is guaranteed if $|\mu_+| > 1$ or $|\mu_-| > 1$, i.e., $|\Delta| > 1 + e^{-c\pi}$. Thus, for fixed c, ε, the instability intervals, i.e., values of λ for which $x \equiv 0$ is unstable for Eq. (1.2), are bounded by the values of λ for which Eq. (1.2) has a π- or 2π-periodic solution.

3. Perturbation equations and solution curves. Fix an integer $m \geq 1$. For the method of Liapunov-Schmidt it is convenient to use the van der Pol transformation on Eq. (1.2), considered as a system. Let

$$\mathbf{y} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \qquad \Psi(t) = \begin{pmatrix} \sin mt & \cos mt \\ m\cos mt & -m\sin mt \end{pmatrix}, \qquad \mathbf{z} = \Psi(t)^{-1}\mathbf{y}.$$

Then $\mathbf{z}$ satisfies

$$\dot{\mathbf{z}} = B_1(t)\mathbf{z}, \tag{3.1}$$

where

$$\begin{aligned} B_1(t; \alpha, \varepsilon, c) = {} & \frac{\alpha}{2m}(-C - (\sin 2mt)A - (\cos 2mt)B) \\ & + \frac{c}{2}(-I - (\cos 2mt)A + (\sin 2mt)B) \\ & + \frac{\varepsilon}{4m}(-2(\cos 2t)C - (\sin(2m-2)t + \sin(2m+2)t)A \\ & \quad - (\cos(2m-2)t + \cos(2m+2)t)B), \end{aligned}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $B_1(\cdot)$ is π-periodic and that the transformation, defined using $\Psi(\cdot)$, is π-periodic (or 2π-periodic) for $m = \text{even}$ (or $m = \text{odd}$, respectively). Thus, Eq. (1.2) has a π-periodic solution if and only if Eq. (3.1) has a π-periodic solution and $m = \text{even}$, and Eq. (1.2) has a 2π-periodic solution, which is not π-periodic, if and only if $m = \text{odd}$ and Eq. (3.1) has either a π-periodic or 2π-periodic solution. Of course, all π-periodic solutions are also 2π-periodic.

To look for π-periodic solutions of Eq. (3.1) we can use the method of Liapunov-Schmidt, i.e., alternative problems [3, 4, 6]. Denote by $\mathscr{P}_\pi$ the space of functions $\mathbf{f}\colon \mathbb{R} \to \mathbb{R}^2$, which are continuous and π-periodic, with the usual norm $|\mathbf{f}| = \max_{0 \le t \le \pi} |\mathbf{f}(t)|$, and denote by $P\mathbf{f}$ the mean value of $\mathbf{f}$, i.e., $P\mathbf{f} = (1/\pi)\int_0^\pi \mathbf{f}$. Note that $P\colon \mathscr{P}_\pi \to \mathscr{P}_\pi$ is a bounded linear operator, in fact, a projection: $P^2 = P$. Let $\mathscr{P}_\pi^1$ be the space of differentiable functions whose derivative is in $\mathscr{P}_\pi$, $L_0\mathbf{f}$ the derivative of $\mathbf{f}$, i.e., $L_0\mathbf{f} = \dot{\mathbf{f}}$, and $L_1(\alpha, \varepsilon, c)\mathbf{f} = B_1(\cdot\,; \alpha, \varepsilon, c)\mathbf{f}$. Then L_0 and L_1 are bounded linear operators: $\mathscr{P}_\pi^1 \to \mathscr{P}_\pi$ and L_1 is small if $|\alpha|, |\varepsilon|, c > 0$ are small. The existence of a π-periodic solution of $\mathbf{z}$ of Eq. (3.1) is equivalent to solving $L_0\mathbf{z} = L_1(\alpha, \varepsilon, c)\mathbf{z}$, $\mathbf{z} \in \mathscr{P}_\pi^1$, and this can be written as a system

$$(I - P)L_0(\mathbf{a} + (I - P)\mathbf{z}) = (I - P)L_1(\mathbf{a} + (I - P)\mathbf{z}), \tag{3.2}$$

$$PL_0(\mathbf{a} + (I - P)\mathbf{z}) = PL_1(\mathbf{a} + (I - P)\mathbf{z}), \tag{3.3}$$

where $\mathbf{a} = P\mathbf{z} \in \mathbb{R}^2$. The operator $(I - P)L_0\colon (I - P)\mathscr{P}_\pi^1 \to (I - P)\mathscr{P}_\pi$ has a right inverse $\mathscr{K}\colon (I-P)\mathscr{P}_\pi \to (I-P)\mathscr{P}_\pi^1$. Explicitly, if $\mathbf{f} \in (I-P)\mathscr{P}_\pi$, i.e., $\mathbf{f}$ is π-periodic with mean value zero, then there is a unique π-periodic function $\mathbf{z}$ with mean value zero such that $\dot{\mathbf{z}} = \mathbf{f}$, namely, $\mathbf{z}$ is the unique indefinite integral of $\mathbf{f}$ with mean value zero, i.e., $\mathbf{z} = (I - P)\int \mathbf{f}$. Noting that $L_0 P \equiv 0$, the "auxiliary equation" (3.2) can be rewritten as

$$(I - P)\mathbf{z} = \mathscr{K}(I - P)L_1(\mathbf{a} + (I - P)\mathbf{z}). \tag{3.4}$$

Since $L_1 = L_1(\alpha, \varepsilon, c)$ is small for small (α, ε, c), one can solve Eq. (3.4) by iteration: Let

$$\mathbf{z}^{(0)} = \mathbf{a}, \quad \mathbf{z}^{(n+1)} = \mathbf{a} + \mathscr{K}(I - P)L_1(\mathbf{a} + (I - P)\mathbf{z}^{(n)}), \qquad n \ge 0. \tag{3.5}$$

We see that $\mathbf{z}^{(n)}$ involves $\alpha^j \varepsilon^k c^l$, where $j, k, l \ge 0$ and $j + k + l \le n$.

The "bifurcation equations" are obtained by substituting the solution of Eq. (3.4), $(I - P)\mathbf{z}$, into Eq. (3.3) after noting that $PL_0\mathscr{K} \equiv 0$:

$$0 = PL_1(\alpha, \varepsilon, c)(\mathbf{a} + (I - P)\mathbf{z}) := D(\alpha, \varepsilon, c)\mathbf{a}, \tag{3.6}$$

where $D(\alpha, \varepsilon, c)$ is a matrix. At each stage of the iteration one can substitute $\mathbf{z}^{(n)}$ into Eq. (3.3) to get truncated bifurcation equations

$$0 = PL_1(\alpha, \varepsilon, c)(\mathbf{a} + (I - P)\mathbf{z}^{(n)}) := D^{(n)}(\alpha, \varepsilon, c)\mathbf{a}, \tag{3.7}$$

where $D^{(n)}(\alpha, \varepsilon, c)$ is a matrix. We see that $D^{(n)}(\alpha, \varepsilon, c)$ involves $\alpha^j \varepsilon^k c^l$, where $j, k, l \geq 0$ and $j + k + l \leq n + 1$. If $D^{(n)}(\alpha, \varepsilon, c)$ contains enough terms so as to be able to determine how many solution curves there are in the (λ, ε)-plane for all small $c > 0$, one can obtain approximate solutions of Eq. (3.6) from

$$0 = \det D^{(n)}(\alpha, \varepsilon, c), \tag{3.8$_n$}$$

because $D(\alpha, \varepsilon, c) = D^{(n)}(\alpha, \varepsilon, c) + \mathscr{O}((|\alpha| + |\varepsilon| + |c|)^{n+2})$, as $|\alpha| + |\varepsilon| + |c| \to 0$.

Note that Eq. (1.2) depends on m. In effect, we are considering an infinite collection of perturbation problems.

For $m = 1$, $\mathbf{z}^{(0)} = \mathbf{a}$, $D^{(0)}(\alpha, \varepsilon, c) = -\alpha C/2 - cI/2 - \varepsilon B/4$ contains enough information because Eq. (3.8_0) is

$$0 = \frac{1}{4} \det \begin{pmatrix} -c & \alpha + \varepsilon/2 \\ \alpha - \varepsilon/2 & -c \end{pmatrix},$$

i.e.,

$$0 = c^2 + \alpha^2 - \left(\frac{\varepsilon}{2}\right)^2.$$

For each small $c > 0$, there are two approximate solution curves $\alpha_\pm \sim \pm((\varepsilon/2)^2 - c^2)^{1/2}$, defined approximately for $|\varepsilon/2| \geq c$, i.e., $|\varepsilon| \geq \varepsilon_1 \sim \hat{\varepsilon}_1 = 2c$. The scaling argument is similar to [3, pp. 435–436]. The value of $\hat{\varepsilon}_1$ agrees with Eq. (1.3).

For $m \geq 2$, $D^{(0)}(\alpha, \varepsilon, c) = \frac{1}{2}(-\alpha C/(2m) - cI/2)$ does not contain enough information; so one needs at least to iterate once in Eq. (3.5).

After some calculations one obtains

$$\begin{aligned} \mathbf{z}^{(1)} = \mathbf{a} + \Bigg[& \frac{-\alpha}{4m^2}(-(\cos 2mt)A + (\sin 2mt)B) - \frac{c}{4m}(\sin(2mt)A + \cos(2mt)B) \\ & + \frac{\varepsilon}{4m}\left(-(\sin 2t)C + \left(\frac{\cos(2m-2)t}{2m-2} + \frac{\cos(2m+2)t}{2m+2}\right)A \right. \\ & \left. - \left(\frac{\sin(2m-2)t}{2m-2} + \frac{\sin(2m+2)t}{2m+2}\right)B\right)\Bigg]\mathbf{a} \end{aligned}$$

and, after more calculations, noting that $AB = C = -BA$, $AC = B = -CA$, one obtains

$$D^{(1)}(\alpha, \varepsilon, c) = -\frac{\alpha}{2m}C - \frac{c}{2}I + \frac{1}{2}\left(\left(\frac{\alpha^2}{4m^2} + \frac{c^2}{4m} + \frac{\varepsilon^2}{8m(m^2-1)}\right)C + \frac{\varepsilon^2}{32}\eta_m B\right), \tag{3.9}$$

where $\eta_m = 1$ if $m = 2$, $\eta_m = 0$ if $m \geq 3$.

For $m = 2$, $D^{(1)}(\alpha, \varepsilon, c)$ contains enough information, because Eq. (3.8_1) is

$$0 = \frac{1}{16}\det\begin{pmatrix} -2c & -(\alpha - \frac{c^2}{4} - \frac{\alpha^2}{16} - \frac{\varepsilon^2}{24}) + \frac{\varepsilon^2}{16} \\ \alpha - \frac{c^2}{4} - \frac{\alpha^2}{16} - \frac{\varepsilon^2}{24} + \frac{\varepsilon^2}{16} & -2c \end{pmatrix},$$

i.e.,

$$0 = 4c^2 + \left(\alpha - \frac{c^2}{4} - \frac{\alpha^2}{16} - \frac{\varepsilon^2}{24}\right)^2 - \left(\frac{\varepsilon^2}{16}\right)^2.$$

For each small $c > 0$, there are two approximate solution curves

$$\alpha_{\pm} \sim \frac{\varepsilon^2}{24} \pm \sqrt{\left(\frac{\varepsilon^2}{16}\right)^2 - 4c^2 + \frac{c^2}{4}},$$

defined for approximately $|\varepsilon^2/16| \geq 2c$, i.e., for $|\varepsilon| \geq \varepsilon_2 \sim \hat{\varepsilon}_2 = \sqrt{32c}$. The value of ε_2 agrees with Eq. (1.3); the corresponding value of $\hat{\alpha}_2 = \frac{4}{3}c + c^2/4$ agrees with Eq. (1.4).

For $m \geq 3$, $D^{(1)}(\alpha, \varepsilon, c)$ does not contain enough information. Rather than attempt to continue the iteration in Eq. (3.5), one can use information about the "width of the instability intervals" for the undamped Mathieu equation, i.e., $c = 0$. In [1, 5] Fourier series are used to show that for the undamped Mathieu equation (1.1), for any $m \geq 1$ there are two solution curves, which are in $(m-1)$st-order contact at $(\alpha, \varepsilon) = (0, 0)$. It follows that the curves must be of the form, for some constants $b_{j,m}$,

$$\alpha_{\pm} \sim \sum_{j=1}^{m} b_{j,m}\varepsilon^j \pm \rho_m \varepsilon^m, \tag{3.10}$$

where $2\rho_m\varepsilon^m$ is the width of the instability interval. It is known from [1] that the instability interval has width

$$\frac{|q|^m}{2^{2m-3}((m-1)!)^2},$$

where $\varepsilon = -2q$ in Bell's notation; hence $\rho_m = \frac{1}{2^{3m-2}((m-1)!)^2}$. It follows from Eq. (3.10) that, up to a multiplicative constant k,

$$k \det D^{(m-1)}(\alpha, \varepsilon, 0) = \left(\alpha - \sum_{j=1}^{m} b_{j,m}\varepsilon^j\right)^2 - \rho_m^2 \varepsilon^{2m}.$$

But, since $D^{(m-1)}(\alpha, \varepsilon, 0) = D^{(1)}(\alpha, \varepsilon, 0) +$ (terms of degrees between 3 and m),

$$\det D^{(m-1)}(\alpha, \varepsilon, 0) = \left(\frac{\alpha}{2m} - \frac{\varepsilon^2}{16m(m^2-1)} - \sum_{j=3}^{m} \frac{b_{j,m}}{2m}\varepsilon^j\right)^2 - \left(\frac{\rho_m}{2m}\right)^2 \varepsilon^{2m} + \mathcal{O}(\alpha^2). \tag{3.11}$$

Furthermore, since $D^{(m-1)}(\alpha, \varepsilon, c)$ is real-analytic in c, Eq. (3.9) implies

$$\det D^{(m-1)}(\alpha, \varepsilon, c) = \det D^{(m-1)}(\alpha, \varepsilon, 0) + \frac{c^2}{4}.$$

It follows that, to lowest order, up to a multiplicative constant, Eq. (3.9) implies

$$\det D^{(m-1)}(\alpha, \varepsilon, c) = \left(\frac{c}{2}\right)^2 + \left(\frac{\alpha}{2m} - \frac{c^2}{8m} - \sum_{j=1}^{m} \frac{b_{j,m}}{2m}\varepsilon^j\right)^2 - \left(\frac{\rho_m}{2m}\right)^2 \varepsilon^{2m}.$$

For each small $c > 0$, there are two approximate solution curves

$$\alpha_{\pm} \sim \frac{c^2}{4} + \frac{\varepsilon^2}{8(m^2-1)} + \sum_{j=3}^{m} b_{j,m}\varepsilon^j \pm \sqrt{\rho_m^2 \varepsilon^{2m} - (mc)^2},$$

defined for $|\varepsilon| \geq \varepsilon_m \sim \hat{\varepsilon}_m = (m\rho_m c)^{1/m} = [2^{3m-2} m((m-1)!)^2 c]^{1/m}$, which agrees with Eq. (1.3). The corresponding value of $\hat{\alpha}_m = \varepsilon_m^2/(8(m^2-1)) + c^2/4$ agrees with Eq. (1.4).

To complete the proof of Theorem 1.1, it suffices to look for 2π-periodic solutions of Eq. (3.1). Now let $P\mathbf{f} = (1/2\pi)\int_0^{2\pi} \mathbf{f}$. Because $B_1(t; \alpha, \varepsilon, c)$ only involves even Fourier terms, for all n, in $L_1(\alpha, \varepsilon, c)(\mathbf{a} + (I-P)\mathbf{z}^{(n)})$ the only terms of nonzero mean value over the interval $0 \leq t \leq 2\pi$ will be exactly the same as the only terms of nonzero mean value over the interval $0 \leq t \leq \pi$, namely, constant terms obtained from products of the form $\sin(2m-2l)t \sin(2m-2l)t$ or $\cos(2m-2l)t\cos(2m-2l)t$. It follows that the truncated bifurcation equations will duplicate those for the search for π-periodic solutions and so will not produce any new 2π-periodic solutions.

4. Comparison with numerical results. In [7, §3.6] one finds some numerical results for the undamped Mathieu equation (1.1), including curves in the (λ, ε)-plane on which the characteristic multipliers $\mu_{\pm}$ satisfy $|\mu_{\pm}| = e^{\pm\nu}$, $\nu > 0$. There is a simple connection between those curves and the curves in the (λ, ε)-plane we obtain, for fixed $c > 0$, for the damped Mathieu equation (1.2). This connection enables us to compare our approximate curves' vertices $(\hat{\alpha}_m, \hat{\varepsilon}_m)$ with numerical results of [7]. We can also compare directly our approximate vertices with numerical results of [8] for the damped Mathieu equation.

The equations

$$\ddot{x} + c\dot{x} + (\lambda + \varepsilon\cos 2t)x = 0, \tag{4.1}$$

$$\ddot{y} + (\tilde{\lambda} + \varepsilon\cos 2t)y = 0 \tag{4.2}$$

are related by $y(t) = e^{ct/2}x(t)$, $\tilde{\lambda} = \lambda - c^2/4$. It follows that the characteristic multipliers $\mu_{\pm}^x$ of Eq. (4.1) are related to the characteristic multipliers $\mu_{\pm}^y$ of (4.2) by $\mu_{\pm}^x e^{c\pi/2} = \mu_{\pm}^y$. We note $\mu_+^y \cdot \mu_-^y = 1$. We know that Eq. (4.1) has a periodic solution if and only if $|\mu_+^x| = 1$ or $|\mu_-^x| = 1$, in which case Eq. (4.2) has a characteristic multiplier $|\mu_+^y| = e^{c\pi/2}$ or $|\mu_-^y| = e^{c\pi/2}$. Since $\mu_+^y \cdot \mu_-^y = 1$, it follows that Eq. (4.1) has a periodic solution if and only if Eq. (4.2) has multipliers of magnitude $e^{\pm c\pi/2}$ for $\tilde{\lambda} = \lambda - c^2/4$. In [7] iso-curves are in the $(\tilde{\lambda}, \varepsilon)$-plane where Eq. (4.2) has multipliers of magnitude $e^{\pm c\pi/2}$. In Table 1 we give some comparisons of the vertices of these curves, obtained from hand measurements of the figures [7, pp. 90–92], with our approximate vertices in Theorem 1.1.

In [8] one finds some numerical results for the damped Mathieu equation. Again, from hand measurements of [8, Figure 1], in Table 2 we compare our results for the approximate vertices in Theorem 1.1.

For both Tables 1, 2 we use the same units as in our paper; for example, in [8] 2γ is our c and 2ε is our ε.

TABLE 1

m	c	$(\hat{\varepsilon}_m, \hat{\alpha}_m - \frac{c^2}{4})$ from (1.3–4)	$(\varepsilon_m, \alpha_m)^*$ from [7]
1	.2	(.4, 0)	(.40, 0)
1	.8	(1.6, −.09)	(1.6, −.25)
2	.2	(2.53, .267)	(2.5, .25)
2	.4	(3.58, .533)	(3.6, .54)
3	.1	(5.36, .451)	(5.3, .44)
3	.2	(6.75, .721)	(6.8, .69)
3	.3	(7.72, .955)	(7.7, .92)

TABLE 2

m	c	$\hat{\varepsilon}_m$ from (1.3–4)	$\varepsilon_m{}^*$ from [8]
1	.2	.40	.4
1	1	2.00	2.2
1	2	4.00	4.6
2	.2	2.53	2.6
2	1	5.66	6.8
2	2	8.00	11.6

5. Remarks. The $\hat{\varepsilon}_m$ provide an asymptotic upper bound on the maximum excitation, which does not cause loss of stability. For each $m \geq 1$, $\hat{\varepsilon}_m$ is an increasing function of c, for small positive c, as one would expect. For each fixed small positive c, with $c < 8$, $\hat{\varepsilon}_m$ is increasing in m, for $m \geq 1$; in fact, for $m \geq 2$

$$\left(\frac{\hat{\varepsilon}_{m+1}}{\hat{\varepsilon}_m}\right)^{m(m+1)} = \frac{4}{c} \cdot \frac{(m+1)^m}{(m-1)!} \cdot \frac{m^m}{m!} \geq \frac{4}{c} m \cdot (m+1)^2$$

and, for $m = 1$, $(\hat{\varepsilon}_2/\hat{\varepsilon}_1)^2 = 8/c$. So $\hat{\varepsilon}_1 = 2c$ provides an *asymptotic* upper bound, which is good for all $m \geq 1$.

Define

$$\lambda_{\pm}(c, \varepsilon) = m^2 + \frac{c^2}{4} + \sum_{j=1}^{m} b_{j,m}\varepsilon^j \pm \sqrt{(\rho_m \varepsilon^m)^2 - (mc)^2},$$

$$\phi_{\pm}(c, \varepsilon) = \lambda_{\pm}(c, \varepsilon) - \lambda_{\pm}(0, \varepsilon) = \frac{c^2}{4} \pm \left(\sqrt{(\rho_m \varepsilon^m)^2 - (mc)^2} - \rho_m \varepsilon^m\right).$$

One can see (Fig. 1) that the curves for $c > 0$ are inside the curves for $c = 0$, at least for $0 < c < 4$, by showing that $\phi_-(c, \varepsilon) > 0 > \phi_+(c, \varepsilon)$. This can be shown after noting that $\pm\frac{\partial \phi_\pm}{\partial \varepsilon}(c, \varepsilon) < 0$ for $|\varepsilon| \geq \hat{\varepsilon}_m$, $c > 0$.

* Only two digits retained from inspections of graphical data.

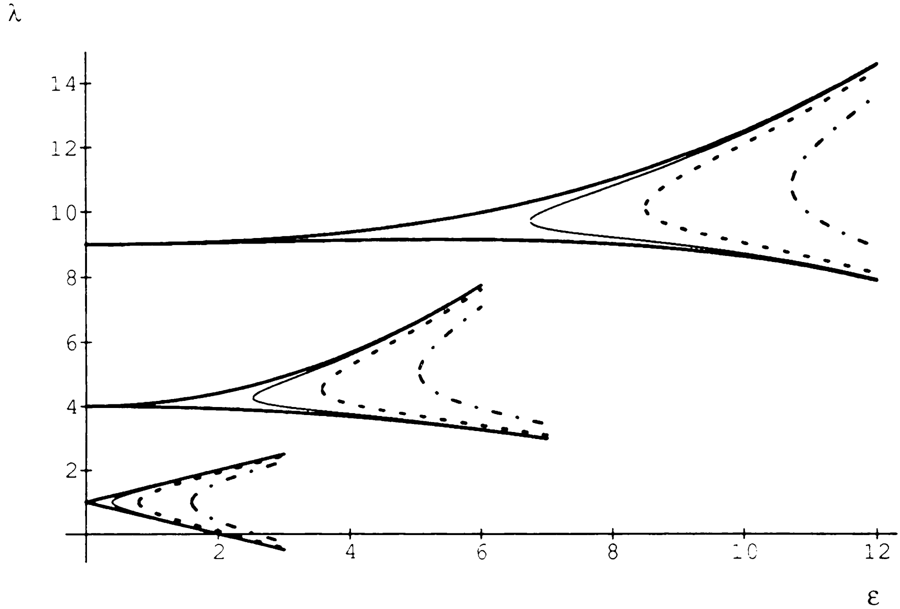

FIG. 1. —— $c = 0$, —— $c = 0.2$, - - - $c = 0.4$, – · – $c = 0.8$.

The characteristic exponents are $1/\pi$ times the branches of the logarithm of the characteristic multipliers, which are complex numbers. If ν denotes the real part of a characteristic exponent, $\nu = c/2$, then the curves in [7] of iso-ν have asymptotic vertices $(\lambda, \varepsilon) = (m^2 + \hat{\alpha}_m - \nu^2, \hat{\varepsilon}_m)$, where from Eqs. (1.3) and (1.4)

$$\hat{\varepsilon}_m = 8 \cdot \left[m!(m-1)!\frac{\nu}{2}\right]^{1/m}, \qquad \hat{\alpha}_m = \frac{8}{m^2 - 1}\left[m!(m-1)!\frac{\nu}{2}\right]^{2/m} + \nu^2,$$

for $m \geq 2$, and $\hat{\varepsilon}_1 = 4\nu$, $\hat{\alpha}_1 = 0$.

The damped Mathieu equation is in some ways more akin to the undamped Mathieu equation than it is to the damped harmonic oscillator equation $\ddot{x} + c\dot{x} + \omega^2 x = 0$. The latter does not have oscillations, i.e., periodic solutions. Also, the latter has damped oscillatory solutions with "quasi-frequency" $\sqrt{\omega^2 - c^2/4}$ for small $c > 0$; no such frequency shift occurs for the damped Mathieu equation.

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