THE DARBOUX TRANSFORMATION OF THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

SHUWEI XU†,JINGSONG HE* †, LIHONG WANG †

† Department of Mathematics, Ningbo University, Ningbo , Zhejiang 315211, P. R. China

Abstract. The n-fold Darboux transformation (DT) is a 22 matrix for the Kaup-Newell (KN) system. In this paper, each element of this matrix is expressed by a ratio of $(n+1)\times(n+1)$ determinant and $n\times n$ determinant of eigenfunctions. Using these formulae, the expressions of the $q^{[n]}$ and $r^{[n]}$ in KN system are generated by n-fold DT. Further, under the reduction condition, the rogue wave,rational traveling solution, dark soliton, bright soliton, breather solution, periodic solution of the derivative nonlinear Schrödinger(DNLS) equation are given explicitly by different seed solutions. In particular, the rogue wave and rational traveling solution are two kinds of new solutions. The complete classification of these solutions generated by one-fold DT is given in the table on page.

Key words: derivative nonlinear Schrödinger equation, Darboux transformation, soliton, rational solution, breather solution, rogue wave.
PACS(2010) numbers: 02.30.Ik, 42.81.Dp, 52.35.Bj, 52.35.Sb, 94.05.Fg
MSC(2010) numbers: 35C08, 37K10, 37K40

1. Introduction

The derivative nonlinear Schrödinger equation,

$$iq_t - q_{xx} + i(q^2q^*)_x = 0, \tag{1}$$

one of the most important integrable systems in the mathematics and physics, is usually called DNLS(or DNLSI) equation. Here"*" denotes the complex conjugation, and subscript of x (or t) denotes the partial derivative with respect to x (or t). This equation is originated from two fields of applied physics. The first is plasma physics in which the DNLS governs the evolution of small but finite amplitude Alfvén waves that propagates quasi-parallel to the magnetic field [1, 2]. Recently, this equation is also used to describe large-amplitude magnetohydrodynamic (MHD) waves in plasmas [3, 4]. Further, it is natural to improve DNLS equation in more practical plasmas. For example, DNLS truncation model [5] and the DNLS with nonlinear Landau damping [6]. In the second area, nonlinear optics, the sub-picosecond or femtosecond pulses in single-mode optical fiber is modeled by the DNLS [7–9].

However, the crucial feature of the DNLS is that the integrability such as the dynamical evolution of the associated physical system can be given analytically by using its exact solution. Under the vanishing boundary condition(VBC), Kaup and Newell(KN) [10] firstly proposed an inverse scattering transform (IST) with a revision in their pioneer works, and got a one-soliton solution. Later, Kawata [11] further solved DNLS under VBC and non-vanishing boundary condition (NVBC) to get two soliton solution, and introduced "paired soliton" which is now

* Corresponding author: hejingsong@nbu.edu.cn,jshe@ustc.edu.cn.

regarded as one kind of breather solution. N-soliton formula [12] of the DNLS with VBC is expressed by determinants with the help of pole-expansion. Further, the IST of the DNLS with VBC is re-considered by Huang's group [13–16] and then the explicit form of the N-soliton is obtained by some algebraic techniques. Now we turn to the DNLS under NVBC, and some special solutions are obtained and the existence of the algebraic soliton is also given [17]. This is followed by paired-soliton of the DNLS from the IST [18]. Wadati etal. [19] have given the stationary solutions of the DNLS under the plane wave boundary and the contributions of the derivative term in the DNLS equation. Recently, to avoid the multi-value problem, Chen and Lam [20] revised the IST for the DNLS under NVBC by introducing an affine parameter, and then got single breather solution, which can be reduced to the dark soltion and bright soliton. Further applications on this method can be found in reference [21]. Cai and Huang [22] found the action-angle variables of the DNLS explicitly by constructing its Hamiltonian formalism.

Similar to many usual soliton equations, the DNLS is also solved by the Hirota method [23] and Darboux transformation(DT) [24, 25] besides IST. By comparing with the corresponding results [26–28] of nonlinear Schrödinger(NLS) equation, the DT [24, 25] of the DNLS has following essential distinctness:

- the kernel of one-fold DT is one-dimensional and it can be defined by one eigenfunction of linear system defined by spectral problem,
- the DNLS will be invariant under one-fold DT associated with a pure imaginary eigenvalue(see the last paragraph of the section 2).

Some solutions [24] including multi-soliton and quasi-periodic solutions are obtained by this DT from a trivial seed: zero solution(or vacuum). Steudel [25] has obtained a general formula of solutions $q^{[N]}$ and $r^{[N]}$ of KN system in terms of Vandermonde-like determinants by N-fold DTs, and then given n-soliton and N-phase solutions from zero seed, N-breather solutions from non-zero seed: monochromatic wave. Unlike the usual DT, Steduel used solutions of Riccati equations, which are transformed from the linear partial differential equations of the spectral problem for the DNLS, to construct the solutions of the DNLS. So the first difficulty of his method is to solve nonlinear Riccati, which is not solvable in general. To overcome this difficulty, Steudel have made an Ansatz(see eq.(51) in reference [25]) and introduced his favorite Seahorse functions. Moreover, the classification of the solutions(see Figure 1 in reference [25]) generated by DT is very interesting and useful. But the conditions of parameters to generate dark soliton and bright soliton of the DNLS are not clear. Therefore, it is natural to question whether the difficult Riccati equations are indeed unavoidable for the DT from non-zero seeds and whether the classification of solutions generated by one-fold DT can be fixed thoroughly or not.

It is interesting that the Ablowitz-Kaup-Newell-Segur(AKNS)system [29] can be mapped to the KN system by a gauge transformation [30]. Moreover, there exists other two kinds of derivative nonlinear Schrödinger equation, i.e,. the DNLSII [31]

$$iq_t + q_{xx} + iqq^* q_x = 0, \tag{2}$$

and the DNLSIII [32]

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2} q^3 q^{*2} = 0, \tag{3}$$

and a chain of gauge transformations between them: DNLSII $\overset{a)}{\Longrightarrow}$ DNLSI $\overset{b)}{\Longrightarrow}$ DNLSIII. Here a) denotes eq.(2.12) in ref. [30], and b) denotes: eq.(4)→ eq.(3) →eq.(6) with $\gamma = 0$ in ref. [23]. But these transformations can not preserve the reduction conditions in spectral problem of the KN system and involve complicated integrations. So each of them deserves investigating separately.

There are two aims of this paper. First aim is to present a detailed derivation of the DT for the DNLS and its determinant representation. Using this representation, the solutions of DNLS can be expressed by the solutions(eigenfunctions) of the linear partial differential equations of the spectral problem of the KN system instead of the solutions of the nonlinear Riccati equations, which shows that the nonlinear Riccati equation and Seahorse functions are indeed avoidable for the DT from nonzero seeds. A second aim is to present a complete classification of the solutions generated by one-fold DT from zero seed, non-zero seeds: constant solution and periodic solution with a constant amplitude.

The organization of this paper is as follows. In section 2, it provides a relatively simple approach to DT for the KN system, and then the determinant representation of the n-fold DT and formulae of $q^{[n]}$ and $r^{[n]}$ expressed by eigenfunctions of spectral problem are given. The reduction of DT of the KN system to the DNLS equation is also discussed by choosing paired eigenvalues and eigenfunctions. In section 3, under specific reduction conditions, several types of particular solutions are given from zero seed, non-zero seeds: constant solution and periodic solution with a constant amplitude. The complete classification of dark soliton, bright soliton, periodic solution are given in a table for one-fold DT of the DNLS equation. In particular, two kinds of new solutions: rational traveling solution and rogue wave are given. The conclusion will be given in section 4.

2. Darboux transformation

Let us start from the first non-trivial flow of the KN system [10],

$$r_t - ir_{xx} - (r^2q)_x = 0, \tag{4}$$

$$q_t + iq_{xx} - (rq^2)_x = 0, \tag{5}$$

which are exactly reduced to the DNLS eq.(1) for $r = -q^*$ while the choice $r = q^*$ would lead to eq.(1) with the sign of the nonlinear term changed. The Lax pairs corresponding to coupled DNLS equations(4) and (5) can be given by the KaupCNewell spectral problem [10]

$$\partial_x\psi = (J\lambda^2 + Q\lambda)\psi = U\psi, \tag{6}$$

$$\partial_t\psi = (2J\lambda^4 + V_3\lambda^3 + V_2\lambda^2 + V_1\lambda)\psi = V\psi, \tag{7}$$

with

$$\psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix},$$

$$V_3 = 2Q, \quad V_2 = Jqr, \quad V_1 = \begin{pmatrix} 0 & -iq_x + q^2r \\ ir_x + r^2q & 0 \end{pmatrix}.$$

Here λ, an arbitrary complex number, is called the eigenvalue(or spectral parameter), and ψ is called the eigenfunction associated with λ of the KN system. Equations(4) and (5) are equivalent to the integrability condition $U_t - V_x + [U, V] = 0$ of (6) and (7).

The main task of this section is to present a detailed derivation of the Darboux transforation of the DNLS and the determinant representation of the n-fold transformation. Based on the DT for the NLS [26–28] and the DNLS [24,25], the main steps are : 1) to find a 2×2 matrix T so that the KN spectral problem eq.(6) and eq.(7) is covariant, then get new solution $(q^{[1]}, r^{[1]})$ expressed by elements of T and seed solution (q, r); 2) to find expressions of elements of T in terms of eigenfunctions of KN spectral problem corresponding to the seed solution (q, r); 3) to get the determinant representation of n-fold DT T_n and new solutions $(q^{[n]}, r^{[n]})$ by n-times iteration of the DT; 4) to consider the reduction condition: $q^{[n]} = -(r^{[n]})^*$ by choosing special eigenvalue λ_k and its eigenfunction ψ_k, and then get $q^{[n]}$ of the DNLS equation expressed by its seed solution q and its associated eigenfunctions $\{\psi_k, k = 1, 2, \cdots, n\}$. However, we shall use the kernel of n-fold DT(T_n) to fix it in the third step instead of iteration.

It is easy to see that the spectral problem (6) and (7) are transformed to

$$\psi^{[1]}{}_x = U^{[1]}\ \psi^{[1]}, \quad U^{[1]} = (T_x + T\ U)T^{-1}. \tag{8}$$

$$\psi^{[1]}{}_t = V^{[1]}\ \psi^{[1]}, \quad V^{[1]} = (T_t + T\ V)T^{-1}. \tag{9}$$

under a gauge transformation

$$\psi^{[1]} = T\ \psi. \tag{10}$$

By cross differentiating (8) and (9), we obtain

$$U^{[1]}{}_t - V^{[1]}{}_x + [U^{[1]}, V^{[1]}] = T(U_t - V_x + [U, V])T^{-1}. \tag{11}$$

This implies that, in order to make eqs.(4) and eq.(5) invariant under the transformation (10), it is crucial to search a matrix T so that $U^{[1]}$, $V^{[1]}$have the same forms as U, V. At the same time the old potential(or seed solution)(q, r) in spectral matrixes U, V are mapped into new potentials (or new solution)($q^{[1]}$, $r^{[1]}$) in transformed spectral matrixes $U^{[1]}$, $V^{[1]}$.

2.1 One-fold Darboux transformation of the KN system

Considering the universality of DT, suppose that the trial Darboux matrix T in eq.(10) is of form

$$T = T(\lambda) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \lambda + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \tag{12}$$

where $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1$ are functions of x, t to need be determined. From

$$T_x + T\ U = U^{[1]}\ T, \tag{13}$$

comparing the coefficients of $\lambda^j, j = 3, 2, 1, 0$, it yields

$$\begin{aligned}
&\lambda^3 : b_1 = 0,\ c_1 = 0,\\
&\lambda^2 : q\ a_1 - 2\ i\ b_0 - q^{[1]}d_1 = 0,\ -r^{[1]}\ a_1 + r\ d_1 + 2\ i\ c_0 = 0,\\
&\lambda^1 : a_{1x} + r\ b_0 - q^{[1]}c_0 = 0,\ d_{1x} + qc_0 - r^{[1]}b_0 = 0, qa_0 - q^{[1]}d_0 = 0, -r^{[1]}a_0 + rd_0 = 0,\\
&\lambda^0 : a_{0x} = b_{0x} = c_{0x} = d_{0x} = 0.
\end{aligned} \tag{14}$$

The last equation shows a_0, b_0, c_0, d_0 are functions of t only. Similarly, from

$$T_t + T\ V = V^{[1]}\ T, \tag{15}$$

comparing the coefficients of $\lambda^j, j = 4, 3, 2, 1, 0$,it implies

$$\begin{aligned}
&\lambda^4 : -2ib_0 - q^{[1]}d_1 + qa_1 = 0,\ 2ic_0 - 2r^{[1]}a_1 + rd_1 = 0,\\
&\lambda^3 : -r^{[1]}q^{[1]}a_1 i - 2q^{[1]}c_0 + a_1 rqi + 2rb_0 = 0,\ qa_0 - q^{[1]}d_0 = 0,\\
&\quad rd_0 - r^{[1]}a_0 = 0, -d_1 rqi + r^{[1]}q^{[1]}d_1 i + 2qc_0 - 2r^{[1]}b_0 = 0,\\
&\lambda^2 : a_0 rq - a_0 r^{[1]}q^{[1]} = 0,\ a_1 rq^2 - r^{[1]}{q^{[1]}}^2 d_1 - b_0 rqi + q^{[1]}_x d_1 i - a_1 q_x i - r^{[1]}q^{[1]}b_0 i = 0,\\
&\quad c_0 rqi - {r^{[1]}}^2 q^{[1]}a_1 + d_1 r^2 q + r^{[1]}q^{[1]}c_0 i + d_1 r_x i - r^{[1]}{}_x a_1 i = 0,\ r^{[1]}q^{[1]}d_0 - rqd_0 = 0,\\
&\lambda^1 : a_{1t} + q^{[1]}{}_x c_0 i + b_0 r^2 q - r^{[1]}{q^{[1]}}^2 c_0 + b_0 r_x i = 0,\ -r^{[1]}{q^{[1]}}^2 d_0 + a_0 rq^2 + q^{[1]}{}_x d_0 i - a_0 q_x i = 0,
\end{aligned}$$

$$d_0 r_x i + d_0 r^2 q - {r^{[1]}}^2 q^{[1]} a_0 - {r^{[1]}}_x a_0 i = 0,\ d_{1t} - c_0 q_x i + c_0 r q^2 - {r^{[1]}}^2 q^{[1]} b_0 - {r^{[1]}}_x b_0 i = 0,$$
$$\lambda^0 : a_{0t} = b_{0t} = c_{0t} = d_{0t} = 0. \tag{16}$$

The last equation shows a_0, b_0, c_0, d_0 are functions of x only. So a_0, b_0, c_0, d_0 are constants.

In order to get the non-trivial solutions, we present a Darboux transformation under the condition $a_0 = 0, d_0 = 0$. Based on eq.(14) and eq.(16) and without losing any generality, let Darboux matrix T be the form of

$$T_1 = T_1(\lambda;\lambda_1) = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}\lambda + \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix}. \tag{17}$$

Here a_1, d_1 are undetermined function of $(x,\ t)$, which will be expressed by the eigenfunction associated with λ_1 in the KN spectral problem. First of all, we introduce n eigenfunctions ψ_j as

$$\psi_j = \begin{pmatrix} \phi_j \\ \varphi_j \end{pmatrix},\quad j = 1, 2, ...n, \phi_j = \phi_j(x,t,\lambda_j),\ \varphi_j = \varphi_j(x,t,\lambda_j). \tag{18}$$

Theorem 1. *The elements of one-fold DT are parameterized by the eigenfunction ψ_1 associated with λ_1 as*

$$d_1 = \frac{1}{a_1},\quad a_1 = -\frac{\varphi_1}{\phi_1},\quad b_0 = c_0 = \lambda_1, \tag{19}$$

$$\Leftrightarrow T_1(\lambda;\lambda_1) = \begin{pmatrix} -\lambda\dfrac{\varphi_1}{\phi_1} & \lambda_1 \\ \lambda_1 & -\lambda\dfrac{\phi_1}{\varphi_1} \end{pmatrix}, \tag{20}$$

and then the new solutions $q^{[1]}$ and $r^{[1]}$ are given by

$$q^{[1]} = (\frac{\varphi_1}{\phi_1})^2 q + 2i\frac{\varphi_1}{\phi_1}\lambda_1, r^{[1]} = (\frac{\phi_1}{\varphi_1})^2 r - 2i\frac{\phi_1}{\varphi_1}\lambda_1, \tag{21}$$

and the new eigenfunction $\psi_j^{[1]}$ corresponding to λ_j is

$$\psi_j^{[1]} = \begin{pmatrix} \dfrac{1}{\phi_1}\begin{vmatrix} -\lambda_j\phi_j & \varphi_j \\ -\lambda_1\phi_1 & \varphi_1 \end{vmatrix} \\ \\ \dfrac{1}{\varphi_1}\begin{vmatrix} -\lambda_j\varphi_j & \phi_j \\ -\lambda_1\varphi_1 & \phi_1 \end{vmatrix} \end{pmatrix}. \tag{22}$$

Proof. Note that $(a_1 d_1)_x = 0$ is derived from the eq.(14), and then take $a_1 = \dfrac{1}{d_1}$ in the followings. By transformation eq.(17) and eq.(14), new solutions are given by

$$q^{[1]} = \frac{a_1}{d_1} q - 2\ i\ \frac{b_0}{d_1}, r^{[1]} = \frac{d_1}{a_1} q + 2\ i\ \frac{c_0}{a_1}. \tag{23}$$

By using a general fact of the DT, i.e., $T_1(\lambda;\lambda_1)|_{\lambda=\lambda_1}\psi_1 = 0$, then eq.(19) is obtained. Next, substituting (a_1, d_1, b_0, c_0) given in eq.(19) back into eq.(23), then new solutions are given as eq. (21). Further, by using the explicit matrix representation eq.(20) of T_1, then $\psi_j^{[1]}$ is given by

$$\psi_j^{[1]} = T_1(\lambda;\lambda_1)|_{\lambda=\lambda_j}\psi_j = \left(\begin{array}{cc} -\lambda\dfrac{\varphi_1}{\phi_1} & \lambda_1 \\ \lambda_1 & -\lambda\dfrac{\phi_1}{\varphi_1} \end{array}\right)\Bigg|_{\lambda=\lambda_j} \left(\begin{array}{c} \phi_j \\ \varphi_j \end{array}\right) = \left(\begin{array}{c} \dfrac{1}{\phi_1}\left|\begin{array}{cc} -\lambda_j\phi_j & \varphi_j \\ -\lambda_1\phi_1 & \varphi_1 \end{array}\right| \\ \\ \dfrac{1}{\varphi_1}\left|\begin{array}{cc} -\lambda_j\varphi_j & \phi_j \\ -\lambda_1\varphi_1 & \phi_1 \end{array}\right| \end{array}\right). \quad (24)$$

Last, a tedious calculation shows that T_1 in eq.(20) and new solutions indeed satisfy eq.(15) or (equivalently eq.(16)). So KN spectral problem is covariant under transformation T_1 in eq.(20) and eq.(21), and thus it is the DT of eq.(4) and eq.(5). □

It is easy to find that T_1 is equivalent to the Imai's result(see eq.(7) of ref. [24]) and to the Steudel's result(see eq.(21) of ref. [25]). Our derivation is more transparent, and new solutions $q^{[1]}$ and $r^{[1]}$ can be constructed by the eigenfunction ψ_1, which is a solution of linear partial different equations eq.(4) and eq.(5). This is simpler than Steudel's method to solve nonlinear Riccati equations. The remaining problem is how to guarantee the validity of the reduction condition, i.e., $q^{[1]} = -(r^{[1]})^*$. We shall solve it at the end of this section by choosing special eigenfunctions and eigenvalues.

2.2 N-fold Darboux transformation for KN system

The key task is to establish the determinant representation of the n-fold DT for KN system in this subsection. To this purpose, set

$$\mathbf{D} = \left\{\left(\begin{array}{cc} a & 0 \\ 0 & d \end{array}\right)\Bigg|\, a, d \text{ are complex functions of } x \text{ and } t\right\},$$
$$\mathbf{A} = \left\{\left(\begin{array}{cc} 0 & b \\ c & 0 \end{array}\right)\Bigg|\, b, c \text{ are complex functions of } x \text{ and } t\right\},$$

as ref. [24].

According to the form of T_1 in eq.(17), the n-fold DT should be the form of [24]

$$T_n = T_n(\lambda;\lambda_1,\lambda_2,\cdots,\lambda_n) = \sum_{l=0}^{n} P_l\lambda^l, \quad (25)$$

with

$$P_n = \begin{pmatrix} a_n & 0 \\ 0 & d_n \end{pmatrix} \in \mathbf{D},\ P_{n-1} = \begin{pmatrix} 0 & b_{n-1} \\ c_{n-1} & 0 \end{pmatrix} \in \mathbf{A},\ P_l \in \mathbf{D}\,(\text{if } l-n \text{ is even}),\ P_l \in \mathbf{A}\,(\text{if } l-n \text{ is odd}). \quad (26)$$

Here P_0 is a constant matrix, $P_i (1 \leq i \leq n)$ is the function of x and t. In particular, $P_0 \in \mathbf{D}$ if n is even and $P_0 \in \mathbf{A}$ if n is odd, which leads to the separate discussion on the determinant representation of T_n in the following by means of its kernel. Specifically, from algebraic equations,

$$\psi_k^{[n]} = T_n(\lambda;\lambda_1,\cdots,\lambda_n)|_{\lambda=\lambda_k}\psi_k = \sum_{l=0}^{n} P_l\lambda_k^l\psi_k = 0, k = 1,2,\cdots,n, \quad (27)$$

coefficients P_i is solved by Cramer's rule. Thus we get determinant representation of the T_n.

Theorem2. (1)For $n = 2k(k = 1, 2, 3, \cdots)$, the n-fold DT of the KN system can be expressed by

$$T_n = T_n(\lambda; \lambda_1, \lambda_2, \cdots, \lambda_n) = \begin{pmatrix} \dfrac{\widetilde{(T_n)_{11}}}{W_n} & \dfrac{\widetilde{(T_n)_{12}}}{W_n} \\ \\ \dfrac{\widetilde{(T_n)_{21}}}{\widetilde{W_n}} & \dfrac{\widetilde{(T_n)_{22}}}{\widetilde{W_n}} \end{pmatrix}, \tag{28}$$

with

$$W_n = \begin{vmatrix} \lambda_1^n\phi_1 & \lambda_1^{n-1}\varphi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \ldots & \lambda_1^2\phi_1 & \lambda_1\varphi_1 \\ \lambda_2^n\phi_2 & \lambda_2^{n-1}\varphi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \ldots & \lambda_2^2\phi_2 & \lambda_2\varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^n\phi_n & \lambda_n^{n-1}\varphi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \ldots & \lambda_n^2\phi_n & \lambda_n\varphi_n \end{vmatrix},$$

$$\widetilde{(T_n)_{11}} = \begin{vmatrix} \lambda^n & 0 & \lambda^{n-2} & 0 & \ldots & \lambda^2 & 0 & \lambda_1\lambda_2\ldots\lambda_n \\ \lambda_1^n\phi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \ldots & \lambda_1^2\phi_1 & \lambda_1\varphi_1 & \lambda_1\lambda_2\ldots\lambda_n\phi_1 \\ \lambda_2^n\phi_2 & \lambda_2^{n-1}\varphi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \ldots & \lambda_2^2\phi_2 & \lambda_2\varphi_2 & \lambda_1\lambda_2\ldots\lambda_n\phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^n\phi_n & \lambda_n^{n-1}\varphi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \ldots & \lambda_n^2\phi_n & \lambda_n\varphi_n & \lambda_1\lambda_2\ldots\lambda_n\phi_1 \end{vmatrix},$$

$$\widetilde{(T_n)_{12}} = \begin{vmatrix} 0 & \lambda^{n-1} & 0 & \lambda^{n-3} & \ldots & 0 & \lambda & 0 \\ \lambda_1^n\phi_1 & \lambda_1^{n-1}\varphi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \ldots & \lambda_1^2\phi_1 & \lambda_1\varphi_1 & \lambda_1\lambda_2\ldots\lambda_n\phi_1 \\ \lambda_2^n\phi_2 & \lambda_2^{n-1}\varphi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \ldots & \lambda_2^2\phi_2 & \lambda_2\varphi_2 & \lambda_1\lambda_2\ldots\lambda_n\phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^n\phi_n & \lambda_n^{n-1}\varphi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \ldots & \lambda_n^2\phi_n & \lambda_n\varphi_n & \lambda_1\lambda_2\ldots\lambda_n\phi_1 \end{vmatrix},$$

$$\widetilde{W_n} = \begin{vmatrix} \lambda_1^n\varphi_1 & \lambda_1^{n-1}\phi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \ldots & \lambda_1^2\varphi_1 & \lambda_1\phi_1 \\ \lambda_2^n\varphi_2 & \lambda_2^{n-1}\phi_2 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \ldots & \lambda_2^2\varphi_2 & \lambda_2\phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^n\varphi_n & \lambda_n^{n-1}\phi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \ldots & \lambda_n^2\varphi_n & \lambda_n\phi_n \end{vmatrix},$$

$$\widetilde{(T_n)_{21}} = \begin{vmatrix} 0 & \lambda^{n-1} & 0 & \lambda^{n-3} & \ldots & 0 & \lambda & 0 \\ \lambda_1^n\varphi_1 & \lambda_1^{n-1}\phi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \ldots & \lambda_1^2\varphi_1 & \lambda_1\phi_1 & \lambda_1\lambda_2\ldots\lambda_n\varphi_1 \\ \lambda_2^n\varphi_2 & \lambda_2^{n-1}\phi_2 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \ldots & \lambda_2^2\varphi_2 & \lambda_2\phi_2 & \lambda_1\lambda_2\ldots\lambda_n\varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^n\varphi_n & \lambda_n^{n-1}\phi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \ldots & \lambda_n^2\varphi_n & \lambda_n\phi_n & \lambda_1\lambda_2\ldots\lambda_n\varphi_1 \end{vmatrix},$$

$$\widetilde{(T_n)_{22}} = \begin{vmatrix} \lambda^n & 0 & \lambda^{n-2} & 0 & \ldots & \lambda^2 & 0 & \lambda_1\lambda_2\ldots\lambda_n \\ \lambda_1^n\varphi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \ldots & \lambda_1^2\varphi_1 & \lambda_1\phi_1 & \lambda_1\lambda_2\ldots\lambda_n\varphi_1 \\ \lambda_2^n\varphi_2 & \lambda_2^{n-1}\phi_2 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \ldots & \lambda_2^2\varphi_2 & \lambda_2\phi_2 & \lambda_1\lambda_2\ldots\lambda_n\varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^n\varphi_n & \lambda_n^{n-1}\phi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \ldots & \lambda_n^2\varphi_n & \lambda_n\phi_n & \lambda_1\lambda_2\ldots\lambda_n\varphi_1 \end{vmatrix}.$$

(2)For $n=2k+1(k=1,2,3,\cdots)$, then

$$T_n=T_n(\lambda;\lambda_1,\lambda_2,\cdots,\lambda_n)=\begin{pmatrix}\dfrac{\widehat{(T_n)_{11}}}{Q_n} & \dfrac{\widehat{(T_n)_{12}}}{Q_n}\\ \\ \dfrac{\widehat{(T_n)_{21}}}{\widehat{Q_n}} & \dfrac{\widehat{(T_n)_{22}}}{\widehat{Q_n}}\end{pmatrix}, \tag{29}$$

with

$$Q_n=\begin{vmatrix}\lambda_1^n\phi_1 & \lambda_1^{n-1}\varphi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \ldots & \lambda_1^3\phi_1 & \lambda_1^2\varphi_1 & \lambda_1\phi_1\\ \lambda_2^n\phi_2 & \lambda_2^{n-1}\varphi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \ldots & \lambda_2^3\phi_2 & \lambda_2^2\varphi_2 & \lambda_2\phi_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^n\phi_n & \lambda_n^{n-1}\varphi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \ldots & \lambda_n^3\phi_n & \lambda_n^2\varphi_n & \lambda_n\phi_n\end{vmatrix},$$

$$\widehat{(T_n)_{11}}=\begin{vmatrix}\lambda^n & 0 & \lambda^{n-2} & 0 & \ldots & \lambda^3 & 0 & \lambda & 0\\ \lambda_1^n\phi_1 & \lambda_1^{n-1}\varphi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \ldots & \lambda_1^3\phi_1 & \lambda_1^2\varphi_1 & \lambda_1\phi_1 & -\lambda_1\lambda_2\ldots\lambda_n\varphi_1\\ \lambda_2^n\phi_2 & \lambda_2^{n-1}\varphi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \ldots & \lambda_2^3\phi_2 & \lambda_2^2\varphi_2 & \lambda_2\phi_2 & -\lambda_1\lambda_2\ldots\lambda_n\varphi_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^n\phi_n & \lambda_n^{n-1}\varphi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \ldots & \lambda_n^3\phi_n & \lambda_n^2\varphi_n & \lambda_n\phi_n & -\lambda_1\lambda_2\ldots\lambda_n\varphi_n\end{vmatrix},$$

$$\widehat{(T_n)_{12}}=\begin{vmatrix}0 & \lambda^{n-1} & 0 & \lambda^{n-3} & ... & 0 & \lambda^2 & 0 & -\lambda_1\lambda_2\ldots\lambda_n\\ \lambda_1^n\phi_1 & \lambda_1^{n-1}\varphi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \ldots & \lambda_1^3\phi_1 & \lambda_1^2\varphi_1 & \lambda_1\phi_1 & -\lambda_1\lambda_2\ldots\lambda_n\varphi_1\\ \lambda_2^n\phi_2 & \lambda_2^{n-1}\varphi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \ldots & \lambda_2^3\phi_2 & \lambda_2^2\varphi_2 & \lambda_2\phi_2 & -\lambda_1\lambda_2\ldots\lambda_n\varphi_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^n\phi_n & \lambda_n^{n-1}\varphi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \ldots & \lambda_n^3\phi_n & \lambda_n^2\varphi_n & \lambda_n\phi_n & -\lambda_1\lambda_2\ldots\lambda_n\varphi_n\end{vmatrix},$$

$$\widehat{Q_n}=\begin{vmatrix}\lambda_1^n\varphi_1 & \lambda_1^{n-1}\phi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \ldots & \lambda_1^3\varphi_1 & \lambda_1^2\phi_1 & \lambda_1\varphi_1\\ \lambda_2^n\varphi_2 & \lambda_2^{n-1}\phi_2 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \ldots & \lambda_2^3\varphi_2 & \lambda_2^2\phi_2 & \lambda_2\varphi_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^n\varphi_n & \lambda_n^{n-1}\phi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \ldots & \lambda_n^3\varphi_n & \lambda_n^2\phi_n & \lambda_n\varphi_n\end{vmatrix},$$

$$\widehat{(T_n)_{21}}=\begin{vmatrix}0 & \lambda^{n-1} & 0 & \lambda^{n-3} & ... & 0 & \lambda^2 & 0 & -\lambda_1\lambda_2\ldots\lambda_n\\ \lambda_1^n\varphi_1 & \lambda_1^{n-1}\phi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \ldots & \lambda_1^3\varphi_1 & \lambda_1^2\phi_1 & \lambda_1\varphi_1 & -\lambda_1\lambda_2\ldots\lambda_n\phi_1\\ \lambda_2^n\varphi_2 & \lambda_2^{n-1}\phi_2 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \ldots & \lambda_2^3\varphi_2 & \lambda_2^2\phi_2 & \lambda_2\varphi_2 & -\lambda_1\lambda_2\ldots\lambda_n\phi_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^n\varphi_n & \lambda_n^{n-1}\phi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \ldots & \lambda_n^3\varphi_n & \lambda_n^2\phi_n & \lambda_n\varphi_n & -\lambda_1\lambda_2\ldots\lambda_n\phi_n\end{vmatrix},$$

$$\widehat{(T_n)_{22}}=\begin{vmatrix}\lambda^n & 0 & \lambda^{n-2} & 0 & \ldots & \lambda^3 & 0 & \lambda & 0\\ \lambda_1^n\varphi_1 & \lambda_1^{n-1}\phi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \ldots & \lambda_1^3\varphi_1 & \lambda_1^2\phi_1 & \lambda_1\varphi_1 & -\lambda_1\lambda_2\ldots\lambda_n\phi_1\\ \lambda_2^n\varphi_2 & \lambda_2^{n-1}\phi_2 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \ldots & \lambda_2^3\varphi_2 & \lambda_2^2\phi_2 & \lambda_2\varphi_2 & -\lambda_1\lambda_2\ldots\lambda_n\phi_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^n\varphi_n & \lambda_n^{n-1}\phi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \ldots & \lambda_n^3\varphi_n & \lambda_n^2\phi_n & \lambda_n\varphi_n & -\lambda_1\lambda_2\ldots\lambda_n\phi_n\end{vmatrix}. \tag{30}$$

Next, we consider the transformed new solutions $(q^{[n]}, r^{[n]})$of KN system corresponding to the n-fold DT. Under covariant requirement of spectral problem of the KN system, the transformed form should be

$$\partial_x\psi^{[n]}=(J\lambda^2+Q^{[n]}\lambda)\psi=U^{[n]}\psi, \tag{31}$$

with

$$\psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Q^{[n]} = \begin{pmatrix} 0 & q^{[n]} \\ r^{[n]} & 0 \end{pmatrix}, \tag{32}$$

and then

$$T_{nx} + T_n\ U = U^{[n]}\ T_n. \tag{33}$$

Substituting T_n given by eq.(25) into eq.(33),and then comparing the coefficients of λ^{n+1}, it yields

$$q^{[n]} = \frac{a_n}{d_n}q - 2i\frac{b_{n-1}}{d_n}, \quad r^{[n]} = \frac{d_n}{a_n}r + 2i\frac{c_{n-1}}{a_n}. \tag{34}$$

Furthermore, taking $a_n, d_n, b_{n-1}, c_{n-1}$ which are obtained from eq.(28) for $n = 2k$ and from eq.(29) for $n = 2k+1$, into (34), then new solutions $(q^{[n]}, r^{[n]})$ are given by

$$q^{[n]} = \frac{\Omega_{11}^2}{\Omega_{21}^2}q + 2i\frac{\Omega_{11}\Omega_{12}}{\Omega_{21}^2}, \quad r^{[n]} = \frac{\Omega_{21}^2}{\Omega_{11}^2}r - 2i\frac{\Omega_{21}\Omega_{22}}{\Omega_{11}^2}. \tag{35}$$

Here, (1)for $n = 2k$,

$$\Omega_{11} = \begin{vmatrix} \lambda_1^{n-1}\varphi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \dots & \lambda_1\varphi_1 & \phi_1 \\ \lambda_2^{n-1}\varphi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \dots & \lambda_2\varphi_2 & \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1}\varphi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \dots & \lambda_n\varphi_n & \phi_n \end{vmatrix}, \tag{36}$$

$$\Omega_{12} = \begin{vmatrix} \lambda_1^{n}\phi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \dots & \lambda_1\varphi_1 & \phi_1 \\ \lambda_2^{n}\phi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \dots & \lambda_2\varphi_2 & \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n}\phi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \dots & \lambda_n\varphi_n & \phi_n \end{vmatrix},$$

$$\Omega_{21} = \begin{vmatrix} \lambda_1^{n-1}\phi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \dots & \lambda_1\phi_1 & \varphi_1 \\ \lambda_2^{n-1}\phi_2 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \dots & \lambda_2\phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1}\phi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \dots & \lambda_n\phi_n & \varphi_n \end{vmatrix},$$

$$\Omega_{22} = \begin{vmatrix} \lambda_1^{n}\varphi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \dots & \lambda_1\phi_1 & \varphi_1 \\ \lambda_1^{n}\varphi_1 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \dots & \lambda_2\phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n}\varphi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \dots & \lambda_n\phi_n & \varphi_n \end{vmatrix};$$

(2) for $n = 2k+1$,

$$\Omega_{11} = \begin{vmatrix} \lambda_1^{n-1}\varphi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \dots & \lambda_1\phi_1 & \varphi_1 \\ \lambda_2^{n-1}\varphi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \dots & \lambda_2\phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1}\varphi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \dots & \lambda_n\phi_n & \varphi_n \end{vmatrix}, \tag{37}$$

$$\Omega_{12} = \begin{vmatrix} \lambda_1^{n}\phi_1 & \lambda_1^{n-2}\phi_1 & \lambda_1^{n-3}\varphi_1 & \dots & \lambda_1\phi_1 & \varphi_1 \\ \lambda_2^{n}\phi_2 & \lambda_2^{n-2}\phi_2 & \lambda_2^{n-3}\varphi_2 & \dots & \lambda_2\phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n}\phi_n & \lambda_n^{n-2}\phi_n & \lambda_n^{n-3}\varphi_n & \dots & \lambda_n\phi_n & \varphi_n \end{vmatrix},$$

$$\Omega_{21}=\begin{vmatrix}\lambda_1^{n-1}\phi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \dots & \lambda_1\varphi_1 & \phi_1\\ \lambda_2^{n-1}\phi_2 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \dots & \lambda_2\varphi_2 & \phi_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^{n-1}\phi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \dots & \lambda_n\varphi_n & \phi_n\end{vmatrix},$$

$$\Omega_{22}=\begin{vmatrix}\lambda_1^{n}\varphi_1 & \lambda_1^{n-2}\varphi_1 & \lambda_1^{n-3}\phi_1 & \dots & \lambda_1\varphi_1 & \phi_1\\ \lambda_1^{n}\varphi_1 & \lambda_2^{n-2}\varphi_2 & \lambda_2^{n-3}\phi_2 & \dots & \lambda_2\varphi_2 & \phi_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \lambda_n^{n}\varphi_n & \lambda_n^{n-2}\varphi_n & \lambda_n^{n-3}\phi_n & \dots & \lambda_n\varphi_n & \phi_n\end{vmatrix}.$$

We are now in a position to consider the reduction of the DT of the KN system so that $q^{[n]}=-(r^{[n]})^*$, then the DT of the DNLS is given. Under the reduction condition $q=-r^*$, the eigenfunction $\psi_k=\begin{pmatrix}\phi_k\\ \varphi_k\end{pmatrix}$ associated with eigenvalue λ_k has following properties [24],

(i): $\phi_k^*=\varphi_k$, $\lambda_k=-\lambda_k{}^*$;
(ii): $\phi_k{}^*=\varphi_l$, $\varphi_k{}^*=\phi_l$, $\lambda_k{}^*=-\lambda_l$, where $k\neq l$.

Notice that the denominator W_n of $q^{[n]}$ is a modulus of a non-zero complex function under reduction condition, so the new solution $q^{[n]}$ is non-singular. For the one-fold DT T_1, set

$$\lambda_1=i\beta_1(\text{a pure imaginary constant}),\quad \text{and its eigenfunction}\quad \psi_1=\begin{pmatrix}\phi_1\\ \phi_1^*\end{pmatrix},\tag{38}$$

then T_1 in theorem 1 is the DT of the DNLS. We note that $q^{[1]}=-(r^{[1]})^*$ holds with the help of eq.(21), $q=-r^*$ and this special choice of ψ_1. This is an essential distinctness of DT between DNLS and NLS, because one-fold transformation of AKNS can not preserve the reduction condition to the NLS. Furthermore, for the two-fold DT, according to above property (ii), set

$$\lambda_2=-\lambda_1^*\quad\text{and its eigenfunction } \psi_2=\begin{pmatrix}\varphi_1^*\\ \phi_1^*\end{pmatrix},\psi_1=\begin{pmatrix}\phi_1\\ \varphi_1\end{pmatrix}\text{associated with eigenvalue}\quad\lambda_1,\tag{39}$$

then $q^{[2]}=-(r^{[2]})^*$ can be verified from eq. (35) and T_2 given by eq.(28) is the DT of the DNLS. Of course, in order to get $q^{[2]}=-(r^{[2]})^*$ so that T_2 becomes also the DT of the DNLS, we can also set

$$\lambda_l=i\beta_l(\text{pure imaginary})\text{ and its eigenfunction }\psi_l=\begin{pmatrix}\varphi_l^*\\ \phi_l^*\end{pmatrix},l=1,2.\tag{40}$$

There are many choices to guarantee $q^{[n]}=-(r^{[n]})^*$ for the n-fold DTs when $n>2$. For example, setting $n=2k$ and $l=1,3,\dots,2k-1$,then choosing following k distinct eigenvalues and eigenfunctions in n-fold DTs:

$$\lambda_l\leftrightarrow\psi_l=\begin{pmatrix}\phi_l\\ \varphi_l\end{pmatrix},\text{and}\lambda_{2l}=-\lambda_{2l-1}^*,\leftrightarrow\psi_{2l}=\begin{pmatrix}\varphi_{2l-1}^*\\ \phi_{2l-1}^*\end{pmatrix}\tag{41}$$

so that $q^{[2k]}=-(r^{[2k]})^*$ in eq.(35). Then T_{2k} with these paired-eigenvalue λ_i and paired-eigenfunctions $\psi_i(i=1,3,\dots,2k-1)$ is reduced to the (2k)-fold DT of the DNLS. Similarly, T_{2k+1} in eq.(29) can also be reduced to the (2k+1)-fold DT of the DNLS by choosing one pure imaginary $\lambda_{2k+1}=i\beta_{2k+1}$(pure imaginary) and k paired-eigenvalues $\lambda_{2l}=-\lambda_{2l-1}^*(l=1,2,\cdots,k)$ with corresponding eigenfunctions according to properties (i) and (ii).

3. Particular solutions

3.1. Darboux transformations applied to zero seed

For $q=r=0$ the equations (6) and (7) are solved by

$$\psi_k=\left(\begin{array}{c}\phi_k\\ \varphi_k\end{array}\right),\quad \phi_k=\exp(i({\lambda_k}^2x+2{\lambda_k}^4t)),\qquad \varphi_k=\exp(-i({\lambda_k}^2x+2{\lambda_k}^4t)). \tag{42}$$

Case 1(N = 1). Under the choice eq.(38), taking ψ_1 in eq.(42) back into eq.(21) with $\lambda_1=i\beta_1$, then one solution of the DNLS is

$$q^{[1]}=-2\beta_1\exp(-2i(-{\beta_1}^2x+2{\beta_1}^4t)), \tag{43}$$

which is not a soliton but a periodic solution with a constant amplitude.

Case 2(N=2). Considering the choice in eq.(40) with $\lambda_1=i(l+m)$, $\lambda_2=i(l-m)$, and taking eigenfunctions in eq. (42) back into T_2,the result of the DT of the DNLS is then simply found from (35),

$$q^{[2]}=-4lm\frac{(m\cos(2G)-il\sin(2G))^3}{((m^2-l^2)\cos(2G)^2+l^2)^2}\exp(2iF), \tag{44}$$

which is a quasi-periodic solution,and here $F=-l^2x+2l^4t+12l^2m^2t-m^2x+2m^4t$, $G=8l^3mt-2lmx+8lm^3t$. Furthermore, considering the choice in eq.(39) with $\lambda_1=\alpha_1+i\beta_1$, $\lambda_2=-\alpha_1+i\beta_1$, and using eigenfunctions in eq. (42), then the solution of the DNLS generated by two-fold DT is simply found from (35),

$$q^{[2]}=4i\alpha\beta\frac{(-i\alpha_1\cosh(2\Gamma)+\beta_1\sinh(2\Gamma))^3}{((-{\alpha_1}^2-{\beta_1}^2)\cosh(2\Gamma)^2+{\beta_1}^2)^2}\exp(2ih), \tag{45}$$

with $h=-{\beta_1}^2x+2{\beta_1}^4t-12{\alpha_1}^2{\beta_1}^2t+{\alpha_1}^2x+2{\alpha_1}^4t$, $\Gamma=-8\alpha_1{\beta_1}^3t+2\alpha_1\beta_1x+8{\alpha_1}^3\beta_1t$. By letting $\alpha_1\to0$ in(45), it becomes a rational solution

$$q^{[2]}=4\beta_1\exp\left(2i{\beta_1}^2(-x+2{\beta_1}^2t)\right)\frac{(4i{\beta_1}^2(4{\beta_1}^2t-x)-1)^3}{(16{\beta_1}^4(4{\beta_1}^2t-x)^2+1)^2}, \tag{46}$$

with an arbitrary real constant β_1. Obviously, the rational solution is a linar soliton, and its trajectory is defined explicitly by

$$x=4{\beta_1}^2t, \tag{47}$$

on $(x-t)$ plane. The solutions $q^{[1]}$ and $q^{[2]}$ of the DNLS equation are consistent with the results of ref. [24, 25] except the rational solution. So the rational solution $q^{[2]}$ in eq.(46) of the DNLS equation is first found in this paper, which is plotted in Figure 1.

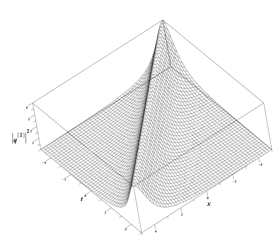

Figure 1. Rational solution $|q^{[2]}|^2$ of the DNLS with $\beta_1=0.5$.

3.2. Darboux transformations applied to non-zero seeds: constant solution and periodic solution.

Set a and c be two complex constants, and take $c>0$ without loss of generality, then $q=c\exp\left(i(ax+(-c^2+a)at)\right)$ is a periodic solution of the DNLS equation, which will be used as a seed solution of the DT. Substituting $q=c\exp\left(i(ax+(-c^2+a)at)\right)$ into the spectral problem eq.(6) and eq.(7), and using the method of separation of variables and the superposition principle, the eigenfunction ψ_k associated with λ_k is given by

$$\begin{pmatrix}\phi_k(x,t,\lambda_k)\\ \varphi_k(x,t,\lambda_k)\end{pmatrix}=\begin{pmatrix}\varpi1(x,t,\lambda_k)[1,k]+\varpi2(x,t,\lambda_k)[1,k]+\varpi1^*(x,t,-\lambda_k^*)[2,k]+\varpi2^*(x,t,-\lambda_k^*)[2,k]\\ \varpi1(x,t,\lambda_k)[2,k]+\varpi2(x,t,\lambda_k)[2,k]+\varpi1^*(x,t,-\lambda_k^*)[1,k]+\varpi2^*(x,t,-\lambda_k^*)[1,k]\end{pmatrix}. \quad (48)$$

Here

$$\begin{pmatrix}\varpi1(x,t,\lambda_k)[1,k]\\ \varpi1(x,t,\lambda_k)[2,k]\end{pmatrix}=\begin{pmatrix}\exp(\frac{\sqrt{s}(x+2{\lambda_k}^2t+(-c^2+a)t)}{2}+\frac{1}{2}(i(ax+(-c^2+a)at)))\\ \frac{ia-2i{\lambda_k}^2+\sqrt{s}}{2\lambda_k c}\exp(\frac{\sqrt{s}(x+2{\lambda_k}^2t+(-c^2+a)t)}{2}-\frac{1}{2}(i(ax+(-c^2+a)at)))\end{pmatrix},$$

$$\begin{pmatrix}\varpi2(x,t,\lambda_k)[1,k]\\ \varpi2(x,t,\lambda_k)[2,k]\end{pmatrix}=\begin{pmatrix}\exp(-\frac{\sqrt{s}(x+2{\lambda_k}^2t+(-c^2+a)t)}{2}+\frac{1}{2}(i(ax+(-c^2+a)at)))\\ \frac{ia-2i{\lambda_k}^2-\sqrt{s}}{2\lambda_k c}\exp(-\frac{\sqrt{s}(x+2{\lambda_k}^2t+(-c^2+a)t)}{2}-\frac{1}{2}(i(ax+(-c^2+a)at)))\end{pmatrix},$$

$$\varpi1(x,t,\lambda_k)=\begin{pmatrix}\varpi1(x,t,\lambda_k)[1,k]\\ \varpi1(x,t,\lambda_k)[2,k]\end{pmatrix},\ \varpi2(x,t,\lambda_k)=\begin{pmatrix}\varpi2(x,t,\lambda_k)[1,k]\\ \varpi2(x,t,\lambda_k)[2,k]\end{pmatrix},$$

$$s=-a^2-4{\lambda_k}^4-4{\lambda_k}^2(c^2-a).$$

Note that $\varpi1(x,t,\lambda_k)$ and $\varpi2(x,t,\lambda_k)$ are two different solutions of the spectral problem eq.(6) and eq.(7), but we can only get the trivial solutions through DT of the DNLS by setting eigenfunction ψ_k be one of them.

What is more ,we can get richer solutions by using (48).

Case 3($N=1$). Under choice in eq. (38) with ψ_1 given by eq.(48) and $\lambda_1=i\beta_1$, the one-fold DT of the DNLS generates

$$|q^{[1]}|^2=c^2-2a+\frac{2(2{\beta_1}^2+a)^2-8c^2{\beta_1}^2}{a+2{\beta_1}^2+2c\beta_1\cosh(K(x-2{\beta_1}^2t+at-c^2t))}, \quad (49)$$

with $K=\sqrt{4c^2{\beta_1}^2-(2{\beta_1}^2+a)^2}$, according to eq.(21). By letting $x\to\infty, t\to\infty$, so $|q^{[1]}|^2\to c^2-2a$. The trajectory is defined implicitly by

$$x-2{\beta_1}^2t+at-c^2t=0. \quad (50)$$

The $q^{[1]}$ in eq.(49) gives a soliton solutions if $4c^2{\beta_1}^2-(2{\beta_1}^2+a)^2>0$, and gives a periodic solution if $4c^2{\beta_1}^2-(2{\beta_1}^2+a)^2<0$. This classification is consistent with Steudel(see Figure 1 of ref. [25]). Further, we find that $q^{[1]}$ in eq.(49) can generate a dark soliton if $c^2-2a>(c-2\beta_1)^2$ and a bright solitons if $c^2-2a<(c-2\beta_1)^2$. Here

$$|q^{[1]}|^2_{extreme}=(c^2-2a)+\frac{2((2\beta_1^2+a)^2-4c^2\beta_1^2)}{a+2\beta_1c+\beta_1^2}=(2\beta_1-c)^2.$$

Note, $\delta = K^2$ has four roots of β_1 and $\delta_0 = (2\beta_1 - c)^2 - (c^2 - 2a)$ has two roots of β_1 in general. Combining the conditions of the bright/dark soliton and periodic solutions, a complete classification of the different solutions generated by one-fold DT is obtained in Table 1. The depth of the dark soliton is $2(-a + 2\beta_1 c - 2{\beta_1}^2)$ and the height of the bright soliton is $2(a - 2\beta_1 c + 2{\beta_1}^2)$. Particularly, for $a = 0$, the seed solution $q = c$ is a positive constant, and then the one fold DT of the DNLS generates a dark soliton under the condition $0 < \beta_1 < c$, the bright soliton under $-c < \beta_1 < 0$, a periodic solution under $\beta_1 < -c$ and $\beta_1 > c$. To illustrate the table, Figure 2 is plotted for the case of $c > 0$ and $a < 0$. Set $y_1 = (c - 2\beta_1)^2$, $y_2 = 4c^2{\beta_1}^2 - (2{\beta_1}^2 + a)^2 = \delta$, $y_3 = c^2 - 2a$ with specific parameters $a = -1.5$ $c = 0.8$. There are four roots of y_2, which are $(\beta_1)_1 > (\beta_1)_2 > (\beta_1)_3 > (\beta_1)_4$. Note the $(\beta_1)_2$ and $(\beta_1)_3$ are also the roots of $y_1 - y_3 = \delta_0$. We can see from Figure 2 that, $q^{[1]}$ in eq.(49) gives the bright soliton when $\beta_1 \in ((\beta_1)_4, (\beta_1)_3)$ because $y_2 > 0$ and $y_1 > y_3$, dark soliton when $\beta_1 \in ((\beta_1)_2, (\beta_1)_1)$ because $y_2 > 0$ and $y_1 < y_3$,periodic solutions for others three cases of β_1 because $y_2 < 0$.

TABLE 1. Classification of the solutions $q^{[1]}$ generated by one-fold DT in case 3 according to the intervals of the eigenvalue $\lambda_1 = i\beta_1$.

Classification of the solutions generated by one-fold DT			
zero seed	$c = 0$	$\forall \beta_1 \in \mathbb{R}$	periodic solution
constant seed	$a = 0, c > 0$	$0 < \beta_1 < c$	dark solitons
		$-c < \beta_1 < 0$	bright solitons
		β_1 belongs to other two intervals	periodic solutions
$\frac{c^2}{2}>a$ periodic seed	$a > 0, c > 0$	$\frac{1}{2}c - \frac{1}{2}\sqrt{c^2 - 2a} < \beta_1 < \frac{1}{2}c + \frac{1}{2}\sqrt{c^2 - 2a}$	dark solitons
		$-\frac{1}{2}c - \frac{1}{2}\sqrt{c^2 - 2a} < \beta_1 < -\frac{1}{2}c + \frac{1}{2}\sqrt{c^2 - 2a}$	bright solitons
		β_1 belongs to other three intervals	periodic solutions
	$a < 0, c > 0$	$-\frac{1}{2}c + \frac{1}{2}\sqrt{c^2 - 2a} < \beta_1 < \frac{1}{2}c + \frac{1}{2}\sqrt{c^2 - 2a}$	dark solitons
		$-\frac{1}{2}c - \frac{1}{2}\sqrt{c^2 - 2a} < \beta_1 < \frac{1}{2}c - \frac{1}{2}\sqrt{c^2 - 2a}$	bright solitons
		β_1 belongs to other three intervals	periodic solutions
$\frac{c^2}{2}\leq a$ periodic seed	$a > 0, c > 0$	$\forall \beta_1 \in \mathbb{R}$	periodic solutions

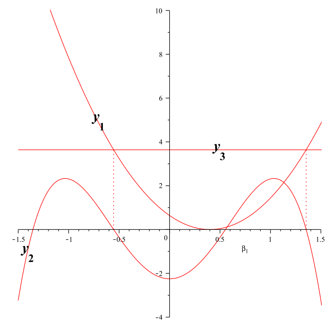

FIGURE 2. Intervals of β_1 in one-fold DT generate different solutions $q^{[1]}$ (dark soliton, bright soliton and periodic solution) under specific parameters $a=-1.5, c=0.8$. Here $y_1=(c-2\beta_1)^2$, $y_2=4c^2{\beta_1}^2-(2{\beta_1}^2+a)^2$, $y_3=c^2-2a$. There are five intervals of β_1 divided by the four roots of y_2. From the left to the right, the second interval and the fourth interval correspond to the bright soliton, dark soliton respectively. The other three intervals correspond to the periodic solutions.

Case 4.($N=2$). Under the choice in eq.(40) with $\lambda_1=i\beta_1, \lambda_2=i\beta_2, \beta_1\neq\beta_2$, the solution of the DNLS equation is generated by two-fold DT from(35) as

$$q^{[2]}=\frac{(\beta_1\phi_1^*\phi_2-\beta_2\phi_1\phi_2^*)^2}{(\beta_1\phi_1\phi_2^*-\beta_2\phi_1^*\phi_2)^2}q-2\frac{({\beta_1}^2-{\beta_2}^2)\phi_1\phi_2(\beta_1\phi_1^*\phi_2-\beta_2\phi_1\phi_2^*)}{(\beta_1\phi_1\phi_2^*-\beta_2\phi_1^*\phi_2)^2}, \tag{51}$$

where ϕ_1 and ϕ_2 are given by eq.(48). Similarly, under the choice in eq.(39) with one paired eigenvalue $\lambda_1=\alpha_1+i\beta_1$ and $\lambda_2=-\alpha_1+i\beta_1$, the two-fold DT eq.(35) of the DNLS equation implies a solution

$$q^{[2]}=\frac{(\lambda_1\varphi_1{\varphi_1}^*-\lambda_2\phi_1{\phi_1}^*)^2}{(-\lambda_2\varphi_1{\varphi_1}^*+\lambda_1\phi_1{\phi_1}^*)^2}q+2i\frac{({\lambda_1}^2-{\lambda_2}^2)\phi_1{\varphi_1}^*(\lambda_1\varphi_1{\varphi_1}^*-\lambda_2\phi_1{\phi_1}^*)}{(-\lambda_2\varphi_1{\varphi_1}^*+\lambda_1\phi_1{\phi_1}^*)^2}, \tag{52}$$

with ϕ_1 and φ_1 given by eq.(48). Two concrete examples of eq.(52) are given below.
(a)For simplicity, let $a=2{\alpha_1}^2-2{\beta_1}^2+c^2$ so that $\mathrm{Im}(-a^2-4{\lambda_1}^4-4{\lambda_1}^2(c^2-a))=0$, then

$$|q^{[2]}|^2=-16\alpha_1\beta_1\frac{w1\cosh(f1)\cos(f2)+w2sinh(f1)\sin(f2)+w3}{w4\cosh(f1)\cos(f2)+w5sinh(f1)\sin(f2)+w6\cos(2f2)+w7\cosh(2f1)+w8}+c^2, \tag{53}$$

$$\begin{aligned}
w1&=c\alpha_1(c^2-4{\beta_1}^2)(c^2+4{\alpha_1}^2),\\
w2&=-c\beta_1(c^2+4{\alpha_1}^2)(c^2-4{\beta_1}^2),\\
w3&=2\alpha_1\beta_1(c^2-4{\beta_1}^2)(4{\alpha_1}^2+c^2),\\
w4&=8c{\alpha_1}^2\beta_1(c^2+4{\alpha_1}^2),\\
w5&=-8c\alpha_1{\beta_1}^2(c^2-4{\beta_1}^2),\\
w6&=c^2{\alpha_1}^2(c^2+4{\alpha_1}^2)+c^2{\beta_1}^2(c^2-4{\beta_1}^2),\\
w7&=16{\alpha_1}^2{\beta_1}^2({\alpha_1}^2+{\beta_1}^2),\\
w8&=c^4({\alpha_1}^2-{\beta_1}^2)+16{\alpha_1}^2{\beta_1}^2({\alpha_1}^2-{\beta_1}^2)+4c^2({\alpha_1}^2+{\beta_1}^2)^2,\\
f1&=K1(4{\alpha_1}^2t-4{\beta_1}^2t+x),\\
f2&=4K1\alpha_1\beta_1t,\\
K1&=\sqrt{16{\alpha_1}^2{\beta_1}^2-4c^2{\alpha_1}^2+4c^2{\beta_1}^2-c^4}.
\end{aligned}$$

By letting $x \to \infty,\ t \to \infty$, so $|q^{[2]}|^2 \to c^2$, the trajectory of this solution is defined explicitly by

$$x = -4{\alpha_1}^2 t + 4{\beta_1}^2 t, \tag{54}$$

from $f_1 = 0$ if $K1^2 > 0$, and by

$$t = 0 \tag{55}$$

from $f_2 = 0$ if $(K1)^2 < 0$. According to eq.(53), we can get the Ma breathers [33](time periodic breather solution) and the Akhmediev breathers [34] (space periodic breather solution) solution. In general, the solution in eq.(53) evolves periodically along the straight line with a certain angle of x axis and t axis. The dynamical evolution of $|q^{[2]}|^2$ in eq.(53) for different parameters are plotted in Figure 3, Figure 4 and Figure 5, which give a visual verification of the three cases of trajectories. Inspired by the extensive research of rogue wave [4, 34] for the nonlinear Schrodinger equation, a limit procedure [34] is used to construct rogue wave of the DNLS equation in the following. By letting $c \to -2\beta_1$ in (52) with $\mathrm{Im}(-a^2 - 4{\lambda_1}^4 - 4{\lambda_1}^2(c^2 - a)) = 0$, it becomes rogue wave

$$q^{[2]}_{rogue\ wave} = \frac{r1r2r3}{r4r5} \tag{56}$$

$$\begin{aligned}
r1 &= 2exp(2i(\alpha_1^2 + \beta_1^2)(2t\alpha_1^2 + x - 2t\beta_1^2))\\
r2 &= \beta_1(16\beta_1^2\alpha_1^2(4t\alpha_1^2 + x)^2 + 16\beta_1^4(4t\beta_1^2 - x)^2 + 8i\beta_1^2(x + 4t\alpha_1^2 - 8t\beta_1^2) + 1)\\
r3 &= 2(16\beta_1^2\alpha_1^2(4t\alpha_1^2 + x)^2 + 16\beta_1^4(4t\beta_1^2 - x)^2 - 8\alpha_1\beta_1(x + 4t\alpha_1^2 - 8t\beta_1^2) + 1)\\
&\quad \times (-\alpha_1 + 16\beta_1(\beta_1^4 - \alpha_1^4)t - 4\beta_1(\alpha_1^2 + \beta_1^2)x + 16i\alpha_1\beta_1^2(\alpha_1^2 + \beta_1^2)t - i\beta_1)\\
&\quad - (16\beta_1^2\alpha_1^2(4t\alpha_1^2 + x)^2 + 16\beta_1^4(4t\beta_1^2 - x)^2 + 8i\beta_1^2(x + 4t\alpha_1^2 - 8t\beta_1^2) + 1)\\
&\quad \times (\alpha_1 + 16\beta_1(\beta_1^4 - \alpha_1^4)t - 4\beta_1(\alpha_1^2 + \beta_1^2)x + 16i\alpha_1\beta_1^2(\alpha_1^2 + \beta_1^2)t + \beta_1 i)\\
r4 &= \alpha_1 + 16\beta_1(\beta_1^4 - \alpha_1^4)t - 4\beta_1(\alpha_1^2 + \beta_1^2)x + 16i\alpha_1\beta_1^2(\alpha_1^2 + \beta_1^2)t + \beta_1 i\\
r5 &= (-16\beta_1^2\alpha_1^2(4t\alpha_1^2 + x)^2 - 16\beta_1^4(4t\beta_1^2 - x)^2 + 8i\beta_1^2(x + 4t\alpha_1^2 - 8t\beta_1^2) - 1)^2
\end{aligned}$$

By letting $x \to \infty,\ t \to \infty$, so $|q^{[2]}_{rogue\ wave}|^2 \to 4\beta_1^2$, the maximum amplitude of $|q^{[2]}_{rogue\ wave}|^2$ occurs at $t = 0$ and $x = 0$ and is equal to $36\beta_1^2$,and the minimum amplitude of $|q^{[2]}_{rogue\ wave}|^2$ occurs at $t = \pm\dfrac{3}{16\sqrt{3(4\alpha_1^2 + \beta_1^2)}\beta_1(\alpha_1^2 + \beta_1^2)}$ and $x = \mp\dfrac{9\alpha_1^2}{4\sqrt{3(4\alpha_1^2 + \beta_1^2)}\beta_1(\alpha_1^2 + \beta_1^2)}$ and is equal to 0. Through Figure 9 and Figure 10 of $|q^{[2]}_{rogue\ wave}|^2$, the main features(such as large amplitude and local property on (x-t) plane) of the rogue wave are shown. We have found that $|q^{[2]}|^2$ in eq.(53) gives the same result of $|q^{[2]}_{rogue\ wave}|^2$ by taking limit of $c \to -2\beta_1$.

(b)When $a = \dfrac{c^2}{2}$, from eq.(48), it is not difficult to find that there are two sets of collinear eigenfunctions,

$$\begin{pmatrix} \varpi1(x,t,\lambda_k)[1,k] \\ \varpi1(x,t,\lambda_k)[2,k] \end{pmatrix} \text{and} \begin{pmatrix} \varpi2^*(x,t,-\lambda_k^*)[2,k] \\ \varpi2^*(x,t,-\lambda_k^*)[1,k] \end{pmatrix}, \tag{57}$$

$$\begin{pmatrix} \varpi2(x,t,\lambda_k)[1,k] \\ \varpi2(x,t,\lambda_k)[2,k] \end{pmatrix} \text{and} \begin{pmatrix} \varpi1^*(x,t,-\lambda_k^*)[2,k] \\ \varpi1^*(x,t,-\lambda_k^*)[1,k] \end{pmatrix}. \tag{58}$$

Therefore, the eigenfunction ψ_k associated with λ_k for this case is given by

$$\begin{pmatrix} \phi_k(x,t,\lambda_k) \\ \varphi_k(x,t,\lambda_k) \end{pmatrix} = \begin{pmatrix} \varpi1(x,t,\lambda_k)[1,k] + \varpi1^*(x,t,-\lambda_k^*)[2,k] \\ \varpi1(x,t,\lambda_k)[2,k] + \varpi1^*(x,t,-\lambda_k^*)[1,k] \end{pmatrix}. \tag{59}$$

Here

$$\begin{pmatrix}\varpi 1(x,t,\lambda_k)[1,k]\\ \varpi 1(x,t,\lambda_k)[2,k]\end{pmatrix}=\begin{pmatrix}\exp(i({\lambda_k}^2x+2{\lambda_k}^4t+\dfrac{1}{2}c^2x-\dfrac{1}{4}c^4t))\\ \dfrac{ic}{2\lambda_k}\exp(i({\lambda_k}^2x+2{\lambda_k}^4t))\end{pmatrix}.$$

Under the choice in eq.(39) with $\lambda_1=\alpha_1+i\beta_1, \lambda_2=-\alpha_1+i\beta_1$, and the ψ_1 given by eq.(59), the solution $q^{[2]}$ is given simply from eq. (35). Figure 6 is plotted for $|q^{[2]}|^2$, which shows the periodical evolution along a straight line on $(x-t)$ plane.
Case 5.$(N=4)$. According to the choice in eq.(41) with two distinct eigenvalues $\lambda_1=\alpha_1+i\beta_1, \lambda_3=\alpha_3+i\beta_3$, substituting ψ_1 and ψ_3 defined by eq. (48) into eq.(35), then the new solution $q^{[4]}$ generated by 4-fold DT is given. Its analytical expression is omitted because it is very complicated. But $|q^{[4]}|^2$ are plotted in Figure 7 and 8 to show the dynamical evolution on $(x-t)$ plane: (a) Let $a=2{\alpha_i}^2-2{\beta_i}^2+c^2, i=1,3$, so that $\mathrm{Im}(-a^2-4{\lambda_i}^4-4{\lambda_i}^2(c^2-a))=0$, then Figure 7 shows intuitively that two breathers may have parallel trajectories;(b) Two breathers have an elastic collision so that they can preserve their profiles after interaction, which is verified in Figure 8.

4. Conclusions

In this paper, a detailed derivation of the DT from the KN system and then the determinant representation of the n-fold case are given in Theorem 1 and Theorem 2. Each element of n-fold DT matrix T_n is expressed by the determinant of eigenfunctions of the spectral problem in eq.(6) and eq.(7). The determinant representations of the new solution $q^{[n]}$ and $r^{[n]}$ of the KN system are also given in eq.(35). Further more, by the special choice of the eigenvalue λ_k and its eigenfunction ψ_k to construct T_n so that $q^{[n]}=-(r^{[n]})^*$, then the T_n is also reduced to the n-fold DT of the DNLS equation and $q^{[n]}$ is a solution of the DNLS. To illustrate our method, solutions of five specific cases are discussed by analytical formulae and figures. In particular, a complete classification of the solutions of the DNLS equation generated by one-fold DT is given in Table 1.

By comparing with known results [24, 25] of the DT for the DNLS equation, our results provide following improvements:

- A detailed derivation of the DT and the determinant representation of T_n. This representation is useful to compute the soliton surfaces of the DNLS equation in the future as we have done for the NLS equation [28]. The rogue wave and rational traveling wave are firstly given about the DNLS equation. The rational solution has been used by us in a separate preprint to construct the rouge wave of the variable coefficient DNLS equation [35].
- A complete and thorough classification of the solution generated by the one-fold DT. The bright soliton and dark soliton is also classified, which is not published before. At the same time, our results show the nonlinear and difficult Riccati equations in ref. [25],which are transformed from the linear equations of the spectral problem, and Seahorse functions are indeed avoidable. Of course, these do not disaffirm the merits of method in ref. [25].
- The general solution eq.(48) of the linear partial differential equations in spectral problem is crucial to get non-trivial solution of the DNLS equation.
- The solution in eq.(53) is a relatively general form of the breather solution of the DNLS, which can evolve periodically along any straight line on $(x-t)$ plane by choosing different

values of parameters α_1, β_1, c. It has two well-known reductions: Ma breather going periodically along t-axis, and Akhmediev breather going periodically along x-axis.

At last, we would like to mention the DT [36] of the DNLSIII. Unlike the DNLS equation, Fan's results show that the kernel of the one-fold DT of the DNLSIII is two dimensional,and then support again the necessity of the separate study of the three kinds of derivative nonlinear Schrödinger equation. So we shall consider the determinant representation of the DT for DNLSII and DNLSIII in the near future. Moreover, we are also interested in the periodic solutions with a variable amplitude of the DNLS equation.

Acknowledgments This work is supported by the NSF of China under Grant No.10971109 and K.C.Wong Magna Fund in Ningbo University. Jingsong He is also supported by Program for NCET under Grant No.NCET-08-0515. Shuwei Xu is also supported by the Scientific Research Foundation of Graduate School of Ningbo University. We thank Prof. Yishen Li(USTC,Hefei, China) for his useful suggestions on the rogue wave.

References

[1] Mjϕlhus, 1976, On the Modulational Instability of Hydromagnetic Waves Parallel to the Magnetic field, J.Plasma.Phys.16, 321-334

[2] S.P.Spangler,in:T.Hada and H.Matsumoto (Eds.), 1997, Nonlinear Waves and Chaos in Space Plasmas, Terrapub,Tokio,p.171

[3] M.S.Ruderman, 2002, DNLS Equation for Large-amplitude Solitons Propagating in an Arbitrary Direction in a High-β Hall plasma, J. Plasma. Phys. 67, 271-276

[4] V. Fedun, M.S.Ruderman and R. Erdélyi, 2008, Generation of Short-lived Large-amplitude Magnetohydrodynamic pulses by dispersive focusing, Phys. Lett.A.372, 6107-6110 ; M.S. Ruderman, 2010, Freak waves in Laboratory and Space plasmas Freak waves in plasmas, The European Physical Jour 185, 57-66

[5] G. Sanchez-Arriaga, J. R. Sanmartin and S. A. Elaskar,2007, Damping models in the truncated Derivative Nonlinear Schrödinger Equation, Physics of Plasmas 14, 082108; G. Sánchez-Arriaga, T. Hada and Y. Nariyuki, 2009, The truncation model of the Derivative Nonlinear Schrödinger Equation, Physics of Plasmas 16, 042302

[6] G. Sánchez-Arriaga, 2010, Alfven soliton and multisoliton dynamics perturbed by Nonlinear Landau damping, Physics of Plasmas 17, 082313

[7] N.Tzoar and M.Jain, 1981, Self-phase Modulation in long-geometry optical waveguide, Phys.Rev.A. 23, 1266-1270

[8] D. Anderson and M. Lisak,1983, Nonlinear Asymmetric Self-phase modulation and self-steepening of pulses in long Optical Waveguides, Phys. Rev. A. 27, 1393-1398

[9] P.A.Govind, Nonlinear Fibers Optics 3rd edn(New York: Adademic,2001)

[10] D. J. Kaup and A. C. Newell, 1978, An exact solution for a Derivative Nonlinear Schrödinger equation J.Math.Phys.19, 798-801

[11] T. Kawata, N. Kobayashi and H.Inoue, 1979, Soliton solution of the Derivative Nolinear Schrödinger equation, J. Phys. Soc. Japan 46,1008-1015

[12] Nian-Ning Huang and Zong-Yun Chen, 1990, Alfven solitons, J.Phys.A: Math.Gen. 23, 439-453

[13] Guo-Quan Zhou and Nian-Ning Huang, 2007, An N-soliton solution to the DNLS equation based on revised inverse scattering transform, J.Phys.A: Math.Theor.40, 13607C13623

[14] Huang Nian Ning, 2007, Marchenko Equation for the Derivative Nonlinear Schrödinger Equation, Chin. Phys. Lett. 24, 894-897

[15] He Jin-Chun, Chen Zong-Yun, 2008, Comment on Revision of Kaup Newell's Works on IST for DNLS Equation, Commun. Theor. Phys. (Beijing, China) 50, 1369-1374

[16] Yang Chun-Nuan, Yu Jia-Lu, Cai Hao and Huang Nian-Ning, 2008, Inverse Scattering Transform for the Derivative Nonlinear Schrödinger Equation, Chin. Phys. Lett. 25, 421-424.

[17] Y.H. Ichikawa and S. Watanabe, 1977, Solitons and envelope solitons in collissionless plasmas, J. Phys. (Paris) 38, C6-C15

[18] T. Kawata and H. Inoue, 1978, Exact solutions of the Derivative Nonlinear Schrödinger Equation under the Nonvanishing Conditions, J. Phys. Soc. Japan 44,1968-1976.

[19] Yoshi H.Ichikawa,Kimiaki Konno, Miki Wadati and Heiji Sanuki 1980, Spiky soliton in Circular Polarized Alfvén wave, J. Phys. Soc. Japan.48, 279-286

[20] Xiang-Jun Chen and Wa Kun Lam, 2004, Inverse scattering transform for the Derivative Nonlinear Schrödinger Equation with nonvanishing boundary conditions, Phys.Rev.E.69, 066604

[21] V.M. Lashkin, 2007, N-soliton solutions and perturbation theory for the Derivative Nonlinear Schrödinger Equation with nonvanishing boundary conditions, J. Phys. A: Math. Theor.40, 6119-6132

[22] Hao Cai and Nian-Ning Huang ,2006, The Hamiltonian formalism of the DNLS equation with a nonvanished boundary value, J.Phys.A: Math.Gen.39, 5007-5014

[23] Kakei S, Sasa N and Satsuma, 1995, Bilinearization of a Generalized Derivative Nonlinear Schrödinger equation, J. Phys. Soc. Japan.64, 1519-1526

[24] Kenji Imai, 1999, Generlization of Kaup-Newell Inverse Scattering Formulation and Darboux Transformation, J.Phys.Soc.Japan.68, 355-359

[25] H Steudel, 2003, The hierarchy of multi-soliton solutions of the Derivative Nonlinear Schrödinger Equation,J.Phys.A: Math.Gen.36, 1931C1946

[26] G.Neugebauer and R.Meinel, 1984, General N-soliton solution of the AKNS class on arbitrary background, Phys.Lett.A.100, 467C470.

[27] V. B. Matveev, M.A. Salle,1991, Darboux Transfromations and Solitons(Springer-Verlag, Berlin)

[28] J.S.He, L.Zhang, Y.Cheng and Y.S.Li, 2006, Determinant representation of Darboux transformation for the AKNS system, Science in China Series A: Mathematics.12, 1867C1878

[29] M. J. Ablowitz,D. J. Kaup, A. C. Newell and H. Segur 1973,Nonlinear Evolution Equations of Physical significance, Phys. Rev. Lett. 31, 125-127; M.J. Ablowitz and P. A. Clarkson, 1991, Solitons, Nonlinear evolution equations and Inverse scattering (Cambridge: Cambridge University Press)

[30] M. Wadati and K. Sogo, 1983, Gauge Transformations in Soliton Theory, J. Phys. Soc. Japan. 52, 394-338

[31] H.H. Chen,Y.C.Lee and C.S. Liu, 1979, Integrability of Nonlinear Hamiltonian Systems by Inverse Scattering Method, Phys.Scr.20, 490-492

[32] V.S.Gerdjikov and I.Ivanov, 1983, A Quadratic Pencil of General Type and Nonlinear Evolution Equations. II. Hierarchies of Hamiltonian structures, J. Phys. Bulgar. 10, 130C143

[33] Y.C. -Ma,1979, The Perturbed plane-wave solutions of the Cubic Schrödinger equation, Stud. Appl. Math.60,43C58

[34] N.N.Akhmediev and V.I.Korneev,1986, Modulation instability and periodic solutions of the Nonlinear Schrödinger equation, Theor. Math. Phys. 69, 1080-1093; N. Akhmediev, J. M. Soto-Crespo and A. Ankiewicz, 2009, Extreme waves that appear from nowhere: On the nature of rogue waves, Phys. Lett. A. 373, 2137-2145.

[35] Jingsong He,Shuwei Xu,Lihong Wang, The Rogue wave of the Variable Coefficient Derivative Nonlinear Schrodinger equation(in preparation 2011)

[36] Engui Fan, 2000, Darboux transformation and Soliton-like solutions for the Gerdjikov-Ivanov equation, J. Phys. A: Math. Gen. 33, 6925C6933

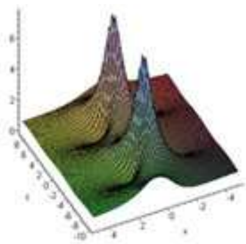

FIGURE 3. The dynamical evolution of $|q^{[2]}|^2$(**time periodic** breather) in eq.(53) on $(x-t)$ plane with specific parameters $\alpha_1 = \beta_1, \beta_1 = 0.5, c = 0.8$. The trajectory is a line $x = 0$.

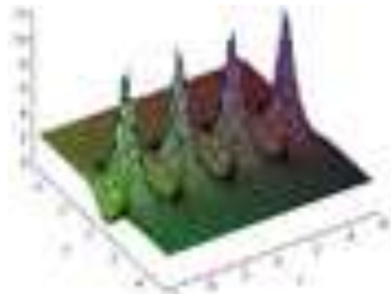

FIGURE 4. The dynamical evolution of $|q^{[2]}|^2$(**space periodic** breather) in eq.(53) on $(x-t)$ plane with specific parameters $\alpha_1 = \beta_1, \beta_1 = 0.5, c = 1.5$. The trajectory is a line $t = 0$.

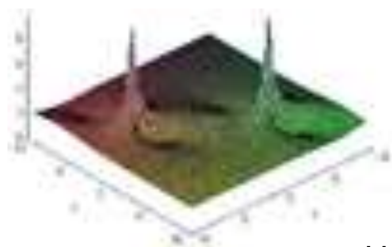

FIGURE 5. The dynamical evolution of solution $|q^{[2]}|^2$ in eq.(53) for case 4(a). It evolves periodically along a straight line with certain angle of x axis and t axis under specific parameters $\alpha_1 = 0.65, \beta_1 = 0.5, c = 0.95$.

FIGURE 6. The dynamical evolution of $|q^{[2]}|^2$ in case 4(b) on $(x-t)$ plane with specific parameters $\alpha_1 = 0.5, \beta_1 = 0.35, c = 0.85$. It evolves periodically along a straight line on $(x-t)$ plane.

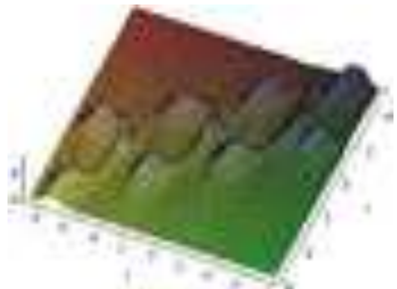

FIGURE 7. The dynamical evolution of periodic breather solution given by case 5(a) on $(x-t)$ plane with specific parameters $\alpha_1 = 0.5, \beta_1 = 0.6, c = 0.5, \alpha_3 = 0.6, \beta_3 = \frac{1}{10}\sqrt{47}$. This picture shows two breathers may parallelly propagate on $(x-t)$ plane.

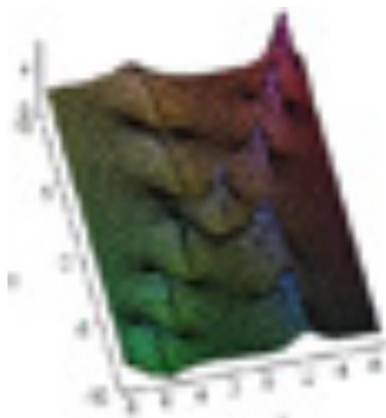

FIGURE 8. The dynamical evolution of periodic breather solution given by case 5(b) on $(x-t)$ plane with specific parameters $a = \frac{c^2}{2}, \alpha_1 = -0.5, \beta_1 = 0.5, \alpha_3 = 0.6, \beta_3 = 0.5, c = 0.95$. This picture shows the elastic interaction of the two breathers.

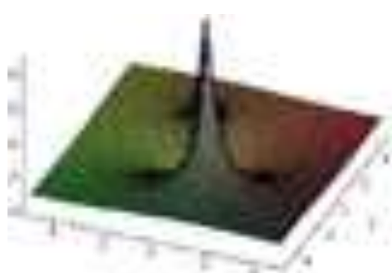

FIGURE 9. The dynamical evolution of $|q^{[2]}_{rogue\ wave}|^2$ given by eq.(56) on $(x-t)$ plane with specific parameters $\alpha_1 = \frac{1}{2}, \beta_1 = \frac{1}{2}$.By letting $x \to \infty,\ t \to \infty$, so $|q^{[2]}_{rogue\ wave}|^2 \to 1$, the maximum amplitude of $|q^{[2]}_{rogue\ wave}|^2$ occurs at $t = 0$ and $x = 0$ and is equal to 9,and the minimum amplitude of $|q^{[2]}_{rogue\ wave}|^2$ occurs at $t = \pm\frac{\sqrt{15}}{10}$ and $x = \mp\frac{3\sqrt{15}}{10}$ and is equal to 0.

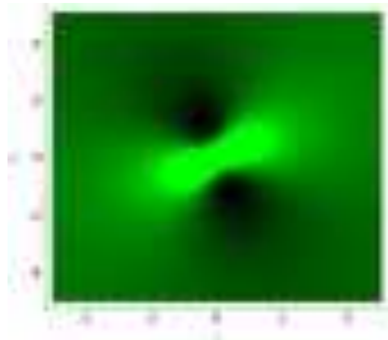

FIGURE 10. Contour plot of the wave amplitudes of $|q^{[2]}_{rogue\ wave}|^2$ in the $(x-t)$ plane is given by eq.(56) for $\alpha_1 = \dfrac{1}{2}, \beta_1 = \dfrac{1}{2}$.