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THE DE SITTER GROUP $L(4,1)$ AND THE BOUND STATES
OF HYDROGEN ATOM

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A B S T R A C T

Assuming that bound states of the non-relativistic hydrogen atom are basis vectors of a unitary irreducible representation of the $4+1$ de Sitter group, the generators of this group are built classically in terms of \vec{r} and \vec{p} using Poisson brackets.

INTRODUCTION

In the relativistic extensions of $SU(6)$ which have been proposed up to now ^{1),2)} some difficulties arise concerning either the number of translation operators or the violation of unitarity ³⁾. For these reasons, it seems preferable to ask if every physical group, containing as a subgroup the rotation group, has to contain all the Poincaré group ⁴⁾. The answer is no : in fact, there exists even in classical physics some problems where an invariance group is good only in the non-relativistic case ; in such problems, the group is of a dynamical origin. If we assume that internal symmetries have also the same origin, it seems interesting to re-examine one of these classical problems, for instance the Kepler problem and this will be the concern of our paper. In the non-relativistic case, for bound states, the orbits are closed : all the geometrical elements of the orbits are invariant in time. Such an invariance corresponds ⁵⁾ to the four-dimensional rotation group $SO(4)$. On the contrary, in the corresponding relativistic problem, the group $SO(4)$ is broken since the orbit, as is well known, is not fixed, but the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ remains constant (the rotation group $SO(3)$ is not broken).

Quantum mechanically, the existence of the invariance group $SO(4)$ corresponds to an accidental degeneracy of levels. If we replace the Schrödinger equation by a relativistic one (for instance the Klein-Gordon equation) the degeneracy is partially broken.

Analogous calculations can be made in another classical (or quantum mechanical) problem, namely the three-dimensional harmonic oscillator ^{6),7)} where the invariance group is $SU(3)$. In both cases the knowledge of these dynamical (and compact) groups does not permit us to get every state from another one. Such a problem is solved for the n dimensional harmonic oscillator using creation and annihilation operators built from the Lie algebra of a non-compact group. It is also solved, but without group theoretical methods, by Infeld and Hull ⁸⁾ using a "factorization procedure". Our purpose was to solve this problem in the case of the bound states of the hydrogen atom.

In Sections 1 and 2, we recall the main properties of the invariants of the Kepler problem limiting ourselves to the case of negative energy. In classical mechanics, one has to use Poisson brackets hereafter denoted $\{ \}$. In quantum mechanics, these brackets have to be replaced by commutators $[\]$. The Hamiltonian is shown to be a function of Casimir operators of $SO(4)$ and this gives the energy spectrum. In Section 3, we try to embed $SO(4)$ into a larger non-compact group. We choose a priori the de Sitter group $L(4,1)$ (i.e., the connected component of Lorentz group with four spacelike directions and one timelike direction) as has been conjectured by many physicists following Fock⁵⁾ and Bargmann⁵⁾. All bound states form a basis for a unique unitary irreducible representation of $L(4,1)$. Unfortunately the problem is solved in terms of Poisson brackets instead of commutators. Due to the ambiguity occurring in replacing the c numbers \vec{r} and \vec{p} by operators, it is not so easy to give a definite solution.

1. HYDROGEN ATOM IN CLASSICAL MECHANICS

A one-particle problem is completely solved if one knows 6 invariants of the motion. For the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{r} \quad (1.1)$$

the 6 invariants can be chosen as the angular momentum

$$\vec{L} = \vec{r} \times \vec{p} \quad (1.2)$$

and the following vector ⁵⁾

$$\vec{K} = \vec{L} \times \vec{p} + \frac{\vec{z}}{r} \quad (1.3)$$

They obey the relations

$$\frac{d\vec{L}}{dt} = \{\vec{L}, H\} = 0 \quad (1.4)$$

$$\frac{d\vec{K}}{dt} = \{\vec{K}, H\} = 0 \quad (1.5)$$

where $\{ \}$ denotes the Poisson brackets defined as follows :

$$\{f(\vec{r}, \vec{p}), g(\vec{r}, \vec{p})\} = \sum_i \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right) \quad (1.6)$$

As is well known, Poisson brackets satisfy the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \quad (1.7)$$

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Equation (1.4) follows from the fact that the Hamiltonian (1.1) is invariant under rotations ; in the same way, Eq. (1.5) corresponds to the invariance of H under canonical transformations generated by \vec{K} . According to (1.7), if f and g are invariants, $\{f, g\}$ is also an invariant (Poisson theorem). Consequently, H is an invariant under all transformations generated by Poisson brackets built on \vec{L} and \vec{K} . In the present case this algebra is

$$\{\vec{L}, \vec{L}\} = \vec{L} \quad (1.8)$$

$$\{\vec{L}, \vec{K}\} = \vec{K} \quad (1.9)$$

$$\{\vec{K}, \vec{K}\} = -2H\vec{L} \quad \text{etc.} \quad (1.10)$$

Here the notation $\{\vec{X}, \vec{Y}\} = \vec{Z}$ means

$$\{X_i, Y_j\} = \varepsilon_{ijk} Z_k \quad (1.11)$$

The algebra (1.8), (1.9), (1.10) is not closed, but if one replaces the vector \vec{K} by the so-called Runge-Lenz vector ⁵⁾

$$\vec{A} = \frac{\vec{K}}{\sqrt{-2H}} \quad (1.12)$$

the algebra is closed

$$\{\vec{L}, \vec{L}\} = \vec{L} \quad (1.13)$$

$$\{\vec{L}, \vec{A}\} = \vec{A} \quad (1.14)$$

$$\{\vec{A}, \vec{A}\} = \vec{L} \quad (1.15)$$

It is the Lie algebra of the four-dimensional rotation group $SO(4)$ as can be verified if one makes the following substitutions :

$$L_i = \frac{1}{2} \varepsilon_{ijk} M_{jk} \quad (1.16)$$

$$A_i = M_{4i} \quad (1.17)$$

It is also the Lie algebra of the group $SU(2) \times SU(2)$ as can be made apparent by using the following basis

$$\vec{X} = \frac{1}{2} (\vec{L} + \vec{A}) \quad (1.18)$$

$$\vec{Y} = \frac{1}{2} (\vec{L} - \vec{A}) \quad (1.19)$$

The Poisson brackets obeyed by \vec{X} and \vec{Y} are

$$\{\vec{X}, \vec{X}\} = \vec{X} \quad (1.20)$$

$$\{\vec{X}, \vec{Y}\} = 0 \quad (1.21)$$

$$\{\vec{Y}, \vec{Y}\} = \vec{Y} \quad (1.22)$$

Obviously, the two Casimir operators of $SO(4)$ are

$$\vec{X}^2 = \frac{1}{4} (\vec{L} + \vec{A})^2 \quad (1.23)$$

$$\vec{Y}^2 = \frac{1}{4} (\vec{L} - \vec{A})^2 \quad (1.24)$$

Since the Hamiltonian H is invariant under $SO(4)$, it can be expressed in terms of \vec{X}^2 and \vec{Y}^2 . In fact, one obtains

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$$-\frac{1}{2H} = (\vec{L} + \vec{A})^2 = (\vec{L} - \vec{A})^2 \quad (1.25)$$

the equality of \vec{X}^2 and \vec{Y}^2 coming from the relation

$$\vec{L}, \vec{A} = 0 \quad (1.26)$$

In the quantum mechanical approach to the Kepler problem, one will get more interesting results.

2. HYDROGEN ATOM IN QUANTUM MECHANICS

We now replace Poisson brackets by commutators. The Runge-Lenz vector becomes the following Hermitian operator

$$\vec{K} = \frac{1}{2} (\vec{L} \times \vec{p} - \vec{p} \times \vec{L}) + \frac{\vec{r}}{r} \quad (2.1)$$

which commutes with the Hamiltonian. As in the classical case, one replaces it by the following normed vector

$$\vec{A} = \frac{\vec{K}}{\sqrt{-2H}} \quad (2.2)$$

where $\sqrt{-2H}$ means the positive square root of the operator $-2H$ acting on negative energy states alone⁹⁾.

Commutation relations are simply, but tediously, derived

$$[\vec{L}, \vec{L}] = i\vec{L} \quad (2.3)$$

$$[\vec{L}, \vec{A}] = i\vec{A} \quad (2.4)$$

$$[\vec{A}, \vec{A}] = i\vec{L} \quad (2.5)$$

They are those of $SO(4)$ as above. The Casimir operators are

$$\frac{1}{4}(\vec{L} + \vec{A})^2 \quad \text{and} \quad \frac{1}{4}(\vec{L} - \vec{A})^2$$

Let $j(j+1)$ and $j'(j'+1)$ be the corresponding equations for a unitary irreducible representation of $SO(4)$. Each state is characterized by a ket

$$|j, j', l, m\rangle \quad (2.6)$$

where $l(l+1)$ and m are eigenvalues of \vec{L}^2 and L_3 , respectively.

As in the classical calculation

$$\vec{L} \cdot \vec{A} = \vec{A} \cdot \vec{L} = 0 \quad (2.7)$$

thus one has the condition

$$j = j' \quad (2.8)$$

The Hamiltonian is expressed in terms of the Casimir operators but instead of (1.25) one obtains :

$$\begin{aligned} -\frac{1}{2H} &= (\vec{L} + \vec{A})^2 + 1 = (\vec{L} - \vec{A})^2 + 1 \\ &= 4j(j+1) + 1 \\ &= (2j+1)^2 \end{aligned} \quad (2.9)$$

When one replaces j by the values $0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ one gets all the values of the energy for the bound levels, namely

$$E_n = -\frac{1}{2n^2} \quad (2.10)$$

where

$$n = 2j + 1 \quad (2.11)$$

3. THE DE SITTER GROUP $L(4,1)$ AND THE HYDROGEN ATOM

Now, we are interested in a group where, besides \vec{L} and \vec{A} , there exist other operators which permit us to transform a level into another one. After Fock's and Bargmann's works ⁵⁾ many physicists have conjectured that the de Sitter group $L(4,1)$ is adequate ¹⁰⁾. The other generators are a vector \vec{B} and a scalar S satisfying the following commutation rules

$$[\vec{L}, \vec{L}] = i\vec{L} \quad (3.1)$$

$$[\vec{L}, \vec{A}] = i\vec{A} \quad (3.2)$$

$$[\vec{L}, \vec{B}] = i\vec{B} \quad (3.3)$$

$$[\vec{L}, S] = 0 \quad (3.4)$$

$$[\vec{A}, \vec{A}] = i\vec{L} \quad (3.5)$$

$$[\vec{A}, \vec{B}] = iS \quad (3.6)$$

$$[S, \vec{A}] = i\vec{B} \quad (3.7)$$

$$[\vec{B}, \vec{B}] = -i\vec{L} \quad (3.8)$$

$$[S, \vec{B}] = i\vec{A} \quad (3.9)$$

where $[\vec{A}, \vec{B}] = iS$ means

$$[A_i, B_j] = i\delta_{ij} S \quad (3.10)$$

The two invariants of the group are

a) the Casimir operator

$$Q = S^2 + \vec{B}^2 - \vec{A}^2 - \vec{L}^2 \quad (3.11)$$

b) a quartic operator

$$W = (S\vec{L} - \vec{A} \times \vec{B})^2 - \frac{1}{4} [\vec{L} \cdot (\vec{A} + \vec{B}) + (\vec{A} + \vec{B}) \cdot \vec{L}]^2 \quad (3.12)$$

A state is completely labelled by the eigenvalues of

$$W, Q, \frac{1}{4}(\vec{L} + \vec{A})^2, \frac{1}{4}(\vec{L} - \vec{A})^2, \vec{L}^2 \text{ and } L_3$$

In order to find explicitly the operators S and \vec{B} in terms of \vec{r} and \vec{p} , it is much simpler to use Poisson brackets instead of commutators because Eqs. (3.1) to (3.9) lead to differential equations which can be integrated in a simple way. Once the expressions of S and \vec{B} are obtained, we have to replace \vec{r} and \vec{p} by the corresponding operators. The differential equations are discussed and integrated in an Appendix. The results are the following :

S and \vec{B} are shown to be derived from a function φ which is

$$\varphi(H, u) = \sqrt{a - \frac{1}{2H}} \sin \left[\sqrt{-2H} u + \theta(H) \right] \quad (3.13)$$

where H is the total (classical) energy and

$$u = \vec{r} \cdot \vec{p} \quad (3.14)$$

In Eq. (3.13), a appears as an arbitrary constant and $\theta(H)$ as an arbitrary function of the energy.

The expressions of S and \vec{B} are

$$S = \frac{1}{\sqrt{-2H}} \left\{ 2Hu\psi - (1+2Hz) \frac{\partial \psi}{\partial u} \right\} \quad (3.15)$$

$$\vec{B} = \left(r \frac{\partial \psi}{\partial u} - u\psi \right) \vec{p} + \frac{\psi}{r} \vec{r} \quad (3.16)$$

Because of the arbitrariness of a and θ , the solution of our system of differential equations is not unique. There exists a useful relation between $\vec{L}, \vec{A}, \vec{B}$ and S , namely

$$\vec{A} \times \vec{B} = S \vec{L} \quad (3.17)$$

from which one gets for the invariants

$$Q = a \quad (3.18)$$

and

$$W = 0 \quad (3.19)$$

Strictly speaking, in so far as we do not give explicitly the operators S and \vec{B} in terms of the operators \vec{r} and \vec{p} , we cannot deduce any conclusion. Unfortunately, we are not able to derive such expressions since there exists an infinite number of Hermitian operators corresponding to the function ψ of Eq. (3.13). Nevertheless, it seems that once we know how to build the operator function ψ , the quartic invariant W will remain zero. So, we have to know what are unitary representations of the de Sitter group $L(4,1)$ characterized by the eigenvalue $W = 0$. This group has been already investigated by Thomas¹¹⁾ and Newton¹²⁾. From the latter paper one learns that there exist two classes of such representations :

Class I : Q is real and strictly positive. Then, the contents of an irreducible representation of this class in terms of unitary representations of the subgroup $SO(4)$ are the $D_{jj'}$, where $j = j'$ and $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ up to infinity. Each representation $D_{jj'}$ occurs only once.

Class II : $Q = -(n-1)(n+2)$ where n is a positive integer. The contents are the $D_{jj'}$, where $j = j'$ and $j = \frac{n}{2}, \frac{n+1}{2}, \dots$ up to infinity.

We know from results of Section 2 that j must take the value zero ; then, one has to discard the second class and we are left with a representation of the first class for which Q is positive. We cannot conclude that in (3.13) and according to (3.18), the constant a cannot be taken equal to zero since our calculation was purely classical. It could happen that $a = 0$ leads to a positive eigenvalue for Q [compare for instance Eqs. (1.25) and (2.9)]. On the other hand, the arbitrary function $\theta(H)$ can vanish but perhaps it does not correspond to the simplest choice. So, the problem is not complete.

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A P P E N D I X

1. POISSON BRACKETS FOR SPHERICAL POTENTIALS

Let $f(\vec{r}, \vec{p})$ be a scalar function of the classical variables \vec{r} and \vec{p} , i.e., satisfying the Poisson bracket relation

$$[\vec{L}, f] = 0 \quad (\text{A.1})$$

then, f can be written as a function of three scalars like r, p and $u = \vec{r} \cdot \vec{p}$ or, equivalently, r, H (energy) and l (length of \vec{L}). We have the relations

$$l^2 = r^2 p^2 - (\vec{r} \cdot \vec{p})^2 \quad (\text{A.2})$$

$$H = \frac{p^2}{2} + V(r) \quad (\text{A.3})$$

$$u = (2Hr^2 - 2Vr^2 - l^2)^{1/2} \quad (\text{A.4})$$

Choosing r, H and l as variables, the following lemmas can be easily derived :

Lemma 1

$$\left\{ f(r, H, l), g(r, H, l) \right\} = \left(\frac{\partial f}{\partial r} \frac{\partial g}{\partial H} - \frac{\partial f}{\partial H} \frac{\partial g}{\partial r} \right) \frac{u}{r}$$

Lemma 2

$$\left\{ f(r, H, l), \vec{r} \right\} = \frac{u}{l} \frac{\partial f}{\partial l} \vec{r} - \left(\frac{\partial f}{\partial H} + \frac{r^2}{l} \frac{\partial f}{\partial r} \right) \vec{p}$$

14.

Lemma 3

$$\{f(r, H, \ell), \vec{p}\} = \left(\frac{1}{2} \frac{\partial f}{\partial r} + \frac{p^2}{\ell} \frac{\partial f}{\partial \ell} + \frac{V'}{2} \frac{\partial f}{\partial H} \right) \vec{r} - \frac{u}{\ell} \frac{\partial f}{\partial \ell} \vec{p}$$

Lemma 4

$$\{f\vec{A}, f\vec{A}\} = f\vec{A} \times \{f, \vec{A}\} + f^2 \{\vec{A}, \vec{A}\}$$

where f is any function of r, H, ℓ .

Lemma 5

S and \vec{A} denoting respectively a scalar and a vector such that

$$\vec{A} = f\vec{p} + g\vec{r}$$

then

$$\{S, \vec{A}\} = \vec{B} = F\vec{p} + G\vec{r}$$

where

$$F = \frac{\partial S}{\partial r} \frac{\partial f}{\partial H} \frac{u}{r} - \frac{\partial S}{\partial H} \left(\frac{\partial f}{\partial r} \frac{u}{r} + g \right) - \frac{1}{\ell} \frac{\partial S}{\partial \ell} (fu + gr^2)$$

$$G = \frac{\partial S}{\partial r} \left(\frac{\partial g}{\partial H} \frac{u}{r} + \frac{f}{r} \right) + \frac{\partial S}{\partial H} \left(f \frac{V'}{r} - \frac{\partial g}{\partial r} \frac{u}{r} \right) + \frac{1}{\ell} \frac{\partial S}{\partial \ell} (gu + fp^2)$$

Lemma 6

For any vector $\vec{A} = f\vec{p} + g\vec{r}$

$$\{\vec{A}, \vec{A}\} = -\sigma \vec{L}$$

with

$$\sigma = \frac{\partial f}{\partial H} \left(f \frac{V'}{r} - \frac{\partial g}{\partial r} \frac{u}{r} \right) + \frac{\partial g}{\partial H} \left(\frac{\partial f}{\partial r} \frac{u}{r} + g \right) + \frac{1}{\rho} \frac{\partial f}{\partial \ell} (gu + fp^2) \\ + \frac{\partial f}{\partial r} \frac{f}{r} + \frac{1}{\rho} \frac{\partial g}{\partial \ell} (fu + gr^2)$$

Note that some expressions are identical to those appearing in Lemma 5.

Lemma 7

If $\vec{A} = f\vec{p} + g\vec{r}$

and $\vec{B} = F\vec{p} + G\vec{r}$

then $\{\vec{A}, \vec{B}\} = gF - Gf$

$$+ \vec{p} \otimes \vec{p} \left[\frac{1}{\rho} \frac{\partial F}{\partial \ell} (gr^2 + fu) - \frac{1}{\rho} \frac{\partial f}{\partial \ell} (Gu + Fu) + \frac{\partial F}{\partial H} \left(\frac{\partial f}{\partial r} \frac{u}{r} + g \right) - \frac{\partial f}{\partial H} \left(\frac{\partial F}{\partial r} \frac{u}{r} + G \right) \right]$$

$$+ \vec{r} \otimes \vec{p} \left[-\frac{1}{\rho} \frac{\partial F}{\partial \ell} (gu + fp^2) - \frac{1}{\rho} \frac{\partial g}{\partial \ell} (Fu + Gr^2) + \frac{\partial F}{\partial H} \left(\frac{\partial g}{\partial r} \frac{u}{r} - f \frac{V'}{r} \right) - \frac{\partial g}{\partial H} \left(\frac{\partial F}{\partial r} \frac{u}{r} + G \right) - \frac{\partial F}{\partial r} \frac{f}{r} \right]$$

$$+ \vec{p} \otimes \vec{r} \left[\frac{1}{\rho} \frac{\partial G}{\partial \ell} (gr^2 + fu) + \frac{1}{\rho} \frac{\partial f}{\partial \ell} (Gu + Fp^2) + \frac{\partial G}{\partial H} \left(\frac{\partial f}{\partial r} \frac{u}{r} + g \right) - \frac{\partial f}{\partial H} \left(\frac{\partial G}{\partial r} \frac{u}{r} - F \frac{V'}{r} \right) + \frac{\partial f}{\partial r} \frac{F}{r} \right]$$

$$+ \vec{r} \otimes \vec{r} \left[-\frac{1}{\rho} \frac{\partial G}{\partial \ell} (gu + fp^2) + \frac{1}{\rho} \frac{\partial g}{\partial \ell} (Gu + Fp^2) + \frac{\partial G}{\partial H} \left(\frac{\partial g}{\partial r} \frac{u}{r} - f \frac{V'}{r} \right) - \frac{\partial g}{\partial H} \left(\frac{\partial G}{\partial r} \frac{u}{r} - F \frac{V'}{r} \right) \right. \\ \left. + \frac{F}{r} \frac{\partial g}{\partial r} - \frac{f}{r} \frac{\partial G}{\partial r} \right]$$

2. GENERATORS OF $L(4,1)$ FOR THE HYDROGEN ATOM

The Poisson brackets corresponding to Eqs. (3.1) to (3.9) are

$$\{\vec{L}, \vec{L}\} = \vec{L} \quad (\text{A.5})$$

$$\{\vec{L}, \vec{A}\} = \vec{A} \quad (\text{A.6})$$

$$\{\vec{L}, \vec{B}\} = \vec{B} \quad (\text{A.7})$$

$$\{\vec{L}, S\} = 0 \quad (\text{A.8})$$

$$\{\vec{A}, \vec{A}\} = \vec{L} \quad (\text{A.9})$$

$$\{S, \vec{A}\} = \vec{B} \quad (\text{A.10})$$

$$\{\vec{A}, \vec{B}\} = S \quad (\text{A.11})$$

$$\{\vec{B}, \vec{B}\} = -\vec{L} \quad (\text{A.12})$$

$$\{S, \vec{B}\} = \vec{A} \quad (\text{A.13})$$

Obviously, Eqs. (A.5) to (A.8) mean that $\vec{L}, \vec{A}, \vec{B}$ are vectors and S is a scalar under the rotation group; then we will learn nothing new from them. Equation (A.9) is satisfied by the Runge-Lenz vector

$$\vec{A} = f\vec{p} + g\vec{r} \quad (\text{A.14})$$

where

$$f = \frac{u}{\sqrt{-2H}} \quad (\text{A.15})$$

and

$$g = -\frac{1}{\sqrt{-2H}} \left(2H + \frac{1}{2} \right) \quad (\text{A.16})$$

Equation (A.10) gives us the vector \vec{B} in terms of S with the aid of lemma 5. We obtain

$$F = \frac{1}{\sqrt{-2H}} \left[\left(\frac{\rho^2}{2rH} - \frac{1}{H} \right) \frac{\partial S}{\partial r} - \frac{1}{\rho} \frac{\partial S}{\partial \rho} (r - \rho^2) \right] \quad (\text{A.17})$$

$$G = \frac{u}{r\sqrt{-2H}} \left[\frac{1}{2Hr} \frac{\partial S}{\partial r} + \frac{1}{\rho} \frac{\partial S}{\partial \rho} \right] \quad (\text{A.18})$$

Equation (A.11) gives us five equations, namely

$$S = gF - Gf \quad (\text{scalar part}) \quad (\text{A.19})$$

$$\frac{1}{\rho} \frac{\partial F}{\partial \rho} (r - \rho^2) + \frac{1}{u} (Gu^2 + Fu) + \frac{1}{2Hu} (2r - \rho^2) \left(\frac{\partial F}{\partial r} \frac{u}{r} + G \right) = 0 \quad (\text{A.20})$$

($\vec{p} \otimes \vec{p}$ part)

$$G \frac{u}{r} \left(1 - \frac{1}{2rH} \right) - \frac{1}{\rho} \frac{\partial F}{\partial \rho} - \frac{1}{2rH} \frac{\partial F}{\partial r} = 0 \quad (\vec{r} \otimes \vec{p} \text{ part}) \quad (\text{A.21})$$

$$\frac{1}{\rho} \frac{\partial G}{\partial \rho} (r - \rho^2) - \frac{1}{u} (Gu + F\rho^2) + \frac{1}{2Hu} (2r - \rho^2) \left(\frac{\partial G}{\partial r} \frac{u}{r} - \frac{F}{r^3} \right) + \frac{1}{u} \left(2H + \frac{1}{2} \right) F = 0$$

($\vec{p} \otimes \vec{r}$ part) (A.22)

$$\frac{F}{2rHu} - \frac{1}{\rho} \frac{\partial G}{\partial \rho} - \frac{1}{2rH} \frac{\partial G}{\partial r} = 0 \quad (\vec{r} \otimes \vec{r} \text{ part}) \quad (\text{A.23})$$

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Eliminating F in the last two equations, we obtain the equation

$$\frac{1}{2} \frac{\partial G}{\partial r} (1 + 2Hr) + \frac{\partial G}{\partial r} + \frac{G}{r} = 0 \quad (\text{A.24})$$

which is readily integrated, putting

$$\boxed{G = \frac{3}{r}} \quad (\text{A.25})$$

Then, ψ is shown to depend only on $u = \vec{r} \cdot \vec{p}$ and H .

From (A.23) and (A.25) we get

$$\boxed{F = r \frac{\partial \psi}{\partial u} - u \psi} \quad (\text{A.26})$$

and (A.19) becomes

$$\boxed{S = \frac{1}{\sqrt{-2H}} \left[2Hu\psi - (1+2Hr) \frac{\partial \psi}{\partial u} \right]} \quad (\text{A.27})$$

Equation (A.27) gives a condition on the function ψ :

$$\frac{\partial^2 \psi}{\partial u^2} = 2H\psi \quad (\text{A.28})$$

It is easy to verify that Eqs. (A.25) to (A.28) are sufficient to satisfy the system of Eqs. (A.17) to (A.23). So, we are left with conditions (A.12) and (A.13). Using lemma 6 for (A.12) or lemma 5 for (A.13) leads to the same following condition

$$4H^2 \psi \frac{\partial \psi}{\partial H} - 2H \frac{\partial \psi}{\partial u} \frac{\partial^2 \psi}{\partial u \partial H} + \left(\frac{\partial \psi}{\partial u} \right)^2 = 1 \quad (\text{A.29})$$

Equations (A.25) to (A.29) give the complete solution of our problem.
First, we integrate (A.28)

$$\boxed{\varphi = \psi(H) \sin [\sqrt{-2H} u + \theta(H)]} \quad (\text{A.30})$$

where $\theta(H)$ is an arbitrary function.

Condition (A.29) imposes us to choose

$$4H^2 \psi \frac{d\psi}{dH} = 1 \quad (\text{A.31})$$

Then

$$\boxed{\psi^2 = a - \frac{1}{2H}} \quad (\text{A.32})$$

where a is an arbitrary constant.

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$$\vec{A} = \frac{\vec{K}}{\sqrt{2H}}$$

and instead of (2.5) one gets

$$[\vec{A}, \vec{A}] = -i\vec{L}$$

The corresponding group is no longer $SO(4)$ but the Lorentz group.

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