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The decay of the spectrum of the gravitational potential and the topography for the Earth

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SUMMARY

The spectrum of the Earth's gravitational potential and topography, as represented by spherical harmonic expansions to degree 180, have been computed. Modelling the decay in the form of $Al^{-\beta}$, values of A and β for several degree (l) ranges were computed. For degree range 5–180, β was 2.54 for the potential and 2.16 for equivalent rock topography. The potential decay was somewhat slower than that (i.e. $\beta = 3$) implied by Kaula's rule. However, at high degree ranges, the β values were larger (3.20 for degrees 101–180) agreeing better with recent determinations from terrestrial gravity data and geoid undulations implied by satellite altimetric data. The values imply that the potential decays faster at higher l values. The values of β for topography were fairly uniform around 2 which agrees with a suggestion made by Vening-Meinesz in 1951. We also found that the β value for the Earth's potential agrees well with the value implied by the topography with Airy isostatic compensation with the depth of compensation equal to 30 km. However, the magnitude of the power implied by the topographic/isostatic potential was approximately one-third of the observed potential.

Key words: Earth, gravity, topography.

INTRODUCTION

The Earth's gravitational potential is usually represented in a spherical harmonic series:

$$V(r, \theta, \lambda) = \frac{kM}{r}\left[1 + \sum_{l=2}^{\infty}\left(\frac{a}{r}\right)^{l}\sum_{m=0}^{l} \times (\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda)\bar{P}_{lm}(\cos\theta)\right] \quad (1)$$

where kM is the product of gravitational constant and mass of the Earth; r, θ, λ are the geocentric coordinates; $\bar{C}_{lm}$, $\bar{S}_{lm}$ are the fully normalized potential coefficients of degree l and order m; and a is the scaling parameter associated with the potential coefficients.

Estimates of the potential coefficients can be made from the analysis of satellite observations or terrestrial gravity data or a combination of both (Rapp 1986; Kaula 1987). From (1) various gravimetric quantities such as gravity anomalies, geoid undulations, etc., can be derived.

The topography of the Earth can be represented in a surface spherical harmonic expansion. Following Rapp (1982) we write:

$$h(\theta, \lambda) = R\sum_{l=0}^{\infty}\sum_{m=0}^{l}(\bar{h}_{clm}\cos m\lambda + \bar{h}_{slm}\sin m\lambda)P_{lm}(\cos\theta) \quad (2)$$

where $\bar{h}_{clm}$, $\bar{h}_{slm}$ are fully normalized spherical harmonic coefficients of the topography; and R is a scaling parameter associated with the elevation coefficients; R is usually a mean earth radius (6371 km).

Balmino, Lambeck & Kaula (1973) reported unscaled (i.e. $R = 1$ m) coefficients based on $5° \times 5°$ mean elevations and depths. These coefficients were given for the actual topography and equivalent rock topography. The latter quantity compresses the ocean to an equivalent density as land masses. This means that the elevation in ocean areas is computed, for harmonic analysis purposes, as:

$$h(\theta, \lambda) = d(\theta, \lambda)(1 - 1.03/2.67) \quad (3)$$

where d is the (negative) ocean depth.

Rapp (1982) computed the harmonic coefficients of a set of $1° \times 1°$ elevations to degree 180. The topographic/isostatic potential implied by this elevation model was compared with the observed gravitational potential implied by a potential coefficient model to degree 180 (Rapp 1981). Rummel *et al.* (1988) have extended this work to a more exact formulation of the topographic/isostatic model equations and consideration of degree-dependent depths of compensation.

Of interest for this paper is the spectrum of the observed potential and the topography. The usual power spectra of the potential and the topography can be written as (Rapp 1982):

$$V_l^2(h) = \sum_{m=0}^{l}(\bar{h}_{clm}^2 + \bar{h}_{slm}^2) \quad (4)$$

$$V_l^2(\Delta V) = \sum_{m=0}^{l}(\bar{C}_{lm}^2 + \bar{S}_{lm}^2). \quad (5)$$

Note that both equations are unitless. Note also that equation (5) implies the power spectrum of the potential on the surface of a sphere of radius a.

Various models have been developed to express the spectrum of the potential. One well-known model is the Kaula rule (Kaula 1966) that gives an estimate of the rms potential coefficient:

$$\sigma(\bar{C}, \bar{S}) = 10^{-5}/l^2. \tag{6}$$

The power spectrum implied by this model is:

$$V_l^2(\Delta V) = (2l+1)(10^{-5}/l^2)^2. \tag{7}$$

Other analyses have estimated the behaviour of $V_l^2(\Delta V)$ from satellite altimeter data (e.g. Rapp 1986) and from terrestrial gravity data (e.g. Forsberg 1984; Vassiliou & Schwarz 1987). In these studies Fourier analysis was carried out to imply potential coefficient behaviour for a local area. This behaviour was studied to degree 960 for altimeter data and to degree 2000 for the terrestrial gravity data.

Following the lead of Kaula's rule, various authors have developed a model for the decay of the potential spectrum. One general form used is:

$$V_l^2(\Delta V) = \frac{A}{l^\beta}. \tag{8}$$

Estimates of A are sensitive to the specific geographic area under study. Values of β seemed to be consistent between altimeter data and the terrestrial gravity data being 3.6 as compared with a value of 3 implied by the Kaula rule. Additional analysis is needed to understand β as a function of degree and of geographic area, if in fact such functional dependence exists.

If we assume an Airy isostatic model, elevations (and depths) imply a potential that can be compared with the observed potential to test the validity of the model. If D is the depth of compensation (specified as an equal mass condition), Rummel *et al.* (1988) show that the topographic isostatic potential coefficients can be expressed as:

$$\begin{aligned} C^{\mathrm{I}}_{l m\alpha} = \frac{3}{2l+1}\frac{\rho_{\mathrm{cr}}}{\rho}\Bigg\{&\left[1-\left(\frac{R-D}{R}\right)^l\right]h_{lm\alpha} \\ &+\frac{l+2}{2}\left[1+\frac{\rho_{\mathrm{cr}}}{\Delta\rho}\left(\frac{R-D}{R}\right)^{l-3}\right]h2_{lm\alpha} \\ &+\frac{(l+2)(l+1)}{6}\left[1-\frac{\rho_{\mathrm{cr}}^2}{\Delta\rho^2}\left(\frac{R-D}{R}\right)^{l-6}\right]h3_{lm\alpha}\Bigg\} \\ &+\dots \end{aligned} \tag{9}$$

where α is 1 for C_{nm} and $\alpha = 2$ for S_{nm}; ρ_{cr} is the crustal density; R is the mean Earth radius; ρ is the mean Earth density; and $\Delta\rho = \rho_{\mathrm{m}} - \rho_{\mathrm{cr}}$; ρ_{m} is the average mantle density.

The coefficients represented by $h_{lm\alpha}$, $h2_{lm\alpha}$, and $h3_{lm\alpha}$ are those found from the spherical harmonic expansion of (h/R), $(h/R)^2$, $(h/R)^3$, respectively, where h is equivalent rock topography. In many applications of (8) (e.g. Rapp 1982), only the first term has been used, but Rummel *et al.* (1988) show that the other terms (especially the second) play a significant role in calculating the topographic/isostatic potential. If we do neglect the second and third terms, the simple power spectrum of the topographic isostatic potential can be written:

$$V_l^2(\Delta V_{\mathrm{T/I}}) = \left(\frac{3}{2l+1}\frac{\rho_{\mathrm{cr}}}{\rho}\right)^2\left[1-\left(\frac{R-D}{R}\right)^l\right]^2 V_l^2(h). \tag{10}$$

A more accurate formulation would yield:

$$V_l^2(\Delta V_{\mathrm{T/I}}) = \sum_{m,\alpha} (C^{\mathrm{I}}_{lm\alpha})^2 \tag{11}$$

where $C^{\mathrm{I}}_{lm\alpha}$ are computed from (9). In a subsequent section of this paper we will consider the spectrum implied by equation (11).

In an alternate spectrum discussion, Turcotte (1987) considers topography and potential theory in terms of fractal theory. The definitions used in this paper are somewhat different from that used in this and other papers. In addition he restricted his discussion to potential and topographic expansions to degree 36. With the availability of new high-degree potential coefficient models (Rapp & Cruz 1986) and substantially improved topographic models (TUG87, Wieser 1987), we wish to study their implications in the estimation of the spectrum of the potential from the topography and its isostatic compensation.

Turcotte (1987) defines the 'variance of the spectra' for topography, in our notation, as follows:

$$V_{\mathrm{T}l} = R^2 \sum_{m=0}^{l} (\bar{h}_{clm}^2 + \bar{h}_{slm}^2) = R^2 V_l^2(h). \tag{12}$$

A corresponding quantity for geoid undulations would be:

$$V_{\mathrm{N}l} = R^2 \sum_{m=0}^{l} (\bar{C}_{lm}^2 + \bar{S}_{lm}^2) = R^2 V_l^2(\Delta V). \tag{13}$$

The power spectral density of the topography is defined as (Turcotte 1987; E597):

$$S_{\mathrm{T}}(k) = \frac{1}{k_0} V_{\mathrm{T}}(k) = \lambda_0 V_{\mathrm{T}}(k) \tag{14}$$

where k_0 is a wavenumber and $\lambda_0 = 1/k_0$ 'is the wavelength on which data are included in the expansions' (ibid.). Since data are given on a sphere, this linear wavelength is $2\pi R$. The wavenumber k depends on the degree l, as follows:

$$k_l = \frac{l}{2\pi R} \tag{15}$$

where the units of k_l would be cycles km^{-1} if R is expressed in km. Using (12), (14) and (15) we have:

$$S_{\mathrm{T}}(k_l) = 2\pi R^3 \sum_{m=0}^{l} (\bar{h}_{clm}^2 + \bar{h}_{slm}^2) = 2\pi R^3 V_l^2(h). \tag{16}$$

The units of S_{T} would be m^2 (cycle^{-1} km) if we write (16) as $2\pi R$ (km) R^2(m)$V_l^2(h)$.

The corresponding expression for the geoid undulation power spectral density would be:

$$S_{\mathrm{N}}(k_l) = 2\pi R^3 \sum_{m=0}^{l} (\bar{C}_{cm}^2 + \bar{S}_{sm}^2) = 2\pi R^3 V_l^2(\Delta V). \tag{17}$$

The units of S_{N} would be m^2 (cycle^{-1} km) if we write (16) as $2\pi R$ (km)R^2(m)$V_l^2(\Delta V)$.

We now postulate the behaviour of the geoid power spectral density to be:

$$S_{\mathrm{N}}(k_l) = \frac{A_{\mathrm{p}}}{k_l^{\beta_{\mathrm{p}}}} \tag{18}$$

where A_{p} and β_{p} are constants associated with the potential.

Table 1. Values of A_p and β_p based on OSU86F potential coefficient model.

Degree range	A_p*	β_p
5–30	3.06×10^{-7}	3.25
31–100	1.32×10^{-3}	2.14
31–180	6.52×10^{-5}	2.62
101–180	2.47×10^{-6}	3.20
5–180	1.04×10^{-4}	2.54

* The units of A_p will yield $S_N(k_l)$ in units of m^2 cycle^{-1} km if R in (15) is in kilometres.

Using (15) we can write:

$$S_N(k_l) = A_p\left(\frac{l}{2\pi R}\right)^{-\beta_p} = A'_p l^{-\beta_p} \tag{19}$$

where A'_p is related to A_p and β_p. Equation (19) is similar in form to equation (8).

We may postulate an analogous behaviour for the topography:

$$S_T(k_l) = A_T k_{l(T)}^{-\beta_T} = A_T\left(\frac{l}{2\pi R}\right)^{-\beta_T} = A'_T l^{-\beta_T}. \tag{20}$$

The value of β_p has been discussed previously in this paper.

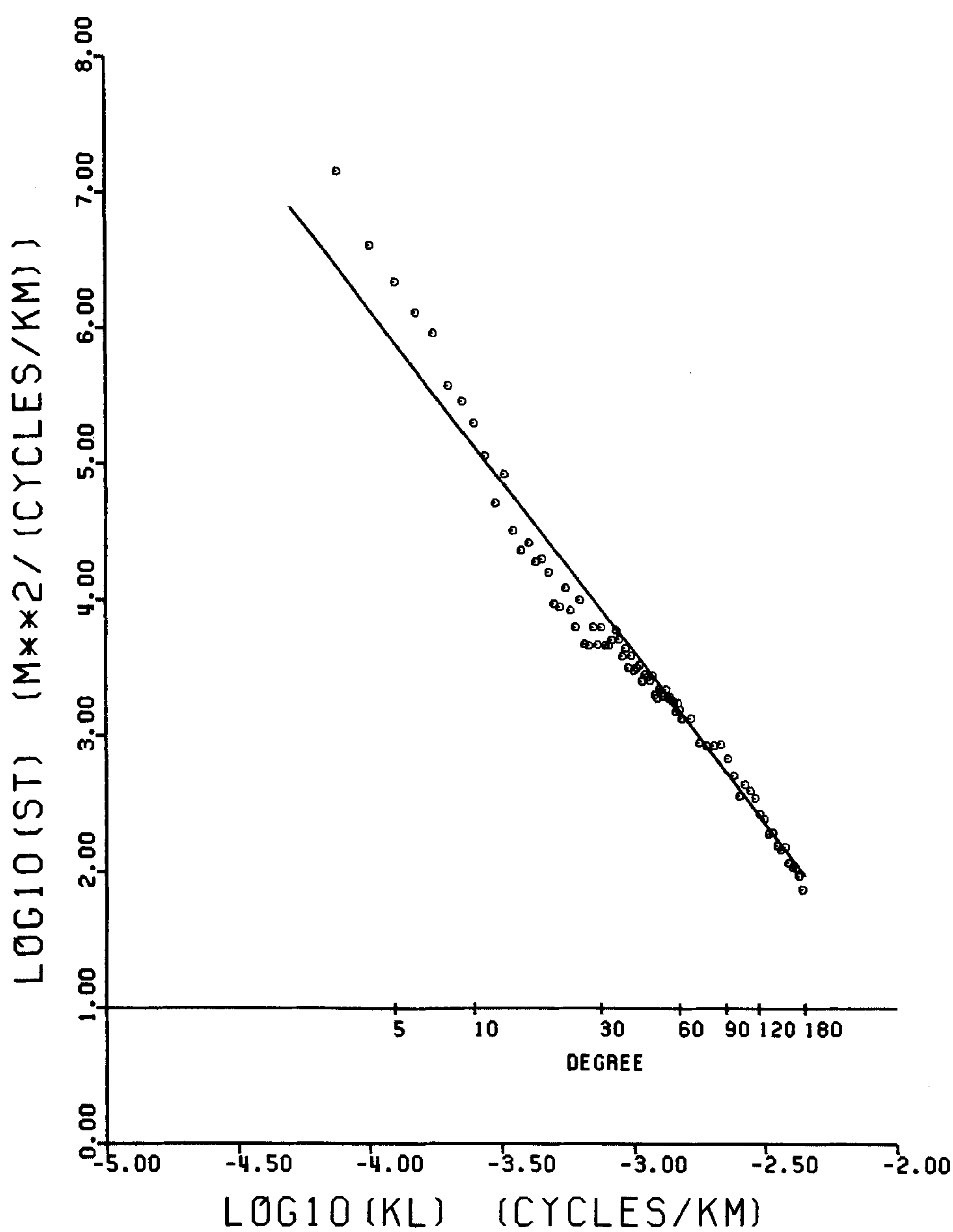

Figure 1. $\mathrm{Log}_{10}[S_N(k_l)$, m^2 cycle^{-1} km] for the OSU86F potential coefficient model versus degree (l) or $\log_{10}$ (k_l, cycle km^{-1}).

Table 2. Value of A_T and β_T based on the harmonic analysis of the TUG87 1°×1° elevation data.

	AT		ERT	
Degree range	A_T*	β_T	A_T*	β_T
5–30	1429.0	1.83	452.3	1.88
31–100	54.8	2.30	17.7	2.35
31–180	119.5	2.17	44.8	2.21
101–180	266.1	2.03	134.5	2.01
5–180	155.4	2.13	59.2	2.16

* The units of A_p are the same as the units of A_p in Table 1.

Turcotte (1987) suggests that a β_T equal to two is reasonable for the Earth based on the Balmino *et al.* (1973) computations. Vening-Meinesz (1951) had made a similiar suggestion based on a topographic expansion to degree 16 made by Prey in 1922. He also found the behaviour valid for an expansion to degree 31 (Vening-Meinesz 1962).

NUMERICAL TESTS

Values of S_N from the OSU86F potential coefficient model were used to estimate A_p and β_p. In carrying out these computations the given potential coefficients were multiplied by the ratio $(6378137/6371000)^l$ so that the spectrum would refer to the radius of the mean Earth. The OSU86F model has been based on the combination of a low ($l=20$) degree potential coefficient model and 30′×30′ mean free-air anomalies. These anomalies were based on surface gravity data and anomalies derived from satellite altimeter data.

Given two $S_N(k_l)$ values computed from equation (17), we have using (18):

$$\beta_p = \frac{\log [S_N(k_{l_1})/S_N(k_{l_2})]}{\log (k_{l_2}/k_{l_1})} \tag{21}$$

where k is computed from equation (15) for the two different degrees l_1 and l_2. Average values for β_p were computed for various degree ranges defined in Table 1. After the β_p was determined a value of A_p was calculated by computing an average of values computed from (19) for the degrees contained in the degree range. The results are shown in Table 1.

A plot of $S_N(k_l)$ is shown in Fig. 1 based on the OSU86F potential coefficient set. The straight line shown in this (and the other figures of this paper) is based on the A and β coefficients for the degree range 5–180.

A similar analysis was carried out to estimate A_T and β_T.

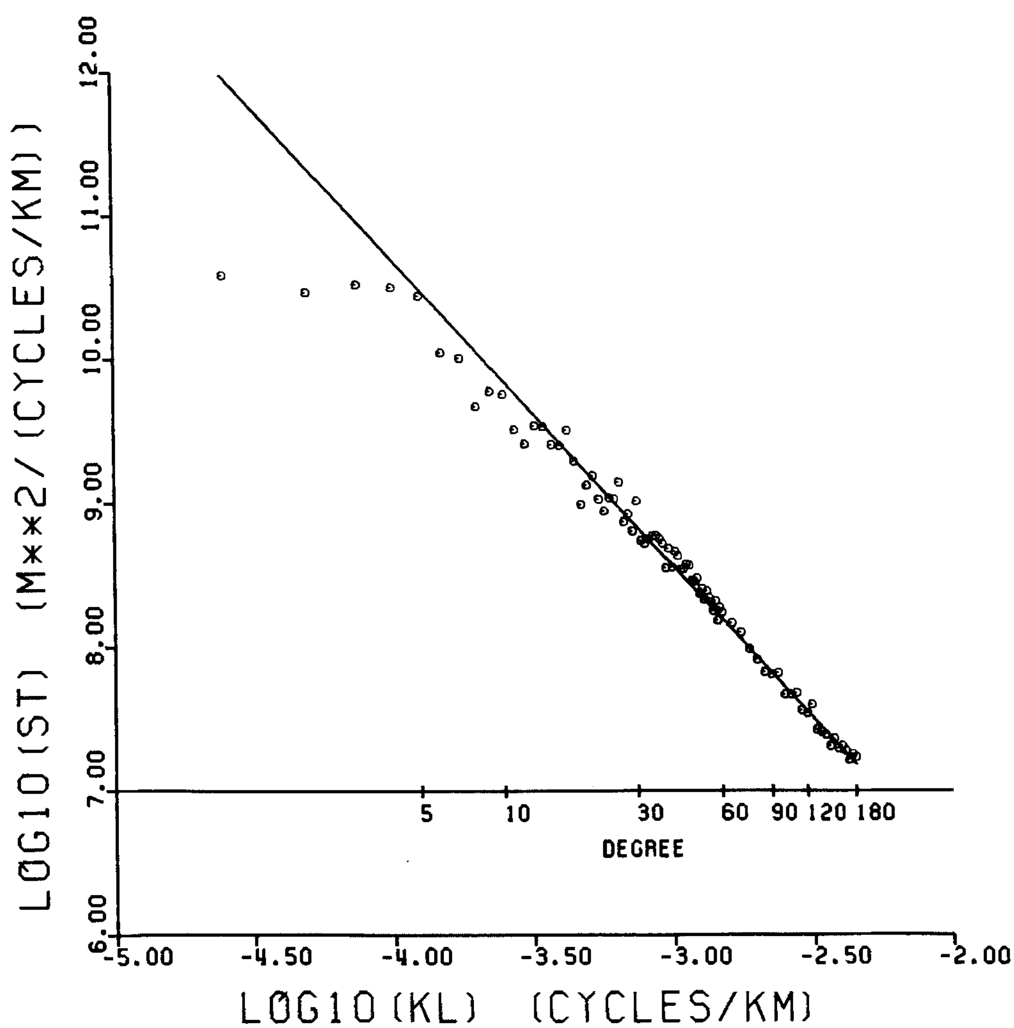

Figure 2. $\text{Log}_{10}\,[S_T(k_l),\ \text{m}^2\ \text{cycle}^{-1}\ \text{km}]$ for the actual TUG87 topography versus degree (l) or $\log_{10}(k_l,\ \text{cycle km}^{-1})$.

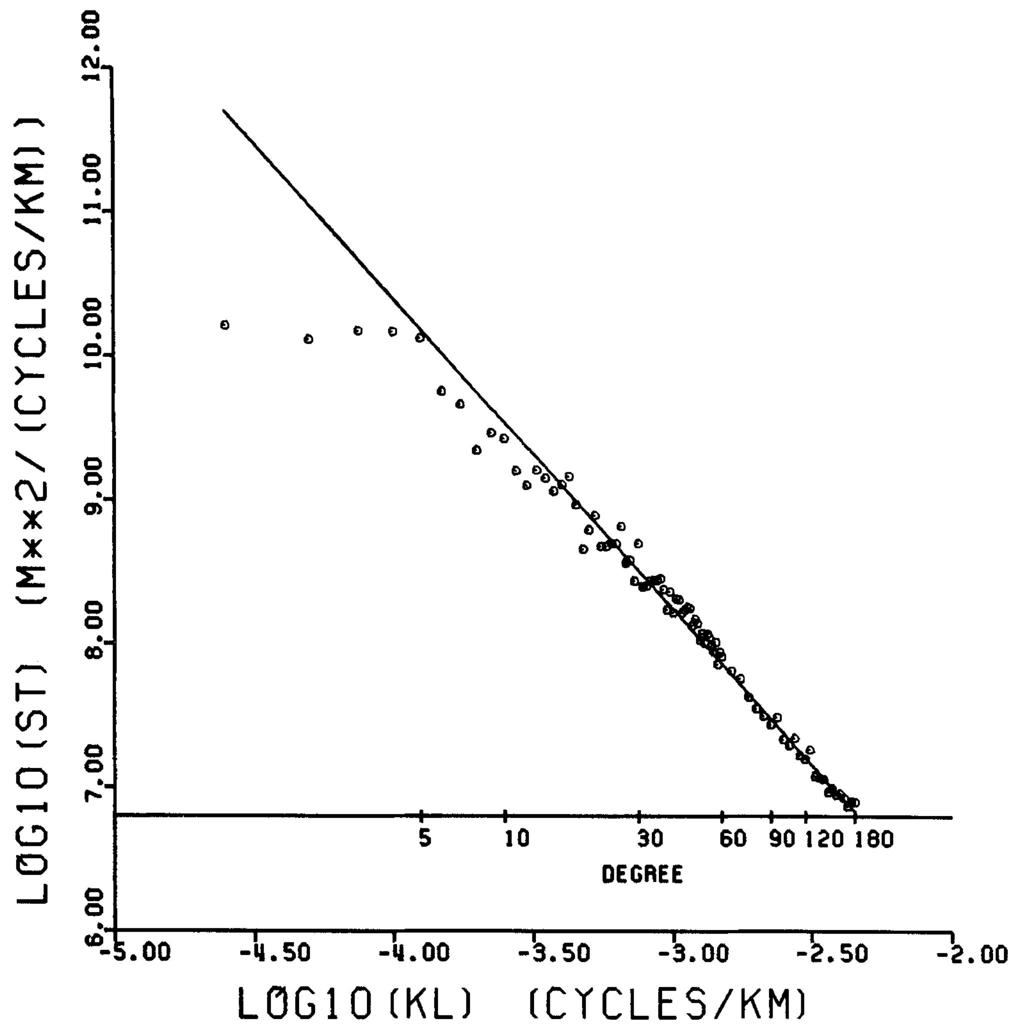

Figure 3. $\mathrm{Log}_{10}[S_T(k_l)$, m^2 cycle^{-1} km] for the equivalent rock topography implied by the TUG87 model versus degree (l) or $\log_{10}(k_l$, cycle km^{-1}).

In this case we considered the harmonic expansion of the actual topography (AT) and the equivalent rock topography (ERT). Values of the parameters are given in Table 2.

A plot $S_T(k_l)$ for the actual topography is shown in Fig. 2 along with the values from the implied model for the degree range 5–180. A plot of $S_T(k_l)$ for the equivalent rock topography is shown in Fig. 3. Similar calculations were carried out using equation (10) for the topographic isostatic potential with $D = 30$ km. The results are shown in Table 3 and Fig. 4.

Table 3. Values of A_p and β_p for the topographic isostatic potential based on the TUG87 elevation model with $D = 30$ km.

Degree range	A_p*	β_p
5–30	4.19×10^{-3}	1.87
31–100	1.71×10^{-5}	2.67
31–180	1.75×10^{-5}	2.66
101–180	3.96×10^{-5}	2.51
5–180	3.46×10^{-5}	2.54

* The units of A_p are the same as the units of A_p in Table 1.

DISCUSSION

The value of β_p changed considerably as different degree ranges were considered. If we take the 5–180 range, the value of 2.54 indicates a slower decay than implied by the studies discussed in the Introduction. However, if the degree range is 101–180, the β_p of 3.2 is in closer agreement to the value implied by the Kaula rule (i.e. 3) or the other studies based on the high degree analysis of altimetric data or terrestrial gravity data (i.e. $\beta_p \approx 3.6$). These results imply that the high degree spectrum decays faster than the spectrum at lower degrees.

In the analysis of the topography (Table 2) there is no significant slope difference between the actual topography and equivalent rock topography. However, the magnitude of the power spectral density differs by about a factor of 3. The average β value of 2.2 for degree range 5–180 is quite similar to the values for the degree range 101–180. We do not see here the substantial change in β seen in the potential for these two degree ranges. The value of 2.2 is close to the value of 2 estimated by Vening-Meinesz (1951) and Turcotte (1987) from lower degree information.

Comparing the slopes of the potential coefficient spectrum implied by the topographic isostatic model, and the actual

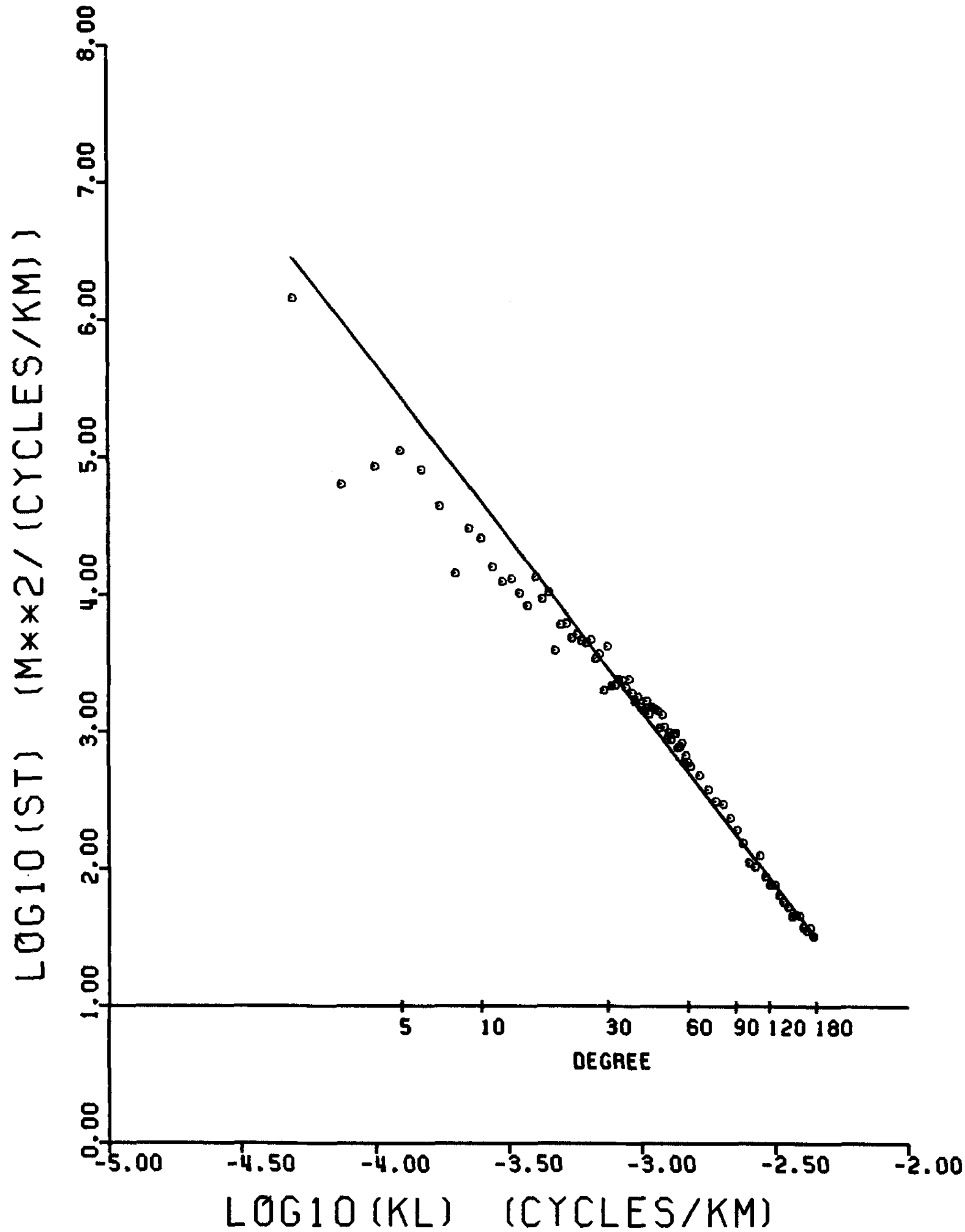

Figure 4. $\text{Log}_{10}[S_N(k_l), \text{m}^2\,\text{cycle}^{-1}\,\text{km}]$ for the topographic isostatic potential ($D = 30$ km) versus degree (l) or $\log_{10}(k_l, \text{cycle km}^{-1})$.

potential implied by the OSU86F model, we see (Tables 1 and 3) excellent agreement for the degree ranges 31–180 and 5–180. The magnitude of the topographic isostatic potential (as represented by A_p) is 0.27 that of the actual potential for degree range 31–180. (The ratio is 0.33 for the degree range 5–180.) Therefore, the topographic isostatic model is able to account for the observed rate of decay of the potential spectrum, but for only a third of the magnitude. The remaining signal must come from unmodelled effects in the simple Airy model. Such effects would include crustal density irregularities, contributions from the mantle and defects in the Airy model assumption with respect to local isostatic behaviour that depend on such parameters as elasticity and age.

The interpretation of the potential power rule has been the subject of several studies. Lambeck (1976) suggested that the power rule (up to degree 20) could be explained by randomly (both horizontally and vertically) distributed density anomalies. Kaula (1977) used a Monte Carlo technique to generate potential information using randomly selected density values of Gaussian distribution for various horizontal spacings, vertical spacings, number of layers, and a decay depth. Although a good fit to the decay exponent was found for some cases, the overall fit was not encouraging. Kaula suggests that more complex geophysical models are required. The current state of our knowledge of the spectra behaviour provides additional information that may be useful for geophysical inferences. The spectra at the lower degrees are better determined than earlier. These values appear to be dependent on the density irregularities in the mantle. At the much higher degrees the spectra are strongly dependent on the topography and its isostatic compensation. The interpretation of the spectra presented here is beyond the scope of this paper. More accurate spectra, at the higher degrees, will be obtained as our global gravity coverage improves. In addition the application of the topographic models needs to be improved through the use of ice thickness information and more reliable topographic and bathymetric information.

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REFERENCES

Balmino, G., Lambeck, K. & Kaula, W. 1973. Spherical harmonic analysis of the Earth's topography, *J. geophys. Res.*, **78,** 2, 478–481.

Forsberg, R., 1984. Local covariance functions and density distributions, *Report No.* 356, Department of Geodetic Science and Surveying, Ohio State University, Columbus.

Kaula, W. M., 1966. *Theory of Satellite Geodesy,* Blaisdell, Waltham, MA.

Kaula, W. M., 1977. Geophysical inferences from statistical analysis of the gravity field, in *The Changing World of Geodetic Science,* vol. 1, pp. 119–141, Department of Geodetic Science and Surveying, Ohio State University, Columbus.

Kaula, W. M., 1987. Satellite measurement of the Earth's gravity field, in *Methods of Experimental Physics,* Vol. 24, Part B, pp. 163–187, Academic Press, New York.

Lambeck, K., 1976. Lateral density anomalies in the upper mantle, *J. geophys. Res.*, **81,** 6333–6340.

Rapp, R. H., 1981. The Earth's gravity field to degree and order 180 using SEASAT altimeter data, terrestrial gravity data, and other data, *Report No.* 322, Department of Geodetic Science and Surveying, Ohio State University, Columbus.

Rapp, R. H., 1982. Degree variances of the Earth's potential, topography, and its isostatic compensation, *Bull. Geodes.*, **56,** 84–94.

Rapp, R. H., 1986. Global geopotential solutions, in *Lecture Notes in Earth Sciences,* Vol. 7, *Mathematical and Numerical Techniques in Physical Geodesy,* ed. Sunkel, H., Springer-Verlag, Berlin, Heidelberg.

Rapp, R. H. & Cruz, J. Y., 1986. Spherical harmonic expansion of the Earth's gravitational potential to degree 360 using 30′ mean anomalies, *Report No.* 376, Department of Geodetic Science and Surveying, Ohio State University, Columbus.

Rummel, R. *et al.,* 1988. Comparisons of global topographic/isostatic models to the Earth's observed gravity field, *Report No.* 388, Department of Geodetic Science and Surveying, Ohio State University, Columbus.

Turcotte, D., 1987. A fractal interpretation of topography and geoid spectra on the Earth, Moon, Venus, and Mars, *J. geophys. Res.*, **92,** E597–E601.

Vassiliou, A. A. & Schwarz, K. P., 1987. Study of the high-frequency spectrum of the anomalous gravity potential, *J. geophys. Res.* **92,** 609–617.

Vening-Meinesz, F. A., 1951. A remarkable feature of the Earth's topography, *Proc. R. ned. Akad. Wet. B, Phys. Sci.,* **54,** 212–228.

Vening-Meinesz, F. A., 1962. Thermal convection in the Earth's mantle, in *Continental Drift,* pp. 145–176, ed. Runcorn, S. K., Academic Press, New York.

Wieser, M., 1987. The global digital terrain model TUG87, *Internal Report,* Institute of Mathematical Geodesy, Technical University of Graz, Austria.