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LOW ENERGY THEOREMS FOR COMPTON SCATTERING ON TARGETS OF ARBITRARY SPIN

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A B S T R A C T

New low energy theorems for Compton scattering on targets of arbitrary spin are derived. The second order (in photon energy ω) low energy theorems derived in the previous paper for spin -1 targets are shown to be valid for targets of any spin, provided we interpret the spin operator $\underset{\sim}{S}$ for spin 1 in these theorems as the spin operator for any spin. Third order low energy theorems are derived. It is shown that the amplitudes are determined to the third order in ω by the electric charge, the dipole magnetic moment, the quadrupole electric moment, the octupole magnetic moment, and eight structure dependent constants. Our method can be used to derive higher order low energy theorems.

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1. INTRODUCTION

It is well known that, to first order in photon energy ω, the Compton scattering amplitudes for spin $-\frac{1}{2}$ targets are determined entirely by the electric charge e and the dipole magnetic moment μ of the target systems [1),2)]. The classical low energy theorems are valid to all orders of strong interaction and to the second order in electromagnetism. The results of Low, Gell-Mann and Goldberger can be generalized to include targets of arbitrary spin by replacing the Pauli matrix $\underset{\sim}{\sigma}$ in their theorems by $2\underset{\sim}{S}$ (where $\underset{\sim}{S}$ is the spin operator of the target)[3),4)]. For spin $-\frac{1}{2}$ targets, the second order $O(\omega^2)$ amplitudes are given by e, μ, and two structure dependent constants; while four additional constants are required to determine the third order amplitudes [5)]. For spin -1 targets, the second order amplitudes are given by e, μ, Q (the quadrupole electric moment), and four constants; while four additional constants are needed to fix the third order amplitudes [6)]. For arbitrary spin targets, Pais [7)] proved the existence of low energy theorems for each multipole moment and Saito [8)] proved that four structure dependent constants are required to determine the second order amplitudes.

In our previous paper [9)], we have studied the Compton scattering on targets of spin 1 and determined the amplitudes to the second order in ω. Our results are expressed in terms of the spin operator. The motivation of this paper is to generalize our previous results to include targets of higher spin. We shall demonstrate that the second order low energy theorems for spin -1 targets are, in fact, true for targets of arbitrary spin, provided we interpret the spin operator $\underset{\sim}{S}$ in these theorems as the spin operator for any spin [10)]. The third order low energy theorems for any spin are derived. Besides the low energy theorem for the octupole magnetic moment μ', which has been given by Pais, new results are obtained. We shall give the scattering amplitudes to third order in ω in terms of e, μ, Q, μ', and eight constants, where four of these constants appear only in the third order terms. In this paper, the P and T non-invariant effects are neglected.

The general properties of the low energy theorems are discussed in the following section. In Section 3, we shall apply the general result of Section 2 to derive the third order low energy theorems.

2. GENERAL FORMALISM

The Compton scattering is described by the amplitude matrix $A_{\mu\nu}(k',k)$ where $k'(k)$ and μ (ν) are the final (initial) photon four-momentum and polarization indices [11]. Gauge invariance implies

$$k'^{\mu} A_{\mu\nu} = A_{\mu\lambda} k^{\lambda} = 0 . \tag{1}$$

It follows from (1) that $(\omega \equiv k^0,\ \omega' \equiv k'^0)$

$$\omega\omega' A_{00} = k'^{i} A_{ij} k^{j} . \tag{2}$$

Equation (2) has been used by Pais to derive low energy theorems for each multipole moment. In order to obtain all possible low energy theorems, we shall use Eq. (1) which is more general than Eq. (2). Crossing symmetry implies

$$A_{\mu\nu}(k',k) = A_{\nu\mu}(-k,-k') . \tag{3}$$

We use the transverse gauge

$$\underset{\sim}{\epsilon}\cdot\underset{\sim}{k} = \underset{\sim}{\epsilon}'\cdot\underset{\sim}{k}' = 0 , \tag{4}$$

where $\underset{\sim}{\epsilon}$ $(\underset{\sim}{\epsilon}')$ is the initial (final) photon polarization vector. The physical amplitude is

$$A = \epsilon'^{i} A_{ij} \epsilon^{j} . \tag{5}$$

The amplitude is divided into two parts

$$A_{\mu\nu} = B_{\mu\nu} + E_{\mu\nu} , \tag{6}$$

where $B_{\mu\nu}$ is the Born (one-particle pole) contribution and $E_{\mu\nu}$ is the contribution from the excited states. $E_{\mu\nu}$ has no singularities as $k \to 0$ because of the minimum excitation energy. The Born contribution is

$$(2\pi)^{-6}[4E(\underset{\sim}{p})E(\underset{\sim}{p}')]^{-1}B_{\mu\nu} = \frac{\langle \underset{\sim}{p}'|j_\mu|\underset{\sim}{p}'+\underset{\sim}{k}'\rangle\langle \underset{\sim}{p}+\underset{\sim}{k}|j_\nu|\underset{\sim}{p}\rangle}{E(\underset{\sim}{p}+\underset{\sim}{k}) - E(\underset{\sim}{p}) - \omega} + \tag{7}$$

$$+ \text{ crossed term,}$$

where $E(\underset{\sim}{p}) = (\underset{\sim}{p}^2+m^2)^{\frac{1}{2}}$, m is the mass of the target, $\underset{\sim}{p}$ ($\underset{\sim}{p}'$) is the initial (final) target three-momentum, ω (ω') is the initial (final) photon energy, $|\underset{\sim}{p}\rangle$ is the one-particle (target) state with momentum $\underset{\sim}{p}$, j_μ is the electromagnetic current, and the crossed term is obtained from the other term by interchanging μ, ω, $\underset{\sim}{k}$ with ν, $-\omega'$, $-\underset{\sim}{k}'$. A summation over all intermediate spin states is implied. Following Singh [12], we write

$$E_{00} = k'^i \Lambda_{ij}(\omega', \underset{\sim}{k}'; \omega, \underset{\sim}{k}) k^j, \tag{8}$$

where Λ_{ij} is even under crossing and free from kinematic singularities.

The multipole expansion of the electromagnetic current matrix element is [13]

$$(2\pi)^3[4E(\underset{\sim}{p})E(\underset{\sim}{p}')]^{\frac{1}{2}}\langle \underset{\sim}{p}'|j_\mu|\underset{\sim}{p}\rangle =$$

$$= D^{-1}(L_{p'})\frac{2m}{P^2}\sum_\ell \left(\left\{\alpha_\ell(q^2)P_\mu + i\beta_\ell(q^2)\varepsilon_{\mu\alpha\beta\gamma}P^\alpha q^\beta \frac{\partial}{\partial q_\gamma}\right\}\mathcal{P}_\ell(W\cdot q)\right)D(L_p) \tag{9}$$

where

$$P = p'+p, \quad q = p'-p, \quad p^2 = p'^2 = m^2,$$

$$D(L_p) = \exp[-\underset{\sim}{S}\cdot\hat{p}\,\varphi],$$

$$\cosh\varphi = p_0/m, \quad \underset{\sim}{p} = |\underset{\sim}{p}|\hat{p},$$

$$W_\mu = (i/2m)\, S_{\mu\nu} P^\nu,$$
$$S_{io} = -S_{oi} = i S_i\ ,\ S_{ij} = \epsilon_{ijk} S^k,\ S_{oo} = 0,$$
$$W\cdot q = (2m)^{-1}(P_o\, \underset{\sim}{S}\cdot \underset{\sim}{q} - q_o\, \underset{\sim}{S}\cdot\underset{\sim}{P} + i\, \underset{\sim}{S}\cdot\underset{\sim}{q}\times\underset{\sim}{P}),$$
$$\mathcal{P}_\ell\,(a_\mu b^\mu) = P_\ell\left(\frac{a\cdot b}{[a^2 b^2]^{\frac{1}{2}}}\right)(a^2 b^2)^{\ell/2}.$$

$P_\ell(x)$ are the Legendre polynomials. The form factors $\alpha_\ell(q^2)$ and $\beta_\ell(q^2)$ are free from kinematic singularities at $q^2 = 0$. It is useful to define

$$\langle \underset{\sim}{p}'|\, j_\mu\, |\underset{\sim}{p}\rangle = \langle \underset{\sim}{p}'|\, \bar{j}_\mu\, |\underset{\sim}{p}\rangle + q^2 \langle \underset{\sim}{p}'|\, j'_\mu\, |\underset{\sim}{p}\rangle, \qquad (10)$$

where

$$(2\pi)^3 [4 p_o p'_o]^{\frac{1}{2}} \langle \underset{\sim}{p}'|\, \bar{j}_\mu\, |\underset{\sim}{p}\rangle$$
$$= D^{-1}(L_{p'})(2m)^{-1}\sum_\ell (\{a_\ell P_\mu + i\, b_\ell\, \varepsilon_{\mu\alpha\beta\gamma} P^\alpha q^\beta \frac{\partial}{\partial q_\gamma}\}(W\cdot q)^\ell)\, D(L_p),$$

$$a_\ell = C\, \alpha_\ell(0)\ ,\quad b_\ell = C\, \beta_\ell(0),$$

C is the numerical coefficient of the term x^ℓ in $P_\ell(x)$, $\langle \underset{\sim}{p}'|\, j'_\mu\, |\underset{\sim}{p}\rangle$ is free from kinematic singularities at $q^2 = 0$, and a_ℓ and b_ℓ are linear combinations of the multipole moments which are defined in the Breit frame. We have [14]

$$\sum_\ell a_\ell\, x^\ell = e \sum_m A_m\, x^m \exp(-x)\ ,\quad m = \text{even},$$

$$\sum_\ell b_\ell\, \ell\, x^\ell = e \sum_n B_n\, n\, x^n \exp(-x),\quad n = \text{odd}, \qquad (11)$$

where A_m (B_m) are proportional to the electric (magnetic) 2^m pole moments. In particular, we have [15)]

$$A_0 = 1 \quad , \quad A_2 = -\tfrac{1}{2} Q \ (\text{in units } e/m^2),$$
$$B_1 = \mu \ (\text{in units } e/2m), \quad B_3 = -3^{-1} 6^{-\frac{1}{2}} \mu' \ (\text{in units } e/2m^3). \tag{12}$$

Equations (11) are identities with respect to the variable x.

The amplitude can now be written in the form :

$$A_{\mu\nu} = U_{\mu\nu} + T_{\mu\nu},$$
$$(2\pi)^{-6} [4E(\underline{p})E(\underline{p}')]^{-1} U_{\mu\nu} =$$
$$= \frac{\langle \underline{p}' | \bar{j}_\mu | \underline{p}' + \underline{k}' \rangle \langle \underline{p} + \underline{k} | \bar{j}_\nu | \underline{p} \rangle}{E(\underline{p}+\underline{k}) - E(\underline{p}) - \omega} + \text{crossed term}, \tag{13}$$
$$T_{00} = k'^i \Lambda_{ij} k^j ,$$

where Λ_{ij} are free from kinematic singularities and even under crossing [12)].

It follows from (1) that ($\rho \equiv \bar{j}_0$)

$$k'^\mu U_{\mu\nu} = - k'^\mu T_{\mu\nu}$$
$$= (2\pi)^6 [4E(\underline{p})E(\underline{p}')] \big(- \langle \underline{p}' | \rho | \underline{p}' + \underline{k}' \rangle \langle \underline{p} + \underline{k} | \bar{j}_\nu | \underline{p} \rangle +$$
$$+ \langle \underline{p}' | \bar{j}_\nu | \underline{p}' - \underline{k} \rangle \langle \underline{p} - \underline{k}' | \rho | \underline{p} \rangle \big), \tag{14}$$
$$U_{\mu\nu} k^\nu = - T_{\mu\nu} k^\nu$$
$$= (2\pi)^6 [4E(\underline{p})E(\underline{p}')] \big(- \langle \underline{p}' | \bar{j}_\mu | \underline{p}' + \underline{k}' \rangle \langle \underline{p} + \underline{k} | \rho | \underline{p} \rangle +$$
$$+ \langle \underline{p}' | \rho | \underline{p}' - \underline{k} \rangle \langle \underline{p} - \underline{k}' | \bar{j}_\mu | \underline{p} \rangle \big).$$

To derive Eq. (14), we have used the condition of current conservation :

$$k'^i \langle \underline{p}' | \bar{j}_i | \underline{p}' + \underline{k}' \rangle = [E(\underline{p}') - E(\underline{p}' + \underline{k}')] \langle \underline{p}' | \rho | \underline{p}' + \underline{k}' \rangle . \tag{15}$$

The physical amplitude is

$$A = \varepsilon'^{i}(U_{ij} + T_{ij})\epsilon^{j}, \tag{16}$$

where T_{ij} are determined from Eqs. (14). If follows from (14) that

$$\begin{aligned}
T_{oj}k^{j} + \omega T_{oo} &= (2\pi)^{6} 4E(\underline{p})E(\underline{p}')\big(\langle \underline{p}'|\rho|\underline{p}'+\underline{k}'\rangle\langle \underline{p}+\underline{k}|\rho|\underline{p}\rangle - \\
&\quad - \langle \underline{p}'|\rho|\underline{p}'-\underline{k}\rangle\langle \underline{p}-\underline{k}'|\rho|\underline{p}\rangle\big), \\
T_{oo} &= k'^{i}\Lambda_{ij}k^{j}, \\
k'^{i}T_{ij} + \omega' T_{oj} &= (2\pi)^{6} 4E(\underline{p})E(\underline{p}')\big(\langle \underline{p}'|\rho|\underline{p}'+\underline{k}'\rangle\langle \underline{p}+\underline{k}|\bar{j}_{j}|\underline{p}\rangle - \\
&\quad - \langle \underline{p}'|\bar{j}_{j}|\underline{p}'-\underline{k}\rangle\langle \underline{p}-\underline{k}'|\rho|\underline{p}\rangle\big).
\end{aligned} \tag{17}$$

Let T^{2}_{ij} be any solution of these equations (which is regular for small values of ω, ω', $\underline{k}$, and $\underline{k}'$). Then the most general solution of Eq. (17) is $T_{ij} = T^{1}_{ij} + T^{2}_{ij}$ where

$$k'^{i}T^{1}_{ij} + \omega' T^{1}_{oj} = T^{1}_{oj}k^{j} + \omega k'^{i}\Lambda_{ij}k^{j} = 0. \tag{18}$$

The most general forms of T^{1}_{ij} and T^{1}_{oj} are :

$$\begin{aligned}
T^{1}_{oj} &= -\omega k'^{i}\Lambda_{ij} + \epsilon_{ijm}k'^{i}k^{m}(T^{s} - T^{a}), \\
T^{1}_{ij} &= \omega\omega'\Lambda_{ij} + \epsilon_{imn}\epsilon_{jm'n'}k'^{m}k^{m'}\bar{\Lambda}_{nn'} + \\
&\quad + T^{s}(\omega\epsilon_{ijm}k'^{m} - \omega'\epsilon_{ijm}k^{m}) + \\
&\quad + T^{a}(\omega\epsilon_{ijm}k'^{m} + \omega'\epsilon_{ijm}k^{m}),
\end{aligned} \tag{19}$$

where Λ_{ij}, $\bar{\Lambda}_{ij}$, and T^{s} are any crossing even functions of (ω, ω', k, k'), and T^{a} is any crossing odd function of (ω, ω', k, k'), which are regular for small values of ω, ω', k, and k'. P and T invariances further restrict the possible forms of Λ_{ij}, $\bar{\Lambda}_{ij}$, T^{s} and T^{a} [16].

3. THIRD ORDER LOW ENERGY THEOREMS

In this section, we shall apply the result of Section 2 to derive low energy theorems for arbitrary spin targets up to the third order. In the previous paper, the second order low energy theorems for spin -1 targets have been derived by using a different and more complicated method. We are able to rederive the second order terms in a much simpler way.

We shall use the laboratory system. We have

$$\omega - \omega' = m^{-1}\omega\omega'(1-\cos\theta),$$

$$E \equiv E(\underset{\sim}{k}),$$

$$(2\pi)^3(4mE)^{\frac{1}{2}}\langle \underset{\sim}{k}|\rho|0\rangle = e\,D^{-1}(L_k)(2m)^{-1}(m+E)\sum_l a_l(\underset{\sim}{S}\cdot\underset{\sim}{k})^l, \tag{20}$$

$$(2\pi)^3(4mE)^{\frac{1}{2}}\langle \underset{\sim}{k}|\underset{\sim}{j}|0\rangle = e\,D^{-1}(L_k)(2m)^{-1}\Big\{\underset{\sim}{k}\sum_l[(a_l+l\,b_l)(\underset{\sim}{S}\cdot\underset{\sim}{k})^l] +$$

$$+ i\sum_l\big[b_l\langle(\underset{\sim}{S}\cdot\underset{\sim}{k})^{l-1}, (m+E)\underset{\sim}{S}\times\underset{\sim}{k} + i\,\underset{\sim}{S}\,\omega^2\rangle\big]\Big\},$$

$$\langle a^n, b\rangle \equiv \sum_{l+m=n} a^l\, b\, a^m, \quad l \geqslant 0,\ m \geqslant 0,$$

$$(2\pi)^3[4E(\underset{\sim}{k}-\underset{\sim}{k}')E(\underset{\sim}{k})]^{\frac{1}{2}}\langle \underset{\sim}{k}-\underset{\sim}{k}'|\rho|\underset{\sim}{k}\rangle =$$

$$e\,D^{-1}(L_{k-k'})\Big\{(2m)^{-1}(m+E+\omega-\omega')\sum_l(a_l C^l) +$$

$$+ i(2m^2)^{-1}\sum_l\big[b_l\langle C^{l-1}, (m+E+\omega-\omega')\underset{\sim}{S}\cdot\underset{\sim}{k}\times\underset{\sim}{k}' + i(2\omega^2-\underset{\sim}{k}\cdot\underset{\sim}{k}')\underset{\sim}{S}\cdot\underset{\sim}{k}' +$$

$$+ i(\omega'^2 - 2\underset{\sim}{k}\cdot\underset{\sim}{k}')\underset{\sim}{S}\cdot\underset{\sim}{k}\rangle\big]\Big\}D(L_k),$$

$$C \equiv -m^{-1}\left[E\,\mathbf{S}\cdot\mathbf{k}' + i\,\mathbf{S}\cdot\mathbf{k}'\times\mathbf{k} + (m-E+\omega-\omega')\,\mathbf{S}\cdot\mathbf{k}\right],$$

$$(2\pi)^3\left[4E(\mathbf{k}-\mathbf{k}')E(\mathbf{k})\right]^{\frac{1}{2}}\langle \mathbf{k}-\mathbf{k}'|\mathbf{j}|\mathbf{k}\rangle = e\,D^{-1}(L_{k-k'})(2m)^{-1}\Big\{$$

$$(2\mathbf{k}-\mathbf{k}')\sum_l\left[(a_l+l\,b_l)C^l\right] + i\,m^{-1}\sum_l\Big[b_l\Big\langle C^{l-1},\ \mathbf{S}\cdot(2\mathbf{k}-\mathbf{k}')(\mathbf{k}\times\mathbf{k}') +$$

$$+(\mathbf{S}\cdot\mathbf{k}'\times\mathbf{k})(2\mathbf{k}-\mathbf{k}') - i\left(E(2\mathbf{k}\cdot\mathbf{k}'-\omega'^2) + (m-E+\omega-\omega')(2\omega^2-\mathbf{k}\cdot\mathbf{k}')\right)\mathbf{S} -$$

$$-E(m+E+\omega-\omega')(\mathbf{S}\times\mathbf{k}') + (2\mathbf{k}\cdot\mathbf{k}'-\omega'^2)(\mathbf{S}\times\mathbf{k})\Big\rangle\Big]\Big\}D(L_k),$$

where θ is the scattering angle and other matrix elements can be obtained from the above ones by crossing.

Let us consider the structure dependent part of the amplitude first. Parity conservation implies [17]

$$T^1_{ij}(\omega',\mathbf{k}',\omega,\mathbf{k},\mathbf{S}) = T^1_{ij}(\omega',-\mathbf{k}',\omega,-\mathbf{k},\mathbf{S}). \tag{21}$$

The restriction imposed by T invariance in the laboratory system is more complicated. In the Breit frame, T invariance implies

$$T^1_{ij}(\omega',\mathbf{k}',\omega,\mathbf{k},\mathbf{S}) = T^1_{ji}(\omega,-\mathbf{k},\omega',-\mathbf{k}',-\mathbf{S}). \tag{22}$$

Using the conditions (21) and (22), Pais [7] has given a general method to determine the complete minimal basis for T^1_{ij}. Note that the lowest order terms of T^1_{ij} are of the second order. Therefore, to get the second order terms, we can use (22) in the laboratory frame as well since any correction must be at least of the third order. We have

$$\begin{aligned}
\Lambda_{ij} &= c_1\,\delta_{ij} + c_2\{S_i,S_j\} + c_3(\omega+\omega')\epsilon_{ijm}S^m + O(\omega^2),\\
T^a &= c_4\,\mathbf{S}\cdot(\mathbf{k}+\mathbf{k}') + O(\omega^2),\\
T^s &= c_5\,\mathbf{S}\cdot(\mathbf{k}-\mathbf{k}') + O(\omega^2),
\end{aligned} \tag{23}$$

where $\{a,b\} \equiv ab+ba$ and c_i are constants. It is now clear that the structure dependent part of the amplitude is

$$A^1 = \epsilon'^i T^1_{ij} \epsilon^j = A^1(2) + O(\omega^3),$$

$$A^1(2) \equiv c_1 \omega\omega' \underset{\sim}{\epsilon}' \cdot \underset{\sim}{\epsilon} + c_2 (\underset{\sim}{\epsilon}' \times \underset{\sim}{k}') \cdot (\underset{\sim}{\epsilon} \times \underset{\sim}{k}) + \tag{24}$$

$$+ c_3 \omega\omega' \{\underset{\sim}{S} \cdot \underset{\sim}{\epsilon}', \underset{\sim}{S} \cdot \underset{\sim}{\epsilon}\} + c_4 \{\underset{\sim}{S} \cdot \underset{\sim}{\epsilon}' \times \underset{\sim}{k}', \underset{\sim}{S} \cdot \underset{\sim}{\epsilon} \times \underset{\sim}{k}\}.$$

To get the third order terms of A^1, let us go back to the Breit frame. We have

$$(A^1)_{Breit} = A^1(2) + (\omega+\omega') A^1(3) + O(\omega^4),$$

$$A^1(3) \equiv c_5 \omega\omega' \underset{\sim}{S} \cdot \underset{\sim}{\epsilon}' \times \underset{\sim}{\epsilon} + c_6 \underset{\sim}{S} \cdot (\underset{\sim}{\epsilon}' \times \underset{\sim}{k}') \times (\underset{\sim}{\epsilon} \times \underset{\sim}{k}) +$$

$$+ c_7 [\underset{\sim}{\epsilon}' \cdot \underset{\sim}{k} (\underset{\sim}{S} \cdot \underset{\sim}{\epsilon} \times \underset{\sim}{k}) - \underset{\sim}{\epsilon} \cdot \underset{\sim}{k}' (\underset{\sim}{S} \cdot \underset{\sim}{\epsilon}' \times \underset{\sim}{k}')] + \tag{25}$$

$$+ c_8 [\underset{\sim}{\epsilon}' \cdot \underset{\sim}{\epsilon} (\underset{\sim}{S} \cdot \underset{\sim}{k}' \times \underset{\sim}{k}) - \underset{\sim}{k}' \cdot \underset{\sim}{k} (\underset{\sim}{S} \cdot \underset{\sim}{\epsilon}' \times \underset{\sim}{\epsilon})].$$

In the laboratory frame, the second order terms in $(A^1)_{Breit}$ introduce correction terms of the third order. Since no spin operator appears in the first and second terms of $A^1(2)$, one can get the correction terms very easily by considering a spin -0 target. Similarly, the correction terms for the third and fourth terms of $A^1(2)$ can be obtained by considering a spin -1 target [6]. We have

$$(A^1)_{Lab} = (A^1)_{Breit} + \frac{1}{4}(\omega+\omega') A_c + O(\omega^4),$$

$$A_c \equiv c_3 (\underset{\sim}{\epsilon}' \cdot \underset{\sim}{k} \{\underset{\sim}{S} \cdot \underset{\sim}{\epsilon}, \underset{\sim}{S} \cdot \underset{\sim}{k}'\} - \underset{\sim}{\epsilon} \cdot \underset{\sim}{k}' \{\underset{\sim}{S} \cdot \underset{\sim}{\epsilon}', \underset{\sim}{S} \cdot \underset{\sim}{k}\}) +$$

$$+ c_4 (\{\underset{\sim}{S} \cdot \underset{\sim}{\epsilon}' \times \underset{\sim}{k}', \underset{\sim}{S} \cdot \underset{\sim}{\epsilon} \times \underset{\sim}{k}'\} - \{\underset{\sim}{S} \cdot \underset{\sim}{\epsilon}' \times \underset{\sim}{k}, \underset{\sim}{S} \cdot \underset{\sim}{\epsilon} \times \underset{\sim}{k}\}). \tag{26}$$

To derive low energy theorems up to the third order for the total amplitude, it is sufficient to use the following approximation : (we set m=1 for simplicity)

$$
\begin{aligned}
&D^{-1}(L_{k-k'}) = 1 + \underset{\sim}{S}\cdot(\underset{\sim}{k}-\underset{\sim}{k}') + \tfrac{1}{2}[\underset{\sim}{S}\cdot(\underset{\sim}{k}-\underset{\sim}{k}')]^2 + \\
&+ \tfrac{1}{6}\underset{\sim}{S}\cdot(\underset{\sim}{k}-\underset{\sim}{k}')\{[\underset{\sim}{S}\cdot(\underset{\sim}{k}-\underset{\sim}{k}')]^2 - (\underset{\sim}{k}-\underset{\sim}{k}')^2\} + O(\omega^4), \\
&D^{-1}(L_{k-k'})\exp(-C)\exp(-\underset{\sim}{S}\cdot\underset{\sim}{k}) \pm \text{crossed term} = A^{\pm}, \\
&A^{+} \equiv 2 + O(\omega^4), \\
&A^{-} \equiv i\,\underset{\sim}{S}\cdot\underset{\sim}{k}'\times\underset{\sim}{k}\,(1 + \tfrac{1}{4}\underset{\sim}{k}\cdot\underset{\sim}{k}' - \tfrac{1}{2}\omega^2) + O(\omega^5).
\end{aligned}
\tag{27}
$$

After a straightforward calculation, we find [18)]

$$
\begin{aligned}
&e^{-2}T^{2}_{0j}k^{j} = \tfrac{i}{4}(Q^2+2Q)\{\underset{\sim}{S}\cdot\underset{\sim}{k}', \{\underset{\sim}{S}\cdot\underset{\sim}{k}, (\underset{\sim}{S}\cdot\underset{\sim}{k}'\times\underset{\sim}{k})\}\} + \\
&+\tfrac{i}{4}(Q\mu - Q + \mu)\{\underset{\sim}{S}\cdot\underset{\sim}{k}'\times\underset{\sim}{k}, (\underset{\sim}{S}\cdot\underset{\sim}{k})^2 + (\underset{\sim}{S}\cdot\underset{\sim}{k}')^2\} + \\
&+\tfrac{1}{2}Q(1-\mu)\underset{\sim}{k}\cdot\underset{\sim}{k}'[(\underset{\sim}{S}\cdot\underset{\sim}{k})^2 - (\underset{\sim}{S}\cdot\underset{\sim}{k}')^2] - i(2\mu-1)\underset{\sim}{S}\cdot\underset{\sim}{k}'\times\underset{\sim}{k} + \\
&+i\bar{\mu}[\langle(\underset{\sim}{S}\cdot\underset{\sim}{k})^2, \underset{\sim}{S}\cdot\underset{\sim}{k}'\times\underset{\sim}{k}\rangle + \langle(\underset{\sim}{S}\cdot\underset{\sim}{k}')^2, \underset{\sim}{S}\cdot\underset{\sim}{k}'\times\underset{\sim}{k}\rangle] - \\
&-\tfrac{i}{4}\underset{\sim}{k}\cdot\underset{\sim}{k}'\,\underset{\sim}{S}\cdot\underset{\sim}{k}'\times\underset{\sim}{k} + O(\omega^5),
\end{aligned}
\tag{28}
$$

where

$$\bar{\mu} \equiv 3^{-1}6^{-\frac{1}{2}}\mu' - 6^{-1}\mu ,$$

$$e^{-2}(k'^i T^2_{ij} + \omega' T^2_{0j}) = \tfrac{1}{2} Q\mu \Big[\{\mathbf{S}\cdot\mathbf{k}, \mathbf{S}\cdot\mathbf{k}'\} k'_j -$$

$$- \mathbf{k}\cdot\mathbf{k}' \{S_j, \mathbf{S}\cdot\mathbf{k}'\}\Big] + \tfrac{1}{2} k'_j \Big\{ 2 + i(\mu-1)\mathbf{S}\cdot\mathbf{k}'\times\mathbf{k} - Q[(\mathbf{S}\cdot\mathbf{k})^2 + (\mathbf{S}\cdot\mathbf{k}')^2]\Big\} -$$

$$- \tfrac{i}{2} k_j (\mu-1)\mathbf{S}\cdot\mathbf{k}'\times\mathbf{k} + \tfrac{1}{2}\mu(\mu-1)\,\epsilon_{ijm} k^m \{S^i, \mathbf{S}\cdot\mathbf{k}'\times\mathbf{k}\} - \qquad (29)$$

$$- \tfrac{i}{2}\mu(\mu-2)\,\mathbf{S}\cdot\mathbf{k}\,\epsilon_{ijm} k^i k'^m + i\mu\omega\,\mathbf{k}\cdot\mathbf{k}'\,\epsilon_{ijm} S^i k'^m +$$

$$+ O(\omega^5),$$

$$e^{-2}\epsilon'^i U_{ij}\epsilon^j = \tfrac{i}{2}(\omega+\omega')U_1 + U_2 + U_3 + O(\omega^4), \qquad (30)$$

where [19]

$$\langle a, b, c\rangle \equiv \{\{a, b\}, c\} + acb + bca,$$

$$U_1 \equiv \mu\Big[\boldsymbol{\epsilon}'\cdot\hat{k}\,(\mathbf{S}\cdot\boldsymbol{\epsilon}\times\hat{k}) - \boldsymbol{\epsilon}\cdot\hat{k}'\,(\mathbf{S}\cdot\boldsymbol{\epsilon}'\times\hat{k}')\Big] +$$

$$+ \mu^2\,\mathbf{S}\cdot(\boldsymbol{\epsilon}\times\hat{k})\times(\boldsymbol{\epsilon}'\times\hat{k}'),$$

$$U_2 \equiv -\tfrac{1}{2}\mu^2 \cos\theta\,\{\mathbf{S}\cdot\boldsymbol{\epsilon}\times\mathbf{k}, \mathbf{S}\cdot\boldsymbol{\epsilon}'\times\mathbf{k}'\} +$$

$$+ \tfrac{i}{2}\mu(2-\cos\theta)\Big[\boldsymbol{\epsilon}\cdot\mathbf{k}'(\mathbf{S}\cdot\boldsymbol{\epsilon}'\times\mathbf{k}') + \boldsymbol{\epsilon}'\cdot\mathbf{k}(\mathbf{S}\cdot\boldsymbol{\epsilon}\times\mathbf{k})\Big],$$

$$U_3 \equiv \tfrac{i}{2}\mu\omega(1-\cos\theta)^2\Big[\boldsymbol{\epsilon}'\cdot\mathbf{k}(\mathbf{S}\cdot\boldsymbol{\epsilon}\times\mathbf{k}) - \boldsymbol{\epsilon}\cdot\mathbf{k}'(\mathbf{S}\cdot\boldsymbol{\epsilon}'\times\mathbf{k}')\Big] +$$

$$+ i\mu\bar{\mu}\,\omega^{-1}\Big\{\langle(\mathbf{S}\cdot\mathbf{k})^2, \mathbf{S}\cdot(\boldsymbol{\epsilon}'\times\mathbf{k}')\times(\boldsymbol{\epsilon}\times\mathbf{k})\rangle +$$

$$+ \langle\mathbf{S}\cdot\mathbf{k}, \mathbf{S}\cdot\boldsymbol{\epsilon}\times\mathbf{k}, \mathbf{S}\cdot(\boldsymbol{\epsilon}'\times\mathbf{k}')\times\mathbf{k}\rangle -$$

$$- \text{c.t.}\Big\} +$$

$$+\tfrac{1}{2}\omega\mu^2\Big[\epsilon'\cdot k\{S\cdot\epsilon, S\cdot k'\}-\{S\cdot\epsilon'\times k, S\cdot\epsilon\times k\}+\tfrac{i}{2}k\cdot k' S\cdot\epsilon'\times\epsilon - c.t.\Big]-$$

$$-i\bar{\mu}\,\omega^{-1}\Big[\epsilon'\cdot k\langle(S\cdot k)^2, S\cdot\epsilon\times k\rangle - c.t.\Big]-$$

$$-i\mu Q(2\omega)^{-1}\Big[\epsilon'\cdot k\,(S\cdot k')^2 S\cdot\epsilon\times k - c.t.\Big]+$$

$$+\mu(\mu-1)(2\omega)^{-1}\Big[\epsilon'\cdot k\,(S\cdot k'\times k)(S\cdot\epsilon\times k) - c.t.\Big]-$$

$$-i\mu(4\omega)^{-1}\Big[\epsilon'\cdot k\{(S\cdot k)^2, S\cdot\epsilon\times k\} - c.t.\Big]+$$

$$+\mu^2(4\omega)^{-1}\Big\{2i(S\cdot k)^2 S\cdot(\epsilon'\times k')\times(\epsilon\times k)-$$

$$-i\,S\cdot k'\times k\{S\cdot\epsilon'\times k', S\cdot\epsilon\times k\}-2i\,k\cdot k' S\cdot\epsilon'\{S\cdot k, S\cdot\epsilon\times k\}+$$

$$+2i\,\epsilon'\cdot k\{S\cdot k, S\cdot k'\}S\cdot\epsilon\times k - c.t.\Big\}.$$

Finally, we obtain the physical amplitude :

$$A = A^1 - e^2m^{-1}\epsilon'\cdot\epsilon + \tfrac{i}{2}e^2m^{-2}(\omega+\omega')\Big[(2\mu-1)S\cdot\epsilon'\times\epsilon + U_1\Big] + e^2m^{-3}B_2 + e^2m^{-4}B_3 + O(\omega^4), \tag{31}$$

where

$$B_2 \equiv \tfrac{1}{2}Q\,\epsilon'\cdot\epsilon\,\Big[(S\cdot k)^2+(S\cdot k')^2\Big]+$$

$$+\tfrac{1}{2}Q\mu\Big[k\cdot k'\{S\cdot\epsilon, S\cdot\epsilon'\}-\epsilon'\cdot\epsilon\{S\cdot k, S\cdot k'\}\Big]-$$

$$-\tfrac{1}{2}\mu^2\cos\theta\{S\cdot\epsilon\times k, S\cdot\epsilon'\times k'\}+$$

$$+\frac{1}{2}\mu(\mu-1)\left[\{\mathbf{S}\cdot\boldsymbol{\epsilon}'\times\mathbf{k},\ \mathbf{S}\cdot\boldsymbol{\epsilon}\times\mathbf{k}\}+c.t.\right]-$$

$$-\frac{1}{2}i(\mu-1)\left[\boldsymbol{\epsilon}'\cdot\mathbf{k}\,(\mathbf{S}\cdot\boldsymbol{\epsilon}\times\mathbf{k}')+c.t.\right]+$$

$$+\frac{1}{2}i\mu(\mu-\cos\theta)\left[\boldsymbol{\epsilon}'\cdot\mathbf{k}\,(\mathbf{S}\cdot\boldsymbol{\epsilon}\times\mathbf{k})+c.t.\right],$$

$$B_3 \equiv U_3+\frac{i}{8}\omega(Q^2+2Q)\left[\{\mathbf{S}\cdot\mathbf{k},\ \{\mathbf{S}\cdot\mathbf{k}',\ \mathbf{S}\cdot\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\}\}-c.t.\right]+$$

$$+\frac{i}{4}\omega(Q-\mu Q-\mu)\{\mathbf{S}\cdot\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon},\ (\mathbf{S}\cdot\mathbf{k})^2+(\mathbf{S}\cdot\mathbf{k}')^2\}-$$

$$-\frac{1}{2}\omega Q(1-\mu)\,\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon}\left[(\mathbf{S}\cdot\mathbf{k})^2-(\mathbf{S}\cdot\mathbf{k}')^2\right]-$$

$$-i\omega\bar{\mu}\left[\{(\mathbf{S}\cdot\mathbf{k})^2,\ \mathbf{S}\cdot\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon}-c.t.\right]-$$

$$-\frac{i}{4}\omega\,\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon}\,(\mathbf{S}\cdot\mathbf{k}'\times\mathbf{k}).$$

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   In Eqs. (5) and (8), all terms of the form $\{\underset{\sim}{S}\times(\underset{\sim}{\epsilon}\times\underset{\sim}{a}),\ \underset{\sim}{S}\times(\underset{\sim}{\epsilon}'\times\underset{\sim}{b})\}$ should be replaced by $-\{\underset{\sim}{S}\cdot\underset{\sim}{\epsilon}\times\underset{\sim}{a},\ \underset{\sim}{S}\cdot\underset{\sim}{\epsilon}'\times\underset{\sim}{b}\}$. The last sentence of Section 3 should be replaced by : if we use the identity $\{S_i, S_j\} = \frac{1}{2}\,\delta_{ij}$ to get rid of all anticommutators.

10) Parts of the second order low energy theorems for arbitrary spin targets are given already by Pais (the quadrupole moment theorem) and Saito (the structure dependent constants). See Refs. 7) and 8).

11) We shall use the metric $k_\mu = (k_0, k_i)$ defined by :
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17) This condition is valid for any Lorentz frame.

18) We ignore all terms which can be absorbed into the structure dependent part of the amplitude.

19) We use c.t. to denote the term (or terms) which can be obtained from the other term (or terms) by crossing.