THE DECOMPOSITION OF A DIFFERENTIABLE MANIFOLD AND ITS APPLICATIONS.

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Introduction. Since G. de Rham [2] proved an interesting theorem concerning the decomposition of simply-connected, complete, reducible Riemannian manifolds, it has been referred to by many authors. The author [3] already attempted an extension of the theorem to an affinely connected manifold. In this note let us treat further to extend it to a differentia le manifold. For this purpose, we shall first introduce the notion of a locally decomposed C^u-manifold with latticed maps (§ 1). In such a manifold, we prove a theorem on its fundamental group and we show that the manifold decomposes globally if it is simply-connected (Theorems 1, 2). Further, as its applications, we prove that locally decomposed, affinely connected manifold and Finsler manifold (§ 4) admit always latticed maps and we show that they decompose globally if they are simply-connected (Theorems 3, 4). All of these results are nothing but extensions of the G. de Rham's theorem, and further note that the idea is analogous to that in [3]. Throughout the whole discussion, let us suppose that the indices run as follows:

$$a,\ b,\ c = 1,\ 2, \ldots\ldots, r;\ i,\ j,\ k = r+1,\ r+2, \ldots\ldots, n;$$
$$\alpha,\ \beta,\ \gamma = 1,\ 2, \ldots\ldots, n.$$

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1. Locally decomposed C^u-manifolds with latticed maps. We take an n-dimensional connected manifold M of class C^u such that the maximal system Ω of admissible coordinate neighbourhoods has the following properties:

1) In each neighbourhood $U \in \Omega$, its coordinate system (x^α) consists of all of (x^α)'s such that $0 < x^\alpha < 1$.

2) For each pair $U, U' \in \Omega$, $U \cap U' \neq 0$, if x is any point of $U \cap U'$ there is a neighbourhood $V \subset U \cap U'$ of x where the transformation from the coordinates x^α in U to the ones x'^α in U' is expressed by decomposed relations

$$x'^a = x'^a(x^1, \ldots\ldots x^r), \quad x'^i = (x^{r+1}, \ldots\ldots, x^n)$$

(x'^a depends on x^a only and x'^i on x^i only).

Let us call such a manifold M a *locally decomposed C^u-manifold.* Now put $s = n - r$. By the property 2) we can introduce into M, apart from the original topology, a newtopology T_R (T_S) which gives to M a structure of an r-(s-) dimensional manifold of class C^u under the coordinate systems induced naturally from M [5]. Under the topology, we denote a maximal connected manifold passing through each point $x \in M$ by $R(x)$ ($S(x)$) and sometimes we use an abbreviated notation R (S). R and S have clearly the structure of submanifolds of M under the original topology. They are called *R- and S-submanifolds* respectively. Here, we shall give the following notion: Given a curve $x(t)$ $(a \leqq t \leqq b)$ in M, by its *partitin* we mean a finite set of arcs $x(t)$ $(t_{\nu-1} \leqq t \leqq t_\nu)$ together with coordinate neighbourhoods $U_\nu \in \Omega$ such that $x(t)$ $(t_{\nu-1} \leqq t \leqq t_\nu) \subset U_\nu$ for $\nu = 1, 2, \ldots\ldots, m$ and $a = t_0 < t_1 < \ldots\ldots < t_m = b$. And, it is denoted by

$$\pi = \{x(t)\,(t_{\nu-1} \leqq t \leqq t_\nu),\ U_\nu \mid \nu = 1, 2, \ldots\ldots, m\}$$

[1]. Using this notion let us prove the following assertion:

Given two curves $x(\sigma)$ $(a_1 \leqq \sigma \leqq b_1)$ and $y(\tau)$ $(a_2 \leqq \tau \leqq b_2)$ in $R(o)$ and $S(o)$ respectively where $x(a_1) = y(a_2) = o \in M$, assume that there exists one continuous map f: a rectangle $\{(\sigma,\tau) \mid a_1 \leqq \sigma \leqq b_1,\ a_2 \leqq \tau \leqq b_2\} \to M$ which satisfies the following conditions:

$$f(\sigma, a_2) = x(\sigma),\ f(a_1,\tau) = y(\tau),\ f(\sigma, \tau) \in R(y(\tau)) \cap S(x(\sigma)).$$

Then, such a map is only one.

Let us call such a map f the *latticed map* with respect to the curves $x(\sigma)$ and $y(\tau)$.

PROOF. Consider first a case where the curves $x(\sigma)$ and $y(\tau)$ are wholly contained in a neighbourhood $U \in \Omega$. When we express $x(\sigma)$ and $y(\tau)$ by $(x^a(\sigma), x^i(a_2))$ and $(x^a(a_1), x^i(\tau))$ respectively in terms of coordinates in U, it is easily seen that a map

$$\{(\sigma, \tau) \mid a_1 \leqq \sigma \leqq b_1,\ a_2 \leqq \tau \leqq b_2\} \to ((\sigma, \tau) \to (x^a(\sigma),\ x^i(\tau)))$$

is one and only one latticed map. Further using this fact we shall prove our assertion in the general case. For simplicity let us sssume $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$. In the other cases too, our proof is quite similarly applied.

Now suppose that another latticed map f' does exist. Then, when we construct a partition of $x(\sigma)$ and use the above fact, we can easily find values $\tau' > 0$ such that $f(\sigma, \tau) = f'(\sigma, \tau)$ for $0 \leqq \sigma \leqq 1$, $0 \leqq \tau \leqq \tau'$. Let τ_0 be the least upper bound of τ'. Then it follows (by the continuity of f, f') that $f(\sigma, \tau) = f'(\sigma, \tau)$ for $0 \leqq \sigma \leqq 1$, $0 \leqq \tau \leqq \tau_0$. If $\tau_0 < 1$, by constructing again a partition of a curve $f(\sigma, \tau_0)$ $(0 \leqq \sigma \leqq 1)$ we can find $h > 0$ such that $f(\sigma, \tau) = f(\sigma, \tau)$ for $0 \leqq \sigma \leqq 1, \tau_0 \leqq \tau \leqq \tau_0 + h$. This contradicts with the fact that τ_0 is maximal. Therefore $\tau_0 = 1$. Hence $f = f'$. So, our assertion is proved.

Under the assumption in the above assertion, let curves $x'(\sigma')\,(a_1' \leqq \sigma' \leqq b_1')$ and $y(\tau')\,(a_2' \leqq \tau' \leqq b_2')$ represent the same ones as $x(\sigma)$ and $y(\tau)$ respectively. Then we can easily obtain: *With respect to the curves $x(\sigma)$ and $y(\tau)$ too, the latticed map f does exist. And further,*

$$f(\{(\sigma, \tau')\,|\,a_1' \leqq \sigma' \leqq b_1',\ a_2' \leqq \tau' \leqq b_2'\}) = f(\{(\sigma, \tau)\,|\,a_1 \leqq \sigma \leqq b_1,\ a_2 \leqq \tau \leqq b_2\})$$

as compact subsets in M.

From now on, we assume that in M there exists always the latticed map with respect to any two (parametrized) curves of class C^u starting from any point $o \in M$, in $R(o)$ and $S(o)$ respectively. Let us call such a manifold M a *locally decomposed C^u-manifold with latticed maps.* Let $x(\sigma)\,(a_1 \leqq \sigma \leqq b_1)$ *and* $y(\tau)\,(a_2 \leqq \tau \leqq b_2)$ be any two curves of class D^u starting from any $o \in M$ in $R(o)$ and $S(o)$ respectively. Then, the following assertion is easily verified: *With respect to the curves $x(\sigma)$ and $y(\tau)$, one and only one latticed map f does exist. Both of curves $f(\sigma, b_2)\,(a_1 \leqq \sigma \leqq b_1)$ and $f(b_1, \tau)\,(a_2 \leqq \tau \leqq b_2)$ thus obtained are of class D^u. Further, the curves themselves are uniquely determined from the given curves $x(\sigma)$, $y(\tau)$ alone, independently of their parametric representations.* The curve $f(\sigma, b_2)\,(a_1 \leqq \sigma \leqq b_1)$ is called the *natural displacement* of $x(\sigma)$ along $y(\tau)$, and we denote it by $D(\sim y)x(\sigma)$ or $D(\sim y(\tau))x(\sigma)$, where $\sigma\ (a_1 \leqq \sigma \leqq b_1)$ denotes also the parameter. Similarly, to the curve $f(b_1, \tau)\,(a_2 \leqq \tau \leqq b_2)$ too, such a definition and such notations will be given.

2. Fundamental groups.

LEMMA 2. 1. *Suppose that two continuous maps $\varphi : \{(\sigma, t)\,|\, a_1 \leqq \sigma \leqq b_1, \alpha \leqq t \leqq \beta\} \to M$ and $\psi : \{(\tau, t)\,|\,a_2 \leqq \tau \leqq b_2, \alpha \leqq t \leqq \beta\} \to M$ are given and satisfy the following conditions:*

1) *For $\alpha \leqq t \leqq \beta$, $\varphi(a_1, t) = \psi(a_2, t)\ (= o(t))$.*

2) *When $t = const.$, $x_t(\sigma)\ (= \varphi(\sigma, t))\ (a_1 \leqq \sigma \leqq b_1)$ and $y_t(\tau)\ (= \psi(\tau, t))\ (a_2 \leqq \tau \leqq b_2)$ are curves of class D^u in $R(o(t))$ and $S(o(t))$ respectively.*

Then a map

$$f_x : \{(\sigma, t)\,|\,a_1 \leqq \sigma \leqq b_1, \alpha \leqq t \leqq \beta\} \to M \quad ((\sigma, t) \to D(\sim y_t)x_t(\sigma))$$

is continuous, and if $\psi(b_2, t)\ (\alpha \leqq t \leqq \beta)$ is a curve in an R-submanifold R, the map f_x is continuous into R. For the similar map f_y, too, the similar results are obtained.

We omit the proof here.

Now we shall adopt the following notations: Given a curve $x(t)\,(a \leqq t \leqq b)$, we denote by $x^{-1}(t)$ or x^{-1} a curve $x(a+b-t)\,(a \leqq t \leqq b)$ which has the opposite orientation as in $x(t)$. Let Q denote a square $\{(\sigma, \tau) | 0 \leqq \sigma, \tau \leqq 1\}$.

On the other hand, let $x(t)$ $(0 \leqq t \leqq 1)$ be a curve of class D^u in M and let $\pi = \{x(t)(t_{\nu-1} \leqq t \leqq t_\nu),\ U_\nu | \nu = 1, 2, \ldots\ldots, m\}$ be any of its partitions. We denote an arc $x(t)$ $(t_{\nu-1} \leqq t \leqq t_\nu)$ by $x_\nu(t)$ or $(x_\nu^\alpha(t))$ in terms of the coordinate system of U_ν. The curve $x(t)$ is then represented by a *product curve* $x_1(t) \cdot x_2(t) \cdot \ldots\ldots \cdot x_m(t)$. We denote arcs $(x_\nu^a(t), x_\nu^i(t_{\nu-1}))$ and $(x_\nu^a(t_{\nu-1}), x_\nu^i(t))(t_{\nu-1} \leqq t \leqq t_\nu)$ by $Rx_\nu(t)$ and $Sx_\nu(t)$ respectively, and arcs $(x_\nu^a(t), x_\nu^i(t_\nu))$ and $(x_\nu^a(t_\nu), x_\nu^i(t))$ $(t_{\nu-1} \leqq t \leqq t_\nu)$ by $R'x_\nu(t)$ and $S'x_\nu(t)$ respectively.
Then,

$$R'x_\nu(t) = D(\sim Sx_\nu)Rx_\nu(t),\ S'x_\nu(t) = D(\sim Rx_\nu(t))Sx_\nu(t).$$

Further, each of product curves $Rx_1(t) \cdot S'x_1(t) \cdot Rx_2(t) \cdot S'x_2(t) \cdot \cdots\cdots \cdot S'x_m(t)$ and $Sx_1(t) \cdot R'x_1(t) \cdot Sx_2(t) \cdot R'x_2(t) \cdot \cdots\cdots R'x_m(t)$ is called a *step-curve* of $x(t)$ with respect to π. Using the step-curve first mentioned, we put

$$\begin{aligned} c_1(t) &= Rx_1(t), & k_1(t) &= S'x_1(t), \\ c_2(t) &= D(\sim k_1^{-1})Rx_2(t), & k_2(t) &= D(\sim c_2)k_1(t) \cdot S'x_2(t), \\ c_3(t) &= D(\sim k_2^{-1})Rx_3(t), & k_3(t) &= D(\sim c_3)k_2(t) \cdot Sx'_3(t), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and let $Rx(t)\,(0 \leqq t \leqq 1)$ denote a product curve $c_1(t) \cdot c_2(t) \cdot \cdots\cdots \cdot x_m(t)$. Similarly, $Sx(t)\,(0 \leqq t \leqq 1)$ will be defined by using the other step-curve.

LEMMA 2. 2. *The curves $Rx(t)$ and $Sx(t)$ are uniquely determined from the given curve $x(t)$ alone, independently of its partition π. Further, when $t = const.$, points $Rx(t)$ and $Sx(t)$ are contained in submanifolds $S(x(t))$ and $R(x(t))$ respectively.*

PROOF. As the latter part is evident from the construction, we prove the former part only. Now, we have $Rx(0) = Sx(0) = x(0)$. And, as the curves $Rx(t)$ and $Sx(t)$ are curves in $R(x(0))$ and $S(x(0))$ respectively, there is one and only one latticed map $f: Q \to M$ which satisfies $f(\sigma, 0) = Rx(\sigma)$, $f(0, \tau) = Sx(\tau)$ and $f(\sigma, \tau) \in R(Sx(\tau)) \cap S(Rx(\sigma))$. By the construction it is obvious that a compact subset $f(Q)$ contains the curve $x(t)$ and the step-curves with respect to π. Further $f(Q)$ contains all of step-curves of $x(t)$ with respect to other partitions. This is easily verified by the property of latticed maps, and shows that our assertion is true.

From now on, each of the curves $Rx(t)$ and $Sx(t)$ is called a *natural projection* of $x(t)$.

LEMMA 2. 3. *Let φ be a continuous map $Q \to M$ which satisfies the following conditions:*

1) *Each curve $x_\sigma(\tau)$ $(=\varphi(\sigma, \tau))$ $(0 \leqq \tau \leqq 1)$ for $\sigma = const.$ is of class D^u.*

2) *For $0 \leqq \sigma \leqq 1$, $\varphi(\sigma, 0)$ is only a point and so is $\varphi(\sigma, 1)$.*

Then a map

$$\varphi_R: Q \to R(o) \; ((\sigma, \tau) \to Rx_\sigma(\tau)),$$

where $o = \varphi(\sigma, 0)$, is continuous and $\varphi_R(\sigma, 1)$ $(0 \leqq \sigma \leqq 1)$ is only a point. For the similar map φ_S, too, the similar results are obtained.

PROOF. For a constant $d (0 \leqq d \leqq 1)$ let $\pi(d) = \{x_d(\tau) \; (\tau_{\nu-1} \leqq \tau \leqq \tau_\nu), U_\nu \mid \nu = 1, 2, \ldots\ldots, m\}$ be a partition of a curve $x_d(\tau)$. Under the same U_ν and τ_ν, we can find $\delta > 0$ such that, for all $\sigma (0 \leqq \sigma \leqq 1)$ satisfying $|\sigma - d| < \delta$, $\pi(\sigma) = \{x_\sigma(\tau) \; (\tau_{\nu-1} \leqq \tau \leqq \tau_\nu), U_\nu \mid \nu = 1, 2, \ldots\ldots, m\}$ become partitions of curves $x_\sigma(\tau)$ $(0 \leqq \tau \leqq 1)$. If we suppose that from the step-curves of $x_\sigma(\tau)$ with respect to $\pi(\sigma)$ their natural projections $Rx_\sigma(\tau)$ are constructed, it is seen by Lemma 2. 1 that the map φ_R is continuous over a domain, $|\sigma - d| < \delta$, $0 \leqq \tau \leqq 1$ *in* Q. Accordingly, φ_R is a continuous map from Q into $R(o)$.

On the other hand, a curve $\varphi_R(\sigma, 1)$ $(0 \leqq \sigma \leqq 1)$ in $R(o)$ is contained in $S(o')$ where $o' = \varphi(\sigma, 1)$, and also is a curve in $S(o')$. In fact, if we consider a map $\varphi'_S: Q \to S(o')$ $((\sigma, \tau) \to Sx_\sigma^{-1}(\tau))$ it is also continuous. Hence $\varphi'_S(\sigma, 1)$ $(0 \leqq \sigma \leqq 1)$ is a curve in $S(o')$, and as seen from the proof of Lemma 2. 2 we obtain $\varphi'_S(\sigma, 1) = \varphi_R(\sigma, 1)$. These imply that $\varphi_R(\sigma, 1)$ is a curve in $R(o)$ and also in $S(o')$. Therefore, the curve $\varphi_R(\sigma, 1)$ is only a point. This is obvious by the local structure of M. So, our lemma has been proved.

LEMMA 2. 4. *Let f be a continuous map of the boundary ∂Q of Q into any of R- and S-submanifolds. If f is homotopic in M to a constant map, then it is homotopic in the submanifold to a constant map.*

PROOF. Let us prove the lemma for an R-submanifold R. First suppose $f(\sigma, 0) = x_0$ and $f(\sigma, 1) = x_1$ for $0 \leqq \sigma \leqq 1$, where $x_0, x_1 \in R$. This assumption does not lose the generality of our assertion. Now, we can find a continuous map $\varphi: Q \to M$ which satisfies the following conditions: 1) $\varphi(\sigma, 0) = x_0$ and $\varphi(\sigma, 1) = x_1$ for $0 \leqq \sigma \leqq 1$, 2) each curve $x_\sigma(\tau)$ $(=\varphi(\sigma, \tau))$ $(0 \leqq \tau \leqq 1)$ for $\sigma =$ const. is of class D^u, 3) $x_0(\tau)$ and $x_1(\tau)$ are curves in R, 4) a map $\varphi \mid \partial Q$ is homotopic in R to the map f. Now, if we consider a map φ_R: $Q \to R$ $((\sigma, \tau) \to Rx_\sigma(\tau))$, the map is continuous and $\varphi_R(\sigma, 1) = x_1$ for $0 \leqq \sigma \leqq 1$ by Lemma 2. 3. Further, since $\varphi_R(\sigma, 0) = x_0$ for $0 \leqq \sigma \leqq 1$ and $\varphi_R(0, \tau) = \varphi(0, \tau)$, $\varphi_R(1, \tau) = \varphi(1, \tau)$ for $0 \leqq \tau \leqq 1$, we have $\varphi_R(\partial Q) = \varphi(\partial Q)$. These facts imply that our assertion is true.

THEOREM 1. *The fundamental group of any of R- and S-submanifolds is* *isomorphic into the fundamental group of M under the homomorphism induced by the inclusion map.*

PROOF. Let us prove the theorem for an R-submanifold R. The inclusion map $i: R \to M$ induces the homomorphism $i_*: \pi_1(R) \to \pi_1(M)$, and it follows by Lemma 2. 4 that the kernel of i_* consists of only the identity element of $\pi_1(R)$. Hence, our theorem is easily seen to be true.

3. Decompositions.. Let $c[x, y]$ denote a curve of class D^u whose initial point is x and whose terminal point is y.

LEMMA 3. 1. *If R and S are any R- and S-submanifolds, then* $R \cap S \neq 0$.

PROOF. Take up a curve $c[x, y]$ where $x \in R$, $y \in S$, and denote by z the terminal point of its natural projection $Rc[x, y]$, then $z \in R(x) \cap S(y)$ from Lemma 2. 2. So, $R \cap S \neq 0$.

LEMMA 3. 2. *Let $S(x_0)$, $x_0 \in M$, be simply-connected and $c[x_0, y_0]$ be a curve given in $R(x_0)$. For each point $x \in S(x_0)$, if c_1 and c_2 are any two curves of class D^u in $S(x_0)$ joining x_0 to x, then*

$$D(\sim c_1)c[x_0, y_0] = D(\sim c_2)\, c[x_0, y_0].$$

PROOF. From the assumption for $S(x_0)$, we can find a continuous map $\psi: Q \to S(x_0)$ which satisfies the following conditions: 1) $\psi(\sigma, 0) = x_0$ and $\psi(\sigma, 1) = x$ for $0 \leqq \sigma \leqq 1$, 2) when $\sigma =$ const., each curve $y_\sigma(\tau)\,(= \psi(\sigma, \tau))$ $(0 \leqq \tau < 1)$ is of class D^u, 3) curves $y_0(\tau)$ and $y_1(\tau)$ represent the same ones as c_1 and c_2 respectively. Then, for a constant $d\,(0 \leqq d \leqq 1)$ there exists $\delta > 0$ such that $D(\sim y_\sigma(\tau))\, c[x_0, y_0] = D(\sim y_d(\tau))\, c[x_0, y_0]$ for any $\sigma\,(0 \leqq \sigma \leqq 1)$ satisfying $|\sigma - d| < \delta$. From this fact we can easily prove our assertion.

LEMMA 3. 3. *Under Lemma* 3. 2, *when $S(y_0)$ is also simply connected and y is the terminal point of $D(\sim c_1)\, c[x_0, y_0]$, then $y \in S(y_0)$ and $S(x_0)$ is C^u-homeomorphic to $S(y_0)$ under a map f: $S(x_0) \to S(y_0)\,(x \to y)$.*

By "*C^u-homeomorphic*" we mean a homeomorphism under a map which is of class C^u together with the inverse. Let us call such a map f the *C^u-homeomorphism with respect to the curve* $c[x_0, y_0]$.

PROOF. It is obvious that $y \in S(y_0)$. Hence f is a map of $S(x_0)$ into $S(y_0)$. On the other hand, for any $y' \in S(y_0)$ if we denote by x' the terminal point of $D(\sim c[y_0', y'])\, c[x_0, y_0]^{-1}$ where $c[y_0, y']$ is any curve in $S(y_0)$, it follows that $f(x') = y'$. Accordingly f is an onto-map. Further, by constructing a partition of $D(\sim c_1)\, c[x_0, y_0]$, we can earily see that the map f is of class C^u(at x) together with the inverse. So, our lemma is proved.

It is quite obvious that Lemmas 3. 2 and 3. 3 hold true, though we exchange the roles of R and S there.

LEMMA 3. 4. *When all the R- and S-submanifolds in M are simply-connected, the product $\widetilde{M} = R \times S$ of any R and S of them is regarded as the universal covering space of M.*

PROOF. Let o be a point of $R \cap S$. We take a point $\widetilde{x} \in \widetilde{M}$, then $\widetilde{x}$ is represented by a pair (y, z) where $y \in R$ and $z \in S$. Let $c[o, y]$ and $c[o, z]$ be any curves in R and S respectively. Let x be the terminal point of a curve $D(\sim c[o, z])c[o, y]$, then $x \in R(z) \cap S(y)$. By Lemma 3. 2, we see that the point x does not depend on the curves $c[o, y]$ and $c[o, z]$, but does depend on the points y and z, namely $\widetilde{x}$. We consider thereby a map

$$f: \widetilde{M} \to M \ (\widetilde{x} \to x).$$

1) Conversely, let x be a point of M. Now take a point $y \in R \cap S(x)$ and let z be the terminal point of a curve $D(\sim c[y, x])\, c[y, o]$, where $c[y, x]$ and $c[y, o]$ are any curves in $S(x)$ and R respectively. Then $z \in S$. If we denote by $\widetilde{x}$ a pair (y, z) as a point of $\widetilde{M}$, it follows that $f(\widetilde{x}) = x$. So, the map f is an onto-map.

2) Again let x be a point of M. We represent a set $R \cap S(x)$ by $\{y_\lambda | \lambda \in J\}$ where J is an index-set. Let z_λ be the point determined from y_λ by the manner in 1). Then $z_\lambda \in R(x) \cap S$. Now let $c[y_\lambda, o]$ and $c[z_\lambda, o]$ be any curves in R and S respectively. When we take a neighbourhood $W(x) \in \Omega$ of x, $W(x)$ is necessarily represented by the product $W_R(x) \times W_S(x)$ of coordinate neighbourhoods $W_R(x)$ and $W_S(x)$ in submanifolds $R(x)$ and $S(x)$ respectively. Let $W_R(y_\lambda)$ and $W_S(z_\lambda)$ be the image of $W_R(x)$ and $W_S(x)$ under the C^u-homeomorphisms with respect to $c[z_\lambda, o]$ and $c[y_\lambda, o]$ respectively. If we denote by $\widetilde{x}_\lambda$ a pair (y_λ, z_λ) as a point of $\widetilde{M}$, then the product $\widetilde{W}_\lambda = W_R(y_\lambda) \times W_S(z_\lambda)$ is regarded as a neighbourhood of $\widetilde{x}_\lambda$ and is C^u-homeomorphic to $W(x)$ under f.

3) In 2), we have $f^{-1}(x) = \bigcup_{\lambda \in J} \widetilde{x}_\lambda$. Now let us verify $\widetilde{W}_\lambda \cap \widetilde{W}_\mu = 0$ for $\lambda, \mu \in J(\lambda \neq \mu)$. In fact, suppose that $\widetilde{W}_\lambda \cap \widetilde{W}_\mu \neq 0$, and let $c[\widetilde{x}_\lambda, \widetilde{x}_\mu]$ be a curve in $\widetilde{W}_\lambda \cup \widetilde{W}_\mu$. Then $f(c[\widetilde{x}_\lambda, \widetilde{x}_\mu])$ is a closed curve with endpoint x contained in $W(x)$. This contradicts with the fact that $\widetilde{W}_\lambda$ is C^u-homeomorphic to $W(x)$ under f. Hence $\widetilde{W}_\lambda \cap \widetilde{W}_\mu = 0$.

Summing up these facts we see that our assertion is true.

THEOREM 2. *When M is simply-connected, M is C^u-homeomorphic to the product $R \times S$ of any R- and S-submanifolds R and S.*

This is immediately proved by Theorem 1 and Lemma 3. 4.

4. Locally decomposed, affinely connected manifolds and Finsler manifolds. Let M_1 and M_2 be locally decomposed C^∞-manifolds defined in § 1. In each of them, too, some notations and words already defined will be used under the same sense, unless defined otherwise.

We assume further that M_1 has an affine connection without torsion of class C^∞ and satisfies the following conditions:

1) In each $U \in \Omega$, if we denote the affine connection by $\Gamma^\alpha_{\beta\gamma}$ in terms of its coordinate system (x^α), Γ^a_{bc} are functions of $x^1, \ldots\ldots, x^r$ only and Γ^i_{jk} of $x^{r+1}, \ldots\ldots, x^n$ only and the remaining $\Gamma^\alpha_{\beta\gamma}$ are all zero.

2) M_1 is complete, i. e. every path may be extended to arbitrarily large values of its affine parameter.

We call such a manifold M_1 a *locally decomposed, affinely connected manifold* [3], and treat all of its R- and S-submanifolds as affinely connected manifolds with the affine connection naturally induced from M_1.

Next we assume that M_2 has a (Finsler) metric such that in each $U \in \Omega$ the square of the infinitesimal distance between two neighboring points (x^α) and $(x^\alpha + dx^\alpha)$ is given by $F_R(x^a, dx^a) + F_S(x^i, dx^i)$, where F_R satisfies as a function in a domain $0 < x^a < 1$, $-\infty < y^a < \infty$ the following conditions:

1) $F_R(x^a, y^a)$ is continuous and $F_R(x^a, \lambda y^a) = \lambda^2 F_R(x^a, y^a)$ for any real number λ,

2) except the point where all of y^a take zero, $F_R(x^a, y^a)$ are of class C^∞ and the matrix $(\partial^2 F_R/\partial y^a \partial y^b)$ is positive definite.

And F_S also satisfies as a function in a domain $0 < x^i < 1$, $-\infty < y^i < \infty$ the similar conditions. Further, let us assume that every geodesic there defined may be extended to arbitrarily large values of its curve-length, i. e. M be complete. We call such a manifold M_2 a *locally decomposed Finsler manifold* and treat all of its R- and S-submanifolds as Finsler manifolds with metrics naturally induced from M_2. For the sake of convenience we call each of geodesics a path, and by its affine parameter we mean one obtained by a linear transformation from its curve-length, as usually defined.

From now on, we denote any of M_1 and M_2 by M_e. The following properties are easily verified: *Any path in an R- or S-submanifold of M_e is also a path in M_e, and a path in M_e through x, whose tangent vector at x is contained in the tangent space of $R(x)$ $(S(x))$ at x, is contained in $R(x)$ $(S(x))$ and is a path in $R(x)$ $(S(x))$. Hence $R(x)$ and $S(x)$ are complete.* Using these properties let us prove:

LEMMA 4. 1. *Let o be a point of M_e. Let $x(\sigma)$ $(a \leqq \sigma \leqq b)$ be a curve of class C^∞ in $R(o)$ and let $y(\tau)$ $(0 \leqq \tau \leqq 1)$ be a path with an affine para-*

meter τ in $S(o)$ where $x(a) = y(0) = o$. Then there exists one and only one latticed map with respect to $x(\sigma)$ and $y(\tau)$.

PROOF. Let us denote a partition of $x(\sigma)$ by $\pi = \{x(\sigma)\ (\sigma_{\nu-1} \leqq \sigma \leqq \sigma_\nu),\ U_\nu | \nu = 1, 2, \ldots\ldots, m\}$ and an arc $x(\sigma)\,(\sigma_{\nu-1} \leqq \sigma \leqq \sigma_\nu)$ by $x_\nu(\sigma)$. In each arc $x_\nu(\sigma)$, we plant at each point of it a vector which has v_ν^α as components in U_ν and satisfies the following conditions:

1) all of v_ν^i are constant and $v_\nu^a = 0$, on each arc $x_\nu(\sigma)$, 2) at each point $x(\sigma_\lambda)\ (\lambda = 1, 2, \ldots\ldots, m-1)$ the vector (v_λ^α) in U_λ coincides with the vector $(v_{\lambda+1}^\alpha)$ in $U_{\lambda+1}$, 3) the vector (v_1^α) at the point o coincides with the initial vector of $y(\tau)$. A vector field $v(\sigma)$ is thereby defined on the curve $x(\sigma)$. Regarding σ as a constant let $g_\sigma(\tau)\ (0 \leqq \tau \leqq 1)$ be a path passing through a point $x(\sigma)$ and having $v(\sigma)$ as its initial vector, where τ denotes its affine parameter. Then, when we consider a map

$$f:\ \{(\sigma, \tau) | a \leqq \sigma \leqq b,\ 0 \leqq \tau \leqq 1\} \to M_e\ ((\sigma, \tau) \to g_\sigma(\tau)),$$

it is one and only one latticed map with respect to the curves $x(\sigma)$ and $y(\tau)$. In fact, if we express paths $g_\sigma(\tau)$ by differential equations, this is easily veri fied. So, our lemma is true.

Next, say, in S, a neighbourhood (open set in S) W such that any two points in W are joined by one and only one path-arc wholly contained in W, is called a *simple convex neighbourhood* in S. And, *at any point of S such a neighbourhood does always exists.* For an affiinely connected manifold, see [4], and for a Finsler manifold too, it is true.

THEOREM 3. *The underlying manifold of M_e is a locally decomposed C^∞-manifold with latticed maps. Hence, Theorem 1 holds true in M_e too.*

PROOF. Let o be any point of M_e, and let $x(\sigma)\ (a_1 \leqq \sigma \leqq b_1)$ and $y(\tau)\ (a_2 \leqq \tau \leqq b_2)$ be any curves of class C^∞ in $R(o)$ and $S(o)$ respectively, where $x(a_1) = y(a_2) = o$. It suffices to show that there exists the latticed map with respect to them.

First let us consider a case where the curve $y(\tau)$ is contained in a simple convex neighbourhood W in $S(o)$. Regarding τ as a constant we join the point o to a point $y(\tau)$ with a path-arc $g_\tau(t)\ (0 \leqq t \leqq 1)$ in W, where $g_\tau(0) = o$, $g_\tau(1) = y(\tau)$ and t is an affine parameter. Then we see that a map

$$\{(\tau, t) | a_2 \leqq \tau \leqq b_2,\ 0 \leqq t \leqq 1\} \to S(o)\ ((\tau, t) \to g_\tau(t))$$

are continuous. On the other hand, we have the latticed map

$$h_\tau:\ \{(\sigma, t) | a_1 \leqq \sigma \leqq b_1,\ 0 \leqq t \leqq 1\} \to M_e$$

with respect to $x(\sigma)$ and $g_\tau(t)\ (0 \leqq t \leqq 1)$ by Lemma 4. 1. Using the map h_τ if we consider a map

$$f: \{(\sigma, \tau) | a_1 \leqq \sigma \leqq b_1, \ a_2 \leqq \tau_1 \leqq b_2\} \to M_e((\sigma, \tau) \to h_\tau(\sigma, 1)),$$

it is the latticed map with respect to the curves $x(\sigma)$ and $y(\tau)$. This is easily verified.

Secondly let us consider the other case. In $S(o)$ we construct a partition of the curve $y(\tau)$, as defined in M. Using simple convex neighbourhoods W_ν we denote it by

$$\pi = \{y(\tau)\,(\tau_{\nu-1} \leqq \tau \leqq \tau_\nu), \ W_\nu | \nu = 1, 2, \ldots\ldots, m\}.$$

Then, with respect to $x(\sigma)$ $(a_1 \leqq \sigma \leqq b_1)$ and $y(\tau)$ $(a_2 \leqq \tau \leqq \tau_1)$ we have the latticed map from the above fact, and denote it by f_1. Next, with respect to curves $f_1(\sigma, \tau_1)$ $(a_1 \leqq \sigma \leqq b_1)$ and $y(\tau)$ $(\tau_1 \leqq \tau \leqq \tau_2)$ of class C^∞ we have also the latticed map and denote it by f_2. By continuing this manner latticed maps f_ν $(\nu = 1, 2, \ldots\ldots, m)$ are obtained. Let f denote their union map. Then f is clearly the latticed map with respect to $x(\sigma)$ and $y(\tau)$. So, the underlying manifold of M_e is a locally decomposed C^∞-manifold with latticed maps and hence Theorem 1 holds also true in M_e.

In two submanifold R and S of M_1, let Γ^a_{bc} and Γ^i_{jk} be the connection coefficients in any W_R and W_S of their coordinate neighbourhoods respectively. Then we may give the product $R \times S$ an affine connection whose connection coefficients $\overline{\Gamma}^\alpha_{\beta\gamma}$ in $W_R \times W_S$ satisfy the following relations: $\overline{\Gamma}^a_{bc} = \Gamma^a_{bc}$, $\overline{\Gamma}^i_{jk} = \Gamma^i_{jk}$ and the remaining $\Gamma^\alpha_{\beta\gamma}$ are all zero. The affinely connected manifold $R \times S$ thus obtained is called the *affine product* of R and S.

Further, in R and S of M_2 let us denote the Finsler metrics in any W_R and W_S of their coordinate neighbourhoods by $ds^2_R = F_R(x^a, dx^a)$ and $ds^2_S = F_S(x^i, dx^i)$ respectively. Then we have a Finsler manifold $R \times S$ whose metric in $W_R \times W_S$ is expressed by $ds^2 = F_R(x^a, dx^a) + F_S(x^i, dx^i)$, and call it the *metric product* of R and S. Under such notions we have:

THEOREM 4. *When M_1 (M_2) is simply connected, M_1 (M_2) is equivalent to the affine* (*metric*) *product $R \times S$ of any R- and S-submanifolds R and S.*

By "*equivalent*" we mean the equivalence as affinely connected manifolds or as Finsler manifolds.

PROOF. Of course, Lemmas 3. 2 is true in M_e, and Lemma 3. 3 holds good in M_e even if we substitute "equivalent" for "C^u-homeomorphic" there. Let us suppose here that the universal covering space of M_1 (M_2) has the affine connection (Finsler metric) naturally induced from it by the covering map. Then, it is easily seen that in Lemma 3. 4 if we substitute "$M_1(M_2)$" and "affine (Finsler) product" for "M" and "product" respectively it holds also true. This fact together with Theorem 3 proves our assertion.

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