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THE DEFORMATION BEHAVIOUR OF THE KODAIRA DIMENSION OF ALGEBRAIC MANIFOLDS

(WITH AN APPENDIX BY K. UENO)

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0. **Introduction.** We consider the following:

PROBLEM. *Is the Kodaira dimension of algebraic manifolds invariant under deformation?*

For curves and surfaces, the answer is affirmative. In the former case the result is clear. In the latter case it was proved by Iitaka [I_2] using the classification of surfaces, whereas for non-algebraic complex manifolds of dimension three, Nakamura [N] produced a counterexample to this problem.

On the other hand, Lieberman-Sernesi [LS] introduced a notion of the relative Kodaira dimension $\kappa(X/Y)$ for a family $f: X \to Y$ of algebraic manifolds, and proved that the Kodaira dimension κ of fibers over a countable intersection of Zariski open sets of Y is equal to $\kappa(X/Y)$ and κ of other fibers are greater than $\kappa(X/Y)$. Using this notion, we formulate our problem in the following way.

CONJECTURE $DF_{n,k}$. *Let $f: X \to Y$ be a family of n-dimensional algebraic manifolds with $\kappa(X/Y) = k$. Then for any fiber $X_y = f^{-1}(y)$ $(y \in Y)$, we have $\kappa(X_y) = k$.*

Note that $DF_{n,n}$ is true by Lieberman-Sernesi's theorem. If all the Conjectures $DF_{n,-\infty}, DF_{n,0}, \cdots, DF_{n,n-1}$ are true, then the deformation invariance of the Kodaira dimension in the algebraic case will be settled.

In this paper, we study Conjecture $DF_{n,k}$ for $1 \leqq k \leqq n-1$. First we describe the geometric structure of every fiber of such a family as follows:

THEOREM I. *Let $f: X \to Y$ be a family of n-dimensional algebraic manifolds with $1 \leqq \kappa(X/Y) \leqq n-1$. Then for any $y \in Y$, the fiber X_y has the following property: There exist a nonsingular model X_y^* of X_y, a variety T and a fiber space $\psi: X_y^* \to T$ such that*

(1) $\dim T = \kappa(X/Y)$

(2) *There is an open set T' of T such that for any $t \in T'$, the*

fiber $\psi^{-1}(t)$ is irreducible nonsingular and is the quasi-specialization of parabolic varieties in the sense of Definition 4.1, i.e., $\psi^{-1}(t)$ is one of the irreducible components of a fiber of a certain degenerating family of varieties of Kodaira dimension zero (of so-called parabolic type).

By this theorem, Conjecture $\mathrm{DF}_{n,k}$ for $1 \leqq k \leqq n-1$ is reduced to a problem on $(n-k)$-dimensional algebraic manifolds of Kodaira dimension zero:

THEOREM II. *Let (n, k) be a pair of positive integers with $1 \leqq k \leqq n-1$. Assume (1) Conjecture $\mathrm{DF}_{n-k,0}$ and (2) the lower-semicontinuity of κ for degenerating families of $(n-k)$-dimensional parabolic manifolds, i.e., the κ of each components of a degenerate fiber of a degenerating family of parabolic manifolds is not larger than zero (see Conjecture PDG_{n-k} in § 5). Then Conjecture $\mathrm{DF}_{n,k}$ is true.*

COROLLARY. *$\mathrm{DF}_{n,n-1}$ is true.*

In § 1, we summarize some known results about the Kodaira dimension and give the definition of the relative Kodaira dimension. § 2 deals with general properties of the relative Kodaira dimension. As a preparation for § 4, § 3 is devoted to the study of a graded C-subalgebra of the canonical ring of an algebraic variety. In § 4, we study families over a nonsingular curve and prove Theorem I in that case. Theorems I and II are compeletely proved in § 5.

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2. **Definitions and notations.** We work in the category of schemes defined over the field of complex numbers C. An algebraic variety is an irreducible reduced C-scheme of finite type. A point of an algebraic variety is a closed point. Open sets are Zariski open sets. Unless otherwise stated, an algebraic manifold is a nonsingular compelete algebraic variety. For a surjective morphism $f: V \to W$ from a variety V to another W, a fiber over a point in a certain countable intersection of nonempty open sets of W is called a *general fiber*. Note that a countable intersection of nonempty open sets of W is dense in W. $f: V \to W$ is called a *fiber space* if f is a surjective morphism from an algebraic variety V to another W such that the general fiber of f are connected. By a family $f: X \to Y$ of algebraic manifolds of dimension n, we mean that f is a proper smooth surjective morphism of relative

dimension n between algebraic varieties X and Y such that every fiber of f is connected.

Let V be an algebraic manifold and ω_V be its canonical sheaf. We put

$$R(V) = \bigoplus_{i \in \boldsymbol{Z}_{\geqq 0}} R(V)_i\ , \quad R(V)_i = H^0(V, \omega_V^{\otimes i})\ ,$$

where $\boldsymbol{Z}_{\geqq 0}$ is the set of non-negative integers. We call $R(V)$ the canonical ring of V. We set

$$R(V)^{[i]} = \boldsymbol{C}[R(V)_i]$$

which is the graded $\boldsymbol{C}$-subalgebra of $R(V)$ generated by $R(V)_i$. We also set

$$N(V) = \{i \in \boldsymbol{Z}_{>0};\ R(V)_i \neq 0\}$$

where $\boldsymbol{Z}_{>0}$ is the set of positive integers. This is a semigroup.

DEFINITION 1.1. The Kodaira dimension of V is

$$\kappa(V) = \begin{cases} \max\limits_{i \in N(V)} \dim \operatorname{Proj} R(V)^{[i]} & \text{if} \quad N(V) \neq \varnothing \\ -\infty & \text{if} \quad N(V) = \varnothing\ . \end{cases}$$

The reader is referred to Ueno [U] for a general discussion of the Kodaira dimension. The proofs of the following theorems can be found in Iitaka [I_1] and [U].

THEOREM 1.2 (The Iitaka estimate). *There exist positive integers m_0, d and positive real numbers α, β such that the following inequality holds for any integer $m \geqq m_0$:*

$$\alpha m^{\kappa(V)} \leqq \dim_{\boldsymbol{C}} H^0(V, \omega_V{}^{\otimes dm}) \leqq \beta m^{\kappa(V)}\ .$$

THEOREM 1.3 (The Iitaka fibration). *Suppose $N(V) \neq \varnothing$. Then there exists an integer m_0 such that for any $m \geqq m_0$ satisfying $m \in N(V)$, the m-th canonical rational map*

$$\varphi_m\colon V \dashrightarrow W = \operatorname{Proj} R(V)^{[m]}$$

has the following properties: Let $\varphi\colon V^{\#} \to W$ be an elimination of the points of indeterminacy of the rational map φ_m. Then we have

(a) $\dim W = \kappa(V)$.

(b) *The general fibers of φ are connected.*

(c) *There is a countable intersection W' of nonempty open sets of W such that for any $w \in W'$, the fiber $V_w = \varphi^{-1}(w)$ is irreducible, nonsingular and $\kappa(V_w) = 0$.*

We call $\varphi\colon V^{\#} \to W$ the Iitaka fibration of V and W' an Iitaka subset of W.

THEOREM 1.4. *Let* $\psi\colon V \to W$ *be a fiber space. Then there exists a nonempty open set* U *of* W *such that we have an inequality*

$$\kappa(V) \leqq \kappa(V_x) + \dim W\,,$$

where $V_x = \psi^{-1}(x)$, *for every point* $x \in U$.

We remark that W may be singular in this theorem.

Next we introduce the notion of the relative Kodaira dimension, which is a generalization of the Kodaira dimension to the relative case. This was originally introduced by Lieberman-Sernesi [LS]. Note that this definition differs from the more recent one of Viehweg [V].

Let $f\colon X \to Y$ be a family of algebraic manifolds, and $\omega_{X/Y}$ be its relative canonical sheaf. We set

$$N(X/Y) = \{i \in \boldsymbol{Z}_{>0}\ ;\ f_*\omega_{X/Y}^{\otimes i} \neq 0\}$$

which is a semigroup.

If $i \in N(X/Y)$, a rational map $g_i\colon X \dashrightarrow \boldsymbol{P}(f_*\omega_{X/Y}^{\otimes i})$ over Y is defined by the homomorphism $f^*f_*\omega_{X/Y}^{\otimes i} \to \omega_{X/Y}^{\otimes i}$, where $\boldsymbol{P}(f_*\omega_{X/Y}^{\otimes i})$ is the projective fiber space over Y associated with the coherent sheaf $f_*\omega_{X/Y}^{\otimes i}$ (see [EGA, Chapter 2]). Z_i is the image of X by g_i, i.e., the closure of $g_i(X \setminus \Sigma_{g_i})$ in $\boldsymbol{P}(f_*\omega_{X/Y}^{\otimes i})$, where Σ_{g_i} is the indeterminacy of g_i. We denote by π_i the restriction to Z_i of the projection $\boldsymbol{P}(f_*\omega_{X/Y}^{\otimes i}) \to Y$. Hence we get the following diagram:

(1.1)
$$\begin{array}{ccc} X & \overset{g_i}{\dashrightarrow} & Z_i \subset \boldsymbol{P}(f_*\omega_{X/Y}^{\otimes i}) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \pi_i} & \\ Y & & \end{array}$$

DEFINITION 1.5. The relative Kodaira dimension $\kappa(X/Y)$ of $f\colon X \to Y$ is

$$\kappa(X/Y) = \begin{cases} \max\limits_{i \in N(X/Y)} \dim Z_i - \dim Y & \text{if } N(X/Y) \neq \varnothing \\ -\infty & \text{if } N(X/Y) = \varnothing\,. \end{cases}$$

Now we slightly generalize the above definition.

DEFINITION 1.6. Let $f\colon X \to Y$ be a generically smooth surjective morphism of algebraic varieties, i.e., f is smooth over a certain open set Y' of Y and assume the fibers over Y' are connected. Then the relative Kodaira dimension $\kappa(X/Y)$ of $f\colon X \to Y$ is defined by

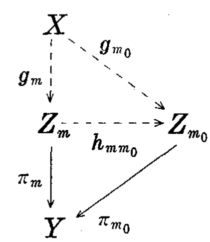
$$\kappa(X/Y) = \kappa(f^{-1}(Y')/Y')\,.$$

This is obviously independent of the choice of Y'.

REMARK 1.7. Let $f: X \to Y$ be as above and assume further that f is a Gorenstein morphism. Let $\omega_{X/Y}$ be its dualizing sheaf. Then the above definition of $\kappa(X/Y)$ coincides with the relative $\omega_{X/Y}$-dimension of [LS].

2. **The properties of the relative Kodaira dimension.** First, we observe the asymptotic behavior of the diagram (1.1) with respect to m. The following lemma and its proof is essentially analogous to that in [U, p. 54].

LEMMA 2.1. *There is a positive integer m_0 such that, for any m satisfying $m \geqq m_0$ and $m \in N(X/Y)$, there exists a birational map $h_{m,m_0}: Z_m \dashrightarrow Z_{m_0}$ which makes up a commutative diagram*



on the locus where all the rational maps are defined.

PROOF. As the problem is local on Y, we can assume $Y = \operatorname{Spec} A$ for a C-algebra A. We put

$$P = \bigoplus_{i \in Z_{\geqq 0}} P_i, \quad P_i = \Gamma(Y, f_* \omega_{X/Y}^{\otimes i}) .$$

This is a graded A-algebra. We set $P^{[i]} = A[P_i]$ which is the graded A-subalgebra of P generated by P_i. Then we have

$$Z_i = \operatorname{Proj} P^{[i]}$$

and the rational function field $C(Z_i)$ of Z_i is

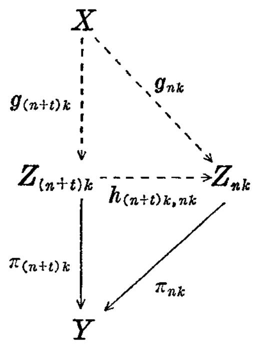
$$C(Z_i) = (S_{P^{[i]}}^{-1} P^{[i]})_0$$

which is the degree 0 part of the graded quotient ring of $P^{[i]}$ with respect to the multiplicatively closed set $S_{P^{[i]}} = \bigcup_{j \geqq 0} (P_j^{[i]} \setminus \{0\})$.

Let $d > 0$ be the greatest common divisor of the integers belonging to $N(X/Y)$. As $N(X/Y)$ is a semigroup, there is a positive integer l_0 such that $l \geqq l_0$ implies $ld \in N(X/Y)$. We set $k = l_0 d$. Now we take a nonzero element $\varphi \in P_k$. For any positive integers n and t, the homomorphism

(2.1) $$P_{nk} \to P_{(n+t)k}$$

sending ψ to $\varphi^t\psi$ induces a homomorphism $P^{[nk]} \to P^{[(n+t)k]}$. This induces a rational map $h_{(n+t)k,nk}\colon Z_{(n+t)k} \dashrightarrow Z_{nk}$ which makes up a commutative diagram



by definition. Further, the inclusion map

$$(S^{-1}_{P[nk]}P^{[nk]})_0 \hookrightarrow (S^{-1}_{P[(n+1)k]}P^{[(n+1)k]})_0$$

is induced by (2.1) with $t = 1$. So there is a chain of function fields

$$C(Y) \subset C(Z_k) \subset \cdots \subset C(Z_{nk}) \subset C(Z_{(n+1)k}) \subset \cdots \subset C(X)\,.$$

The field extention $C(X)$ over $C(Y)$ is finitely generated. Thus there exists a positive integer n_0 such that

$$C(Z_{n_0k}) = C(Z_{(n_0+1)k}) = C(Z_{(n_0+2)k}) = \cdots\,.$$

Now we set $m_0 = (n_0 + 1)k$. Then we can easily check that for any $m \geqq m_0$ satisfying $m \in N(X/Y)$, $C(Z_m) = C(Z_{m_0})$ holds. Hence h_{m,m_0} is a birational map. q.e.d.

We immediately get the following:

COROLLARY 2.2. *There is a positive integer m_0 such that, for any $m \geqq m_0$ satisfying $m \in N(X/Y)$, we have*

$$\dim Z_m - \dim Y = \kappa(X/Y)\,.$$

Next for later use, we review the following consequence of Grothendieck's base change theorem.

THEOREM 2.3. *Let $f\colon X \to Y$ be a proper morphism of (reduced connected) algebraic varieties, F be a coherent sheaf on X which is $\mathcal{O}_Y$-flat, and q be any nonnegative integer. Then,*

(1) *The function*

$$\begin{array}{rcl} d_q\colon Y & \to & Z \\ \cup\!\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\!\shortmid \\ y & \mapsto & \dim_C H^q(X_y, F_y) \end{array}$$

is upper-semicontinuous on Y, *where* $F_y = F \otimes_{\mathcal{O}_Y} C_y$ *and* C_y *is the residue field of the local ring* $\mathcal{O}_{Y,y}$. *Especially, the subset* U *of* Y *on which* d_q *takes the smallest value is open.*

(2) *The following conditions are equivalent:*

(a) d_q *is a constant function.*

(b) $R^q f_* F$ *is locally free on* Y *and the natural map*

$$R^q f_* F \otimes_{\mathcal{O}_Y} C_y \to H^q(X_y, F_y)$$

is an isomorphism for any $y \in Y$.

For the proof of this theorem, see for instance Mumford [M, p. 50].

We study the relation between the relative Kodaira dimension and the Kodaira dimension of its fibers. The following theorem is a special case of Lieberman-Sernesi's theorem [LS, p. 83], although there seems to be a gap in their proof [LS, p. 84]. Hence we give here the proof of the required special case.

THEOREM 2.4. *Let* $f\colon X \to Y$ *be a family of algebraic manifolds. Then for any* $y \in Y$, *we have*

$$\kappa(X_y) \geqq \kappa(X/Y) .$$

Moreover, set

$$\mathbf{LS}(X/Y) = \{y \in Y; \kappa(X_y) = \kappa(X/Y)\} .$$

Then $\mathbf{LS}(X/Y)$ *is a countable intersection of nonempty open sets of* Y.

Before the proof of the above theorem, we give a lemma.

LEMMA 2.5. *We consider the following commutative diagram of algebraic varieties* X, Y *and* Z:

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & Z \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \pi} & \\ Y & & \end{array}$$

where f *and* π *are surjective morphisms and* g *is a generically surjective rational map. Let* Σ *be the indeterminacy of* g. *Then there exists a nonempty open set* Y' *of* Y *such that* (1) π *is flat over* Y' *and* (2) *for any* $y \in Y'$, Σ *does not contain* $X_y = f^{-1}(y)$ *and the closure of* $g(X_y \setminus X_y \cap \Sigma)$ *coincides with* $Z_y = \pi^{-1}(y)$.

PROOF. We put $U = X \setminus \Sigma$ and consider the dominant morphism $g: U \to Z$. There is an open set Z' of Z such that $Z' \subset g(U)$. Let $A_1, A_2, \cdots, A_k$ be the reduced models of the irreducible components of $Z \setminus Z'$. We consider $\pi|_{A_i}: A_i \to \pi(A_i) \subset Y$ for each i $(1 \leqq i \leqq k)$, and put

$$Y_i = \{y \in Y; \dim(\pi|_{A_i})^{-1}(y) < \dim Z - \dim Y\}$$

which is clearly a nonempty open set of Y. Now we take an open set Y'' of Y such that π is flat over Y'', and put

$$Y' = Y'' \cap \left(\bigcap_{i=1}^{k} Y_i\right).$$

This is also a nonempty open set of Y. Since the dimension of any irreducible component of Z_y is equal to $\dim Z - \dim Y$ for any $y \in Y''$, the closure of $Z'_y = (\pi|_{Z'})^{-1}(y)$ coincides with Z_y for any $y \in Y'$. q.e.d.

PROOF OF THEOREM 2.4. *Step* 1. Let Y^* be the set of points $y \in Y$ such that $\dim H^0(X_y, \omega_{X_y}^{\otimes m})$ is minimal for each $m \in \boldsymbol{Z}_{>0}$. Then by the base change theorem, Y^* is a countable intersection of nonempty open sets of Y. Furthermore,

$$f_* \omega_{X/Y}^{\otimes m} \otimes_{\mathcal{O}_Y} \boldsymbol{C}_y \cong H^0(X_y, \omega_{X_y}^{\otimes m})$$

holds for any $y \in Y^*$ and any $m \in \boldsymbol{Z}_{>0}$. Then we first claim

$$\kappa(X_y) = \kappa(X/Y)$$

for any $y \in Y^*$.

When $\kappa(X/Y) = -\infty$, the assertion is clear. Thus we may assume $\kappa(X/Y) \geqq 0$. By Lemma 2.5, there exists a point $y \in Y^*$ such that for any $m \in N(X/Y)$, the following conditions hold.

(1) The closure of $g_m(X_{y_0} \setminus X_{y_0} \cap \Sigma_{g_m})$ coincides with $\pi_m^{-1}(y_0)$.

(2) $\dim \pi_m^{-1}(y_0) = \dim Z_m - \dim Y$

where the notations are as in (1.1). Thus by letting m sufficiently large, we obtain

$$\kappa(X_{y_0}) = \dim \operatorname{Proj} R(X_{y_0})^{[m]} = \pi_m^{-1}(y_0) = \dim Z_m - \dim Y = \kappa(X/Y).$$

Next we take any point $y \in Y^*$. The Iitaka estimates with respect to X_y and X_{y_0} say that there are positive real numbers $\alpha, \alpha', \beta, \beta'$ and a positive integer d such that for any sufficiently large integer m, we have

$$\alpha m^{\kappa(X_y)} \leqq \dim H^0(X_y, \omega_{X_y}^{\otimes dm}) \leqq \beta m^{\kappa(X_y)}$$
$$\alpha' m^{\kappa(X_{y_0})} \leqq \dim H^0(X_{y_0}, \omega_{X_{y_0}}^{\otimes dm}) \leqq \beta' m^{\kappa(X_{y_0})}.$$

On the other hand, by the definition of Y^*,

$$\dim H^0(X_y, \omega_{X_y}^{\otimes m}) = \dim H^0(X_{y_0}, \omega_{X_{y_0}}^{\otimes m})$$

holds for any $m \in \boldsymbol{Z}_{>0}$. Thus we have

$$\kappa(X_y) = \kappa(X_{y_0}) .$$

Hence combining this with the previous equality, we obtain

$$\kappa(X_y) = \kappa(X/Y) \quad \text{for any} \quad y \in Y^* .$$

Step 2. We secondly claim $\kappa(X_y) \geqq \kappa(X/Y)$ for any $y \in Y \setminus Y^*$. By the definition of Y^*,

$$\dim H^0(X_y, \omega_{X_y}^{\otimes m}) \geqq \dim H^0(X_{y_0}, \omega_{X_{y_0}}^{\otimes m})$$

is satisfied for any $y \in Y \setminus Y^*$, $y_0 \in Y^*$ and any $m \in \boldsymbol{Z}_{>0}$. Thus by the same argument using the Iitaka estimates with respect to X_y and X_{y_0}, we obtain

$$\kappa(X_y) \geqq \kappa(X_{y_0}) = \kappa(X/Y) .$$

Consequently, we have the first assertion of the theorem.

Step 3. Now we prove the latter statement of the theorem by induction on the dimension of the base space Y. If $\dim Y = 0$, the assertion is obvious. Next suppose the statement is true if the dimension of the base space is smaller than n, and consider the case of $\dim Y = n > 0$. By Step 1, there exists a countable intersection Y^* of nonempty open sets of Y such that

$$\kappa(X_y) = \kappa(X/Y)$$

is satisfied for $y \in Y^*$. $Y \setminus Y^*$ can be written set-theoretically as the union of countably many irreducible reduced lower dimensional closed subvarieties W_i $(i \in I)$ for a countable set I. Then the base change

$$f_i : X \times_Y W_i \to W_i$$

is a family of algebraic manifolds with $\dim W_i < n$. Thus by the induction assumption, the set

$$\mathbf{LS}_{f_i}(W_i) = \{w \in W_i : \kappa(X_w) = \kappa(X \times_Y W_i / W_i)\}$$

is a countable intersection of nonempty open sets of W_i for each i. Let I' be the subset of I consisting of $i \in I$ with $\kappa(X \times_Y W_i / W_i) = \kappa(X/Y)$. Then we obtain

$$\mathbf{LS}(X/Y) = Y^* \cup \{\bigcup_{i \in I'} \mathbf{LS}_{f_i}(W_i)\} = Y \setminus \{(\bigcup_{i \in I \setminus I'} W_i) \cup [\bigcup_{j \in I'} \{W_j \setminus \mathbf{LS}_{f_j}(W_j)\}]\} .$$

This is a countable intersection of nonempty open sets of Y. q.e.d.

For later use, we slightly generalize the definition of $\mathbf{LS}(X/Y)$.

DEFINITION 2.6. Let $f\colon X \to Y$ be a generically smooth surjective morphism of algebraic varieties of relative dimension n such that the general fibers of f are connected. Define the Lieberman-Sernesi set $\mathbf{LS}(X/Y)$, or denoted $\mathbf{LS}_f(Y)$, of Y with respect to f to be the set of points $y \in Y$ such that the fiber X_y is irreducible, nonsingular, n-dimensional and $\kappa(X_y) = \kappa(X/Y)$.

Now we come back to the diagram (1.1). We set m sufficiently large, set $Z = Z_m$, $g = g_m$, $\pi = \pi_m$ and consider an elimination of the points of indeterminacy of g,

$$(2.3) \qquad \begin{array}{ccc} X^{\#} & & \\ {\scriptstyle\tau}\downarrow & \searrow^{h} & \\ X & \overset{g}{\dashrightarrow} & Z \\ {\scriptstyle f}\downarrow & \swarrow_{\pi} & \\ Y & & \end{array}$$

where $X^{\#}$ is nonsingular and h is the induced surjective morphism (see [H]). In this situation,

$$\dim Z - \dim Y = \kappa(X/Y)$$

is satisfied. Let

$$(2.2) \qquad \begin{array}{ccc} X_y^{\#} & & \\ {\scriptstyle\tau_y}\downarrow & \searrow^{h_y} & \\ X_y & \overset{g_y}{\dashrightarrow} & Z_y \\ {\scriptstyle f_y}\downarrow & \swarrow_{\pi_y} & \\ \{y\} & & \end{array}$$

be the restriction of (2.2) over $y \in Y$, where $X_y^{\#} = (f \cdot \tau)^{-1}(y)$, $Z_y = \pi^{-1}(y)$, $h_y = h|_{X_y}$ and so on. g_y is defined if the indeterminacy Σ_g of g does not contain X_y.

Then we have simultaneous Iitaka fibrations for general fibers as follows:

PROPOSITION 2.7. *Let $f\colon X \to Y$ be a family of algebraic manifolds of dimension n with $1 \leqq \kappa(X/Y) \leqq n-1$. And consider* (2.2). *Then*

(a) *The fibers of h over normal points of Z are connected.*

(b) *There exists a subset $Y^{\flat}$ in Y which contains a countable intersection of nonempty open sets of Y such that we have $\kappa(X_y) = \kappa(X/Y)$*

for any $y \in Y^\flat$. *Furthermore,*

$$h_y: X_y^\sharp \to Z_y$$

is the Iitaka fibration of X_y *for any* $y \in Y^\flat$.

PROOF. Let $Y^\flat$ be the set of points $y \in Y$ which satisfies the following conditions.

(1) For any $i \in \boldsymbol{Z}_{>0}$,

$$f_*\omega_{X/Y}^{\otimes i} \underset{\mathcal{O}_Y}{\otimes} \boldsymbol{C}_y \cong H^0(X_y, \omega_{X_y}^{\otimes i}) .$$

(2) For any $i \in N(X/Y)$ satisfying $i \geqq m_0$ with m_0 as in Lemma 2.1, the birational map $h_{i,m_0}: Z_i \dashrightarrow Z_{m_0}$ induces a birational map between the fibers $\pi_i^{-1}(y)$ and $\pi_{m_0}^{-1}(y)$.

(3) $f \circ \tau$ is a smooth morphism over a neighborhood of y.

$Y^\flat$ clearly is a countable intersection of nonempty open sets of Y. And for $y \in Y^\flat$, $\kappa(X_y) = \kappa(X/Y)$ is clear by (1) and Step 1 of the proof of Theorem 2.4. Further for any $y \in Y^\flat$, $h_y: X_y^\sharp \to Z_y$ is birationally equivalent to the m-th canonical rational map of X_y by (1) and (3). Hence by (2), h_y is the Iitaka fibration of X_y. Thus the statement (b) is true.

Next let $X^\sharp \xrightarrow{h} Z' \xrightarrow{\mu} Z$ be the Stein factorization of h. Then by (b) and Theorem 1.3 (b), μ is generically one to one. Thus by Zariski's main theorem, μ is an isomorphism over normal points of Z. This proves (a). q.e.d.

We remark that h is a generically smooth morphism by Sard's theorem applied to the nonsingular variety $X^\sharp$. Thus $\kappa(X^\sharp/Z)$ is well-defined.

COROLLARY 2.8. $\kappa(X/Z^\sharp) = 0$.

PROOF. Since $\pi^{-1}(Y^\flat)$ and $\mathrm{LS}(X^\sharp/Z)$ are countable intersections of nonempty open sets of Z, there exists a $y \in Y^\flat$ such that Z_y intersect with $\mathrm{LS}(X^\sharp/Z)$. Let Z_y' be an Iitaka subset of Z_y (see Theorem 1.3). As $Z_y' \cap \mathrm{LS}(X/Z^\sharp)$ is clearly a countable intersection of nonempty open sets of Z_y, this is not empty. So if we take a point $z \in Z_y' \cap \mathrm{LS}(X^\sharp/Z)$, then

$$\kappa(X^\sharp/Z) = \kappa(h^{-1}(z)) = 0 . \qquad \text{q.e.d.}$$

3. A graded subring of the canonical ring. For later use, we study graded rings. We fix an algebraic manifold V. Let R be the canonical ring of V and let

$$P = \bigoplus_{i \in Z_{\geqq 0}} P_i\,, \quad (\text{where } P_0 = C)$$

be a graded C-subalgebra of R. We also denote $P^{[i]} = C[P_i]$ in the same way as in § 1. We set

$$N(P) = \{i \in Z_{>0};\, P_i \neq 0\}\,.$$

This is a semigroup.

DEFINITION 3.1. The number $\kappa(P)$ is defined by

$$\kappa(P) = \begin{cases} \max\limits_{i \in N(P)} \dim \operatorname{Proj} P^{[i]} & \text{if} \quad N(P) \neq \varnothing \\ -\infty & \text{if} \quad N(P) = \varnothing\,. \end{cases}$$

REMARK 3.2. If $P = R$ itself, $\kappa(P)$ is equal to the Kodaira dimension of V.

Now we study the properties of $\kappa(P)$.

LEMMA 3.3. *If $N(P) \neq \varnothing$, then*

$$\kappa(P) = \operatorname{tr.deg}_C P - 1\,.$$

LEMMA 3.4. *There is a positive integer m_0 such that, for any $m \geqq m_0$ satisfying $m \in N(P)$, we have*

$$\dim \operatorname{Proj} P^{[m]} = \kappa(P)\,.$$

The above two lemmas are easily seen in the same method as in the case of the Kodaira dimension. Our aim of this section is the following:

PROPOSITION 3.5. *Assume that there exist a non-negative integer t, positive real numbers α, β and a positive integer d such that for any sufficiently large m,*

$$\alpha m^t \leqq \dim_C P_{dm} \leqq \beta m^t$$

holds. Then we have $t = \kappa(P)$.

PROOF. We first prove $t \geqq \kappa(P)$. We put $k = \kappa(P)$. Let $x_0, x_1, \cdots, x_k$ be a transcendental basis of P over C. We may assume that every x_i $(0 \leqq i \leqq k)$ belongs to P_{n_0} for some positive integer n_0. We put

$$P^{(n_0)} = \bigoplus_{i \in Z_{\geqq 0}} P_{in_0}\,,$$

where the homogeneous part $P_i^{(n_0)}$ of degree i is P_{in_0}. Then the polynomial ring $C[x_0, x_1, \cdots, x_k]$ is a graded C-subalgebra of $P(n_0)$. Thus we have

$$\dim_C P_{n_0 m} \geqq m^k/k!\,.$$

Hence $t \geqq k$ is clear.

It remains to prove $t \leqq \kappa(P)$. We put $r = \kappa(R) - \kappa(P) = \text{tr.deg}_C R - \text{tr.deg}_C P$ and let $y_1, y_2, \cdots, y_r$ be a transcendental basis of R over P. We may assume that every y_i $(1 \leqq i \leqq r)$ belongs to $R_{n'}$ for some n'. We put

$$R' = R^{(n')} = \bigoplus_{i \in \mathbf{Z}_{\geqq 0}} R_{in'} , \quad P' = P^{(n')} = \bigoplus_{i \in \mathbf{Z}_{\geqq 0}} P_{in'} .$$

Then it is clear that $P'[y_1, y_2, \cdots, y_r] \subset R'$.

First we consider the graded ring $P'[y_1]$. By the grading of $P'[y_1]$, we have $\dim P'[y_1]_m = \dim P'_m + \dim P'_{m-1} + \cdots + \dim P'_1 + 1$. Thus by assumption, we have $\alpha' m^{t+1} \leqq \dim P'[y_1]_m$ for any sufficiently large m and some $\alpha' > 0$. Repeating the above arguement r-times, we obtain

$$\alpha'' m^{t+r} \leqq \dim P'[y_1, y_2, \cdots, y_r]_m \leqq \dim R'_m$$

for $m \gg 0$ and some $\alpha'' > 0$.

On the other hand, the Iitaka estimate of the Kodaira dimension says

$$\dim R_{d'm} \leqq \beta' m^{\kappa(R)}$$

for $m \gg 0$, some $\beta' > 0$ and $d' > 0$. Thus we have $t + r \leqq \kappa(R)$ and we are done. q.e.d.

4. Families over a nonsingular curve. In §§ 4 and 5, nonsingular curves may not be complete.

In this section, we study families over a nonsingular curve. First, we give a definition.

DEFINITION 4.1. For a variety V, we say that V is the quasi-specialization of parabolic varieties if there exist a nonsingular variety M, a nonsingular curve C and a proper surjective morphism $\varphi: M \to C$ such that

(1) $\kappa(M/C) = 0$.

(2) V is equal to the associated reduced scheme of one of the irreducible components of $\varphi^{-1}(0)$ for some $0 \in C$.

REMARK 4.2. A parabolic variety V (see [U, § 11]) trivially is a quasi-specialization of parabolic varieties since the second projection $p: V \times C \to C$ satisfies the required property.

Our aim of this section is to prove the following:

THEOREM 4.3. *Let $f: X \to S$ be a family of n-dimensional algebraic manifolds over a nonsingular curve S such that $1 \leqq \kappa(X/S) \leqq n-1$.*

Then for any $s \in S$, the fiber X_s has the following property: There exist a nonsingular model X_s^* of X_s, a variety T and a fiber space $\psi: X_s^* \to T$ such that

(1) $\dim T = \kappa(X/S)$.

(2) *There is an open set T' of T such that, for any $t \in T'$, the fiber $\psi^{-1}(t)$ is irreducible nonsingular and is the quasi-specialization of parabolic varieties.*

In the situation of Theorem 4.3, we consider the following diagram putting N sufficiently large.

$$\begin{array}{ccl} U & & \\ \cap\downarrow & & \\ X^{\#} & \xrightarrow{t} & W \\ \tau\downarrow & \searrow^{h} & \downarrow\mu \\ X & \overset{g}{\dashrightarrow} & Z \subset P(f_*\omega_{X/S}^{\otimes N}) \\ f\downarrow & \swarrow_{\pi} & \\ S & & \end{array} \tag{4.1}$$

where $X^{\#} \xrightarrow{t} W \xrightarrow{\mu} Z$ is the Stein factorization of h, U is the pull back of the defining locus of g by τ and others are the same as in (2.2) with $Y = S$. We may assume each component of any fiber of $f \circ \tau$ is nonsingular by Hironaka's theorem.

From now on, we fix a point $s \in S$. We consider the fiber space

$$\tilde{t}_s: \tilde{X}_s^{\#} \to \tilde{W}_s , \tag{4.2}$$

where $\tilde{X}_s^{\#}$ is the strict transform of X_s with respect to τ, i.e., the closure of $U_s = U \cap X_s^{\#}$, $\tilde{t}_s$ is the restriction of t to $\tilde{X}_s^{\#}$ and $\tilde{W}_s$ is the image of $\tilde{X}_s^{\#}$ by $\tilde{t}_s$. As $\tilde{X}_s^{\#}$ is one of the irreducible components of $X_s^{\#}$, $\tilde{X}_s^{\#}$ is nonsingular.

Now as a preparation for Lemma 4.5, we review the following, which was proved by Lieberman-Sernesi [LS, p. 79].

PROPOSITION 4.4. *Let X be a variety, S be a nonsingular curve, $f: X \to S$ be a proper morphism and F be an $\mathcal{O}_S$-flat coherent sheaf on X. Then,*

(1) *f_*F is a locally free sheaf of finite rank.*

(2) *For any $s \in S$, the natural map*

$$t_s^0 : f_*F \underset{\mathcal{O}_S}{\otimes} C_s \to H^0(X_s, F_s)$$

is injective.

In the situation before Proposition 4.4, we have

LEMMA 4.5. $\dim \tilde{W}_s = \kappa(X/S)$.

PROOF. For any point $u \in S$, we set

$$P(X_u) = \bigoplus_{i \in Z_{\geq 0}} (f_* \omega_{X/Y}^{\otimes i} \bigotimes_{\mathcal{O}_S} C_u) .$$

This is a graded C-subalgebra of the canonical ring $R(X_u)$ by Proposition 4.4 (2).

Now we set $\tilde{Z}_s = h(\tilde{X}_s^{\sharp})$. Since by Proposition 4.4 it is easy to show that the indeterminacy Σ_g does not contain X_s, $\tilde{Z}_s$ coincides with the closure of $g(X_s \setminus X_s \cap \Sigma_g)$. Then we have $\tilde{Z}_s = \operatorname{Proj} P(X_s)^{[N]}$. As N is sufficiently large, we have

$$\dim \tilde{Z}_s = \kappa(P(X_s))$$

where $\kappa(P(X_s))$ is as defined in § 3.

On the other hand, for a general point $u \in S$, we have $R(X_u) = P(X_u)$ and $\kappa(X_u) = \kappa(X/S)$. As $\dim P(X_u)_i = \dim P(X_s)_i$ holds for any $i \in \boldsymbol{Z}_{\geq 0}$ by Proposition 4.4 (1), the Iitaka estimate with respect to X_u says

$$\alpha m^{\kappa(X/S)} \leqq \dim P(X_s)_{dm} \leqq \beta m^{\kappa(X/S)}$$

for some $\alpha > 0$, $\beta > 0$, $d \in \boldsymbol{Z}_{>0}$ and any sufficiently large m. Hence by Proposition 3.5, we have

$$\kappa(P(X_s)) = \kappa(X/S) .$$

Thus we have $\dim \tilde{Z}_s = \kappa(X/S)$. As $\dim \tilde{W}_s = \dim \tilde{Z}_s$ is clear, we are done. q.e.d.

We consider (4.2). As $\tilde{X}_s^{\sharp}$ is nonsingular, there is an open set $\tilde{W}_s'$ of $\tilde{W}_s$ such that (1) $\tilde{t}_s$ is a smooth morphism over $\tilde{W}_s'$, (2) $\tilde{W}_s' \subset t(U)$ and (3) the closure of $U_w = (t|_U)^{-1}(w)$ coincides with $(\tilde{X}_s^{\sharp})_w$ for any $w \in \tilde{W}_s'$. The assertion (3) is possible by Lemma 2.5. We take and fix any point $w \in \tilde{W}_s'$.

LEMMA 4.6. *There exists an irreducible curve C on W passing through w such that*

(a) $\mathbf{LS}_t(W) \cap C$ *is a countable intersection of nonempty open sets of C.*

(b) *There is an irreducible component $\tilde{X}$ of $t^{-1}(C)$ such that*

(1) *the induced morphism $\tilde{t} = t|_{\tilde{X}}: \tilde{X} \to C$ is surjective.*

(2) *$(\tilde{X}_s^{\sharp})_w = (\tilde{t}_s)^{-1}(w)$ is one of the irreducible components of the fiber $\tilde{X}_w = \tilde{t}^{-1}(w)$.*

PROOF. We take points $p \in \mathbf{LS}_t(W) \cap t(U)$ satisfying $p \neq w$, $x \in t^{-1}(w) \cap U$ and $y \in t^{-1}(p) \cap U$. By a lemma in Mumford [M, p. 56], we can take an irreducible curve D on X passing through x and y. We set $C = t(D)$. Since C is a curve passing through p, the desired property (a) is clearly satisfied.

Next we consider the proper morphism $t|_{t^{-1}(C)}: t^{-1}(C) \to C$. As the general fibers of $t|_{t^{-1}(C)}$ are irreducible by (a), there exists only one irreducible component $\widetilde{X}$ of $t^{-1}(C)$ which dominates C. We set $\widetilde{U} = \widetilde{X} \cap U$ and consider the dominant morphism

$$t|_{\widetilde{U}}: \widetilde{U} \to C .$$

$t|_{\widetilde{U}}$ is an equidimensional morphism of relative dimension $n - \kappa(X/S)$ since $\widetilde{U}$ is irreducible, C is a curve and by (a). Then the closure of the fiber $U_w = (t|_{\widetilde{U}})^{-1}(w)$ coincides with $(\widetilde{X}^{\sharp}_s)_w$, because $\widetilde{U}_w$ is contained in $U_w = (t|_U)^{-1}(w)$, the closure of U_w coincides with $(\widetilde{X}^{\sharp}_s)_w$, $\dim \widetilde{U}_w = \dim (\widetilde{X}^{\sharp}_s)_w = n - \kappa(X/S)$ and $(\widetilde{X}^{\sharp}_s)_w$ is irreducible. As the closure of $\widetilde{U}$ coincides with $\widetilde{X}$, the closure of $\widetilde{U}_w$ is contained in the fiber $\widetilde{X}_w$. Thus the property (b) is clearly satisfied. q.e.d.

Now we prove Theorem 4.3. For any $s \in S$, we show that the fiber space $\tilde{t}_s: \widetilde{X}^{\sharp}_s \to \widetilde{W}_s$ has the desired property. The statement (1) was proved in Lemma 4.5. For any $w \in \widetilde{W}'_s$, we consider $\tilde{t}: \widetilde{X} \to C$. Let $\hat{C} \to C$ be the normalization, $\hat{X} \to \widetilde{X}$ be a desingularization and $\hat{t}: \hat{X} \to \hat{C}$ be the induced fiber space. By Corollary 2.8, Lemma 4.6 (a) and our construction, we have $\kappa(\hat{X}/\hat{C}) = 0$. Thus Lemma 4.6 (b) says that $\tilde{t}_s^{-1}(w)$ is the quasi-specialization of parabolic varieties for any $w \in \widetilde{W}'_s$. q.e.d.

5. Proofs of main theorems. We prove Theorem I. We take any point $y \in Y$. By a lemma in [M, p. 56], we can take an irreducible curve C on Y passing through y such that $C \cap \mathbf{LS}(X/Y)$ is a countable intersection of nonempty open sets of C. Let $\tau: \hat{C} \to C$ be the normalization and $\hat{f}: \hat{X} = X \times_Y \hat{C} \to \hat{C}$ be the induced family. Since $\tau\{\mathbf{LS}(\hat{X}/\hat{C})\} \cap \mathbf{LS}(X/Y) \neq \varnothing$, we have

$$\kappa(\hat{X}/\hat{C}) = \kappa(X/Y) .$$

As X_y is isomorphic to $\hat{f}^{-1}(\hat{y})$ for some $\hat{y} \in \hat{C}$, our assertion is clear by Theorem 4.3. q.e.d.

Next we formulate the lower-semicontinuity of the Kodaira dimension of a degenerating family of parabolic varieties.

CONJECTURE PDG_n. *Let $f: M \to S$ be a proper surjective morphism with connected fibers of relative dimension n from an algebraic mani-*

fold M to a nonsingular curve S. Assume that there is a set $P = \{p_1, p_2, \cdots, p_r\}$ *of points of S such that* (1) f *is a smooth morphism over* $S \setminus P$. (2) *For any* $s \in S \setminus P$, *we have* $\kappa(M_s) = 0$. *Let*

$$M_{p_i} = \bigcup_j m_{ij} M_{p_i}^{(j)}$$

be the decomposition into irreducible components of the fiber $M_{p_i} = f^{-1}(p_i)$ *for* $p_i \in P$. *Then we have*

$$\kappa(M_{p_i}^{(j)}) \leqq 0$$

for any $p_i \in P$ *and* j.

REMARK. When $n = 1$, Conjecture PDG_1 is true because any component of a degenerate fiber of a degenerating family of elliptic curves is elliptic or rational (see Kodaira [K]).

When $n = 2$, every known example of degenerating families of surfaces of parabolic type satisfies the statement of Conjecture PDG_2 (see for example Persson [P]).

We prove Theorem II. For any $y \in Y$, let $\psi: X_y^* \to T$ be the fibration defined in Theorem I. For a point $t \in T'$, let $\varphi: M \to C$ be the fiber space which induces on $\psi^{-1}(t)$ the quasi-specialization of parabolic varieties. Then Conjectures $\mathrm{DF}_{n-k,0}$ and PDG_{n-k} clearly imply $\kappa(\psi^{-1}(t)) \leqq 0$. Thus by Theorem 1.4, we have

$$\kappa(X_y) = \kappa(X_y^*) \leqq \kappa(\psi^{-1}(t)) + \dim T \leqq \kappa(X/Y) .$$

On the other hand, $\kappa(X_y) \geqq \kappa(X/Y)$ is also true by Theorem 2.4. Hence we obtain $\kappa(X_y) = \kappa(X/Y)$. q.e.d.

Since Conjectures $\mathrm{DF}_{1,0}$ and PDG_1 are true, we have:

COROLLARY. *Conjecture* $\mathrm{DF}_{n,n-1}$ *is true.*

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Appendix

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In this appendix we shall show that PDG_2 is true. Since $DF_{2,0}$ is true by virtue of the classification of algebraic surfaces, Theorem II implies the validity of $DF_{n,n-2}$.

The conjecture PDG_2 is true by the following.

Proposition. *Let $f: M \to D$ be a proper surjective morphism of a three dimensional complex manifold M to a disk $D = \{z \in \boldsymbol{C} | \, |z| < \varepsilon\}$ with connected fibers. Assume that f is smooth at each point on $f^{-1}(D^*)$, $D^* = D - \{0\}$ and f is projective. Then for each irreducible component E of the fiber $f^{-1}(0)$, we have*

$$\kappa(E) \leqq \kappa(M_x) ,$$

where $M_x = f^{-1}(x)$ is a general fiber of f.

Note that κ is invariant under smooth deformations for surfaces. The above proposition was proved by Persson [P] except a few cases. By virtue of Kulikov [Ku], Persson-Pinkham [P²], we can prove the proposition in these cases. For reader's convenience, we give a detailed proof.

Proof of Proposition. If $\kappa(M_x) = 2$, the proposition is trivially true. Therefore, first assume $\kappa(M_x) = 1$, that is, M_x is an elliptic surface of general type. Then, all fibers M_y, $y \in D^*$ are elliptic surfaces of general type and there is a positive integer m such that the m-canonical mapping $\Phi_{mK}: M_y \to C_y = \Phi_{mK}(M_y)$ associated with the m-canonical system $|mK_{M_y}|$ of M_y is a morphism and gives the structure of an elliptic surface for each M_y, $y \in D^*$. (See [K] and [I_2].) Let

$$\begin{array}{ccc} M & \overset{g}{\dashrightarrow} & Z \subset \boldsymbol{P}(f_*\omega_{M/D}^{\otimes m}) \\ {\scriptstyle f}\big\downarrow & \swarrow & \\ D & & \end{array}$$

be a diagram similar to that in § 1. The meromorphic mapping g is holomorphic on $M^* = f^{-1}(D^*)$. Let $h: \hat{M} \to M$ be obtained by a succession of monoidal transformations along non-singular centers such that $\hat{g} = g \circ h: \hat{M} \to Z$ is a morphism. If the proposition is true for the family $\hat{f} = f \circ h: \hat{M} \to D$, then it is also true for the original family $f: M \to D$. Hence, we can assume that g is a morphism. For the same reason, we can assume that Z is a two dimensional complex manifold. By our construction, every regular fiber of g is a non-singular elliptic curve. Let us consider an irreducible component E of the fiber $f^{-1}(0)$. By taking a finite succession of monoidal transformations of M and Z along non-singular centers, we can assume that E is non-singular and $g(E)$ is a non-singular curve on Z. Since general fibers of g are connected, every fiber of $g|_E: E \to g(E)$ is connected. Let $p \in g(E)$ be a general smooth analytic curve in Z passing through p such that $g^{-1}(C)$ is non-singular. Then each general fiber of $g_C: g^{-1}(C) \to C$ is a non-singular elliptic curve. Since $g|_E{}^{-1}(p)$ is contained in $g_C^{-1}(p)$ and any irreducible component of $g_C^{-1}(p)$ is an elliptic curve or a rational curve, a general fiber of $g|_E: E \to g(C)$ is an elliptic curve or a rational curve. Hence, by Theorem 1.4, we have

$$\kappa(E) \leqq 1 .$$

This is the desired result.

Note that in the proof we need not assume that f is projective so that the proposition is valid for any fibration $f: M \to D$ with $\kappa(M_x) = 1$.

Next we consider the case in which $\kappa(M_x) = 0$. First we will show that we can assume that all fibers over D^* are minimal.

Let ω be a non-zero element in $H^0(M, \omega_{M/D}^{\otimes 12})$. (Since $f_*\omega_{M/D}^{\otimes 12}$ is torsion free, hence free over D, we can always find such an element.) If a regular fiber M_x contains an exceptional curve of the first kind, by [W, Proposition 2.4, p. 291], all regular fibers over D^* contain exceptional curves of the first kind. Moreover, all exceptional curves of the first kind on each regular fiber do not intersect each other. Therefore, if necessary, taking a finite ramified covering $\hat{D}$ of D ramified at the origin and a non-singular model $\hat{M}$ of $M \times_D \hat{D}$, we may assume that there exists an irreducible divisor E appearing in the divisor (ω) such that the intersection $E \cap M_x$ is an exceptional curve of the first kind on each fiber M_x, $x \in D^*$. (Note that by the argument given below if the proposition is valid for $\hat{f}\colon \hat{M} \to D$, it is also valid for our original family $f\colon M \to D$.) Let F be a relatively ample divisor of $f\colon M \to D$. We let m be the intersection number $(F \cap M_x)\cdot(E \cap M_x)$ on M_x. Let us consider an invertible sheaf $O_M(n(F + mE))$ for a sufficiently large positive integer n and consider a diagram

$$\begin{array}{ccl} M & \overset{h}{\dashrightarrow} & M' \subset \boldsymbol{P}(f_* O_M(n(F + mE))) \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle f'} \\ & & M \quad . \end{array}$$

Then h is a morphism on $M^* = f^{-1}(D^*)$, $h_y\colon M_y \to M'_y$, $y \in D^*$ is a contraction morphism (that is $h_y(E \cap M_y)$ is a point) and M' is smooth over D^*. We let M_1 be a nonsingular model of M' obtained by a finite succession of monoidal transformations along non-singular centers contained in the fibers over the origin. Since h is bimeromorphic, the proposition is true for the family $f\colon M \to D$ if and only if it is true for $f_1\colon M_1 \to D$. Applying this process finitely many times, we can assume that all regular fibers of $f\colon M \to D$ do not contain exceptional curves of the first kind.

Next we will show that it is enough to consider the case in which $p_g(M_x) = 1$ for a regular fiber. Assume $p_g(M_x) = 0$. Then there exists a positive integer $m \geqq 2$ which is a divisor of 12 such that $P_m(M_x) = 1$, $P_l(M_x) = 0$, $l = 1, 2, \cdots, m-1$. Then, using a divisor (ω) for a non-zero element $\omega \in H^0(M, \omega_{M/D}^{\otimes m})$, we can construct an m-sheeted covering $g\colon \hat{M} \to M$ which may ramify along divisors contained in the fiber over the origin. (See, for example, [U, p. 176–177].) Note that $g_y\colon \hat{M}_y \to M_y$ is an m-sheeted covering with $p_g(\hat{M}_y) = 1$ for each $y \in D^*$. Let $\tilde{M}$ be a non-singular model of $\hat{M}$ obtained by a finite succession of monoidal transformations along non-singular centers contained in the fiber over the origin. Then for an irreducible component E of $f^{-1}(0)$, there is an irreducible component $\tilde{E}$ of the fiber over the origin of $\tilde{f}\colon \tilde{M} \to D$ such that $\tilde{E} \to E$ is a generically finite morphism. Since $\kappa(\tilde{E}) \geqq \kappa(E)$, it is enough to prove the proposition under the assumption that each regular fiber is a minimal surface with $\kappa = 0$, $p_g = 1$, hence an abelian surface or a $K3$ surface.

Now by [N] (for a family of abelian surfaces), [K] and [P[2]] (for a family of $K3$ surfaces), there exists a finite ramified covering $\hat{D} \to D$ ramified along the origin and a fibration $\hat{f}\colon \hat{M} \to \hat{D}$ which is bimeromorphically equivalent to $M \times_D \hat{D} \to \hat{D}$ such that our proposition is true for the family $\hat{f}: \hat{M} \to \hat{D}$. Then, for the same reason as above, it is also true for our family $f: M \to D$.

Finally we consider the case in which $\kappa(M_x) = -\infty$ for a general fiber M_x. Since f is projective, M_x is a rational surface or a ruled surface. Therefore the surface M_x contains a rational curve l_x with self-intersection number $l_x^2 = 1$ or 0 in M_x. Moreover if $l_x^2 = 1$, then M_x is a rational surface. Let us consider the relative Hilbert scheme $\mathrm{Hilb}_{M/D}$. By the stability theorem ([K$_2$]), the component F of $\mathrm{Hilb}_{M/D}$ containing the point corresponding to l_x is of positive dimension. Since the natural morphism $p: F \to D$ is proper, there is a holomorphic map $\pi: E = \{t \in \boldsymbol{C} \mid |t| < \varepsilon'\} \to F$ such that $\pi \circ p: E \to D$ is a finite ramified covering. We may assume that $\pi \circ p$ is ramified only over the origin. Let $\hat{f}: \hat{M} \to E$ be a non-singular model of $M \times_D E \to E$ and L be the pull back of the universal family $\mathscr{L}$ over F to E. Then the image $\hat{L}$ of L to $\hat{M}$ is a divisor on $\hat{M}$ such that for a general fiber $\hat{M}_y$ of $\hat{f}$, $\hat{L} \cap \hat{M}_y = \hat{l}_y$ is a non-singular rational curve with $\hat{l}_y^2 = 1$ or 0 in $\hat{M}_y$. Let us consider a diagram

$$\begin{array}{ccl} \hat{M} & \overset{g}{\dashrightarrow} & S \subset \boldsymbol{P}(\hat{f}_* O_{\hat{M}}(\hat{L})) \\ {\scriptstyle \hat{f}}\downarrow & \swarrow & \\ \hat{E}. & & \end{array}$$

For the same reason as above, we can assume that g is a morphism. If $\hat{l}_y^2 = 1$, then $\hat{f}_* O_{\hat{M}}(\hat{L})$ is a locally free sheaf of rank 3, since $\hat{M}_y$ is rational. Moreover, in this case, it is easy to show that $S = \boldsymbol{P}(\hat{f}_* O_{\hat{M}}(\hat{L}))$ and g is bimeromorphic, since $\Phi_{\hat{L}_y}: \hat{M}_y \to \boldsymbol{P}^2$ is birational. Hence any component G of the fiber of f over the origin is a rational or ruled surface. Hence $\kappa(G) = -\infty$. This implies our proposition. Assume $\hat{l}_y^2 = 0$. Then, $\Phi_{\hat{L}_y}: \hat{M}_y \to C_y = \Phi_{L_y}(\hat{M}_y)$ gives the structure of a ruled surface, if we blow down all exceptional curves contained in fibers of $\Phi_{\hat{L}_y}$. Hence, in this case, S is a surface and general fibers of g are $\boldsymbol{P}^1$'s. Hence by an arguement similar to that in the case of $\kappa(M_x) = 1$, we conclude that each component of the fibers of f over the origin is rational or ruled. Thus we obtain the desired result.

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