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THE DEGREE OF CONVERGENCE FOR ENTIRE FUNCTIONS

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ABSTRACT

Consider a point set C in the complex plane whose complement K is connected and regular (i.e. K possesses a Green's function with pole at infinity). Let d_∞ denote the transfinite diameter of C. Recall that $d_\infty = 1/|\phi'(\infty)|$ where $\phi(z)$ maps K onto the exterior of the unit circle. Equivalently, $d_\infty = \lim_{n\to\infty}[||T_n(z)||_C]^{1/n}$ where $||g(z)||_C = \max|g(z)|, z\varepsilon C$ and $T_n(z)$ is the Tchebycheff polynomial (or Faber polynomial) for C. Given f(z) defined on C, let $P_n^*(z)$ be the best polynomial approximation to f(z) on C i.e., $||f(z)-P_n(z)||_C$ is minimized for polynomials $P_n(z)$ of degree n. The purpose of this paper is to prove the

<u>THEOREM We have</u>

$$\lim_{n\to\infty} n^{1/\rho}||f(z) - P_n^*(z)||^{1/n} = d_\infty(e\rho\tau)^{1/\rho}$$

<u>if and only if</u> f(z) <u>is entire of order</u> $\rho > o$ <u>and type</u> $o < \tau < \infty$

The method of proof essentially combines basic techniques from the theory of entire functions with machinery used to establish degree of convergence theorems for polynomial approximations to analytic functions. Thus it is shown that this degree of convergence is achieved (with a factor $1+\varepsilon$, arbitrary $\varepsilon > o$) by a polynomial expansion of the form

$$f(z) \quad \sum_{k=1}^{\infty} q_k(z)\, p(z)^{k-1}$$

where p(z) is a polynomial of degree λ and $q_k(z)$ is of degree $\lambda-1$. The level curves of $|p(z)|$ define a lemniscate which approximates the boundary of C. A number of lemmas are established which relate the nature of the coefficient polynomials $q_k(z)$ to the order and type of f(z).

THE DEGREE OF CONVERGENCE FOR ENTIRE FUNCTIONS

JOHN R. RICE*

Dedicated to J. L. Walsh

I. INTRODUCTION. The main result of this paper is to characterize the set of entire functions of order $\rho > 0$ and type $0<\tau<\infty$ in terms of their degree of convergence on rather general sets. The preliminary results include extensions of some classical properties of entire functions. A generalization is indicated at the end for the approximation of a function with a finite number of singularities by sequences of rational functions. It is assumed that the reader is familiar with the terminology and results of [1] and [4].

One may obtain results about the degree of approximation on disks by the direct application of techniques from the theory of entire functions. Let $P_n(z)$ be the best approximation to $f(z)$ on a set D. Set

$$E_n = ||f(z)-P_n(z)||_D = \max_{z\varepsilon D}|f(z)-P_n(z)|$$

Then for $D = \{z|\ |z|\leq r\}$ one sees that $f(x)$ is entire of order ρ and type τ if and only if

$$\lim_{n\to\infty} \sqrt[n]{E_n} = \frac{1}{r}\left(\frac{e\rho\tau}{n}\right)^{1/\rho}$$

These same techniques apply to more general sets, but the constants obtained are no longer sharp, see [3].

Bernstein [2] obtains similar results for approximation on $[-1,1]$ by considering the expansion of $f(z)$ in terms of Tchebycheff polynomials of best least squares approximation with weight $(1-x^2)^{1/2}$. He, however, does not obtain a sharp result in the sense that the constants are determined.

In this paper we establish the following

THEOREM 1. Let C be a closed bounded point set whose complement is connected and regular and let $d_\infty(C)$ be the transfinite diameter of C. Given $f(z)$ defined on C then for the sequence of polynomials $P_n^*(z)$ of degree n of best approximation to $f(z)$ on C we have

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$$\lim_{\to\infty} n^{\frac{1}{\rho}} ||f(z)-P^*_n(z)||_C^{1/n} = d_\infty(C)(e\rho\tau)^{1/\rho}$$

if and only if $f(z)$ is entire of order $\rho>0$ and type $0<\tau<\infty$.

2. PRELIMINARY RESULTS. Let $p(z)$ be a polynomial of degree λ and Γ_R the lemniscate

$$\Gamma_R = \{ z| \quad |p(z)| = R\}.$$

The following result may be established about the length $||\Gamma_R||$ of Γ_R.

LEMMA 1 As $R \to \infty$, $||\Gamma_R|| = 2\pi R^{1/\lambda}(1 + o(1))$.

We consider expansions of the function $f(z)$ in terms of a power series in $p(z)$ with polynomial coefficients, e.g.

$$f(z) = \sum_{k=1}^{\infty} q_k(z)[p(z)]^{k-1} \tag{1}$$

where $q_k(z)$ is a uniquely determined polynomial of degree λ-1 or less. The following notation is useful:

$$M(\Gamma_R) = ||f(z)||_{\Gamma_R} .$$

We now obtain estimates of the coefficients in the expansion (1) in terms of properties of $f(z)$.

LEMMA 2. Let $f(z)$ be analytic in Γ_R. Then there exists a polynomial $Q(z)$ of degree λ-1, independent of n and R, such that for $\alpha < R$,

$$||q_n(z)||_{\Gamma_\alpha} \le \frac{||\Gamma_R|| M(\Gamma_R)}{2\pi R^n} ||Q(z)||_{\Gamma_R}. \tag{2}$$

Proof. We have

$$f(z) - \sum_{k=1}^{n} q_k(z)[p(z)]^{k-1} = \frac{1}{2\pi i} \int_\Gamma \frac{[p(z)]^n f(t)\, dt}{[p(t)]^n (t-z)} , \quad z \text{ interior } \Gamma_R,$$

which implies

$$q_n(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{[p(t)-p(z)]f(t)dt}{[p(t)]^n (t-z)} \quad , z \text{ interior } \Gamma_R \quad .$$

Now $p(t) - p(z)$ is a polynomial in t of degree λ with a zero at $t = z$. Hence $p(t) - p(z) = (t-z)Q(t,z)$ and we have

$$||q_n(z)||_{\Gamma_\alpha} \equiv ||\frac{1}{2\pi i} \int_{\Gamma_R} \frac{Q(t,z)f(t)dt}{[p(t)]^n}||_{\Gamma_\alpha} \leq \frac{||\Gamma_R||\, M(\Gamma_R)}{2\pi R^n} ||Q(t,t)||_{\Gamma_R}.$$

<u>LEMMA 3. $f(z)$ is an entire function of order $\rho > o$ and type $0<\tau<\infty$ if and only if</u>

(3) $$\overline{\lim_{R\to\infty}} \frac{\log M(\Gamma_R)}{R^{\rho/\lambda}} = \tau$$

<u>Proof</u>: From Lemma 1 we have that $z\varepsilon\Gamma_R$ implies that $|z| = R^{1/\lambda}(1+o(1))$ as R tends to ∞. Thus we have

$$\overline{\lim_{R\to\infty}} \frac{\log M(\Gamma_R)}{R^{\rho/\lambda}} = \overline{\lim_{S\to\infty}} \quad \log \frac{M(|z|=S)}{S^{\rho}(1+o(1))^{\rho}}$$

if we set $S = R^{1/\lambda}(1+o(1))$. The factor $(1+o(1))^{\rho}$ tends to 1 and it is well known [1] that the limit on the right is τ if and only if $f(z)$ is of order ρ and type τ.

<u>COROLLARY $f(z)$ is an entire function of order ρ if and only if</u>

(4) $$\overline{\lim_{R\to\infty}} \frac{\log \log M(\Gamma_R)}{\log R} = \rho/\lambda$$

<u>LEMMA 4. Let α be fixed. Then $f(z)$ is entire of order $\rho>0$ if and only if</u>

(5) $$\rho = \lambda \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \frac{1}{||q_n(z)||_{\Gamma_\alpha}}}$$

Proof. Let μ denote the lim sup and let ρ denote the order of $f(z)$ as an entire function. We first show that $\mu \leq \rho/\lambda$. We may assume $\mu > 0$. Given $\mu/2 > \varepsilon > 0$, (5) implies that there is an infinity of n such that

$$\log ||q_n(z)||_{\Gamma_\alpha} \geq - \frac{n \log n}{\mu-\varepsilon} .$$

It follows from (2) that, for $R \geq \alpha$,

$$\log \left[\frac{||\Gamma_R|| \; M(\Gamma_R)}{2\pi R^n} ||Q||_{\Gamma_R} \right] \geq \log||q_n(z)||_{\Gamma_\alpha} \geq \frac{-n \log n}{\mu-\varepsilon}$$

or

$$\log M(\Gamma_R) \geq n \log R - \frac{n}{\mu-\varepsilon} \log n - \log \frac{||\Gamma_R||}{2\pi} ||Q||_{\Gamma_R}.$$

Take $R = e\, n^{\frac{1}{\mu-\varepsilon}}$. Then, for n sufficiently large, we have $R > \alpha$ and

$$\log M(\Gamma_R) \geq (eR)^{\mu-\varepsilon} - \log \frac{||\Gamma_R||}{2\pi} ||Q||_{\Gamma_R}.$$

Thus, for these values of R, we have

$$\log \log M(\Gamma_R) \geq (\mu-\varepsilon) \log R + \mu-\varepsilon - \log [1-(eR)^{\varepsilon-\mu}\log \frac{||\Gamma_R||}{2\pi} ||Q||_{\Gamma_R}].$$

It follows from Lemmas 1 and 2 that, as $R\to\infty$,

$$\log \frac{||\Gamma_R||}{2\pi} \; ||Q||_{\Gamma_R} = \log \, [R(1 + o(1))].$$

Hence, as $R\to\infty$,

$$\frac{\log \log M(\Gamma_R)}{\log R} \geq \mu-\varepsilon + o(1)$$

It follows from Lemma 3 that $\rho/\lambda \geq \mu-\varepsilon + \theta(1)$ as $R\to\infty$ and completes the first part of the proof.

We now show that $\mu\geq\rho/\lambda$. We may assume that $\mu<\infty$. Given $\varepsilon>o$, there is an N_o such that for all $n>N_o$

$$||q_n(z)||_{\Gamma_\alpha} \leq n^{\frac{n}{\mu+\varepsilon}} .$$

The above inequality implies in (1) that $f(z)$ is entire. We may assume that this inequality holds for all n by modifying $f(z)$ by a polynomial (this does not affect the order of $f(z)$ as an entire function). We have, by [4, Lemma p. 77] that

$$M(\Gamma_R) \leq \sum_{n=1}^{\infty} ||q_n(z)||_{\Gamma_R} ||p(z)||_{\Gamma_R}^n < \sum_{n=1}^{\infty} ||q_n(z)||_{\Gamma_\alpha} R^{n+\lambda-1}$$

The maximum value of $n^{-\frac{n}{\mu+\varepsilon}} R^n$ occurs for $e^{-1} R^{\mu+\varepsilon} -1 < N_1 \leq e^{-1} R^{\mu+\varepsilon}$.

We have

$$R^{\lambda-1} \sum_{n=1}^{N_1} n^{-\frac{n}{\mu+\varepsilon}} R^n \leq R^{n_1+\lambda-1} \sum_{n=1}^{\infty} n^{-\frac{n}{\mu+\varepsilon}} \leq K_2 R^{N_1+\lambda-1}$$

where K_2 is a constant. Also

$$R^{\lambda-1}\sum_{n=N_1+1}^{\infty} n^{-\frac{n}{\mu+\varepsilon}} R^n \leq R^{\lambda-1} .$$

This implies that $M(\Gamma_R) \leq K R^{N_1}$ for some constant K or

$$\frac{\log\log M(\Gamma_R)}{\log R} \leq \mu+\varepsilon - \frac{1}{\log R} + \frac{\log\log R}{\log R} + \frac{\log[1+\frac{\log K}{N_1 \log R}]}{\log R}$$

It follows from (4) that given $\eta>o$ there is a sequence $R\to\infty$ such that

$$\rho/\lambda - \eta \leq \frac{\log\log M(\Gamma_R)}{\log R} < \mu+\varepsilon + o(1).$$

This implies that $\rho/\lambda \leq \mu$ and concludes the proof.

With the aid of this Lemma we can now establish a sharper result concerning the nature of the coefficient polynomials $q_n(z)$.

LEMMA 5. f(z) is an entire function of order $\rho>o$ and type $o<\tau<\infty$ if and only if

$$(6) \qquad \overline{\lim_{n\to\infty}}\ n(||q_n(z)||_{\Gamma_\alpha})^{\frac{\rho}{\lambda n}} = \frac{e\rho\tau}{\lambda}$$

Proof: First assume that f(z) is entire of order $\rho>o$ and type $o<\tau<\infty$. Throughout this proof we use the notation

$$\mu = \overline{\lim_{n\to\infty}}\ n(||q_n(z)||_{\Gamma_\alpha})^{\frac{\rho}{\lambda n}}$$

and thus we are to show that $\mu=e\rho\tau/\lambda$. From Lemma 2 we have

$$M(\Gamma_R) \geq \frac{2\pi R^n ||q_n(z)||_{\Gamma_\alpha}}{||\Gamma_R|| \; ||Q(z)||_{\Gamma_R}} .$$

Let $\varepsilon>o$ be given, then for infinitely many values of n we have

(7) $$n(||q_n(z)||_{\Gamma_\alpha})^{\frac{\rho}{\lambda n}} \geq \mu-\varepsilon$$

and for these values of n

$$M(\Gamma_R) \geq \frac{2\pi(R^{\frac{1}{\lambda}}\frac{\mu-\varepsilon}{n})^{\frac{\lambda n}{\rho}}}{||\Gamma_R||\ ||Q(z)||_{\Gamma_R}}$$

Choose a corresponding sequence of values of R so that $R^{\frac{\rho}{\lambda}} = ne/(\mu-\varepsilon)$, then we have for these R

$$M(\Gamma_R) \geq \frac{2\pi e^{\frac{\lambda n}{\rho}}}{||\Gamma_R||\ ||Q(z)||_{\Gamma_R}}$$

and hence

$$\frac{\log M(\Gamma_R)}{R^{\rho/\lambda}} \geq \frac{\lambda n}{\rho} R^{-\frac{\rho}{\lambda}} + o(1) = \frac{\lambda(\mu-\varepsilon)}{\rho e} + o(1)$$

As R tends to infinity the limit on the left is the type τ by Lemma 3 and hence $\mu\leq\rho e\tau/\lambda+\varepsilon$. A similar argument can be carried through by noting that for n sufficiently large we have

(8) $$n(||q_n(z)||_{\Gamma_\alpha})^{\frac{\rho}{\lambda n}} \leq \mu+\varepsilon$$

and we obtain $\mu\geq\rho e\tau/\lambda-\varepsilon$. Since ε is arbitrary, this establishes the only if portion of the lemma.

To establish the if portion note that give $\varepsilon>0$ we have (8) for n sufficiently large and hence

$$\log \frac{1}{||q_n(z)||_{\Gamma_\alpha}} \geq \frac{\lambda n}{\rho} \log \frac{n}{\mu+\epsilon}$$

or

$$\frac{\rho}{\lambda} \geq \frac{n \log \frac{n}{\mu+\epsilon}}{\log \frac{1}{||q_n(z)||_{\Gamma_\alpha}}} = \frac{(n \log n)(1+o(1))}{\log \frac{1}{||q_n(z)||_{\Gamma_\alpha}}}$$

It follows from Lemma 4 that $f(z)$ is of order at most ρ. Similarly for infinitely many values of n we have (7) and hence

$$\frac{\rho}{\lambda} \geq \frac{(n \log n)(1+o(1))}{\log \frac{1}{||q_n(z)||_{\Gamma_\alpha}}}$$

It follows from Lemma 4 that the order of $f(z)$ is at least ρ and hence $f(z)$ is of order ρ.

To determine the type τ of $f(z)$ we have (8) for n sufficiently large. We may, in fact, assume that (8) holds for all n because we can add a polynomial to $f(z)$ without affecting its order or type. Then for $z\epsilon\Gamma_R$ we have

$$|f(z)| \leq \sum_{n=1}^{\infty} |q_n(z)| \; R^{n-1} \leq \sum_{n=1}^{\infty} ||q_n(z)||_{\Gamma_\alpha} R^{n+\lambda-2}$$

$$\leq R^{\lambda-2} \sum_{n=1}^{\infty} \left(\frac{\mu+\epsilon}{n}\right)^{\frac{\lambda n}{\rho}} R^n$$

The maximum term in this series occurs for $n_1 = (\mu+\epsilon)R^{\lambda/\rho}/e$. Choose N_1 so that $N_1 \leq n_1 < N_1+1$ and observe that

$$R^{\lambda-2} \sum_{n=1}^{N_1} \left(\frac{\mu+\varepsilon}{n}\right)^{\frac{\lambda n}{\rho}} R^n \le N_1 R^{\lambda-2} e^{\frac{\lambda}{\rho}[(\mu+\varepsilon)R^{\lambda/\rho}/e]}$$

$$R^{\lambda-2} \sum_{n=N_1+1}^{\infty} \left(\frac{\mu+\varepsilon}{n}\right)^{\frac{\lambda n}{\rho}} R^n \le R^{\lambda-2}$$

Thus

$$M(\Gamma_R) = ||f(z)||_{\Gamma_R} \le R^{\lambda-2}\left[(\mu+\varepsilon)R^{\lambda/\rho} \quad e^{\frac{\lambda}{\rho}[(\mu+\varepsilon)R^{\lambda/\rho}/e]-1} +1\right]$$

and it follows that

$$\frac{\log M(\Gamma_R)}{R^{\rho/\lambda}} \le \frac{\lambda(\mu+\varepsilon)}{\rho e} + o(1)$$

From Lemma 3 we have then $\rho e\tau/\lambda \le \mu+\varepsilon$,

For an infinite sequence of values of n (7) holds and for each such value we choose R so that $R^{\lambda/\rho} = en/(\mu-\varepsilon)$. Then from Lemma 2 we have for this R and n

$$\left(\frac{\mu-\varepsilon}{n}\right)^{\frac{\lambda n}{\rho}} \le ||q_n(z)||_{\Gamma_\alpha} \le \frac{||\Gamma_R|| \; M(\Gamma_R)}{2\pi R^n} \; ||Q(z)||_{\Gamma_R}$$

or

$$\log M(\Gamma_R) \ge \frac{\lambda n}{\rho} \log\left(\frac{\mu-\varepsilon}{e}R^{\rho/\lambda}\right) + \log \frac{2\pi}{||\Gamma_R|| \; ||Q(z)||_{\Gamma_R}}$$

$$= \frac{\lambda n}{\rho} + \log \frac{2\pi}{||\Gamma_R|| \; ||Q(z)||_{\Gamma_R}}$$

Thus

$$\frac{\log M(\Gamma_R)}{R^{\lambda/\rho}} \geq \frac{\lambda(\mu-\varepsilon)}{\rho e} + o(1)$$

and we have that $\tau \geq \lambda(\mu-\varepsilon)/(\rho e)$. Since ε is arbitrary in the inequalities

$$\mu-\varepsilon \leq \frac{\rho e\tau}{\lambda} \leq \mu+\varepsilon$$

we have established that $\mu = \rho e\tau/\lambda$ and the proof is complete.

The next lemma is a restatement of results established in [4, pages 68 through 76].

LEMMA 6. Let C be a closed limited point set whose complement is connected and regular. Then given $\varepsilon > o$ there is a polynomial $p_m(z)$ degree m so that the interior of the l

$$\Gamma_\alpha = \{z \,|\, |p_m(z)| = \alpha\}$$

contains C and the distance from Γ_α to C is less than ε. Furthermore

$$[d_\infty(C)]^m \leq \alpha \leq [d_\infty(C)(1+\varepsilon)]^m$$

where $d_\infty(C)$ is the transfinite diameter of the set C. If $||q_m(z)||_C \leq L$ then $||q_m(z)||_{C_R} \leq LR^m$ where m is the degree of the polynomial $q_m(z)$ and C_R is the level curve of the modulus of the mapping function associated with C.

The proof is not repeated here. The intuitive content of this lemma is that the boundary of C can be approximated to within ε by a lemniscate and, further this lemniscate is related to the transfinite diameter of C.

LEMMA 7. Suppose

$$f = \sum_{k=o}^{\infty} b_k$$

and

$$\left(\frac{c'}{n}\right)^{b'n} \leq |f-\sum_{k=o}^{n} b_k| \leq \left(\frac{c}{n}\right)^{bn}$$

where $b,b',c,c'>o$. *Then given* $\varepsilon>o$ *we have for n sufficiently large*

$$|b_{n+1}| \leq \left(\frac{c}{n}\right)^{bn}(1+\varepsilon)$$

and for an infinite sequence of values of n

$$|b_{n+1}| \geq \left(\frac{c'}{n}\right)^{b'n}(1-\varepsilon)$$

The proof of this lemma is a straightforward exercise in manipulating infinite series.

3. THE MAIN RESULT

THEOREM 1. *Let C be a closed bounded point set whose complement is connected and regular and let* $d_\infty(C)$ *be the transfinite diameter of C. Given f(z) defined on C, then for the sequence of polynomials* $P_n^*(z)$ *of degree n of best approximation to f(z) on C we have*

$$\text{(9)} \qquad \lim_{n\to\infty} n^{1/\rho}\,||f(z) - P_n^*(z)||_C^{1/n} = d_\infty(C)(\varepsilon\rho\tau)^{1/\rho}$$

if and only if f(z) is entire of order $\rho>o$ *and type* $o<\tau<\infty$.

That is to say that f(z) may be analytically extended to the whole complex plane so as to be entire of order ρ and type τ.

Proof. We establish the if part first. Let C_R denote the level curves of the modulus of the mapping function associated with the domain C.

Given $\varepsilon>o$ there is a lemniscate $\Gamma_\alpha = \{\,||p(z)| = \alpha\}$ so that

a) Γ_α is interior to $C_{1+\varepsilon}$ and exterior to C_I

b) $\alpha \leq [d_\infty(C)(1+\varepsilon)]^\lambda$

where λ is the degree of $p(z)$ and C_1 corresponds to the boundary of C. Since $||g(z)||_C \le ||g(z)||_{\Gamma_\alpha}$, we need only consider approximation on Γ_α. Let

$$S_n(f,z) = \sum_{k=1}^{n-1} q_k(z)\, p(z)^{k-1}$$

We have from Lemma 5 that

$$||f(z) - S_n(f,z)||_{\Gamma_\alpha} \le \sum_{k=n}^{\infty} ||q_k(z)||_{\Gamma_\alpha} \alpha^{k-1} \le \sum_{k=n}^{\infty} \left(\frac{\varepsilon\rho\tau}{\lambda k}\right)^{\frac{\lambda k}{\rho}} \alpha^{k-1}$$

$$\le \alpha^{n-1} \left(\frac{\varepsilon\rho\tau}{\lambda n}\right)^{\frac{\lambda n}{\rho}} \sum_{k=o}^{\infty} \left[\left(\frac{\varepsilon\rho\tau}{\lambda(k+n)}\right)^{\frac{\lambda k}{\rho}} \left(\frac{n}{n+k}\right)^{n}\right]$$

The ratio test shows that the infinite series on the right converges, and is $1+o(1)$ as n tends to infinity. Thus we have

$$||f(z) - S_n(f,z)||_{\Gamma_\alpha} \le \alpha^{n-1} \left(\frac{\varepsilon\rho\tau}{\lambda n}\right)^{\frac{\lambda n}{\rho}} \quad (1+o(1))$$

Note that $S_n(f,z)$ is a polynomial $P_m(z)$ of degree $m = \lambda n$ and that

$\alpha^n \le [d_\infty(C)]^{\lambda n}(1+\varepsilon)^{\lambda n}$ and hence

$$||f(z) \le P_m(z)||_{\Gamma_\alpha} \le [d_\infty(C)]^m \left(\frac{\varepsilon\rho\tau}{m}\right)^{\frac{m}{\rho}} (1+o(1))(1+\varepsilon)^m \alpha^{-1}$$

This implies that

$$\lim_{n\to\infty} m^{1/\rho} ||f(z)-P_m(z)||_{\Gamma_\alpha}^{1/n} \le d_\infty(C)(\varepsilon\rho\tau)^{\frac{1}{\rho}} (1+\varepsilon)$$

Observe that as ε tends to zero then the norm on Γ_α converges to that on C and the right side of this relation converges to the right side of (9).

In order to complete the first part of the proof, we must show that the limit in (9) is not a smaller constant than stated, say

(10) $$c = (\varepsilon\rho\tau[d_\infty(C)]^\rho)^{\frac{1}{\rho}}(1-\eta)$$

where $0<\eta\leq 1$. Suppose there is a sequence $\{P_m(z)\}$ of polynomials of degree m which achieves the constant (10). We have then

(11) $$f(z) = \sum_{k=o}^{\infty} (P_{k+1}(z)-P_k(z))$$

and

$$||f(z) - \sum_{k=o}^{m} (P_{k+1}(z)-P_k(z))||_C \leq (\frac{c}{m})^{\frac{m}{\rho}}$$

We apply Lemma 7 to see that this series converges absolutely and we may rewrite it as follows

$$\sum_{k=o}^{\lambda n-1} P_{k+1}(z) - P_k(z) = \sum_{k=1}^{n} q_{k,n}(z)p(z)^{k-1}$$

We see that, given $\varepsilon>o$, for n sufficiently large we have

$$||(q_{k,n}(z)-q_k(z)) \quad p(z)^{k-1}||_C \leq (\frac{c}{\lambda n})^{\frac{\lambda n}{\rho}} (1+\varepsilon)\alpha^{-1}$$

Consider then the inequality

(12) $$||f(z) - \sum_{k=1}^{n} q_k(z)p(z)^{k-1}||_C \leq ||f(z) - \sum_{k=o}^{\lambda n-1} (P_{k+1}(z)-P_k(z))||_C$$

$$+ ||\sum_{k=1}^{n} (q_{k,n}(z)-q_k(z))p(z)^{k-1}||_C$$

The right hand side is smaller than $[1+(n+1)(1+\varepsilon)\alpha^{-1}]\,(\frac{c}{\lambda n})^{\frac{\lambda n}{\rho}}$. The left hand side may be estimated from the fact that f(z) is entire of order ρ and type τ by means of Lemmas 5, 6 and 7.

From Lemma 5 we estimate the size of the terms $||q_n(z)||_{\Gamma_\alpha}$. We then choose λ sufficiently large that

$$||q_n(z)||_C \geq ||q_n(z)||_{\Gamma_\alpha} \quad (1-\varepsilon)^\lambda$$

(see Lemma page 77 of [4]). We use Lemma 6 to estimate p(z) similarly and obtain $||p(z)||_C \geq d_\infty(C)^\lambda(1-\varepsilon)^\lambda$ We now apply Lemma 7 to see that the left side of (12) is greater than

$$\left(\frac{\varepsilon\rho\tau}{\lambda n}\right)^{\frac{\lambda n}{\rho}} \left[(1-\varepsilon)^2\, d_\infty(C)\right]^{\lambda n}\alpha^{-1}$$

Thus we have, comparing these upper ad lower bounds,

$$(1-\varepsilon)^{2\lambda n}\alpha^{-1} \leq (1-\eta)^{\frac{\lambda n}{\rho}} \quad [1+(n+1)(1+\varepsilon)\ \alpha^{-1}]$$

Recall that η is fixed, but ε is arbitrary. Clearly we can choose ε so that $(1-\varepsilon)^2 > (1-\eta)^{\frac{1}{\rho}}$ and a contradiction is reached. This concludes the first part of the proof.

We now show that (9) implies that f(z) is entire of order ρ and type t. The analysis is similar to that starting at (11) above and we have, similar to (12),

$$||f(z) - \sum_{k=1}^{n} q_k(z)p(z)^{k-1}||_C \leq ||f(z) - \sum_{k=o}^{\lambda n-1} (P^*_{k+1}(z)-P^*_k(z))||_C$$

$$+ ||\sum_{k=1}^{n} (q_{k,n}(z)-q_k(z))p(z)^{k+1}||_C$$

Let $c = (\varepsilon\rho\tau\,[d_\infty(C)]^\rho)^{\frac{1}{\rho}}$ and by hypothesis the first term on the right is less than $\left(\frac{c}{\lambda n}\right)^{\frac{\lambda n}{\rho}}$. The estimate above shows that, for n and λ sufficiently large, we have, for given $\varepsilon > o$,

$$||(q_{n,k}(z)-q_k(z))p(z)^{k-1}||_{\Gamma_\alpha} \leq \left(\frac{c}{\lambda n}\right)^{\frac{\lambda n}{\rho}} (1+\varepsilon)\alpha^{-1}$$

1

and hence

$$||f(z) - \sum_{k=1}^{n} q_k(z)p(z)^{k-1}||_C \leq [1+(n+1)(1+\epsilon)\alpha^{-1}] \left(\frac{c}{\lambda n}\right)^{\frac{\lambda n}{\rho}} ;$$

It follows from Lemma 7 then, for k sufficnelty large,

$$||q_n(z)p(z)^{n-1}||_{\Gamma_\alpha} < \left(\frac{c}{\lambda n}\right)^{\frac{\lambda n}{\rho}} \frac{[1+(n+1)(1+\epsilon)\alpha^{-1}]}{(1-\epsilon)^{\lambda n+\lambda}}$$

We have from this

$$\overline{\lim_{n\to\infty}} \; n(||q_n(z)||_{\Gamma_\alpha})^{\frac{\rho}{\lambda n}} < \frac{\epsilon\rho\tau}{\lambda(1-\epsilon)^\rho}$$

Since η is arbitrary, it follows from Lemma 5 that f(z) is of order ρ', τ' where $\rho' \leq \rho$ and, if $\rho' = \rho$, $\tau \leq \tau$.

We see that if $\rho' < \rho$ we have directly from Lemma 4 and Lemma 7 that

$$||f(z) - \sum_{k=1}^{n} q_n(z)p(z)^{k-1}||_C = O[\left(\frac{c'}{\lambda n}\right)^{\frac{\lambda n}{\rho'}}]$$

and contradicts the definition of $P_m^k(z)$ which do not achieve this order of convergence. Thus we have $\rho' = \rho$. Now a direct application of Lemma 5 and Lemma 7 shows that if $\tau' < \tau$ we again contradict the definition of $P_m^*(z)$ as best approximations to f(z) on C. This concludes the proof.

4. EXTENSION TO RATIONAL APPROXIMATION OF FUNCTIONS WITH ESSENTIAL SINGULARITIES.

In this final section we point out that this analysis can be extended to cover rational approximation of functions with certain kinds of essential singularities. No attempt is made to obtain results as sharp as Theorem 1, we merely indicate the degree of convergence possible.

Let z_o be an isolated singularity of f(z), set $S_\epsilon = \{z | |z-z_o|=\epsilon\}$ and $M(\epsilon) = ||f(z)||_{S_\epsilon}$. The following definition generalizes the concept of order of an entire function.

DEFINITION. The order of $f(z)$ at z_0 is

$$\rho = \overline{\lim_{\varepsilon \to 0}} \frac{\log \log M(\varepsilon)}{\log \varepsilon}$$

An analysis similar to parts of the proof of Theorem 1 leads to

THEOREM 2. Assume $f(z)$ has a finite number of singularities of order ρ or less, $\rho > 0$, none of which lie in C. C is as in Theorem 1. Then there exists a sequence of rational functions $R_n(z)$ of total degree n and a constant A so that

$$||f(z) - R_n(z)||_C^{1/n} \leq An^{-1/\rho}.$$

5. REMARKS ON APPLICATIONS. The most common application occurs when C is [a,b], a case already considered in [2]. We note that the transfinite diameter of an interval is one quarter of its length, i.e. $(b-a)/4$. We may use the above results to compare the best approximations with the Taylor's series expansions. Assume $f(z)$ is entire of order ρ and let

$$E_{T,n} = ||f(z) - \sum_{k=0}^{n} P_{T,n}(z)||_{[a,b]}$$

$$E_n^* = ||f(z) - \sum_{k=0}^{n} P_n^*(z)||_{[a,b]}$$

where $P_{T,n}(z)$ is the term of degree n in the taylor's series expansion about $(a+b)/2$. We have then

COROLLARY

$$\lim_{n \to \infty} \left[\frac{E_{T,m}}{E_n^*} \right]^{\frac{1}{n}} = 2$$

Thus the best approximations do better than the Taylor's series expansion by a factor of the order of 2^{-n}. This difference is exhibited for even very small values of n for the common entire functions e.g. sin(x), e^x.

We see for approximations on disks that power series expansions give the same degree of approximation as the best approximations.

The next most interesting region for applications is the rectangle. Theorem 1 implies that one may obtain the best degree of convergence by simple expansion in terms of a polynomial which defines a lemniscate which approximate well the rectangle. Thus a practical procedure would seem to be to obtain such a lemniscate and associated polynomial and then obtain the expansion in terms of this polynomial by telescoping the Taylor's series expansion. A little reflection shows that this approach is equally applicable to the approximation of analytic functions in general.

Finally, we note that Theorem 2 implies that there are rational approximations of total degree n which approximate $\cosh(\frac{1}{z})$ on the interval [1,2] with degree of approximation of order $(\frac{e}{4n})^n$.

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