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# THE DEGREE OF PIECEWISE MONOTONE INTERPOLATION

ELI PASSOW AND LOUIS RAYMON

ABSTRACT. Let $0 = x_0 < x_1 < \cdots < x_k = 1$ and let $y_0, y_1, \cdots, y_k$ be real numbers such that $y_{j-1} \neq y_j$, $j = 1, 2, \cdots, k$. Estimates are obtained on the degree of an algebraic polynomial $p(x)$ that interpolates the given data piecewise monotonely; i.e., such that (i) $p(x_j) = y_j$, $j = 0, 1, \cdots, k$, and such that (ii) $p(x)$ is increasing on $I_j = (x_{j-1}, x_j)$ if $y_j < y_{j-1}$, and decreasing on $I_j$ if $y_j < y_{j-1}$, $j = 1, 2, \cdots, k$. The problem is seen to be related to the problem of monotone approximation.

Let $0 = x_0 < x_1 < \cdots < x_k = 1$ and let $y_0, y_1, \cdots, y_k$ be real numbers such that $y_{j-1} \neq y_j$, $j = 1, 2, \cdots, k$. It is a result of Wolibner [7], Kammerer [2], and Young [8] that there exists an algebraic polynomial $p(x)$ such that:

(i) $p(x_j) = y_j$, $j = 0, 1, \cdots, k$, and

(ii) $p(x)$ is increasing on $I_j = (x_{j-1}, x_j)$ if $y_j > y_{j-1}$, and decreasing on $I_j$ if $y_j < y_{j-1}$, $j = 1, 2, \cdots, k$.

A polynomial $p(x)$ with properties (i) and (ii) is said to interpolate *piecewise monotonely*; in case $y_j > y_{j-1}$ for all $j$ (or if $y_j < y_{j-1}$ for all $j$), $p(x)$ is simply said to interpolate *monotonely*. The smallest degree of a polynomial that interpolates the values $Y = \{y_0, y_1, \cdots, y_k\}$ at the points $X = \{x_0, x_1, \cdots, x_k\}$ (piecewise) monotonely is called the *degree of (piecewise) monotone interpolation of Y with respect to X*, and is denoted by $N = N(X; Y)$. Rubinstein has obtained estimates for the degree of monotone interpolation for the special case $k = 2$ [6]. We seek general estimates on $N(X; Y)$. Let

$$\Delta = \Delta(Y) = \min_{1<j\le k} |y_j - y_{j-1}|, \quad \text{and} \quad M = M(X; Y) = \max_{1\le j\le k} \left| \frac{y_j - y_{j-1}}{x_j - x_{j-1}} \right|.$$

The estimates on $N(X; Y)$ are found (not too surprisingly) to be related

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to the degree of monotone and comonotone approximation. Let $f \in C[0, 1]$, with a finite number of relative extrema $0 = x_0 < x_1 < x_2 < \cdots < x_{j+1} = 1$ (such a function is called a *piecewise monotone function*). The relative extrema $x_1, x_2, \cdots, x_j$ are called the *peaks* of $f$. The *degree of comonotone approximation* of $f$ by algebraic polynomials of degree $\leq n$ is defined by

$$E_n^*(f) = \inf\{\|f - p\| : p \in P(n; f)\} \quad \text{(sup norm on } [0, 1]),$$

where $P(n; f)$ is the set of algebraic polynomials of degree $\leq n$, monotone on each of the subintervals $(x_{i-1}, x_i)$, $i = 1, 2, \cdots, j+1$, with the same monotonicity as $f$ on these intervals. If $f$ is monotone on $[0, 1]$, then $E_n^*(f)$ is called the *degree of monotone approximation* of $f$.

Let $S_j$, $j = 0, 1, 2, \cdots$, be the set of all piecewise monotone functions $f$ with $j$ peaks such that

$$\sup_{0 \leq x, y \leq 1} \left| \frac{f(x) - f(y)}{x - y} \right| \leq 1. \tag{1}$$

The *degree of comonotone approximation to* $S_j$ is defined by

$$E_n^*(S_j) = \sup\{E_n^*(f) : f \in S_j\}.$$

It is known that $\lim_{n \to \infty} E_n^*(S_j) = 0$ [3], [4]. The smallest degree $n$ such that $E_n^*(S_j) < \delta$ will be denoted by $n_j(\delta)$.

With given data $X$ and $Y$ we associate a piecewise linear function $L(x) = L(X; Y; x)$ defined by $L(x_i) = y_i$, $i = 0, 1, \cdots, k$. Let $j$ be the number of peaks of $L$. We state our main results:

**Theorem 1.** $N(X; Y) \leq n_j(\Delta/12M)$.

**Theorem 2.** *If* $y_0 < y_1 < \cdots < y_k$, *then there exists an absolute constant* $A$ *such that* $N(X; Y) \leq AM/\Delta$.

We will first prove Theorem 1. Theorem 2 will follow from Theorem 1 and estimates on the degree of monotone approximation. The proof is based on an idea of Kammerer [2, Theorem 4.1], later used by Ford and Roulier [1].

**Proof of Theorem 1.** Let $\epsilon = \frac{1}{4}\Delta$, and let $S$ be the set of $2^{k+1}$ piecewise linear functions $f$ such that $f(x_i) = y_i + \epsilon$ or $y_i - \epsilon$, $i = 0, 1, \cdots, k$. We enumerate the functions in $S$ and denote the $i$th function in $S$ by $f_i$. Note that our choice of $\epsilon$ guarantees that each $f_i$ is comonotone with $L(x)$ and that $|f_i(x) - f_i(y)| \leq 3M|x - y|/2$ for all $x, y \in [0, 1]$. Thus $2f_i/3M \in S_j$, so that there exists $p_i \in P[n_j(\Delta/12M); L]$ such that $\|2f_i/3M - p_i\| < \Delta/12M$. Let $q_i = 3Mp_i/2$. Then $q_i \in P[n_j(\Delta/12M); L]$ and $\|f_i - q_i\| < (3M/2)(\Delta/12M) = \Delta/8 = \epsilon/2$.

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The vector $(y_0, y_1, \cdots, y_k)$ is thus contained in the convex hull of the vectors $(q_i(x_0), q_i(x_1), \cdots, q_i(x_k))$, $i = 1, 2, \cdots, 2^{k+1}$, and therefore there exists a convex linear combination of the $q_i$'s which will give rise to a polynomial $p$ which interpolates $Y$ piecewise monotonely. Since the degree of each $q_i \leq n_j(\Delta/12M)$, we have $N(X; Y) \leq n_j(\Delta/12M)$.

**Lemma** (Lorentz and Zeller [3, Theorem 2]). *There exists a constant $c$ such that $E_n^*(S_0) \leq c/n$.*

**Proof of Theorem 2.** From the Lemma we obtain $n_0(\Delta/12M) \leq 12c(M/\Delta)$. It follows from this and Theorem 1 that $N(X; Y) \leq AM/\Delta$.

While Theorem 2 may fail to give the exact value of $N(X; Y)$ for a particular configuration (e.g., if all the points lie on a straight line), it is best possible in a classwide sense, as we shall show in Theorem 3.

Let $G(M; \Delta)$ be the set of all $(X; Y)$ such that $M(X; Y) \leq M$ and $\Delta(Y) \geq \Delta$. Let $N(G) = \sup\{N(X; Y): (X; Y) \in G\}$.

**Theorem 3.** *Let $y_0 < y_1 < \cdots < y_k$. Then there exist constants $c_1$, $c_2 > 0$ such that*

$$c_1 M/\Delta \leq N(G) \leq c_2 M/\Delta. \tag{2}$$

**Proof.** For each $(X; Y) \in G$, $N(X; Y) \leq AM(X; Y)/\Delta(Y)$, by Theorem 2. Thus the upper bound in (2) holds with $c_2 = A$.

For the lower bound, note that Theorem 2 was proved using estimates on $E_n^*(S_0)$ and that this theorem, in turn, may be used to give estimates on $E_n^*(S_0)$. Indeed, if $f \in C[0, 1]$ is monotone and satisfies (1), we may choose $X = \{0 = x_0, x_1, \cdots, x_k = 1\}$ such that

$$1/n \leq f(x_i) - f(x_{i-1}) \leq 2/n, \qquad i = 1, 2, \cdots, k. \tag{3}$$

Let $Y = \{f(x_0), f(x_1), \cdots, f(x_k)\}$. Then $\Delta(Y) \geq 1/n$ and $M(X; Y) \leq 1$ (by (1)), so that, by Theorem 2, there exists a polynomial $p \in P(An; f)$ such that $p(x_i) = f(x_i)$, $i = 0, 1, \cdots, k$. Since $p$ is a monotone interpolation of $Y$ and $f$ satisfies (3), we have $\|f - p\| \leq 2/n$. Thus $E_{An}^*(f) \leq 2/n$, so that $E_n^*(S_0) \leq A_1/n$, which is the result of Lorentz and Zeller contained in the Lemma. If $N(G) = o(M/\Delta)$, it would follow that there exists a sequence of polynomials $q_n \in P(n; f)$ which would satisfy $\|f - q_n\| = o(1/n)$. Since the result of Lorentz and Zeller is essentially unimprovable, this is impossible. Thus, there exists $c_1 > 0$ such that $N(G) \geq c_1 M/\Delta$.

**Remark.** It follows from [4] and [5] that there exists $c_3 > 0$ such that

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$$n_j(\Delta/12M) \leq \min_{P \geq j+2} [c_3 P^2 2^P M/\Delta]^{(P+1)/(P-j-1)}. \tag{4}$$

From this result we can obtain an estimate on $N(X; Y)$ in the general case of $j$ peaks. We have, in particular, that for any $\epsilon > 0$ there exists a constant $A_{j,\epsilon}$ such that

$$N(X; Y) \leq A_{j,\epsilon}(M/\Delta)^{1+\epsilon}. \tag{5}$$

We believe, however, that this estimate is not the best possible one, since (4) is probably short of best possible. In fact, the proof of the lower bound (2) in Theorem 3 can be modified to show that in the general case of $j$ peaks, $N(G) \geq c_4 M/\Delta$ for some constant $c_4$. We therefore conjecture that there exists $d_j$ such that $N(X; Y) \leq d_j M/\Delta$, where $j$ is the number of peaks of $L(X; Y; x)$. It may even be possible to replace $d_j$ by an absolute constant $d$.

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DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19122

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