Tôhoku Math. J.
46 (1994), 13–26

THE DEGREE OF THE BEST APPROXIMATION IN BANACH SPACES

TOSHIHIKO NISHISHIRAHO

(Received August 10, 1992, revised September 25, 1992)

Abstract. Direct theorems of Jackson type on estimating the degree of the best approximation in Banach spaces are obtained by means of the moduli of continuity of higher orders of elements having certain smoothness properties.

1. Introduction. Let $C_{2\pi}$ denote the Banach space of all 2π-periodic, continuous functions f defined on the real line $\boldsymbol{R}$ with the norm

$$\|f\|_{\infty}=\max\{|f(t)|:|t|\leq\pi\}.$$

Let $\boldsymbol{N}$ be the set of all natural numbers, and put $\boldsymbol{N}_0=\boldsymbol{N}\cup\{0\}$. For each $n\in\boldsymbol{N}_0$, we denote by $\mathscr{T}_n$ the set of all trigonometric polynomials of degree at most n. For a given $f\in C_{2\pi}$, we define

$$E_n(C_{2\pi};f)=\inf\{\|f-g\|_{\infty}:g\in\mathscr{T}_n\},$$

which is called the best approximation of degree n to f with respect to $\mathscr{T}_n$.

The Weierstrass approximation theorem simply states that $E_n(C_{2\pi};f)$ converges to zero as n tends to infinity for all $f\in C_{2\pi}$. It does not say how fast $E_n(C_{2\pi};f)$ tends to zero. The following fundamental direct estimates due to Jackson (cf. [9]) assert that $E_n(C_{2\pi};f)$ approaches zero much faster when f is smooth: For all $f\in C_{2\pi}$ and all $n\in\boldsymbol{N}$,

$$E_n(C_{2\pi};f)\leq K\omega(C_{2\pi};f,1/n),$$

where K is a positive constant independent of f and n, and

$$\omega(C_{2\pi};f,\delta)=\sup\{\|f(\cdot-t)-f(\cdot)\|_{\infty}:|t|\leq\delta\}\qquad(\delta\geq 0)$$

denotes the modulus of continuity of f. If $f\in C_{2\pi}$ has a continuous r-th derivative $f^{(r)}$ for some $r\in\boldsymbol{N}$, then for all $n\in\boldsymbol{N}$

$$E_n(C_{2\pi};f)\leq K_r n^{-r}\omega(C_{2\pi};f^{(r)},1/n),$$

where K_r is a positive constant depending only on r.

Similar estimates also hold for the Banach space $L^p_{2\pi}$ consisting of all 2π-periodic, p-th power Lebesgue integrable functions f on $\boldsymbol{R}$ with the norm

---

1991 *Mathematics Subject Classification*. Primary 41A65.

Partly supported by the Grants-in-Aid for Scientific as well as Co-operative Research, The Ministry of Education, Science and Culture, Japan.

$$\|f\|_p=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^p\,dt\right)^{1/p}\qquad(1\le p<\infty)$$

using the integral modulus of continuity (see, e.g., [1], [4], [21]).

The purpose of this paper is to extend these results to arbitrary Banach spaces, and in particular, homogeneous Banach spaces (cf. [10], [15], [19]) which include $C_{2\pi}$ and $L^p_{2\pi}$, $1\le p<\infty$, as particular cases. For this purpose, we consider the following setting:

Let X be a complex Banach space with norm $\|\cdot\|_X$, and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Let $\boldsymbol{Z}$ denote the set of all integers, and let $\{P_j\}_{j\in\boldsymbol{Z}}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

(P-1) The projections P_j, $j\in\boldsymbol{Z}$, are mutually orthogonal, i.e., $P_jP_n=\delta_{j,n}P_n$ for all j, $n\in\boldsymbol{Z}$, where $\delta_{j,n}$ denotes Kronecker's symbol.

(P-2) $\{P_j\}_{j\in\boldsymbol{Z}}$ is fundamental, i.e., the linear span of $\bigcup_{j\in\boldsymbol{Z}}P_j(X)$ is dense in X.

(P-3) $\{P_j\}_{j\in\boldsymbol{Z}}$ is total, i.e., if $f\in X$ and $P_j(f)=0$ for all $j\in\boldsymbol{Z}$, then $f=0$.

For each $n\in\boldsymbol{N}_0$, let M_n be the linear span of $\{P_j(X);\ |j|\le n\}$. Note that M_n is a closed linear subspace of X. For a given $f\in X$, we define

$$E_n(X;f)=\inf\{\|f-g\|_X:\ g\in M_n\},$$

which is called the best approximation of degree n to f with respect to M_n. Obviously,

$$E_0(X;f)\ge E_1(X;f)\ge\cdots\ge E_n(X;f)\ge\cdots\ge 0,$$

and Condition (P-2) implies that

$$\lim_{n\to\infty}E_n(X;f)=0\qquad\text{for every}\quad f\in X.$$

In this paper, we relate the rapidity with which $E_n(X;f)$ approaches zero to certain smoothness properties of f, which can be described in terms of its moduli of continuity of higher orders with respect to a strongly continuous group of multiplier operators on X associated with Fourier series expansions corresponding to $\{P_j\}$.

2. Moduli of continuity. Let $\{T_t: t\in\boldsymbol{R}\}$ be a uniformly bounded strongly continuous group of operators in $B[X]$, i.e., a family of operators in $B[X]$ satisfying the following conditions:

(T-1) $A=\sup\{\|T_t\|_{B[X]}: t\in\boldsymbol{R}\}<\infty$.

(T-2) $T_0=I$ (I=identity operator).

(T-3) $T_{s+t}=T_sT_t$ for all s, $t\in\boldsymbol{R}$.

(T-4) For each $f\in X$, the mapping $t\mapsto T_t(f)$ is strongly continuous on R, i.e., $\lim_{t\to u}\|T_t(f)-T_u(f)\|_X=0$ for all $u\in\boldsymbol{R}$.

We define

$$G(f)=\lim_{t\to 0}\frac{T_t(f)-f}{t}, \tag{1}$$

whenever the limit exists in the sense of strong convergence, and let $D(G)$ denote the set of all $f\in X$ for which the strong limit in (1) exists. Evidently, $D(G)$ is a linear subspace of X and G is a linear operator of $D(G)$ into X. This operator G is called the infinitesimal generator of the group $\{T_t\}$. For $r=0, 1, 2, \ldots$, the operator G^r is inductively defined by the relations

$$G^0=I, \quad G^1=G,$$

$$D(G^r)=\{f: f\in D(G^{r-1}), \quad G^{r-1}(f)\in D(G)\}$$

and

$$G^r(f)=G(G^{r-1}(f)) \quad (f\in D(G^r), r=1, 2, 3, \ldots).$$

Then for each $r\in \boldsymbol{N}$, $D(G^r)$ is a dense linear subspace of X and G^r is a closed linear operator with domain $D(G^r)$ (cf. [3, Propositions 1.1.4 and 1.1.6]). For further extensive list of properties of semigroups of operators on Banach spaces, we refer to [3], [6], [7] and [8].

For each $r\in \boldsymbol{N}_0$ and $t\in \boldsymbol{R}$, we define

$$\Delta_t^0=I, \quad \Delta_t^r=(T_t-I)^r=\sum_{m=0}^{r}(-1)^{r-m}\binom{r}{m}T_{mt} \qquad (r\geq 1),$$

which stands for the r-th iteration of T_t-I. Clearly, Δ_t^r belongs to $B[X]$ and

$$\|\Delta_t^r\|_{B[X]}\leq A_r,$$

where

$$A_r=\min\{(A+1)^r, 2^r A\}.$$

If $r\in \boldsymbol{N}_0$, $f\in X$ and $\delta\geq 0$, then we define

$$\omega_r(X; f, \delta)=\sup\{\|\Delta_t^r(f)\|_X: |t|\leq\delta\},$$

which is called the r-th modulus of continuity of f with respect to the family $\{T_t\}$. This quantity has the following properties:

LEMMA 1. *Let $r\in \boldsymbol{N}$ and $f\in X$.*

(a) $$\omega_r(X; f, \delta)\leq A_r\|f\|_X$$

for all $\delta\geq 0$.

(b) *$\omega_r(X; f, \cdot)$ is a non-decreasing function on $[0, \infty)$ and $\omega_r(X; f, 0)=0$.*

(c) $$\omega_{r+s}(X; f, \delta)\leq A_r\omega_s(X; f, \delta)$$

for all $s \in N_0$ *and all* $\delta \geq 0$. *In particular, we have*

$$\lim_{\delta \to +0} \omega_r(X; f, \delta) = 0 .$$

(d) $$\omega_r(X; f, \xi\delta) \leq A(1+\xi)^r \omega_r(X; f, \delta)$$

for all $\xi, \delta \geq 0$.

(e) *If* $0 < \delta \leq \xi$, *then*

$$\omega_r(X; f, \xi)/\xi^r \leq 2^r A \omega_r(X; f, \delta)/\delta^r .$$

(f) *If* $f \in D(G^r)$, *then*

$$\omega_{r+s}(X; f, \delta) \leq A\delta^r \omega_s(X; G^r(f), \delta)$$

for all $s \in N_0$ *and all* $\delta \geq 0$.

PROOF. Statements (a) and (b) are obvious. (c) follows from the semigroup property of Δ_t^r and (a). (d) is well-known if Δ_t^r is fefined by the translation group, and the present case is proved similarly. Since

$$\Delta_t^r(f) = \int_0^t \int_0^t \cdots \int_0^t T_{(u_1+u_2+\cdots+u_r)}(G^r(f))du_1 du_2 \cdots du_r$$

(cf. [3, Proposition 1.1.6]), we have

$$\Delta_t^{r+s}(f) = \int_0^t \int_0^t \cdots \int_0^t T_{(u_1+u_2+\cdots+u_r)}\Delta_t^s(G^r(f))du_1 du_2 \cdots du_r ,$$

and so

$$\|\Delta_t^{r+s}(f)\|_X \leq A|t|^r \|\Delta_t^s(G^r(f))\|_X ,$$

which gives (f). q.e.d.

For $r \in N$ and $\alpha > 0$, an element $f \in X$ is said to satisfy an r-th Lipschitz condition of order α with constant $M, M > 0$, or to belong to the class $\mathrm{Lip}_r(X; \alpha, M)$ if $\omega_r(X; f, \delta) \leq M\delta^\alpha$ for all $\delta \geq 0$. Also, for $r \in N$ and $\alpha > 0$ the class $\mathrm{Lip}_r(X; \alpha)$ consists of all $f \in \mathrm{Lip}_r(X; \alpha, M)$ for some constant $M > 0$. Note that $D(G^r) \subset \mathrm{Lip}_r(X; r)$ for each $r \in N$ and that if $\alpha > r$, then $f \in \mathrm{Lip}_r(X; \alpha)$ if and only if $\omega_r(X; f, \delta) = o(\delta^r)$ as $\delta \to +0$.

3. Multiplier operators and convolution operators. For any $f \in X$, we associate its (formal) Fourier series expansion (with respect to $\{P_j\}$)

(2) $$f \sim \sum_{j=-\infty}^{\infty} P_j(f) .$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence

$\{\tau_j\}_{j\in \boldsymbol{Z}}$ of complex numbers such that for every $f\in X$,

$$T(f)\sim \sum_{j=-\infty}^{\infty} \tau_j P_j(f)\,,$$

and the following notation is used:

$$T\sim \sum_{j=-\infty}^{\infty} \tau_j P_j \tag{3}$$

(cf. [5], [15], [16], [22]).

REMARK 1. The expansion (2) is a generalization of the concept of Fourier series in a Banach space X with respect to a fundamental, total, biorthogonal system $\{f_j, f_j^*\}_{j\in \boldsymbol{Z}}$. Here $\{f_j\}_{j\in \boldsymbol{Z}}$ and $\{f_j^*\}_{j\in \boldsymbol{Z}}$ are sequences of elements in X and X^* (the dual space of X), respectively such that the linear span of $\{f_j\colon j\in \boldsymbol{Z}\}$ is dense in X (fundamental), $f_j^*(f)=0$ for all $j\in \boldsymbol{Z}$ implies $f=0$ (total), and $f_j^*(f_n)=\delta_{j,n}$ for all j, $n\in \boldsymbol{Z}$ (biorthogonal). Then (2) reads

$$f\sim \sum_{j=-\infty}^{\infty} f_j^*(f)f_j$$

(cf. [2], [13], [20]).

Let $M[X]$ denote the set of all multiplier operators on X, which is a commutative closed subalgebra of $B[X]$ containing the identity operator I. Let $\{T_t\colon t\in \boldsymbol{R}\}$ be a family of operators in $M[X]$ satisfying Condition (T-1) and having the expansions

$$T_t\sim \sum_{j=-\infty}^{\infty} \exp(\lambda_j t)P_j \quad (t\in \boldsymbol{R})\,, \tag{4}$$

where $\{\lambda_j\}_{j\in \boldsymbol{Z}}$ is a sequence of complex numbers. Then $\{T_t\colon t\in \boldsymbol{R}\}$ becomes a strongly continuous group of operators in $B[X]$ and there holds

$$G^r(f)\sim \sum_{j=-\infty}^{\infty} \lambda_j^r P_j(f) \qquad (f\in D(G^r))$$

(cf. [15, Proposition 2]). Let φ: $\boldsymbol{R}\to\boldsymbol{R}$ be a continuous function. If k is a function in $L^1_{2\pi}$ having the Fourier series expansion

$$k(t)\sim \sum_{j=-\infty}^{\infty} k^\wedge(j)\exp(ijt)$$

with its Fourier coefficients

$$k^\wedge(j)=\frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)\exp(-ijt)dt \qquad (j\in \boldsymbol{Z})$$

and if $T\in B[X]$, then we define the convolution operator $(k*T)(\varphi;\,\cdot)$ by

$$(k*T)(\varphi; f)=\frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)T_{\varphi(t)}(T(f))dt \qquad (f\in X),$$

which exists as a Bochner integral (cf. [15]). Clearly, $(k*T)(\varphi; \cdot)$ belongs to $B[X]$ and

$$\|(k*T)(\varphi; \cdot)\|_{B[X]}\leq B\|k\|_1\|T\|_{B[X]},$$

where

$$B=\sup\{\|T_{\varphi(t)}\|_{B[X]}: |t|\leq\pi\}.$$

LEMMA 2. *Let $k\in L^1_{2\pi}$ and let T be an operator in $M[X]$ having the expansion* (3). *Then $(k*T)(\varphi; \cdot)$ belongs to $M[X]$ and*

$$(5) \qquad (k*T)(\varphi; \cdot)\sim\sum_{j=-\infty}^{\infty} c_j(\varphi; k)\tau_j P_j(\cdot),$$

where

$$c_j(\varphi; k)=\frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)\exp(\lambda_j\varphi(t))dt \qquad (j\in \boldsymbol{Z}).$$

PROOF. Let $j\in\boldsymbol{Z}$ and $f\in X$. Then we have

$$\begin{aligned}P_j((k*T)(\varphi; f))&=\frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)P_j(T_{\varphi(t)}(T(f)))dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)\exp(\lambda_j\varphi(t))P_j(T(f))dt\\&=c_j(\varphi; k)P_j(T(f))=c_j(\varphi; k)\tau_jP_j(f),\end{aligned}$$

which implies (5). q.e.d.

For each $n\in\boldsymbol{N}_0$, we set

$$\pi_n(\varphi)=\{k\in L^1_{2\pi}: c_j(\varphi; k)=0 \quad \text{whenever } |j|>n\},$$

which is a closed linear subspace of $L^1_{2\pi}$. For $a\in\boldsymbol{R}$, we define $\varphi_a(t)=at$, $t\in\boldsymbol{R}$, and put

$$(k*T)_a(\cdot)=(k*T)(\varphi_a; \cdot), \quad \pi_{n,a}=\pi_n(\varphi_a).$$

Let $r\in\boldsymbol{N}$, $k\in L^1_{2\pi}$ and consider the following linear combination of the convolution operators $(k*I)_j$, $1\leq j\leq r$,

$$L_{k,r}=\sum_{j=1}^{r}(-1)^{j+1}\binom{r}{j}(k*I)_j.$$

Then we have the following key estimate for the operator $L_{k,r}$.

LEMMA 3. *Let* $r \in \boldsymbol{N}$, $k \in L^1_{2\pi}$, $k^\wedge(0)=1$ *and* $f \in X$. *Then*

$$\|L_{k,r}(f)-f\|_X \le A\omega_r(X; f, \delta)\sum_{j=0}^{r}\binom{r}{j}\delta^{-j}\mu(k; j) \qquad (\delta>0),$$

where

$$\mu(k; j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}|t|^j|k(t)|dt$$

denotes the j-th absolute moment of k.

PROOF. Since

$$L_{k,r}(f)-f=\frac{(-1)^{r+1}}{2\pi}\int_{-\pi}^{\pi}k(t)\Delta_t^r(f)dt$$

and

$$\|\Delta_t^r(f)\|_X \le \omega_r(X; f, |t|) \le A\left(1+\frac{|t|}{\delta}\right)^r \omega_r(X; f, \delta)$$

by Lemma 1 (d), we have

$$\|L_{k,r}(f)-f\|_X \le A\omega_r(X; f, \delta)\sum_{j=0}^{r}\binom{r}{j}\delta^{-j}\frac{1}{2\pi}\int_{-\pi}^{\pi}|t|^j|k(t)|dt,$$

which implies the desired inequality. q.e.d.

4. Direct theorems. Recall that $\mathscr{T}_n$ is the set of all trigonometric polynomials of degree at most n. In this section we suppose that

$$\mathscr{T}_n \subset \bigcap_{m=1}^{\infty} \pi_{n,m} \qquad \text{for each} \quad n \in \boldsymbol{N}_0 . \tag{6}$$

REMARK 2. Let $\{\lambda_j\}_{j\in\boldsymbol{Z}}=\{-ij\}_{j\in\boldsymbol{Z}}$.

(a) $\mathscr{T}_n \subset \bigcap_{m\in\boldsymbol{Z}\setminus\{0\}}\pi_{n,m}$ for every $n \in \boldsymbol{N}_0$, and so (6) always holds.

(b) If $\varphi=\varphi_m$, $m \in \boldsymbol{Z}\setminus\{0\}$, then (5) reduces to

$$(k*T)_m \sim \sum_{j=-\infty}^{\infty} k^\wedge(jm)\tau_j P_j ,$$

and in particular if $k \in \mathscr{T}_n$, then

$$(k*T)_m = \sum_{|j|\le[n/|m|]} k^\wedge(jm)\tau_j P_j ,$$

where $[\lambda]$ denotes the largest integer not exceeding $\lambda \geq 0$.

Now we have the following general estimate:

THEOREM 1. *Let $r \in N$. Then for all $f \in X$ and all $n \in N_0$,*

$$E_n(X; f) \leq \inf\{\|L_{k,r}(f) - f\|_X : k \in \mathscr{T}_n^1\}$$
$$\leq A \inf\{\omega_r(X; f, \delta) \sum_{j=0}^{r} \binom{r}{j} \delta^{-j} \mu(k; j) : \delta > 0, k \in \mathscr{T}_n^1\},$$

where

$$\mathscr{T}_n^1 = \{k \in \mathscr{T}_n : k^\wedge(0) = 1\}.$$

PROOF. Let $k \in \mathscr{T}_n$ and $f \in X$. Then by Condition (6) and Lemma 2 we have

$$(k * I)_m(f) = \sum_{j=-n}^{n} c_j(\varphi_m; k) P_j(f)$$

for all $m \in N$, and so $L_{k,r}(f)$ belongs to M_n. Therefore, we are done by Lemma 3. q.e.d.

Here we consider the generalized Jackson kernel given by

$$J_{n,m}(t) = c_{n,m} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^{2m} \qquad (n, m \in N),$$

where the normalizing constant $c_{n,m} > 0$ is taken in such a way that

$$J_{n,m}^\wedge(0) = \frac{1}{\pi} \int_0^\pi J_{n,m}(t) dt = 1$$

(cf. [12]). Note that

$$J_{n,1}(t) = F_n(t) = \sum_{j=1-n}^{n-1} \left(1 - \frac{|j|}{n}\right) \exp(ijt)$$

is the Fejér kernel, and so $J_{n,m}(t) = c_{n,m} n^m F_n^m(t)$ is a non-negative, even trigonometric polynomial of degree $m(n-1)$. Also, we have

$$J_{n,2}(t) = \frac{3}{n(2n^2+1)} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^4,$$

which is the Jackson kernel (cf. [9], [14]).

LEMMA 4. *We have*

$$2^{-1}(2m-1)^{1/2m} \left(\frac{2}{\pi n} \right)^{2m-1} \leq c_{n,m} \leq 2^{-1} \pi \left(\frac{\pi}{2n} \right)^{2m-1}.$$

PROOF. By definition,

$$\frac{1}{c_{n,m}}=\frac{1}{\pi}\int_0^\pi\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^{2m}dt\,.$$

The lemma follows from the well-known inequality

$$(2/\pi)x\le\sin x\le x\qquad(0\le x\le\pi/2)\,. \tag{7}$$

q.e.d.

LEMMA 5. *We have*

$$J_{n,m}(t)\le\min\left\{\left(\frac{\pi}{2}\right)^{4m}n,\left(\frac{\pi^2}{2nt}\right)^{2m}n\right\}\qquad(0<|t|<\pi)\,.$$

PROOF. This follows from (7) and Lemma 4. q.e.d.

LEMMA 6. *We have*

$$\mu(J_{n,m};j)\le 2^{j+1}\left(\frac{\pi}{4}\right)^{4m}\frac{1}{j+1}\left(\frac{j+1}{2m-j-1}\right)^{(j+1)/2m}n^{-j}\,.$$

PROOF.

$$\mu(J_{n,m};j)=\frac{1}{\pi}\int_0^\pi t^jJ_{n,m}(t)dt=\frac{1}{\pi}\left(\int_0^{a/n}+\int_{a/n}^\pi\right)=\frac{1}{\pi}(\alpha+\beta)\,,$$

say. By Lemma 5 we have

$$\alpha\le\left(\frac{\pi}{2}\right)^{4m}n\int_0^{a/n}t^jdt=\left(\frac{\pi}{2}\right)^{4m}\frac{1}{j+1}a\left(\frac{a}{n}\right)^j$$

and

$$\beta\le\pi^{4m}2^{-2m}n^{1-2m}\int_{a/n}^\infty t^{j-2m}\,dt=\pi^{4m}2^{-2m}\frac{1}{2m-j-1}a^{-2m+1}\left(\frac{a}{n}\right)^j.$$

Choose a so that $a=2((j+1)/(2m-j-1))^{1/2m}$. q.e.d.

We are now in a position to establish the following Jackson-type result:

THEOREM 2. *Let $r\in N$. Then for all $f\in X$ and all $n\in N$,*

$$E_n(X;f)\le AC_r\omega_r(X;f,1/n)\,, \tag{8}$$

where $C_r=2^{-3}(\pi/2)^{2r+4}$.

PROOF. Let $m=[(r+3)/2]$ and $q=[n/m]+1$. Then $J_{q,m}$ belongs to $\mathscr{T}_n^1$. Therefore, by Theorem 1 we have

$$E_n(X; f) \le A\omega_r(X; f, 1/n) \sum_{j=0}^{r} \binom{r}{j} n^j \mu(J_{q,m}; j) .$$

By Lemma 6 we get our estimate. q.e.d.

COROLLARY 1. (a) *If* $f \in \mathrm{Lip}_r(X; \alpha, M)$ *for some* $r \in \boldsymbol{N}$, *then for all* $n \in \boldsymbol{N}$

$$E_n(X; f) \le AMC_r n^{-\alpha} .$$

(b) *If* $f \in D(G^r)$ *for some* $r \in \boldsymbol{N}$, *then for all* $n \in \boldsymbol{N}$

$$E_n(X; f) \le A^2 C_r \|G^r(f)\|_X n^{-r} . \tag{9}$$

(c) *If* $f \in \bigcap_{r=1}^{\infty} D(G^r)$, *then for every* $\lambda > 0$

$$\lim_{n \to \infty} n^{\lambda} E_n(X; f) = 0 .$$

The following result gives an improvement of the estimate (9) in terms of the moduli of continuity of higher orders.

THEOREM 3. *Let* $r \in \boldsymbol{N}$ *and* $f \in D(G^r)$. *Then for all* $n \in \boldsymbol{N}$ *and all* $s \in \boldsymbol{N}_0$,

$$E_n(X; f) \le A^2 C_{r+s} n^{-r} \omega_s(X; G^r(f), 1/n) .$$

PROOF. By Theorem 2 and Lemma 1 (f), we have

$$E_n(X; f) \le AC_{r+s}\omega_{r+s}(X; f, 1/n) \le A^2 C_{r+s} n^{-r} \omega_s(X; G^r(f), 1/n) .$$

q.e.d.

As an immediate consequence of Theorem 3 we have the following.

COROLLARY 2. *Let* $r \in \boldsymbol{N}$ *and* $f \in D(G^r)$. *If* $G^r(f)$ *belongs to* $\mathrm{Lip}_s(X; \alpha, M)$ *for some* $s \in \boldsymbol{N}$, *then for all* $n \in \boldsymbol{N}$

$$E_n(X; f) \le A^2 MC_{r+s} n^{-(\alpha+r)} .$$

For $r=1$ and $s=1, 2$ the above-mentioned results may be compared with our previous results in [17] and [18], in which we employed the Fejér-Korovkin kernel given by

$$k_n(t) = \Lambda_n \left| \sum_{j=0}^{n} \lambda_n(j) e^{ijt} \right|^2 \qquad (n \in \boldsymbol{N}_0,\ t \in \boldsymbol{R}) ,$$

where

$$\lambda_n(j) = \sin((j+1)\pi/(n+2)) \qquad (j = 0, 1, 2, \dots, n)$$

and

$$\Lambda_n = (\lambda_n^2(0) + \lambda_n^2(1) + \lambda_n^2(2) + \cdots + \lambda_n^2(n))^{-1}$$

(cf. [11]). Note that $k_n \in \mathscr{T}_n^1$ and

$$k_n(t) = 1 + 2 \sum_{m=1}^{n} \theta_n(m) \cos mt ,$$

where

$$\theta_n(m) = \Lambda_n \sum_{j=0}^{n-m} \lambda_n(j)\lambda_n(m+j) \qquad (m = 1, 2, \ldots, n)$$

with

$$\theta_n(1) = \cos(\pi/(n+2)) .$$

Theorem 2 also yields other results on the best approximation as well as the convergence of Fourier series (2):

THEOREM 4. *Let $\{U_n\}_{n \in N_0}$ be a sequence of operators in $B[X]$ satisfying $U_n(g) = g$ for every $g \in M_n$, and let $r \in N$. Then for all $f \in X$ and all $n \in N$,*

$$\begin{aligned} \| U_n(f) - f \|_X &\leq (\| U_n \|_{B[X]} + 1) E_n(X; f) \\ &\leq A C_r (\| U_n \|_{B[X]} + 1) \omega_r(X; f, 1/n) . \end{aligned}$$

PROOF. If g is an arbitrary element in M_n, then

$$\begin{aligned} \| U_n(f) - f \|_X &\leq \| U_n(f-g) \|_X + \| g - f \|_X \\ &\leq (\| U_n \|_{B[X]} + 1) \| f - g \|_X , \end{aligned}$$

which implies

$$\begin{aligned} \| U_n(f) - f \|_X &\leq (\| U_n \|_{[X]} + 1) E_n(X; f) \\ &\leq A C_r (\| U_n \|_{B[X]} + 1) \omega_r(X; f, 1/n) \end{aligned}$$

by virtue of (8). q.e.d.

COROLLARY 3. *Let $\{U_n\}_{n \in N_0}$ be as in Theorem 4, and let $f \in X$. If $\lim_{n \to \infty} \| U_n \|_{B[X]} E_n(X; f) = 0$, then*

(10) $$\lim_{n \to \infty} \| U_n(f) - f \|_X = 0 .$$

In particular, if $\lim_{n \to \infty} \| U_n \|_{B[X]} \omega_r(X; f, 1/n) = 0$ for some $r \in N$, then (10) *holds.*

Let $\{S_n\}_{n \in N_0}$ be the sequence of the n-th partial sum operators associated with the Fourier series (2), that is,

$$S_n = \sum_{j=-n}^{n} P_j \qquad (n \in N_0) .$$

Then by Theorem 4 we have the following Lebesgue-type estimate for the n-th partial sum operators S_n.

THEOREM 5. *Let $r\in \boldsymbol{N}$. Then for all $f\in X$ and all $n\in \boldsymbol{N}$,*

$$\begin{aligned}\|S_n(f)-f\|_X &\leq (\|S_n\|_{B[X]}+1)E_n(X; f)\\ &\leq AC_r(\|S_n\|_{B[X]}+1)\omega_r(X; f, 1/n)\,.\end{aligned}$$

COROLLARY 4. *If $\lim_{n\to\infty}\|S_n\|_{B[X]}E_n(X; f)=0$, then the Fourier series of f converges to f, i.e.,*

$$\left\|f-\sum_{j=-n}^{n} P_j(f)\right\|_X \to 0 \qquad as \quad n\to\infty\,. \tag{11}$$

In particular, if $\lim_{n\to\infty}\|S_n\|_{B[X]}\omega_r(X; f, 1/n)=0$ for some $r\in \boldsymbol{N}$, then (11) *holds.*

Let σ_n, $n\in \boldsymbol{N}_0$, be the n-th Cesàro mean operators, that is,

$$\sigma_n=(S_0+S_1+\cdots+S_n)/(n+1)=\sum_{j=-n}^{n}(1-|j|/(n+1))P_j\,,$$

and let V_n be the de la Vallée-Poussin operator

$$V_n=(S_n+S_{n+1}+\cdots+S_{2n-1})/n=2\sigma_{2n-1}-\sigma_{n-1} \qquad (n\in \boldsymbol{N})\,.$$

Suppose that $\{\sigma_n\}_{n\in \boldsymbol{N}_0}$ is uniformly bounded, i.e.,

$$C=\sup\{\|\sigma_n\|_{B[X]}: n\in \boldsymbol{N}_0\}<\infty\,.$$

Applying Theorem 4 to the case $U_n=V_n$, we derive the following de la Vallée-Poussin-type estimate:

THEOREM 6. *Let $r\in \boldsymbol{N}$. Then for all $f\in X$ and all $n\in \boldsymbol{N}$,*

$$\begin{aligned}E_{2n-1}(X; f)\leq \|V_n(f)-f\|_X &\leq (3C+1)E_n(X; f)\\ &\leq A(3C+1)C_r\omega_r(X; f, 1/n)\,.\end{aligned}$$

5. Applications to homogeneous Banach spaces. Here we restrict ourselves to the case where X is a homogeneous Banach space, i.e.,

(H-1) X is continuously embedded in $L^1_{2\pi}$, i.e., there exists a constant $M>0$ such that $\|f\|_1\leq M\|f\|_X$ for all $f\in X$.

(H-2) X is a Banach space with norm $\|\cdot\|_X$.

(H-3) The translation operator T_t defined by

$$T_t(f)(\cdot)=f(\cdot-t) \qquad (f\in X)\,,$$

is isometric on X for each $t\in \boldsymbol{R}$.

(H-4) For each $f\in X$, the mapping $t\mapsto T_t(f)$ is strongly continuous on $\boldsymbol{R}$.

Typical examples of homogeneous Banach spaces are $C_{2\pi}$ and $L^p_{2\pi}$, $1 \leq p < \infty$. For other examples see [15] (cf. [10], [19]).

Now we define the sequence $\{P_j\}_{j \in \boldsymbol{Z}}$ of projection operators in $B[X]$ by

$$P_j(f)(\cdot) = f^\wedge(j)\exp(ij\cdot) \qquad (f \in X),$$

which satisfies Conditions (P-1), (P-2) and (P-3) just as in Section 1 (cf. [10], [15]). Note that each T_t has the expansion (4) with $\lambda_j = -ij$, and so for $\varphi = \varphi_m$, $m \in \boldsymbol{Z}$, the expansion (5) reduces to

$$(k * T)_m \sim \sum_{j=-\infty}^{\infty} k^\wedge(jm)\tau_j P_j,$$

and

$$M_n = \mathscr{T}_n \subset \bigcap_{q \in \boldsymbol{Z} \setminus \{0\}} \pi_{n,q}$$

for each $n \in \boldsymbol{N}_0$ (cf. Remark 2). Furthermore, for $f \in X$ we have

$$\Delta_t^0(f) = f, \quad \Delta_t^r(f)(\cdot) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(\cdot - jt) \quad (r \geq 1).$$

Consequently, in the above setting all the results obtained in the preceding sections hold with $A = 1$. In particular, Theorems 2 and 5 and Corollary 4 for $r = 1$ and Corollary 1 (b) include Theorems 9.3.3.1 and 9.3.4.2 and Corollary 9.3.4.3 and Theorem 9.3.3.2 in [19], respectively.

References

[1] N. I. Achieser, Theory of Approximation, Frederick Ungar, New York, 1956.

[2] H. Buchwalter, Saturation dans un espace normé, C. R. Acad. Sci. Paris 250 (1960), 651–653.

[3] P. L. Butzer and H. Berens, Semi-Groups of Operators and Approximation, Springer-Verlag, Berlin/Heidelberg/New York, 1967.

[4] P. L. Butzer and R. J. Nessel, Fourier Analysis and Approximation, Vol. I, Academic Press, New York, 1971.

[5] P. L. Butzer, R. J. Nessel and W. Trebels, On summation processes of Fourier expansions in Banach spaces. I. Comparison theorems, Tôhoku Math. J. 24 (1972), 127–140; II. Saturation theorems, ibid., 551–569; III. Jackson- and Zamansky-type inequalities for Abel-bounded expansions, ibid. 27 (1975), 213–223.

[6] E. B. Davies, One-Parameter Semigroups, Academic Press, New York, 1980.

[7] J. A. Goldstein, Semigroups of Linear Operators and Applications, Oxford Univ. Press, New York, 1985.

[8] E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloq. Publ., Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

[9] D. Jackson, The Theory of Approximation, Amer. Math. Soc. Colloq. Publ., Vol. 11, Amer. Math. Soc., New York, 1930.

[10] Y. KATZNELSON, An Introduction to Harmonic Analysis, John Wiley, New York, 1968.
[11] P. P. KOROVKIN, Linear Operators and Approximation Theory, Hindustan Pubi. Corp., Delhi, 1960.
[12] G. G. LORENTZ, Approximation of Functions, 2nd. ed., Chelsea, New York, 1986.
[13] V. D. MILMAN, Geometric theory of Banach spaces I; The theory of bases and minimal systems, Russian Math. Surveys 25 (1970), 111–170.
[14] I. P. NATANSON, Constructive Function Theory, Vol. I; Uniform Approximation, Frederick Ungar, New York, 1964.
[15] T. NISHISHIRAHO, Quantitative theorems on linear approximation processes of convolution operators in Banach spaces, Tôhoku Math. J. 33 (1981), 109–126.
[16] T. NISHISHIRAHO, Saturation of multiplier operators in Banach spaces, Tôhoku Math. J. 34 (1982), 23–42.
[17] T. NISHISHIRAHO, Direct theorems for best approximation in Banach spaces, Approximation, Optimization and Computing (IMACS, 1990; A. G. Law and C. L. Wang, eds.), pp. 155–158, North-Holland, Amsterdam, 1990.
[18] T. NISHISHIRAHO, The order of best approximation in Banach spaces, Proc. 13-th Symp. Appl. Funct. Analysis (H. Umegaki and W. Takahashi, eds.), pp. 90–104, Tokyo Inst. Technology, Tokyo, 1991.
[19] H. S. SHAPIRO, Topics in Approximation Theory, Lecture Notes in Math. 187, Springer-Verlag, Berlin/Heidelberg/New York, 1971.
[20] I. SINGER, Bases in Banach Spaces I, Springer-Verlag, Berlin/Heidelberg/New York, 1970.
[21] A. F. TIMAN, Theory of Approximation of Functions of a Real Variable, Macmillan, New York, 1963.
[22] W. TREBELS, Multipliers for (C, α)-Bounded Fourier Expansions in Banach Spaces and Approximation Theory, Lecture Notes in Math. 329, Springer-Verlag, Berlin/Heidelberg/New York, 1973.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE RYUKYUS
NISHIHARA-CHO, OKINAWA, 903-01
JAPAN